Some Elementary Properties of Hardy-Littlewood Homogeneous Spaces

by

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Introduction

The notion of Hardy-Littlewood varieties was first introduced by Borovoi and Rudnick ([B-R]) for counting integer points in affine algebraic varieties defined over \( \mathbb{Q} \). Let \( W \) be an affine space and \( X \) be a closed subvariety of \( W \). We choose an open subset \( B \) in the adele space \( X(\mathbb{A}) \) whose infinite component \( B_\infty \) is a connected component of \( X(\mathbb{R}) \) and finite component \( B_f \) is open compact in \( X(\mathbb{A}_f) \). Then an important problem of number theory is to count or estimate the number of rational points in \( B \). Define the counting function

\[
N(T, X ; B) = \sharp (X(\mathbb{Q}) \cap (B_\infty^T \times B_f)),
\]

where \( B_\infty^T \) is the set of \( x \in B_\infty \) whose Euclidean norm is less than or equal to a positive real number \( T \). A variety \( X \) is called a Hardy-Littlewood variety if there exists a locally constant non-negative function \( \delta \) on \( X(\mathbb{A}) \) such that, for any \( B \) as above, one has

\[
N(T, X ; B) \sim \int_{B_\infty^T \times B_f} \delta(x) d\omega_\mathcal{X}(x) \quad (T \to \infty).
\]

Here \( \omega_\mathcal{X} \) denotes the Tamagawa measure associated with a gauge form on \( X \). A typical example of such varieties is the quadric \( \{ x : Q(x) = a \} \) defined from an indefinite quadratic form \( Q \) in \( n \) variables, \( n \geq 4 \), and \( a \in \mathbb{Z} - \{0\} \). In fact, Borovoi and Rudnick proved that many affine homogeneous spaces are Hardy-Littlewood. For instance, based on the work of [D-R-S] or [E-M-S], an affine symmetric space \( X = G/H \) is Hardy-Littlewood if \( G \) is a semisimple and simply connected group having the strong approximation property and \( H \) has no nontrivial \( \mathbb{Q} \)-rational characters. Several interesting examples were also given in [B-R].

In this paper, we give some general properties satisfied by Hardy-Littlewood homogeneous spaces. Our tool is a mean value theorem in adele geometry (cf. [O], [M], [M-W]). Morishita showed that the uniformity holds for some wide class of

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homogeneous spaces. It is stated as the following integral formula;
\[ \int_{G(\mathbb{A})X(\mathbb{Q})} f(x)\omega^\mathbb{Q}_1(x) = \frac{\tau(G, X)}{\tau(G)} \int_{G(\mathbb{A})/G(\mathbb{Q})} \sum_{z \in X(\mathbb{Q})} f(gz)\omega^\mathbb{Q}_1(g) \]
for \( X = G/H \) and any compactly supported continuous function \( f \) on \( G(\mathbb{A})X(\mathbb{Q}) \), where \( G \) is a connected algebraic group defined over \( \mathbb{Q} \), \( H \) a closed \( \mathbb{Q} \)-subgroup of \( G \) and it is assumed that both \( G \) and \( H \) have nontrivial \( \mathbb{Q} \)-rational characters. Constants \( \tau(G) \) and \( \tau(G, X) \) denote Tamagawa numbers of \( G \) and \( X \), respectively. Applying this formula to Hardy-Littlewood homogeneous spaces, we will prove the following theorem.

**Theorem.** If \( X = G/H \) is a Hardy-Littlewood variety with density function \( \delta \), then one has
\[ \frac{1}{\tau(G)} \int_{G(\mathbb{A})/G(\mathbb{Q})} \delta(g^{-1}x)\omega^\mathbb{Q}_1(g) = \begin{cases} \tau(G, X)^{-1} & (x \in G(\mathbb{A})X(\mathbb{Q})) \\ 0 & (x \notin G(\mathbb{A})X(\mathbb{Q})) \end{cases} \]

When \( G \) has the strong approximation property, Borovoi and Rudnick gave a formula of the density function \( \delta \) under some conditions, (see Theorem 2). The same formula of \( \delta \) follows immediately from the above theorem.

**Corollary.** Let \( X = G/H \) be as above. Assume that \( G \) has the strong approximation property. Then
\[ \delta(x) = \begin{cases} |\pi_1(H)_{\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}| & (x \in G(\mathbb{A})X(\mathbb{Q})) \\ 0 & (x \notin G(\mathbb{A})X(\mathbb{Q})) \end{cases} \]
where \( \pi_1(H)_{\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} \) denotes the coinvariant quotient of Borovoi's fundamental group of \( H \) under \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \)-action.

Furthermore, we obtain a geometric sufficient condition for the strongly Hardy-Littlewood property.

**Corollary.** Let \( X = G/H \) be as above. Assume that \( G \) has the strong approximation property and the second homotopy group of the complex manifold \( X(\mathbb{C}) \) vanishes. Then \( \delta \) is identically equal to 1 on \( X(\mathbb{A}) \), i.e. \( X \) is strongly Hardy-Littlewood.

In Section 2, we will exactly give a definition of Hardy-Littlewood varieties and will recall the results of Borovoi and Rudnick. The results stated above will be proved in Section 3.

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**Notations.** Let \( k \) be an algebraic number field and \( \mathcal{V} \) the set of all places of \( k \). We write \( \mathcal{V}^\infty \) and \( \mathcal{V}_f \) for the sets of all infinite places and all finite places, respectively. For \( v \), \( k_v \) stands for the completion of \( k \) at \( v \). If \( v \) is a finite place, \( \mathcal{O}_v \) denotes the ring of integers in \( k_v \). The ring of adeles and the ring of finite adeles of \( k \) are denoted by \( \mathbb{A} \) and \( \mathbb{A}_f \), respectively. We set \( k_\infty = \prod_{v \in \mathcal{V}^\infty} k_v \). If \( S \) is a finite set of \( \mathcal{V} \) containing
\( \mathcal{V}_\infty \), then \( \mathcal{O}(S) \) denotes the ring of \( S \)-integers in \( k \).

Let \( X \) be a non-singular algebraic variety defined over \( k \). For any \( k \)-algebra \( R \), \( X(R) \) denotes the set of \( R \)-rational points of \( X \). The set \( X(\mathbb{A}) \) of adele points of \( X \) is a locally compact topological space. A real-valued function \( \delta \) on \( X(\mathbb{A}) \) is said to be locally constant if the restriction of \( \delta \) to \( B_\infty \times B_f \) is a constant for any topological connected component \( B_\infty \) of \( X(k_\infty) \) and any sufficiently small open compact subset \( B_f \) in \( X(\mathbb{A}_f) \). By a gauge form on \( X \), we mean a nowhere zero regular differential form of degree \( \dim X \). If \( X \) has a gauge form \( \omega_X \), then one associates a measure \( \omega_{X(k_v)} \) on \( X(k_v) \) for any place \( v \in \mathcal{V} \). Since \( X \) is defined over \( \mathcal{O}(S) \) for some finite set \( S \) of \( \mathcal{V} \) containing \( \mathcal{V}_\infty \), one can consider the set \( X(\mathcal{O}(v)) \) of \( \mathcal{O}(v) \)-rational points and its volume \( \omega_{X(k_v)}(X(\mathcal{O}(v))) \) for any \( v \notin S \). If the infinite product \( \prod_{v \notin S} \omega_{X(k_v)}(X(\mathcal{O}(v))) \) converges absolutely, then we define the Tamagawa measure on \( X(\mathbb{A}) \) by

\[
\omega^X_{\mathbb{A}} = |\Delta_k|^{-\dim X/2} \prod_{v \in \mathcal{V}} \omega_{X(k_v)},
\]

where \( \Delta_k \) is the discriminant of \( k \). When \( \prod_{v \notin S} \omega_{X(k_v)}(X(\mathcal{O}(v))) \) does not converge absolutely, we choose a family \( \{\lambda_v\} \) of convergence factors for \( X \) and define the Tamagawa measure derived from \( \{\lambda_v\} \) by

\[
\omega^X_{\mathbb{A}} = |\Delta_k|^{-\dim X/2} \prod_{v \in \mathcal{V}} \lambda_v^{-1} \omega_{X(k_v)}
\]

(cf. [We, 2.3]).

Let \( G \) be a connected affine algebraic group defined over \( k \). We denote by \( G^u \) the unipotent radical of \( G \) and by \( G^{red} \) the quotient group \( G/G^u \). In this paper, we will identify \( G^{red} \) with a Levi subgroup of \( G \), so that \( G = G^u G^{red} \) is a Levi-Chevalley decomposition of \( G \). If \( G^{red} \) is semisimple and simply connected, then \( G \) is said to be simply connected. Let \( X(G) \) and \( X_l(G) \) be the free \( \mathbb{Z} \)-modules consisting of all rational characters and all \( k \)-rational characters of \( G \), respectively. The absolute Galois group \( \text{Gal}(\bar{k}/k) \) acts on \( X(G) \). The representation of \( \text{Gal}(\bar{k}/k) \) in the space \( X(G) \otimes \mathbb{Q} \) is denoted by \( \rho_G \) and the corresponding Artin \( L \)-function is denoted by \( L(s, \rho_G) = \prod_{v \in \mathcal{V}} L_v(s, \rho_G) \). Borovoi defined the algebraic fundamental group \( \pi_1(G) \) of \( G \). (See [B], [M] for its definition.) As an abstract group, \( \pi_1(G) \) is canonically isomorphic to the topological fundamental group of the complex Lie group \( G(\mathbb{C}) \).

The difference between these two fundamental groups is that \( \pi_1(G) \) has an additional \( \text{Gal}(\bar{k}/k) \)-module structure. That is, \( \pi_1(\cdot) \) defines an exact functor from the category of connected affine \( k \)-groups to the category of \( \text{Gal}(\bar{k}/k) \)-modules generated finitely over \( \mathbb{Z} \). We denote by \( \pi_1(G)_{\text{Gal}(\bar{k}/k)_{tor}} \) the torsion part of the coinvariant quotient of \( \pi_1(G) \) under \( \text{Gal}(\bar{k}/k) \) action. Notice that if \( X_l(G) = 0 \), then \( \pi_1(G)_{\text{Gal}(\bar{k}/k)} \) is finite, i.e. \( (\pi_1(G)_{\text{Gal}(\bar{k}/k)})_{\text{tor}} = \pi_1(G)_{\text{Gal}(\bar{k}/k)} \) (cf. [M, Theorem 3.4]).

Let \( G \) be a unimodular connected affine algebraic group defined over \( k \). Then \( G \) admits an invariant gauge from \( \omega_G \). There is a finite set \( S \) of \( \mathcal{V} \) containing \( \mathcal{V}_\infty \) so that \( G \) is defined over \( \mathcal{O}(S) \). It is known by [O] that the product \( \prod_{v \notin S} \omega_{G(k_v)}(G(\mathcal{O}(v))) \) converges absolutely if and only if \( X(G) = 0 \). If \( X(G) \neq 0 \), we take (and fix) a family
of convergence factors for \( G \) as \( \{ L_e(1, \rho_G)^{-1} \} \) and normalize the Tamagawa measure on \( G(\mathbb{A}) \) as follows.

\[
\omega_G^G = \prod_{v \in \mathcal{V}_\infty} \omega_{G(\mathbb{Q}_v)},
\]

(0.1)

\[
\omega_f^G = |\Delta_k|^{-\dim G/2} r_G^{-1} \prod_{v \in \mathcal{V}_f} L_e(1, \rho_G)^{-1} \omega_{G(\mathbb{Q}_v)},
\]

\[
\omega_A^G = \omega_G^G \omega_f^G.
\]

Here the constant \( r_G \) is defined to be

\[
r_G = \lim_{s \to 1} (s-1)^{\text{rank} X_0(G)} L(s, \rho_G).
\]

For non-negative non-decreasing functions \( f(t) \) and \( g(t) \) defined on the set of positive real numbers, the notation \( f(t) \sim g(t) \ (t \to \infty) \) means that \( f(t) \) is identically zero if \( g(t) \) is identically zero, otherwise \( \lim_{t \to \infty} f(t)/g(t) = 1 \).

1. Special homogeneous spaces

In this section, we define \( k \)-special homogeneous spaces. The terminology "special homogeneous space" was first introduced by Ono in [O].

**DEFINITION.** An affine algebraic group \( G \) is said to be \( k \)-special if \( G \) is defined over \( k \), connected and \( X_k(G) = 0 \).

The condition \( X_k(G) = 0 \) is equivalent to that \( G^{\text{red}} \) has no nontrivial central \( k \)-split torus. If \( G \) is \( k \)-special, then \( G \) is unimodular, i.e. \( G \) admits an invariant gauge form \( \omega_G \). The associated Tamagawa measure \( \omega_A^G \) on \( G(\mathbb{A}) \) is normalized as in (0.1).

**DEFINITION.** An algebraic variety \( X \) is called a \( k \)-special homogeneous space if \( X \) satisfies the following two conditions;

(S1) There exist a \( k \)-special group \( G \) and a \( k \)-special closed subgroup \( H \) of \( G \) such that \( X \) is \( k \)-isomorphic to the homogeneous space \( G/H \).

(S2) \( X(k) \) is nonempty.

We denote by \( \mathcal{S}_k \) the set of all \( k \)-special homogeneous spaces.

Notice that if we assume the condition (S1), then the condition (S2) is equivalent to that \( X(k_v) \) is nonempty for all \( v \in \mathcal{V} \) and the Brauer-Manin obstruction to the Hasse principle for \( X \) is trivial ([B2, Corollary 2.5]).

**DEFINITION.** Let \( X \) be a \( k \)-special homogeneous space and \( x_0 \in X(k) \) be a base point. A pair \( (G, H) \) consisting of a \( k \)-special algebraic group \( G \) and its \( k \)-special closed subgroup \( H \) is called a realization of \( (X, x_0) \) if there is a \( k \)-isomorphism \( f \) from \( G/H \) onto \( X \) such that \( f(H) = x_0 \). We denote by \( \mathcal{R}_k(X, x_0) \) the set of all realizations of \( (X, x_0) \).

If \( (G, H) \in \mathcal{R}_k(X, x_0) \), then \( X \) has an action of \( G \) through an isomorphism
$f: G/H \to X$. If no confusion arise, we omit $f$ and the action of $g \in G$ to $x \in X$ will be simply written as $gx$. The next proposition follows from [R, Theorem 3 and Lemma 2].

**Proposition 1.** A $k$-special homogeneous space $X$ is a quasi-affine variety defined over $k$. For every $(G, H) \in \mathcal{R}(X, x_0)$, there exists a $k$-embedding $\iota$ of $X$ into an affine space $W$ (onto a closed subset of an affine space $W$ if $X$ is affine) on which $G$ operates $k$-linearly.

If $x \in S_k$ and $(G, H) \in \mathcal{R}(X, x_0)$, then $X$ admits the canonical gauge form $\omega_X$ so that the matching $\omega_G = \omega_X \omega_H$ holds. We normalize the Tamagawa measure on $X(\mathbb{A})$ as follows.

$$\omega_{\infty}^X = \prod_{v \in V_{\infty}} \omega_{X(k_v)} ,$$

$$\omega_f^X = |\Delta_k|^{\dim X/2} \prod_{v \in V_f} \frac{L_v(1, \rho_G)}{L_v(1, \rho_H)} \omega_{X(k_v)} ,$$

$$\omega_A^X = \omega_{\infty}^X \omega_f^X .$$

Then the Tamagawa measures $\omega_X^A$, $\omega_f^A$, and $\omega_A^H$ match together topologically (cf. [B-R, Lemma 1.6.5]). We note that the Tamagawa measure $\omega_A^X$ is independent of the choice of a realization of $X$ (cf. [B-R, Remark 1.6.6]).

### 2. Hardy-Littlewood varieties

We give a definition of Hardy-Littlewood varieties. Let $X$ be a non-singular quasi-affine algebraic variety defined over $k$. We assume that $X$ has a gauge form $\omega_X$ and the Tamagawa measure $\omega_A^X$ derived some family of convergence factors for $X$. We consider a triple $(X, \iota, W)$ consisting of above $X$, a $k$-affine space $W$ and a $k$-embedding $\iota : X \to W$. The set $W(k_{\infty})$ is regarded as a finite dimensional real vector space. Let $\Omega$ be any $o$-symmetric bounded convex body in $W(k_{\infty})$ which contains the origin $o$ as an inner point. By an $o$-symmetric set, we mean a set which is symmetric with respect to the origin. Notice that there is an one to one correspondence between the set of norms of $W(k_{\infty})$ and the set of $o$-symmetric bounded convex body having $o$ as an inner point (cf. [L, p. 7, Theorem 4]). For a positive real number $T$, we set

$$\Omega(T) = \{ T \omega : \omega \in \Omega \} .$$

Let $B_\infty$ be a topological connected component of $X(k_{\infty})$ and $B_f$ an open compact subset of $X(\mathbb{A}_f)$. Then $B = B_\infty \times B_f$ is an open subset of $X(\mathbb{A})$. We set

$$B_{\infty}(T) = \{ x \in B_\infty : \iota(x) \in \Omega(T) \} , \quad B_f(T) = B_\infty(T) \times B_f .$$

The counting function $N(\Omega(T), X ; B)$ is defined to be

$$N(\Omega(T), X ; B) = \sharp(X(k) \cap B(T)) .$$
For \((X, \mathfrak{i}, W)\), we assume the following condition;

(HL0) The volume \(\omega_x^X(i^{-1}(\Omega(T)))\) is finite for any \(T > 0\) and any \(\Omega\) as above. This condition is clearly satisfied if \(i(X)\) is Zariski closed in \(W\).

**DEFINITION.** A triple \((X, \mathfrak{i}, W)\) is called relatively Hardy-Littlewood with respect to \(\omega_x\) if there exists a non-negative function \(\delta : X(\mathbb{A}) \to \mathbb{R}\) which satisfies the following two conditions:

(HL1) \(\delta\) is not identically zero and is locally constant.

(HL2) For any \(\Omega\), any \(B_o\) and any \(B_f\) as above, one has

\[
N(\Omega(T), X ; B) \sim \int_{B_{o}(T)} \delta(x) d\omega_x^X(x) \quad (T \to \infty).
\]

Furthermore, in addition to the above conditions, if \(\delta\) is identically 1 on \(X(\mathbb{A})\), then \((X, \mathfrak{i}, W)\) is called strongly Hardy-Littlewood.

It is easy to see that the non-negative function \(\delta\) is uniquely determined by the conditions (HL1) and (HL2).

If the context is clear, we often omit the notations \(i, W, \omega_x\), and we will say that \(X\) is a relatively Hardy-Littlewood variety with density function \(\delta\), or more simply, \((X, \delta)\) is relatively Hardy-Littlewood.

We note that the condition (HL2) is a little stronger than the corresponding condition of Definition 2.3 in [B-R]. In [B-R], a norm of the vector space \(W(k_\infty)\) was fixed at first. Namely, an only one \(o\)-symmetric bounded convex body \(\Omega_o\) having \(o\) as an inner point was fixed, once and for all. Therefore, it seems that the definition of Hardy-Littlewood varieties in [B-R] depends on the choice of \(\Omega_o\). In order to avoid this dependence, we adopt the condition (HL2). Although our definition of Hardy-Littlewood varieties is slightly different from that in [B-R], the results in [B-R] still remain true.

**THEOREM 1 ([B-R, Propositions 2.4–2.7]).** Let \((X, \delta)\) be relatively Hardy-Littlewood. Then \(X\) satisfies the following;

1. There is a finite set \(S\) of \(\mathcal{V}\) containing \(\mathcal{V}_\infty\) such that \(X\) is defined over \(\mathcal{O}(S)\) and \(X(\mathcal{O}(S))\) is dense in \(\prod_{v \in S} X(\mathcal{O}_v)\).
2. \(X\) is geometrically simply connected, i.e. \(\pi_1(X(\mathbb{C})) = 1\). In particular, if \(X\) is a quasi-affine homogeneous space of a connected group \(G\), then the stabilizer \(H\) of a base point \(x_0 \in X\) must be connected.
3. A gauge form on \(X\) is unique up to scalar factors. Hence the Tamagawa measure \(\omega_x^X\) does not depend on the choice of a gauge form \(\omega_x\) if a family of convergence factors for \(X\) is fixed.

Furthermore, if \((X, \delta)\) is strongly Hardy-Littlewood, then \(X\) has the strong approximation property, i.e. \(X(k)\) is dense in \(X(\mathbb{A}_f)\).

The following is the main theorem of [B-R].

**THEOREM 2 ([B-R, Theorems 5.3, 5.4]).** Let \(X\) be a \(k\)-special homogeneous space. Assume that \((X, \mathfrak{i}, W)\) satisfies the following three conditions.
(i) There exists a realization \((G, H) \in \mathcal{R}(X, x_0)\) such that \(G\) is a semisimple group having the strong approximation property and \(H\) is reductive. (So that \(X\) is an affine variety.)

(ii) \(G\) operates \(k\)-linearly on the affine space \(W\) and \(i(X)\) is a closed \(G\)-orbit in \(W\). (Such \(W\) exists at least one by Proposition 1.)

(iii) The following asymptotic count holds; For any \(\Omega\) as above, any arithmetic subgroup \(\Gamma\) of \(G(k_\infty)\) and any \(x \in X(k)\), one has

\[
\#(\Gamma \cap i^{-1}(\Omega(T))) \sim \frac{\omega^h_x(H_x(k_\infty)/\Gamma \cap H_x(k_\infty))}{\omega^x_\infty(G(k_\infty)/\Gamma)} \frac{\omega^x_\infty(G(k_\infty) \cap i^{-1}(\Omega(T)))}{\omega^x_\infty(G(k_\infty)/\Gamma)} \quad (T \to \infty),
\]

where \(H_x\) denotes the stabilizer of \(x\) in \(G\).

Then \((X, i, W)\) is relatively Hardy-Littlewood, with density function

\[
\delta(x) = \begin{cases} 
|\pi_1(H)_{\text{Gal}(\bar{k}/k)}| & (x \in G(\mathbb{A})X(k)) \\
0 & (x \notin G(\mathbb{A})X(k)).
\end{cases}
\]

In addition, if \(\pi_1(H)_{\text{Gal}(\bar{k}/k)} = 1\), then \((X, i, W)\) is strongly Hardy-Littlewood.

To prove this theorem, Borovoi and Rudnick used the finiteness theorem for the orbits of arithmetic subgroups ([B-H, Theorem 6.9]). The assumption (ii) is required for that reason. [B-R, Corollary 5.5] also remains true and all examples studied in [B-R] are still Hardy-Littlewood under our definition.

3. Some properties of Hardy-Littlewood special homogeneous spaces

Throughout this section, we assume that \(X\) is a \(k\)-special homogeneous space and \((X, i, W)\) is relatively Hardy-Littlewood with density function \(\delta\). We certainly assume (HL0) holds for \((X, i, W)\), but we need not assume that \(G\) acts on \(W\) for a realization \((G, H) \in \mathcal{R}(X, x_0)\).

First, we give general properties satisfied by \((X, \delta)\). In the following, we denote by \(\Omega_w\) the set of all \(\alpha\)-symmetric bounded convex body in \(W(k_\infty)\) which contains \(\alpha\) as an inner point.

**Lemma 1.** Let \((G, H) \in \mathcal{R}(X, x_0)\) and \(g \in G(\mathbb{A})\). For any \(B \subset X(\mathbb{A})\) as in Section 2 and any \(\Omega \in \Omega_w\), define a counting function by

\[
N(\Omega(T), X; B; g) = \#(gX(k) \cap B^{\Omega(T)}), \quad (T > 0).
\]

Then one has

\[
N(\Omega(T), X; B; g) \sim \int_{B^{\Omega(T)}} \delta(g^{-1}x) d\omega_x^\infty(x) \quad (T \to \infty).
\]

**Proof.** This is obvious by the condition (HL2).

**Lemma 2.** Let \((G, H) \in \mathcal{R}(X, x_0)\). Then one has
\[ \delta(gx) = \delta(x) \]

for all \( g \in G(k) \) and \( x \in X(\mathbb{A}) \).

**Proof.** If we set \( \delta_1(x) = \delta(gx) \), then both \( \delta \) and \( \delta_1 \) satisfy the conditions (HL1) and (HL2). From uniqueness, it follows \( \delta = \delta_1 \). \( \square \)

**Lemma 3.** Let \((G, H) \in \mathfrak{R}(X, x_0)\) and \(G(k) \times G(k)\) be the topological identity connected component of \(G(k)\). Then one has

\[ \delta((g_\infty, g_f)x) = \delta(1, g_f)x \]

for all \((g_\infty, g_f) \in G(k) \times G(\mathbb{A}_f)\) and \(x \in X(\mathbb{A})\).

**Proof.** Let \( x = (x_\infty, x_f) \in X(k) \times X(\mathbb{A}_f)\). Since the orbit \(G(k) \times x_\infty\) is connected in \(X(k)\), there is the connected component \(B_\infty\) of \(X(k)\) which contains \(G(k) \times x_\infty\). By the condition (HL1), the restriction of \(\delta\) to \(B_\infty \times B_f\) identically equals \(\delta(1, g_f)x\). \( \square \)

**Lemma 4.** For any \((G, H) \in \mathfrak{R}(X, x_0)\), one has

\[ \operatorname{supp} \delta \subseteq G(\mathbb{A})X(k) \]

**Proof.** For \( y = (y_\infty, y_f) \in X(\mathbb{A}_f) = X(k_\infty) \times X(\mathbb{A}_f)\), the orbit \(G(\mathbb{A})y\) is open and closed in \(X(\mathbb{A})\). We take the connected component \(B_\infty\) of \(X(k)\) containing \(y_\infty\) and a sufficiently small open compact neighbourhood \(B_f\) of \(y_f\) such that \(B_\infty \times B_f\) is contained in \(G(\mathbb{A})y\) and \(\delta\) is identically constant on \(B_\infty \times B_f\). Then, for any \(\Omega \in \Omega_X\), we have

\[ N(\Omega(T), X : B) \sim \int_{B \times T} \delta(x)d\omega_A^X(x) = \omega_A^X(B^{\Omega(T)})\delta(y) \quad (T \to \infty). \]

If \(\delta(y) \neq 0\), then \(B\), and hence \(G(\mathbb{A})y\), contains \(k\)-rational points. Therefore \(y\) is contained in \(G(\mathbb{A})X(k)\). \( \square \)

**Lemma 5.** Let \((G, H) \in \mathfrak{R}(X, x_0)\). Assume that \(G\) has the strong approximation property. Then, for any \(x \in X(k)\), the restriction of \(\delta\) to \(G(\mathbb{A})x\) is identically equal to \(\delta(x)\). Furthermore, in this case, one has \(\operatorname{supp} \delta = G(\mathbb{A})X(k)\).

**Proof.** Let \(G = G^aG^\text{red}\) be a Levi-Chevalley decomposition of \(G\). Since \(G\) has the strong approximation property, \(G^\text{red}\) must be semisimple and simply connected (cf. [P-R, 7.4]). Then \(G(k)\) is topologically connected (cf. Corollaire (4.7) in [B-T]). By Lemma 3, we have

\[ \delta((g_\infty, g_f)x) = \delta(1, g_f)x \]

for any \((g_\infty, g_f) \in G(\mathbb{A})\). Since \(\delta\) is locally constant, there is an open compact subgroup \(K_x\) of \(G(\mathbb{A}_f)\) such that \(\delta((1, g_f)x) = \delta(x)\) for all \(g_f \in K_x\). Combining this with Lemma 2, we have \(\delta(gx) = \delta(x)\) for all \(g \in G(k)G(k)K_x\). The strong approximation property implies \(G(\mathbb{A}) = G(k)G(k_x)K_x\). The second assertion is obvious by the condition (HL2). \( \square \)
Next, we investigate a relation of the Hardy-Littlewood property and the uniformity of $X$ (cf. [M], [M-W]).

**Lemma 6.** For any $(G, H) \in \mathcal{A}_k(X, x_0)$ and any $x \in X(\mathbb{A})$, the integral

$$
\int_{G(\mathbb{A})/G(k)} \delta(g^{-1}x) d\omega^G_\mathbb{A}(g)
$$

converges.

**Proof.** Let $D = D_\infty \times D_f$ be a fundamental set of $G(\mathbb{A})/G(k)$ such that $D_\infty$ is open in $G(k_\infty)$ and $D_f$ is compact in $G(k_f)$ (cf. [P-R, Theorem 4.17, Proposition 5.9]). It follows from $X_k(G) = 0$ that the volume of $D_\infty$ is finite. Let $h$ be the index of $G(k_\infty)^0$ in $G(k_\infty)$ and $\{g_1, \cdots, g_h\}$ be a complete set of representatives for $G(k_\infty)/G(k_\infty)^0$. Note that $h$ is finite by a theorem of Whitney ([Wh, Theorem 3]). We set $D^i_\infty = D_\infty \cap g_i G(k_\infty)^0$ for each $i$. Then, by Lemma 3, we have

$$
\int_{G(\mathbb{A})/G(k)} \delta(g^{-1}x) d\omega^G_\mathbb{A}(g) \leq \int_D \delta(g^{-1}x) d\omega^G_\mathbb{A}(g)
$$

$$
= \sum_{i=1}^h \omega^G_\mathbb{A}(D^i_\infty) \int_{D^i_f} \delta(g_i^{-1}g_f^{-1}x) d\omega^G_\mathbb{A}(g_f) < + \infty \quad \square
$$

For each $(G, H) \in \mathcal{A}_k(X, x_0)$, we set

$$
F_{(G, H)}(x) = \frac{1}{\tau(G)} \int_{G(\mathbb{A})/G(k)} \delta(g^{-1}x) d\omega^G_\mathbb{A}(g), \quad (X \in X(\mathbb{A})).
$$

Here $\tau(G)$ stands for the Tamagawa number of $G$. The main theorem of this paper is the following.

**Theorem 3.** For any $(G, H) \in \mathcal{A}_k(X, x_0)$, one has

$$
F_{(G, H)}(x) = \begin{cases} \tau(G, X)^{-1} & (x \in G(\mathbb{A})X(k)) \\ 0 & (x \notin G(\mathbb{A})X(k)) \end{cases},
$$

where $\tau(G, X)$ denotes the Tamagawa number of $X$.

**Proof.** We fix $(G, H) \in \mathcal{A}_k(X, x_0)$ and set $F = F_{(G, H)}$. It is obvious by Lemma 4 that $F(x) = 0$ if $x \notin G(\mathbb{A})X(k)$. Thus, we assume $x \in G(\mathbb{A})X(k)$. Since $F$ is $G(\mathbb{A})$-invariant, we may assume $x \in X(k)$. Let $B_\infty$ be the connected component containing $G(k_\infty)^0 x$ and $B_f$ be an open compact subset of $G(\mathbb{A})_f X$ containing $x$. Set $B = B_\infty \times B_f$. It is easy to see that the restriction of $F$ to $B_\infty \times G(\mathbb{A})_f x$ is identically equal to $F(x)$. Therefore, we have

$$
(3.1) \quad \int_{B(0)(T)} F(y) d\omega^A_\mathbb{A}(y) = \omega^A_\mathbb{A}(B(0)(T)) F(x)
$$

for any $\Omega \in \Omega_\mathbb{W}$ and $T > 0$. By changing order of integrations, we obtain
\[
\int_{B^m(T)} F(y) d\omega_{\lambda}(y) = \frac{1}{\tau(G)} \int_{G(A)/G(k)} \int_{B^m(T)} \delta(g^{-1} y) d\omega_{\lambda}(y) d\omega_{\lambda}(g).
\]

It follows from Lemma 1 that, for any given \( \varepsilon > 0 \) and \( \Omega \in \Omega_w \), there exists a sufficiently large \( T > 0 \) such that
\[
(1 - \varepsilon) \int_{B^m(T)} \delta(g^{-1} y) d\omega_{\lambda}(y) \leq N(\Omega(T), X; B; g) \leq (1 + \varepsilon) \int_{B^m(T)} \delta(g^{-1} y) d\omega_{\lambda}(y).
\]

Therefore,
\[
(3.2) \quad (1 - \varepsilon) \int_{B^m(T)} F(y) d\omega_{\lambda}(y)
\]
\[
\leq \frac{1}{\tau(G)} \int_{G(A)/G(k)} N(\Omega(T), X; B; g) d\omega_{\lambda}(g) \leq (1 + \varepsilon) \int_{B^m(T)} F(y) d\omega_{\lambda}(y).
\]

If \( \chi \) is a characteristic function of \( B^m(T) \), then the counting function is written as
\[
N(\Omega(T), X; B; g) = \sum_{z \in X(k)} \chi(gz).
\]

By the uniformity ([M, Theorem 3.2]), we have
\[
\frac{1}{\tau(G)} \int_{G(A)/G(k)} N(\Omega(T), X; B; g) d\omega_{\lambda}(g)
\]
\[
= \frac{1}{\tau(G, X)} \int_{G(A)/X(k)} \chi(y) d\omega_{\lambda}(y) = \frac{\omega_{\lambda}(B^m(T))}{\tau(G, X)}.
\]

Combining this with (3.1) and (3.2), we have
\[
(1 - \varepsilon)F(x) \leq \frac{1}{\tau(G, X)} \leq (1 + \varepsilon)F(x).
\]

This concludes \( F(x) = \tau(G, X)^{-1} \). \( \square \)

By [M, Theorem 3.1], the Tamagawa numbers \( \tau(G) \) and \( \tau(G, X) \) are described as
\[
\tau(G) = \frac{|\pi_1(G)_{\text{Gal}(\bar{k}/k)}|}{|\text{Ker}^1(k, G)|},
\]
\[
\tau(G, X) = \frac{\tau(G)}{\tau(H)|\text{Ker}^1(k, H) \to \text{Ker}^1(k, G)|},
\]
where \( \text{Ker}^1(k, G) \) denotes the Tate-Shafarevich group
\[
\text{Ker}^1(k, G) = \text{Ker}(H^1(k, G) \to \prod_{v \in \mathcal{V}} H^1(k_v, G)).
\]

Furthermore, by [M, Theorem 3.4], if \( \text{Ker}^1(k, G) = 1 \), one has
\[
\tau(G, X) = \frac{\left| \pi_1(G)_{\text{Gal}(\overline{k}/k)} \right|}{\left| \pi_1(H)_{\text{Gal}(\overline{k}/k)} \right|}.
\]

Therefore, we obtain the following Corollary.

**Corollary.** Let \((G, H) \in \mathcal{R}_k(X, x_0)\). If \(G\) has the strong approximation property, then

\[
F_{(G, X)}(x) = \delta(x) = \begin{cases} 
\left| \pi_1(H)_{\text{Gal}(\overline{k}/k)} \right| & (x \in G(\mathbb{A})X(k)) \\
0 & (x \notin G(\mathbb{A})X(k)).
\end{cases}
\]

**Corollary.** Let \((G, H) \in \mathcal{R}_k(X, x_0)\). Assume that \(G\) has the strong approximation property and the second homotopy group of the complex manifold \(X(\mathbb{C})\) vanishes. Then \(\delta\) is identically equal to 1 on \(X(\mathbb{A})\), i.e. \((X, \delta)\) is strongly Hardy-Littlewood.

**Proof.** By the assumption of \(X\), Theorem 1 (2) and [M, Theorem 3.5], we have \(\tau(G, X) = \left| \pi_1(H)_{\text{Gal}(\overline{k}/k)} \right| = 1\). Then \(X(\mathbb{A}) = G(\mathbb{A})X(k)\) follows from [B-R, Corollary 3.7].

\[\Box\]

Finally, we remark that the condition \(x \in G(\mathbb{A})X(k)\) can be rewritten in terms of the Kotwitz invariant. Let \((G, H) \in \mathcal{R}_k(X, x_0)\). When \(G\) is simply connected, Borovoi and Rudnick constructed the mapping \(\kappa : X(\mathbb{A}) \to (\pi_1(H)_{\text{Gal}(\overline{k}/k)})_{\text{tor}}\) such that \(\kappa(x) = 0\) is equivalent to \(G(\mathbb{A})x \cap X(k) \neq \emptyset\) for \(x \in X(\mathbb{A})\) ([B-R, Theorem 3.6]). Morishita pointed out the author that the argument of Borovoi and Rudnick is extendable to the case where the derived group \(G^s\) of \(G^\text{red}\) is simply connected and the quotient map \(H \to G^\text{red}/G^s\) is surjective. Therefore, in this case, \(x \in G(\mathbb{A})X(k)\) is equivalent to \(\kappa(x) = 0\). We notice that any \(k\)-special homogeneous space \(X\) has a realization \((G, H) \in \mathcal{R}_k(X, x_0)\) which \(G^s\) is simply connected ([B2, Lemma 5.1]).

**References**


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