Fundamental Invariants of Weyl Groups and Excellent Families of Elliptic Curves

by

Tetsuji Shioda and Hisashi Usui

(Received 13 July, 1992)

1. Introduction

As is well-known, the ring of invariants of a finite reflection group is a polynomial ring (Chevalley's theorem: see e.g. [B]). To be more precise, if \( G \) is such a group acting on a finite-dimensional vector space \( V \) over a field \( F \) (with coordinates \( u_1, \cdots, u_n \)), then the ring of invariants in \( F[u_1, \cdots, u_n] \) is a graded polynomial ring generated by \( n \) homogeneous elements, say \( p_1, \cdots, p_n \):

\[
F[u_1, \cdots, u_n]^G = F[p_1, \cdots, p_n],
\]

provided that the order of \( G \) is not divisible by \( \text{char}(F) \). The set of the weights (or degrees) of the fundamental invariants \( p_1, \cdots, p_n \) is uniquely determined by \( G \).

Geometrically, this can be translated as follows: the group \( G \) acts on the affine space \( A^n \) (with coordinates \( u_1, \cdots, u_n \)) in such a way that the quotient space \( A^n/G \) becomes an affine space again (with coordinates \( p_1, \cdots, p_n \)). In other words, \( X = A^n \) is realized as a finite Galois covering of another \( Y = A^n \) with Galois group \( G \):

\[
\pi : A^n \rightarrow A^n/G \simeq A^n.
\]

Further there is a compatible action of the multiplicative group \( G_m \) on the affine spaces \( A^n \) both upstairs and downstairs.

The most classical example of the above situation is the case where \( G = S_n \) is the symmetric group on \( n \) letters \( u_1, \cdots, u_n \). Then the fundamental theorem on symmetric functions says that

\[
F[u_1, \cdots, u_n]^{S_n} = F[e_1, \cdots, e_n],
\]

where \( e_i \) denotes the \( i \)-th elementary symmetric function of \( u_1, \cdots, u_n \). (\( F \) can be any field in this case, and indeed it can be replaced by the ring of integers \( \mathbb{Z} \).) The action of \( G \) restricted to the \( (n-1) \)-dimensional subspace \( \sum_{i=1}^n u_i = 0 \) of \( V \) identifies \( S_n \) with \( W(A_{n-1}) \), the Weyl group of the root system of type \( A_{n-1} \) (cf. [B], [CS]), and (1.1) reduces to

\[
F[u_2, \cdots, u_n]^{W(A_{n-1})} = F[e_2, \cdots, e_n].
\]
In general, the Weyl group $W(R)$ of a root system $R$ is a finite reflection group, for which the weights of the fundamental invariants are well-known: they are the exponents increased by 1 (cf. [B]). For example, the weights for $W(R)$ are given as follows when $R$ is of type $A, D, E$:

\[
\begin{align*}
W(A_{n-1}) & \{2, 3, \cdots, n\} & (n \geq 2) \\
W(D_n) & \{2, 4, \cdots, 2(n-1), n\} & (n \geq 4) \\
W(E_6) & \{2, 5, 6, 8, 9, 12\} \\
W(E_7) & \{2, 6, 8, 10, 12, 14, 18\} \\
W(E_8) & \{2, 8, 12, 14, 18, 20, 24, 30\} \\
\end{align*}
\]

(1.4)

Now let us introduce the notion of an excellent family with Galois group $G$. For the sake of simplicity, we take $F = Q$ in what follows, but a suitable modification works in a more general case. Suppose that $\{X_\lambda\}$ is a family of algebraic varieties (possibly with some extra structure) depending on the parameter

$$
\lambda = (p_1, \cdots, p_n) \in \mathbb{A}^n.
$$

The generic member $X_\lambda$ is a variety defined over the field $k_0 = Q(\lambda)$ which is a purely transcendental extension of $Q$ of dimension $n$. Furthermore, suppose given $G(X_\lambda)$, a suitable group of algebraic cycles on $X_\lambda$, such that (i) it spans a finite dimensional vector space isomorphic to $V$ and (ii) the Galois group $\text{Gal}(k/k_0)$ acts on it ($k$ is the algebraic closure of $k_0$). Let $\rho_\lambda$ denote the Galois representation:

$$
\rho_\lambda: \text{Gal}(k/k_0) \to \text{Aut}(G(X_\lambda)) \subseteq \text{Aut}(V).
$$

(1.5)

Let $\mathcal{K}_\lambda$ be the Galois extension of $k_0$ corresponding to $\text{Ker}(\rho_\lambda)$, i.e. with Galois group $\text{Gal}(\mathcal{K}_\lambda/k_0) = \text{Im}(\rho_\lambda)$. It is equal to the smallest extension of $k_0$ such that $G(X_\lambda)$ is generated by $\mathcal{K}_\lambda$-rational cycles. We call $\mathcal{K}_\lambda$ the splitting field of $G(X_\lambda)$.

Now recall that $G$ is a finite reflection group acting on $V$ so that it is a subgroup of $\text{Aut}(V)$.

**DEFINITION.** We call $\{X_\lambda\}$ an excellent family with Galois group $G$ if the following conditions hold:

1. the image of $\rho_\lambda$ is equal to $G$.
2. there is a $\text{Gal}(k/k_0)$-equivariant evaluation map

$$
s: G(X_\lambda) \to k ,
$$

3. there exists a basis $\{Z_1, \cdots, Z_n\}$ of $G(X_\lambda)$ such that if we set $u_i = s(Z_i)$, then $u_1, \cdots, u_n$ are algebraically independent over $Q$, and
4. 

$$
Q[u_1, \cdots, u_n]^G = Q[p_1, \cdots, p_n] .
$$

It follows from the definition that the splitting field $\mathcal{K}_\lambda$ is then equal to $Q(u_1, \cdots, u_n)$, a purely transcendental extension of $Q$, such that
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\[ \text{Gal}(\mathcal{O}(u_1, \cdots, u_n)/\mathcal{O}(p_1, \cdots, p_n)) \cong G. \]

We call \( u_1, \cdots, u_n \) the splitting variables or variables upstairs (in view of (1.2)).

**Example 1.** The classical case (1.3) for \( G = \mathcal{S}_n = W(A_{n-1}) \) can be interpreted as follows. Let \( X_\lambda \) be the 0-dimensional variety defined by the equation
\[
x^n + \epsilon_2 x^{n-2} + \cdots + (-1)^n \epsilon_n = 0
\]
over \( k_0 = \mathcal{O}(\lambda) \), \( \lambda = (\epsilon_2, \cdots, \epsilon_n) \). Let \( u_i \) (\( i = 1, \cdots, n \)) be the \( n \) roots of (1.6). Then we have
\[ X_\lambda \otimes k_0 = \{ u_1, \cdots, u_n \} \quad (u_1 + \cdots + u_n = 0) \]
and each \( u_i \) defines a 0-cycle \([u_i]\) on \( X_\lambda \). Take
\[
\mathfrak{Z}(X_\lambda) = \sum_{i=1}^n Z^\ast [u_i] \mod \sum_{i=1}^n [u_i] = 0
\]
and define the evaluation map by \( s : [u_i] \rightarrow u_i \), extended by linearity; obviously this is a Galois equivariant map. The Galois group \( \text{Gal}(k/k_0) \) acts as permutations of \([u_i]\), so the image of \( \rho_2 \) is equal to \( G = \mathcal{S}_n \). In view of (1.3), \( \{ X_\lambda \} \) forms an excellent family with Galois group \( \mathcal{S}_n \).

**Example 2.** In the recent work on Mordell-Weil lattices (see especially [S5], [S6]), it has been shown that certain family of elliptic curves over a rational function field (or that of rational elliptic surfaces) forms an excellent family with Galois group \( W(E_r) \) \( (r = 6, 7, 8) \). For example, for the case \( r = 8 \), consider the elliptic curve \( E_8 \)
\[ y^2 = x^3 + x \left( \sum_{i=0}^3 p_{20 - 6i} t^i \right) + \left( \sum_{j=0}^3 p_{30 - 6j} t^j + t^5 \right), \]
over \( k_0(t) \) where we set \( k_0 = \mathcal{O}(\lambda) \) with
\[ \lambda = (p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30}). \]

Let
\[ E(E_8) = E_8(k(t)) \]
be the Mordell-Weil group of \( k(t) \)-rational points on \( E_8 \), \( k \) being the algebraic cosure of \( k_0 \). By the theory of Mordell-Weil lattices (MWL), it has the structure of the root lattice \( E_8 \) for generic \( \lambda \). Further there is a natural evaluation map
\[ s_{p_{\infty}} : E(k(t)) \rightarrow G_0(k) = k, \]
(the specialization homomorphism at the singular fibre over \( t = \infty \)). Then the fundamental theorems for type \( E_8 \) (cf. [S5], §8) show that \( \{ E_\lambda \} \) forms an excellent family of elliptic curves with Galois group \( W(E_8) \) (cf. §4 below).

This fact and its variant for \( E_6 \) and \( E_7 \) are the motivating examples for introducing the notion of an excellent family of elliptic curves. In view of its potential applications
to arithmetic, algebra and geometry, \ldots, as shown for $E$, in [S5] or [S6], it will be an important problem to establish the existence of an excellent family of elliptic curves with Galois group $G$ for a wider class of Weyl groups $G$.

The purpose of this paper is to construct such a family when $G = W(L)$ is the Weyl group of (the root system in) a root lattice $L$, which is a sublattice of $E_8$ of relatively high rank. More precisely, we consider the Mordell-Weil lattice of a rational elliptic surface, the structure of which has been classified into 74 types (see [OS] and the next section). Among them, there are exactly 31 "admissible" types for which the narrow Mordell-Weil lattice $L$ is a root lattice of positive rank. In this paper, we treat about half of these admissible types; we construct an excellent family for each type where the rank of $L$ is greater than 4 and for a few more types. This will extend the previous work [S5] for $L = E_8$, $E_7$, $E_6$, $D_4$, $A_2$ and the recent one [U] for $D_5$. The remaining cases will be treated in a forthcoming paper [S7], where the configuration of singular fibres can be more complicated as the rank of $L$ gets smaller. Together with it, the existence of excellent families of elliptic curves (over the rational function field) will be established for all admissible types (see also [S8]).

The paper is organized as follows. First, in §2, we state the main results and exhibit an excellent family of elliptic curves for each type in question (Theorem 1). Then we give a necessary and sufficient condition for the nondegeneracy of MWL (Theorem 2). In §3, we explain a general idea of proof and then our strategy for finding (or constructing) a good candidate for such a family; there are two main ideas: magic of weights and degeneration of Mordell-Weil lattices. In §4, we review the results for the case $L = E_8$, $E_7$, $E_6$ from [S5]. The rest of the paper is devoted to verifying that the families given in §2 are excellent. Naturally this process requires a case by case examination, but during the course of it, we obtain further useful informations. Thus, in each case, the results will include:

(I) classifying data and the defining equation  
(II) discriminant and singular fibres  
(III) minimal (or short) vectors in the MWL  
(IV) the fundamental algebraic equation  
(V) explicit formula for the fundamental invariants of the Weyl group  
(VI) generators of the Mordell-Weil group  
(VII) non-degeneracy condition of MWL  
(VIII) a $Q$-split example.

Finally it should be remarked that, once an excellent family is given, we have some standard applications such as (i) construction of Galois extensions (or representations) of $Q$ with Galois group $W(L)$ or (ii) deformation of rational double points, etc. (cf. [S6], [S4]). Apart from these, the defining equations of excellent families (given in §2 and [S7]) could be regarded as a new kind of normal forms of elliptic curves for which the generators of the rational points of infinite order have explicit description, just as the classical Legendre (or Hessian) normal form describes elliptic curves with torsion points of order 2 (or 3). As such, it is not hard to imagine that they should have some further interesting applications.
2. Main results

First we fix the notation used throughout the paper, recalling some facts on Mordell-Weil lattices (MWL); we refer to [S1], [S2], [OS] for more details.

General notation:

\[ k: \] an algebraically closed field

\[ K=k(t): \] the rational function field over \( k \)

\[ E/K: \] an elliptic curve over \( K \)

\[ E(K): \] the Mordell-Weil group, i.e. the group of \( K \)-rational points of \( E \) with the identity \( O \), which is finitely generated under a mild assumption

\[ \langle P, Q \rangle: \] the height pairing on \( E(K) \), defined in [S1, I] or [S2], §8. This is a symmetric bilinear pairing which is positive-definite modulo torsion.

\[ \text{the MWL } E(K): \] We abuse this terminology, the Mordell-Weil lattice \( E(K) \), to mean the structure of Mordell-Weil group given with the height pairing.

\[ E(K)^0: \] the narrow Mordell-Weil lattice of \( E/K \); this is a certain subgroup of finite index in \( E(K) \), which becomes an even integral lattice with respect to the height pairing.

\[ f: S \to C = P^1: \] the associated elliptic surface (the Kodaira-Néron model) of \( E/K \).

\[ (P): \] a \( K \)-rational point \( P \in E(K) \) is identified with a section of \( f \).

\[ R: \] the set of \( v \in C \) such that \( f^{-1}(v) \) is a reducible singular fibre of \( f \) (cf. [K], [N], [T]).

\[ \Theta_{v,j}: \] irreducible components of \( f^{-1}(v) (v \in R) \), with \( j = 0 \) corresponding to the identity component

\[ T_v: \] the lattice generated by \( \Theta_{v,j} (j > 0) \) with the sign changed; this is a root lattice of type \( A, D, E \) determined by the type of reducible fibre \( f^{-1}(v) \) (cf. [S2], (7.6)).

\[ T = \bigoplus_{v \in R} T_v: \] the trivial lattice

Now we are interested in the case where \( L = E(K)^0 \) is a root lattice of positive rank. Note that this will be the case only if the elliptic surface \( S \) is a rational elliptic surface. In fact, the minimal norm of \( E(K)^0 \) has, in general, a lower bound \( 2\chi \), where \( \chi \) is the arithmetic genus of \( S \) (see [S2], Th. 8.7). Since the minimal norm of a root lattice is 2, we must have \( \chi = 1 \), which implies that \( S \) is a rational elliptic surface.

The structure of the Mordell-Weil lattice of a rational elliptic surface has been determined in [S2], §10, and classified in [OS]. Namely, given such an \( E/K = k(t) \), let
\[
\begin{align*}
T & = \bigoplus_{v \in R} T_v \subset E_8 \quad \text{(trivial lattice)} \\
L & = E(K)^0 \quad \text{(narrow MWL)} \\
M & = E(K) \quad \text{(MWL)}.
\end{align*}
\]

By [S2], Th. 10.3 (with the notation modified), \( L \) is equal to the orthogonal complement of \( T \) in \( E_8 \) (which depends on the embedding of \( T \) in \( E_8 \)), and \( M \) is the direct sum of the dual lattice \( L^* \) of \( L \) and the torsion subgroup \( E(K)_{\text{tor}} \cong T'/T, \) \( T' \) being the primitive closure of \( T \) in \( E_8 \). Further, by the main result of [OS], the structure of the triple \( \{T, L, M\} \) can be classified into 74 types (No. 1, \ldots, No. 74). Among them, there are exactly 31 "admissible" types for which the narrow Mordell-Weil lattice \( L \) is a root lattice of positive rank:

\[
L = E_8, \; E_7, \; E_6, \; D_6, \; D_5, \; A_5, \; D_4 \oplus A_1, \; A_4, \; D_4, \; A_3 \oplus A_1, \; A_2 \oplus A_2, \; \ldots, \; A_1. 
\]

The main aim of this paper is to prove the following theorem, i.e., to construct explicitly an excellent family of elliptic curves for each admissible type with \( \text{rk} \; L > 4 \) and for a few more types derived from the case \( L = A_4 \). In the following statement, \( W(L) \) denotes the Weyl group of a root lattice \( L \), and for the fundamental invariants \( p_w, \ldots \) of \( W(L) \), the subscripts \( w, \ldots \) will indicate the weights of the invariants.

**Theorem 1 (Existence).** For each admissible type \( \{T, L, M\} \) below with \( L \) a root lattice, there exists an excellent family of elliptic curves (over the rational function field) with Galois group \( W(L) \). More precisely, let \( E_\lambda \) be the elliptic curve defined below by a generalized Weierstrass equation with \( r \) parameters \( \lambda = (p_w, \ldots) \) (\( r = \text{rk} \; L \)). Then, for \( \lambda \) generic over \( Q \), (2.1) holds as isomorphy of lattices, where \( E = E_\lambda \) and \( K = k(t), k \) being the algebraic closure of \( k_0 = Q(\lambda) \), and \( \{E_\lambda\} \) defines an excellent family of elliptic curves with Galois group \( W(L) \). In other words, for \( \lambda \) generic, we have

(i) the image of the Galois representation

\[
\rho_\lambda : \text{Gal}(k/k_0) \rightarrow \text{Aut}(E(k(t))^0) \cong \text{Aut}(L)
\]

is exactly \( W(L) \).

(ii) Let \( u_1 = sp_1(Q) \) for a suitable choice of rational points \( Q \in E(k(t)) \) and a place \( v \) of \( k(t) \) (\( sp_v \) : the specialization map). Then \( u_1, \ldots, u_r \) are algebraically independent over \( Q \) and

\[
Q[u_1, \ldots, u_r]^W = Q[p_w, \ldots].
\]

(iii) In particular, the coefficients \( p_w, \ldots \) of the elliptic curve \( E_\lambda \) form a set of fundamental invariants of the Weyl group \( W(L) \), while \( \mathcal{E}_\lambda = Q(u_1, \ldots, u_r) \) gives the splitting field of MWL, i.e. the smallest extension of \( k_0 \) such that \( E_\lambda(k(t)) = E_\lambda(\mathcal{E}_\lambda(t)) \).

**No. 1:** \( T = \{0\}, \; L = E_8, \; M = E_8 \)
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\[ y^2 = x^3 + x \left( \sum_{i=0}^{3} p_{20 - 6i} t^i \right) + \left( \sum_{i=0}^{3} p_{30 - 6i} t^i + t^2 \right) \]
\[ \lambda = (p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30}) \]

No. 2: \( T = A_1, L = E_7, M = E_7^* \)
\[ y^2 = x^3 + x(p_{12} + p_8 t + t^3) + \left( \sum_{i=0}^{4} p_{18 - 4i} t^i \right) \]
\[ \lambda = (p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18}) \]

No. 3: \( T = A_2, L = E_6, M = E_6^* \)
\[ y^2 = x^3 + x \left( \sum_{i=0}^{2} p_{8 - 3i} t^i \right) + \left( \sum_{i=0}^{2} p_{12 - 3i} t^i + t^4 \right) \]
\[ \lambda = (p_2, p_5, p_6, p_8, p_9, p_{12}) \]

No. 4: \( T = A_3^0, L = D_6, M = D_6^* \)
\[ y^2 = x^3 + x^2 r_6 + x(p_8 t^2 + p_4 t^2 + t^3) + (p_{10} t^2 + p_6 t^3 + p_2 t^4) \]
\[ \lambda = (p_2, p_4, p_6, r_6, p_8, p_{10}) \]

No. 5: \( T = A_3, L = D_5, M = D_5^* \)
\[ y^2 + p_5 xy = x^3 + x^2 (p_4 t) + x(p_8 t^2 + p_2 t^3) + (p_6 t^4 + t^5) \]
\[ \lambda = (p_2, p_4, p_5, p_6, p_8) \]

No. 6: \( T = A_2 \oplus A_1, L = A_5, M = A_5^* \)
\[ y^2 + p_3 xy + p_5 ty = x^3 + x^2 (p_2 t) + x(p_6 t^2 + p_2 t^3) + (p_6 t^4) \]
\[ \lambda = (p_2, p_3, p_4, p_5, p_6) \]

No. 7: \( T = A_3^0, L = D_4 \oplus A_1, M = D_4^* \oplus A_1^* \)
\[ y^2 = x^3 + x^2 (p_6 + p_2 t) + x(t - p_4)(t - q_4) + q_2 t^2 (t - p_4) \]
\[ \lambda = (p_2, p_4, q_4, p_6, q_2) \]

No. 8: \( T = A_4, L = A_4, M = A_4^* \)
\[ y^2 + p_3 xy + p_3 t^2 y = x^3 + x^2 (p_4 t) + x(p_2 t^3) + t^5, \quad \lambda = (p_2, p_3, p_4, p_5) \]

No. 9: \( T = D_4, L = D_4, M = D_4^* \)
\[ y^2 = x^3 + x(p_4 - t^2) + (q_6 + q_4 t + q_2 t^2), \quad \lambda = (q_2, q_4, p_4, q_6) \]

No. 10: \( T = A_3 \oplus A_1, L = A_3 \oplus A_1, M = A_3^* \oplus A_1^* \)
\[ y^2 + p_3 xy = x^3 + x^2 (p_2 t) + x(p_4 t^2 + t^3) + q_2 t^4, \quad \lambda = (p_2, p_3, p_4, q_2) \]
No. 11:  \( T = A_2^2, \ L = A_2^{\oplus 2}, \ M = A_2^{\oplus 2} \)

\[ y^2 + p_2 xy + p_3 ty = x^3 + x(q_2 t^2) + (q_3 t^3 + t^4), \quad \lambda = (p_2, p_3, q_2, q_3) \]

No. 15:  \( T = A_2, \ L = A_2 \oplus A_1, \ M = A_2^+ \oplus A_1^+ \)

\[ y^2 + q_2 p_3 xy + p_3 t^2 y = x^3 + x^2((p_2 q_2 t) + x(p_2 + q_2) t^3 + t^5 \]

\[ \lambda = (p_2, p_3, q_2) \]

No. 16:  \( T = A_3, \ L = A_3, \ M = A_3^+ \)

\[ y^2 + p_3 t^2 y = x^3 + x^2(p_4 t) + x(p_2 t^3) + t^5, \quad \lambda = (p_2, p_3, p_4) \]

No. 26:  \( T = D_6, \ L = A_1^{\oplus 2}, \ M = A_1^{\oplus 2} \)

\[ y^2 = x^3 + x^2(p_2 q_2 t) - x(p_2 + q_2) t^3 + t^5, \quad \lambda = (p_2, q_2) \]

No. 27:  \( T = D_6, \ L = A_2, \ M = A_2^{\oplus 2} \)

\[ y^2 + p_3 t^2 y = x^3 + x(p_2 t^3) + t^5, \quad \lambda = (p_2, p_3) \]

No. 43:  \( T = E_7, \ L = A_1, \ M = A_1^+ \)

\[ y^2 = x^3 + x(p_2 t^3) + t^5, \quad \lambda = p_2 \]

**Remark.** (i) An excellent family is not unique in general (see §6 where two such families are given for No. 6). But the existence of an explicit family is sufficient for usual arithmetic applications such as (a) construction of Galois representations (or Galois extensions) with Galois group \( W(L) \) or (b) construction of elliptic curves over \( \mathcal{O}(t) \) such that \( \text{E}(\mathcal{O}(t))^0 \cong L \).

(ii) Actually Theorem 1 is valid for every admissible type \( \{ T, L, M \} \); the remaining cases will be proven in the paper [S7] (in preparation). As for the cases No. 12, No. 17, ... of [OS] missing above and in [S7], the narrow Mordell-Weil lattice \( L \) is either a non-root lattice or \( \{ 0 \} \).

Next we describe the condition for the Mordell-Weil lattice to be non-degenerate when the parameter \( \lambda \) is specialized. It is known that the ramification locus of the quotient map \( \pi \) (see (1.2)) is defined by \( \delta(\lambda) = 0 \) where \( \delta(\lambda) \) denotes the square of the basic anti-invariant of \( \text{W}(L) \) ([B] Ch. 5, §5.4, Prop. 5).

**Theorem 2** (Nondegeneracy of MWL). Suppose \( \{ E_\lambda \} \) is an excellent family of elliptic curves as above. Let \( \lambda \rightarrow \lambda' \) be any specialization and let \( k' \) be an algebraically closed field containing \( \mathbb{Q}(\lambda') \). Then the MWL is nondegenerate, i.e.,

\[ E_\lambda(k'(t)) \cong E_\lambda(k(t)) = M, \]

if and only if

\[ \delta(\lambda') \neq 0 \quad \text{and} \quad J(\lambda') \neq 0, \]

where \( J(\lambda) \) is a certain invariant of \( \text{W}(L) \), called the frame invariant (cf. 3.3(a)), whose
explicit form will be given later in each case.

Both invariants $\delta(\lambda)$ and $J(\lambda)$ can be expressed as simple polynomials (such as a product of certain linear forms) in terms of the splitting variables $u=(u_1, \cdots, u_x)$ via $\lambda = \pi(u)$, the latter relation being in effect given by the explicit formula of the fundamental invariants of $W(L)$ (Theorem 1 (ii) or the step (V) in 3.1).

**Corollary 3** (Construction of $Q$-split examples). *Given an admissible type $T$, $L, M$ with $rk L = r$, take any $r$-tuple $u^0 = (u_i^0) \in Q'$ satisfying $\delta(u^0) \neq 0$ and $J(u^0) \neq 0$, and let $\lambda^0 = \pi(u^0) \in Q'$. Then $E = E_{20}$ is an elliptic curve defined over $Q(t)$ with rank $r$ such that $E(Q(t))^0 \simeq L$, $E(Q(t))^0 \simeq M$. Moreover a set of explicit generators $\{Q_i\}$ of $E(Q(t))^0$ can be given in terms of the prescribed values $u_i^0$.***

3. **Idea of proof**

3.1. **Verification.** The first three cases (No. 1, No. 2, No. 3) have been treated in [S5], which will serve as a guide post for other cases. Given a good candidate of an excellent family $E = E_4$ over $K = k(t)$ as in Theorem 1, we can verify its excellence with the help of the theory of Mordell-Weil lattices. This will be done in the following steps:

(I) The data to be achieved is the triple $T, L, M$, where $T$ is a sublattice of $E_8$ which is a direct sum of root lattices of type $A, D$ or $E_6$; $L$ is a root lattice orthogonal to $T$ and $M$ is equal to the dual lattice $L^*$ up to a torsion subgroup.

(II) Determine the reducible fibres of the associated elliptic surface

$$f: S \to P^1$$

for this, we can freely use the results of Kodaira, Néron, Tate ([K], [N], [T]). Confirm that the trivial lattice so obtained is isomorphic to the given lattice $T$. In most cases, the structure of $L = T^\perp$ is unique, but in certain cases where $T^\perp$ has two possibilities, $L$ is determined by the 2-torsion part of $M$ (cf. [OS]).

Once the lattice structure of $L, M$ is fixed, we have a complete information on the minimal norm, the number of minimal vectors and generators of $M$. In most cases, this is elementary, or we can refer to [CS], Ch. 4.

(III) Determine minimal (or short) vectors in $E(K) \simeq M$. An element $P \in E(K)$ is called a *minimal vector* if its norm $\langle P, P \rangle$ takes minimal positive value, and a *short vector* if $\langle PO \rangle = 0$ (cf. [S2], Lemma 10.2). The following formula for the height pairing will be constantly used:

\begin{equation}
\langle P, P \rangle = 2 + 2(PO) - \sum_{v \in R} \text{contr}_v(P)
\end{equation}

\begin{equation}
\langle P, Q \rangle = 1 + (PO) + (QQ) - (PQ) - \sum_{v \in R} \text{contr}_v(P, Q)
\end{equation}
See [S2], Th. 8.6 for the notation and the value of $\text{contr}_v(P)$ (p. 229).

(IV) Get the fundamental algebraic equation which describes minimal (or certain short) vectors (cf. [S5], [S6]). The relation of roots and coefficients of this equation should give (V).

(V) Explicit formula for the fundamental invariants of the Weyl group $W(L)$. This will prove that the elliptic curve $E/K$ defines an excellent family for the case in question, hence Theorem 1 in this case.

Furthermore we give:

(VI) Generators of the Mordell-Weil group $M = E(K)$

(VII) Nondegeneracy condition of MWL (Theorem 2)

(VIII) A $\mathbb{Q}$-split example, i.e. an example of elliptic curve $E/\mathbb{Q}(t)$ such that $E(\mathbb{Q}(t)) = M$, together with explicit generators.

Thus, once a candidate of an excellent family is explicitly given, we can prove or disprove its excellence by the above method. The reader might then be interested in knowing how we find a good candidate of an excellent family for a given admissible type. This is indeed the hardest and perhaps the most interesting step, though logically it is not necessary for the proof of Theorem 1. There are two main ideas for this:

(A) Magic of weights, (B) Degeneration of Mordell-Weil lattices

3.2. Magic of weights. Given a triple $T$, $L$, $M$, we have to find some elliptic curve $E/K = k(t)$ for which the trivial lattice (the direct sum of root lattices $T_v, v \in R$ ranging over the reducible fibres) is equal to $T$, and $L = E(K)^0$ and $M = E(K)$. By assumption, $L$ is a root lattice with the Weyl group $W(L)$. Let $w(L) = \{w_1, \cdots, w_r\}$ be the set of weights of the fundamental invariants of $W(L)$, which will be called in short the weight-set of $L$.

We start from the known cases (No. 1, No. 2, No. 3). In the case No. 1, we have $L = E_8$ and $T = \{0\}$, and the defining equation

$$y^2 = x^3 + \cdots + t^5$$

is a weighted homogeneous one with total weight 30. Thus $x$, $y$, $t$ have weights 10, 15, 6 respectively, and they determine the weights of other coefficients $p_{wr}$, etc., which coincide with the weight-set $w(E_8)$ (cf. (1.4)). (N.B. This equation has been studied in some other context, i.e. as the semi-universal deformation of $E_8$-singularity. The method of MWL is closely related to that of Milnor lattice in the singularity theory; see [S4]).

Let us consider a general Weierstrass equation

$$(3.3) \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

where each $a_d$ is a polynomial of degree at most $d$ in $t$:

$$a_d = \sum_{i=0}^{d} a_{d,i} t^i.$$
Suppose that (3.3) has the same weighted homogeneity (of total degree 30) as before. Then each $a_{ij}$ has weight $5d$ so that the weight of $a_{d,i}$ is equal to $5d-6i$. Omitting $a_{d,i}$ with negative weight, we obtain the following table, which will be called the weight-table of $E_8$ type:

\[
\begin{array}{cccccc}
5 & 15 & 10 & 20 & 30 \\
9 & 4 & 14 & 24 \\
3 & 8 & 18 \\
2 & 12 & 6 \\
0 & \end{array}
\]

(3.4)

Now assume that the weight-set $wt(L)$ of a given root lattice can be embedded into the above weight-table (3.4). For example, for \textbf{No. 5} and \textbf{No. 8}, we have $wt(D_5) = \{2, 4, 5, 6, 8\}$ and $wt(A_4) = \{2, 3, 4, 5\}$, both of which satisfy this assumption. Then, in the equation (3.3), we leave only those terms $a_{d,i}t^i$ with the relevant weights and $t^5$. In this way, we get a candidate for an excellent family of elliptic curves with Galois group $W(L)$. Then we apply the method described in 3.1. First, in the step (II), we compute the discriminant to determine the trivial lattice, which turns out to be (like magic!) the desired lattice $T$ in many cases, and then we can complete the proof of excellence by following other steps in 3.1. In such a case, we say that \textit{magic of weights works}. For example, this is the case for \textbf{No. 5}, \textbf{8}, \textbf{16}, \textbf{27}, \textbf{43}, as listed in Theorem 1 (see §4, §8 below). We note that this magical phenomenon has been first observed for \textbf{No. 5} in [U] other than the cases \textbf{No. 1}, \textbf{2}, \textbf{3}, \textbf{9} treated in [S5].

Of course, “magic” does not always work. For instance, although the weight-set $wt(A_5)$ can be embedded into the weight-table (3.4), the resulting family does not give an excellent family for \textbf{No. 6}, since it has $T = A_3$, $L = D_5$ rather than the desired $T = A_2 \oplus A_1$, $L = A_5$. Another drawback is that the weight-table of type $E_8$ (3.4) is rather restrictive in the sense that each weight occurs with multiplicity at most 1, and so it is impossible to embed a weight-set $wt(L)$ having some multiple weights.

However we can remedy this to certain extent by considering the variant for $E_7$ and $E_6$. The defining equation for \textbf{No. 2}, \textbf{3}

\[y^2 = x^3 + xt^3 + \cdots, \quad \text{or} \quad y^2 = x^3 + \cdots + t^4\]

has total weight 18 or 12 (the Coxeter number), and the weight-table of type $E_7$ or $E_6$ is given as follows:

\[
\begin{array}{cccccccc}
3 & 9 & 6 & 12 & 18 \\
5 & 2 & 8 & 14 \\
1 & 4 & 10 & 16 \\
2 & 6 & 4 & 8 & 12 \\
6 & 3 \\
0 & \end{array}
\]

(3.5)

(3.6)
For example, the weight-set \( wt(D_4) = \{2, 4, 6, 8, 10\} \) (for No. 4) embeds into (3.5), and \( wt(A_4) = \{2, 3, 4, 5, 6\} \) (for No. 6) embeds into both (3.5) and (3.6). The resulting families for these cases turn out to be excellent. Incidentally, note that we get 2 distinct families for No. 6, which shows the non-uniqueness of an excellent family of elliptic curves in general.

3.3. Degeneration of Mordell-Weil lattices. Another main idea for constructing a family with a new triple \( \{T', L', M'\} \) is to consider a suitable specialization of some established family with \( \{T, L, M\} \) where \( L' \) is a lattice of smaller rank than \( L \). In general, the reducible fibres of an elliptic surface will “increase” under specialization so that the original trivial lattice \( T \) will be enlarged to a new trivial lattice \( T' \); accordingly the Mordell-Weil lattice \( L \) (or \( M \)) will degenerate to \( L' \) (or \( M' \)) of smaller rank (if not equal).

The problem here is how to specialize parameters in the original family in order to achieve a desired new triple \( \{T', L', M'\} \), in particular, how to control parameters to produce a desired set of reducible fibres. The key ideas are the following:

(a) the frame and the frame invariant

(b) vanishing roots (analogy of vanishing cycles).

The former will be sufficient for proving Theorem 1, while the latter will play a key role in [S7].

(a) Given an excellent family \( \mathcal{F} = \{E_\lambda\} \) with the data \( \{T, L, M\} \), we have \( T = \bigoplus_{v \in R} T_v \) for \( \lambda \) generic (see the notation at the beginning of §2). We call \( T \) the frame of the family \( \mathcal{F} \).

When we specialize \( \lambda \) to some \( \lambda' \), the trivial lattice \( T' \) for the elliptic surface \( f : S_{\lambda'} \to \mathbb{P}^1 \) has a similar description: \( T' = \bigoplus_{v' \in R'} T'_{v'} \). The given specialization defines a map of \( R \) into \( R' \), say \( \phi : R \to R' \).

DEFINITION. The frame is said to be preserved or unbroken under the specialization in question if the map \( \phi : R \to R' \) is injective and if, for each \( v \in R \), we have \( T_v \cong T'_{\phi(v)} \) with \( v' = \phi(v) \). Otherwise the frame is said to be broken under the specialization in question.

There is an invariant of the Weyl group \( W(L) \), say \( J(\lambda) \), such that the frame \( T \) is broken under a specialization \( \lambda \mapsto \lambda' \) if and only if \( J(\lambda') = 0 \). Such an invariant will be called a frame invariant of the family.

N.B. The type of the root lattice \( T_v \) determines the type of the reducible fibre \( f^{-1}(v) \) if the rank of \( T_v \) is greater than 2, but this is not true for lower rank. Thus \( T_v = A_1 \) corresponds to the singular fibre of type \( I_2 \) or \( III \), and \( T_v = A_2 \) to that of type \( I_3 \) or \( IV \). It is natural to regard the frame to be preserved when a singular fibre of type \( I_2 \) (or \( I_3 \)) is specialized to that of type \( III \) (or \( IV \)), leaving the lattice structure \( T_v \) to be unchanged. For this, see e.g. No. 4.

Observe that several excellent families in Theorem 1 are related to each other via specialization where the frame is broken. For instance, for No. 8, we have \( T = A_4, L = A_4, M = A_4^* \) and
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\[ E_\lambda: y^2 + p_3 xy + p_4 t^2 y^2 = x^3 + x^2(p_4 t) + x(p_2 t^3) + t^5 \]
\[ \lambda = (p_2, p_3, p_4, p_5). \]

In this case, the successive specializations

[Table: No. 8, No. 16, No. 27, No. 43]

correspond to the following change of the frames:

\[ T = A_4 \to D_5 \to E_6 \to E_7. \]

Other example is given by

[Table: No. 26, No. 15, No. 27]

(b) Now we turn to a more systematic way to study the degeneration of MWL of an excellent family, by using parameters upstairs (or splitting variables) \( \{u_1, \ldots, u_r\} \) rather than that of parameters downstairs \( \lambda \) (cf. (1.2)). To fix the idea, assume that there is a singular fibre of additive type over \( t = \infty \) and let

\[ sp_\infty : E(k(t)) \to G_4(k) = k \]

be the specialization homomorphism which was denoted \( sp'_{\infty} \) in [S5].

For \( \lambda \) generic, it defines an isomorphism of \( E(k(t))/(tor) \cong L^* \) onto \( Zu_1 + \cdots + Zu_r \) where \( u_i = sp_\infty(P_i) \) for suitable generators \( \{P_i\} \) of \( E(k(t)) \) modulo torsion. We choose \( \{P_i\} \) among minimal or short vectors. In particular, the "roots" in the root lattice \( L \) \( (\cong L^*) \) are mapped under \( sp_\infty \) to certain \( Z \)-linear combinations of \( u_1, \ldots, u_r \), say \( \alpha_1, \ldots, \alpha_N, \) \( N \) being the number of roots in \( L \). Let

\[ \delta(\lambda) = \prod_{j=1}^{N} \alpha_j \in Z[u_1, \ldots, u_r]; \]

note that this is an invariant of the Weyl group \( W(L) \) which is equal to the square of the basic anti-invariant up to a constant.

(N.B. If \( \{P_i\} \) are only assumed to be independent elements generating a subgroup of finite index in \( E(k(t)) \), then we can modify the above by replacing \( Z \) by \( Q \) so that \( \alpha_j, \delta(\lambda) \) have \( Q \)-coefficients.)

Now the proof of Theorem 2 (stated at the end of §2) reduces to the following:

**Lemma 3.1.** Suppose that the frame is unbroken under a given specialization \( \lambda \to \lambda', \) i.e. \( J(\lambda') \neq 0 \). Then the Mordell-Weil lattice does not degenerate \( (E_\lambda(k(t)) \cong E_{\lambda'}(k(t)) \) if and only if \( \delta(\lambda') \neq 0 \).

**Proof.** Let \( T', L' \) denote the trivial lattice and the narrow MWL for \( E' = E_{\lambda'}. \)
The $T$ for generic $\lambda$ is naturally embedded into $T'$ under a given specialization, and
the MWL does not degenerate if and only if $T'=T$. First assume this. Then we have
$L'=L$, so any root $P\in E(K)^0=L$ specializes to a root $P'\in E'(K)^0=L'$. Hence the
section $(P')$ is disjoint from the zero section $(O)$, and in particular, we have $sp_x(P')\neq 0$,
which implies $\delta(\lambda')\neq 0$.

Conversely, assume $T\neq T$; by assumption, there is a new reducible fibre
$f^{-1}(v), v\in R' - \phi(R)$. Take an irreducible component, say $\Theta$, other than the identity
component. It has the self-intersection number $-2$, and is orthogonal to $T$ as well
as $(O)$ and any fibre in NS($S_\lambda$). Thus it defines a root of $L$, which vanishes under
the specialization in question, which implies $\delta(\lambda')=0$.

We can reformulate the above result by introducing the following terminology.
It is obvious that any specialization upstairs
$$u=(u_1, \cdots, u_n)\mapsto u'=(u'_1, \cdots, u'_n)$$
uniquely determines a specialization downstairs $\lambda\to\lambda'$. A root $\alpha_i$ of $L$ will be called
a vanishing root if it vanishes under the said specialization. In spirit, this is very close
to the idea of vanishing cycles in the deformation of singularities (cf. [S4]).

**Corollary 3.2 (Principle of vanishing roots).** Under any specialization $u\to u'$
with a vanishing root, the Mordell-Weil lattice $M=E_8(k(t))$ degenerates.

The mode of degeneration of MWL depends upon the behavior of vanishing
roots. For instance, if there is only one vanishing root (upto sign) under a
frame-preserving specialization, then the rank of MWL decreases by one.

This principle will be used in [S7] to construct a new excellent family from
known ones. Actually it can be used in a more general situation where the narrow
MWL may be no longer a root lattice: indeed this method is applicable for the
existence proof of all the 74 types of $T, L, M$ in [OS], which will be treated
elsewhere (cf. [S8]).

4. Review for No. 1, No. 2, No. 3 ($L=E_8, E_7, E_6$)

Here let us briefly review the results for No. 1, 2, 3 from [S5] for later reference.
(For instance, (No. 1, V) will refer to No. 1, (V).) Unless otherwise mentioned, $\lambda$
is assumed to be generic, i.e. its components are algebraically independent over $Q$.

4.1. No. 1 ($L=E_8$).

(I) No. 1: $T=\{0\}, L=E_8, M=E_8$ (cf. [S5], §4, §8)

$$y^2=x^3+x\left(\sum_{i=0}^3 p_{20-6i}t^i\right) + \left(\sum_{j=0}^3 p_{30-6j}t^j + t^5\right)$$

$$\lambda=(p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30}).$$

(II) For any $\lambda$, the elliptic surface $f:S_\lambda\to P^1$ has an irreducible singular fibre
of type II (a rational curve with a cusp) at \( t = \infty \). For general \( \lambda \), it has no reducible fibres at all, since the discriminant \( \Delta(t) \) is a polynomial of degree 10 in \( t \) which has no multiple factors in general. In particular, for \( \lambda \) generic over \( \mathcal{Q} \), the trivial lattice is \( \{0\} \) so that \( L = M = E_8 \), as required.

(III) The root lattice \( E_8 \) has minimal norm 2 and the number of minimal vectors (= roots in this case) is 240. Accordingly there are 240 rational points \( P \in E(k(t)) \) with \( \langle P, P \rangle = 2 \), or equivalently (by (3.1)), with \( (PO) = 0 \), and they are of the form (cf. [S5], §10, Lemma 10.9)

\[
P = (x, y), \quad x = gt^2 + at + b, \quad y = ht^3 + ct^2 + dt + e.
\]

(IV) By the specialization map

\[
sp_{\infty} : E(k(t)) \to f^{-1}(\infty)^{\#} = G_2
\]

(\# denotes the smooth part), the above \( P \) is mapped to

\[
u := sp_{\infty}(P) = g/h \quad ([S5], \text{Lemma 8.2}).
\]

Substitute \( x, y \) into the defining equation to get the relations among \( g, h, a, \cdots, e \). Then an explicit elimination leads to an algebraic equation of degree 240 in \( u \):

\[
u^{240} + 60p_2u^{238} + \cdots = 0;
\]

the left hand side \( \Phi(u, \lambda) \) is a monic polynomial in \( u \) with coefficients in \( \mathbb{Z}[\lambda] = \mathbb{Z}[p_2, \cdots, p_{30}] \), which was called the universal polynomial of type \( E_8 \) ([S5], Th. 8.3.). The above equation (and the similar ones in the sequel) will be called the fundamental algebraic equation for the case under consideration. This terminology should be justified by the next step.

(V) Let \( \{P_1, \cdots, P_8\} \) be a basis of \( E(k(t)) \cong E_8 \) in the sense of root systems, and let \( u_i = sp_{\infty}(P_i) \). Then the 240 roots of the fundamental algebraic equation are given by \( u_1, \cdots, u_8 \) and certain integral linear combinations of them. Then the relation of roots and coefficients for \( u^w \), \( w \) ranging over the weights of \( \mathbb{W}(E_8) \) in (1.4), gives an explicit expression of \( p_\lambda \) as a fundamental invariant of weight \( w \) for \( \mathbb{W}(E_8) \) (see [S5], Th. (E_8), p. 681; note that the notation \( p_i \) or \( q_{ij} \) there correspond to \( p_{20-i} \) or \( p_{30-j} \) here). It follows that \( \{E_\lambda\} \) forms an excellent family for No. 1 (see [S5], Th. 8.3, 8.4).

(VI) Generators. The above \( \{P_1, \cdots, P_8\} \) forms a set of generators of the Mordell-Weil group \( E(k(t)) = E(\mathcal{Q}(u_1, \cdots, u_8)(t)) \). Each \( P_i \) is of the form

\[
P_i = (t^2/u_i^2 + a_it + b_i, \ t^3/u_i^3 + \cdots + e_i)
\]

where \( a_i, \cdots, e_i \) are rational functions in \( u_1, \cdots, u_8 \) ([S5], Th. 8.5).

(VII) Nondegeneracy condition. The MWL of a specialized elliptic curve \( E_\lambda \) is nondegenerate (\( \cong E_8 \)) if and only if \( \delta(\lambda') \neq 0 \), where \( \delta(\lambda) = \Phi(0, \lambda) \). (Since \( T = 0 \), the frame invariant can be taken as the constant 1.)

(VIII) For an example of \( E(\mathcal{Q}(t)) \cong E_8 \), together with explicit generators, see [S3] or [S5], p. 685.
4.2. No. 2 \((L=E_7)\).

(I) No. 2: \(T=A_1, L=E_7, M=E_7^+\) (cf. \([S5], \S9\)).

\[
y^2 = x^3 + x(p_{12} + p_6t + t^3) + \left( \sum_{i=0}^4 p_{18 - 4i}t^i \right)
\]

\[
\hat{\lambda} = (p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18}).
\]

(II) For any \(\lambda\), there is a reducible singular fibre of type \(III\) at \(t = \infty\), consisting of two smooth rational curves tangent at a single point; thus \(T_{\infty} = A_1\). For general \(\lambda\), there are no other reducible fibres so that the trivial lattice \(T\) is \(A_1\), and we have \(L = E_7, M = E_7^+\), as required.

(III) The lattice \(E_7^+\) has minimal norm \(3/2\) and the number of minimal vectors is 56. Accordingly there are 56 rational points \(P \in E(k(t))\) with \(\langle P, P \rangle = 3/2\); by (3.1), this is the case if and only if the section \((P)\) is disjoint from the zero-section \((O)\) and intersects the non-identity component \(\Theta_{\infty,1}\) of \(f^{-1}(\infty)\). Such a \(P\) is given by

\[
P = (x, y), \quad x = at + b, \quad y = ct^2 + dt + e
\]

([S5], Lemma 9.1).

(IV) In this case, we use the specialization map \(sp'_\infty\) which is the map:

\[
sp_{\infty}: E(k(t)) \to f^{-1}(\infty)^+ = G_a \times \mathbb{Z}/2\mathbb{Z},
\]

composed with the projection to the \(G_a\)-component. Then we have for the above \(P\)

\[
sp'_\infty(P) = -c \quad ([S5], \text{Lemma 9.2}).
\]

By the elimination method as in (No. 1, VI), we obtain the fundamental algebraic equation for No. 2:

\[
\Phi(c, \lambda) = c^{56} - 36p_2c^{54} + \cdots = 0.
\]

(V) Noting that \(E_7^+\) is generated by minimal vectors (as easily checked), we can choose a basis \(\{P_1, \cdots, P_7\}\) of \(E(k(t)) \simeq E_7^+\) among minimal vectors. Let
$u_i = -sp'_{\infty}(P_i)$. As in (No. 1, V), the relation of roots and coefficients for the above equation gives the explicit formula of $p_\omega$ as the fundamental invariants of $W(E_7)$ (see [S5], Th. (E_7), p. 680; the $p_i$ or $q_i$ there correspond to $p_{14-a_i}$ or $p_{18-a_i}$ here). We conclude that our family defines an excellent family for No. 2 ([S5], Th. 9.3, 9.4).

(VI) The above \{P_1, \cdots, P_7\} generate the MWL $E(k(i))$. We have

$$P_i = (a_i t + b_i, u_i t^2 + d_i t + e_i) \quad (i = 1, \cdots, 7)$$

where $a_i, b_i, d_i, e_i$ are rational functions of $u_1, \cdots, u_7$ ([S5], Th. 9.5).

(VII) The MWL of a specialized elliptic curve $E_{\lambda'}$ is nondegenerate ($\simeq E_7^*$) if and only if $\delta(\lambda') \neq 0$, where $\delta(\lambda)$ is the product of all the 126 roots $\alpha_i$ of $E_7$, expressed as integral linear combinations of $u_1, \cdots, u_7$ (it was denoted by $\delta_0$ in [S5].) The frame invariant can be taken as the constant 1, since the frame is unbroken ($T_\infty = A_1$) for any $\lambda'$.

(VIII) An example of $E(Q(i)) \simeq E_7^*$, together with explicit generators, can be found in [S5], p. 684.

4.3. No. 3 ($L = E_6$).

(I) No. 3: $T = A_2, L = E_6, M = E_6^*$ (cf. [S5], §10)

$$y^2 = \chi^3 + x \left( \sum_{i=0}^{2} p_{8 - 3i} t^i \right) + \left( \sum_{i=0}^{2} p_{12 - 3i} t^i + t^4 \right)$$

$$\lambda = (p_2, p_5, p_6, p_8, p_9, p_{12}).$$

(II) For any $\lambda$, there is a reducible singular fibre of type IV at $t = \infty$, consisting of 3 smooth rational curves transversally meeting at a single point; thus $T_\infty = A_2$. For general $\lambda$, there are no other reducible fibres so that the trivial lattice $T$ is $A_2$, and we have $L = E_6, M = E_6^*$.

(III) The lattice $E_6^*$ has 54 minimal vectors of minimal norm 4/3, which divides into two orbits under the Weyl group $W(E_6)$. Accordingly there are 54 rational points $P \in E(k(i))$ with $\langle P, P \rangle = 4/3$. By (3.1), they are characterized by the condition that $\langle PO \rangle = 0$ and that the section $(P)$ intersects one of the two non-identity components $\Theta_{\infty, i} (i = 1, 2)$ of $f^{-1}(\infty)$; the two $W(E_6)$-orbits correspond to $i = 1$ or 2. Thus there are 27 $P$'s such that

$$P = (x, y), \quad x = at + b, \quad y = t^2 + dt + e.$$

([S5], Lemma 10.1).

(IV) As in (No. 2, IV), we define the specialization map $sp'_{\infty}$ as the composite of the map:

$$sp_{\infty} : E(k(i)) \to f^{-1}(\infty)^4 = G_\infty \times \mathbb{Z}/3 \mathbb{Z},$$

and the projection to the $G_\infty$-component. Then we have for the above $P$

$$sp'_{\infty}(P) = -\frac{a}{2}$$

([S5], Lemma 10.2).
By the elimination method, we obtain the fundamental algebraic equation for No. 3:
\[ \Phi(a, \lambda) = a^{27} + 12p_2a^{25} + \cdots = 0. \]

(V) In this case, we can choose a basis \( \{ P_1, \cdots, P_6 \} \) of \( E(k(t)) \simeq E^*_8 \) among 27 minimal vectors of the above form. Let \( u_i = -2sp'_\infty(P_i) \). Again, the relation of roots and exponents for the above equation gives the explicit formula of \( p_w \) as the fundamental invariants of \( W(E_6) \) (see [S5], Th. (E6), p. 679; the \( p_i \) or \( q_i \) there correspond to \( p_{8-3i} \) or \( p_{12-3j} \) here). It follows that our family defines an excellent family for No. 3 ([S5], Th. 10.3, 10.4).

(VI) The above \( \{ P_1, \cdots, P_6 \} \) are generators of the MWL \( E(k(t)) \) such that
\[ P_i = (u_i + b_i, t^2 + d_i + e_i) \quad (i = 1, \cdots, 6) \]
where \( b_i, d_i, e_i \) are rational functions of \( u_1, \cdots, u_6 \) ([S5], Th. 10.5).

(VII) A specialized elliptic curve \( E_{\lambda'} \) has a nondegenerate MWL \( \simeq E^*_8 \) if and only if \( \delta(\lambda') \neq 0 \), where \( \delta(\lambda) \) is the product of all the 72 roots \( \alpha_i \) of \( E_6 \), expressed as integral linear combinations of \( u_1, \cdots, u_6 \) (it was denoted by \( \delta_0 \) in [S5]). The frame invariant can be taken as the constant 1, since the frame is unbroken \( (T_{\infty} = A_2) \) for any \( \lambda' \).

(VIII) Examples of \( E(k(t)) \simeq E^*_8 \), together with explicit generators, can be found in [S5], p. 683 or [S6], p. 487.

5. No. 4 and No. 5 \((L = D_6, D_5)\)

5.1. No. 4 \((L = D_6)\).

(I) No. 4: \( T = A_1^{\oplus 2}, L = D_6, M = D_5^* \)
\[ E_{\lambda}: y^2 = x^3 + x^2r_6 + x(p_8t + p_4t^2 + t^3) + (p_{10}t^2 + p_6t^3 + p_2t^4) \]
\[ \lambda = (p_2, p_4, p_6, r_6, p_8, p_{10}) \]

(II) The discriminant is computed as follows:
\[ \Delta = 16r_6^2(p_8^2 - 4p_{10}r_6) + 2 + (-64p_8^3 + r_6 + \cdots) + \cdots - 64t^9. \]

Let us see that the singular fibres over \( t = 0 \) and \( t = \infty \) are reducible with two irreducible components:

\[
f^{-1}(v) = \Theta_{v,0} + \Theta_{v,1} \quad (v = 0, \infty)
\]

under the condition

\[
J(\lambda) = p_8^2 - 4p_{10}r_6 \neq 0.
\]

Indeed, the singular fibre at \( t = \infty \) is of type III for any \( \lambda \), just as in No. 2. On the other hand, the singular fibre at \( t = 0 \) is of type either I_2 or III according to whether \( r_6 \neq 0 \) or \( = 0 \) under the said condition. For, if \( r_6 \neq 0 \), then \( \operatorname{ord}_{r_6}(A) = 2 \) and the above Weierstrass equation reduces to a nodal cubic \( y^2 = x^3 + x^2r_6 \) (with the node at \( (0, 0) \)) as \( t \to 0 \). By the well-known algorithm ([N], [T]), this shows that we have a singular fibre of type I_2 (two smooth rational curves meeting transversally at 2 points). Next, if \( r_6 = 0 \) (but \( J \neq 0 \)), then \( \operatorname{ord}_{r_6}(A) = 3 \) and the defining equation reduces to the cuspidal cubic \( y^2 = x^3 \) (with the cusp at \( (0, 0) \)) as \( t \to 0 \). This shows that the singular fibre is of type III.

For \( \lambda \) generic, there are no other reducible fibres (this is easily checked by looking at some special case; cf. example in (VIII)) and so the trivial lattice is \( T = A_1^{\otimes 2} \), which implies that \( L = D_6, M = D_6^* \) ([S2], Th. 10.4, or [OS]). Note that the above \( J \) is an example of what we called the frame invariant in §3, 3.3(a), with the property that the frame is unbroken precisely when \( J \neq 0 \).

(III) First recall basic facts on the root lattice \( D_6 \) and its dual lattice \( D_6^* \) (cf. [CS]). A standard realization of \( D_6 \) is the sublattice of \( \mathbb{Z}^6 \subseteq \mathbb{R}^6 \) consisting of those \( (x_1, \cdots, x_6) \) with \( x_1 + \cdots + x_6 \in 2\mathbb{Z} \). Then we have \( D_6 \subseteq \mathbb{Z}^6 \subseteq D_6^* \) with each index 2. The minimal norm of \( D_6^* \) is 1 and the minimal vectors are the unit vectors \((1, 0, \cdots, 0)\), etc. up to sign (12 in number). There are \( 2^6 = 64 \) vectors of norm 3/2 such as \((1/2, \cdots, 1/2)\), any one of which generates \( D_6^*/\mathbb{Z}^6 \).

The Weyl group \( W(D_6) \) is generated by the permutations of coordinates \( x_i \) and the sign change at 2 coordinates \( x_i, x_j \). It acts transitively on the set of 12 minimal vectors as well as the set of 60 roots of \( D_6 \) (such as \((\pm 1, \pm 1, 0, \cdots, 0)\)), but the set of vectors of norms 3/2 in \( D_6^* \) are divided into 2 orbits under \( W(D_6) \). The weights of the fundamental invariants are \( \{2, 4, 6, 6, 8, 10\} \), as reflected in the chosen parameter \( \lambda = (p_2, p_4, p_6, r_6, p_8, p_{10}) \).

By the formula of the height pairing (3.1), \( P \in M = E(K) \) has norm

\[
\langle P, P \rangle = 2 + 2(PO) - \begin{cases} \frac{1}{2} & \text{if } \langle P \Theta_{0,1} \rangle = 1 \\ \frac{1}{2} & \text{if } \langle P \Theta_{v,1} \rangle = 1 \\ 0 & \text{otherwise} \end{cases}
\]

Here \( \langle P \Theta_{v,1} \rangle = 1 \) means that the section \( (P) \) passes through the non-identity component.
\( \Theta_{v,1} \) of \( f^{-1}(v) \). Hence we have \( \langle P, P \rangle = 1 \) if and only if \( (PO) = 0 \) and \( P \) passes through the non-identity components at \( t = 0 \) and \( t = \infty \). In terms of the Weierstrass equation, this means that \( P = (x, y) \) as in (No. 2, III) should satisfy \( b = e = 0 \) to meet the node or cusp \((0, 0)\) at \( t = 0 \), i.e.

\[
P = (at, ct^2 + dt).
\]

(IV) Using the same notation as in (No. 2, IV), we have \( sp_{\infty}(P) = -c \) for such a \( P \). Now the above \( P \) satisfies the defining equation if and only if

\[
a = c^2 - p_2, \quad 2cd = a^3 + ap_4 + p_6, \quad d^2 = p_{10} + ap_8 + a^2 r_6.
\]

Eliminating \( a, d \), we obtain the fundamental algebraic equation for No. 4:

\[
\Phi(c, \lambda) = c^{12} - 6p_2c^{10} + c^8(15p_2^2 + 2p_4)
\]
\[
+ c^6(2p_6^2 - 4r_6 - 20p_2^3 - 8p_2p_4)
\]
\[
+ c^4(-4p_8 + 15p_2^4 + 12p_2^2p_4 + p_2^2 - 6p_2p_6 + 8p_2r_6)
\]
\[
+ c^2(-4p_{10} + 6p_2^5 - 8p_2^3p_4 - 2p_2p_6^2 + 6p_2^2p_6 + 2p_4p_6 + 4p_2p_8 - 4p_2^3r_6)
\]
\[
+ (p_2^2 + p_2p_4 - p_6)^2 = 0.
\]

(V) The 12 roots of this equation can be written as \( \pm u_i \) \((i = 1, \ldots, 6)\). Writing down the relation of roots and coefficients, we obtain the following formulas:

\[
\begin{align*}
p_2 &= \epsilon_2'/6 \\
p_4 &= -(15p_2^2 - \epsilon_4)/2 \\
p_6 &= p_2^3 + p_2p_4 - \epsilon_6 \\
r_6 &= (-20p_2^3 - 8p_2p_4 + 2p_6 + \epsilon_6')/4 \\
p_8 &= (15p_2^4 + 12p_2^2p_4 + p_2^2 - 6p_2p_6 + 8p_2r_6 - \epsilon_8)/4 \\
p_{10} &= (-6p_2^5 - 8p_2^3p_4 - 2p_2p_6^2 + 6p_2^2p_6 + 2p_4p_6 + 4p_2p_8 - 4p_2^3r_6 + \epsilon_1)/4.
\end{align*}
\]

Here \( \epsilon_2'd \) denotes the \( d \)-th elementary symmetric function of \( u_1^2, \ldots, u_6^2 \) and \( \epsilon_6 = u_1 \cdots u_6 \), which are obviously invariants of \( W(D_6) \).

This shows that if \( p_2, p_4, p_6, r_6, p_8, p_{10} \) are algebraically independent over \( \mathbb{Q} \), so are \( \epsilon_2', \ldots, \epsilon_6' \) and \( \epsilon_6 \), and hence both of them give a set of the fundamental invariants of \( W(D_6) \), since they have the right weights. It follows that \( u_1, \ldots, u_6 \) are also algebraically independent over \( \mathbb{Q} \), and that the above formulas express \( p_i \) and \( r_6 \) explicitly as the fundamental invariants of \( W(D_6) \). Passing to the quotient fields, this implies that the extension \( \mathbb{Q}(u_1, \ldots, u_6)/\mathbb{Q}(p_2, \ldots, p_{10}) \) is a Galois extension with Galois group \( W(D_6) \), and therefore the Galois representation \( \rho_2 \) has the image \( W(D_6) \) with the splitting field \( \mathcal{X} = \mathbb{Q}(u_1, \ldots, u_6) \). Thus we have proven the excellency of our family.

(VI) The 6 rational points

\[
Q_i = ((u_i^2 - p_2)t, u_i t^2 + d(t)) \quad (i = 1, \ldots, 6)
\]

where \( d_i \) is rationally determined by \( u_i \) over \( \mathbb{Q}(\lambda) \) as in (IV), generate the sublattice
of $E(K) \cong D_6^*$ of index 2 corresponding to $\mathbb{Z}^6$. The full MWL is generated by $Q_i$, and one more rational point of the form

$$Q_0 = (at + b, ct^2 + dt + e) \quad \text{with} \quad c = \frac{1}{2} (u_1 + \cdots + u_6).$$

(VII) The MWL is nondegenerate if and only if $\delta \neq 0$ and $J \neq 0$. In terms of the splitting variables $u_1, \cdots, u_6$, we have

$$\delta = \prod_{i<j} (\pm u_i \pm u_j)$$

and

$$J = \pm \prod_{i<j} \frac{1}{2} (u_1 + \cdots + u_5 - u_6)$$

where the latter product is taken over 16 linear forms obtained from the written one under $W(D_6)/\{ \pm 1 \}$.

(VIII) Example 1. Let $u_i = i - 1$ for $i = 1, \cdots, 6$, i.e.

$$\{u_i\} = \{0, 1, 2, 3, 4, 5\}.$$

Determine $p_2, \cdots, p_6$ by the formulas in (V), which define the elliptic curve $E(Q(t)):

$$y^2 = x^3 + \frac{2475}{32} x^2 + \left( \frac{152625}{256} - \frac{2849}{24} t^2 + t^3 \right) x + \frac{1313825}{512} - t^2$$

$$- \frac{137335}{432} + \frac{55}{6} t^4$$

Then we have $T = A_2^{\oplus 2}$, $L = D_6$, $M = D_6^*$ for this example by checking the above non-degeneracy condition (or by directly checking that $\Delta(t)$ has order 2 zero at $t = 0$, and 7 simple zeroes at $t \neq 0, \infty$).

The generators of $E(Q(t)) \cong D_6^*$ are given as follows. First the 6 points

$$Q_1 = \left( \frac{-55}{6}, -60t \right), \quad Q_2 = \left( \frac{-49}{6}, t^2 + \frac{855}{16} t \right),$$

$$Q_3 = \left( \frac{-31}{6}, 2t^2 + \frac{315}{8} t \right), \quad Q_4 = \left( \frac{-1}{6}, 3t^2 - \frac{795}{16} t \right),$$

$$Q_5 = \left( \frac{41}{6}, 4t^2 - \frac{405}{4} t \right), \quad Q_6 = \left( \frac{95}{6}, 5t^2 + \frac{2835}{16} t \right).$$

generate the index 2 subgroup $(\cong \mathbb{Z}^6)$. Observe that $t^2$-coefficient of the $y$-coordinate of $Q_i$ is 0, 1, 2, 3, 4, 5 as prescribed. Next, corresponding to $c = \frac{1}{2} (u_1 + \cdots + u_6) = 15/2$, we find the following point:
\[ Q_0 = \left( \frac{565}{12} + \frac{675675}{64} t + \frac{15}{2} t^2 + \frac{116295}{16} t + \frac{557431875}{512} \right). \]

These points are related by \(2Q_0 = Q_1 + \cdots + Q_6\), and the full Mordell-Weil group \( E(Q(t)) \) is generated by \( \{Q_0, Q_1, \cdots, Q_3\} \).

**Example 2.** If one prefers some examples of the same type but without denominators (both in the defining equation and in the coordinates of generators), here is an example. Take
\[ \{u_i\} = \{0, 2, 4, 6, 8, 18\}. \]
We leave it as an exercise to verify the above statement.

**5.2. No. 5 \((L = D_3)\).**

(I) **No. 5:** \(T = A_3, L = D_5, M = D_5^*\) (cf. [U] for details)
\[ y^2 + p_5 xy = x^3 + x^2(p_4 t) + x(p_8 t^2 + p_2 t^3) + (p_6 t^4 + t^5) \]
\[ \lambda = (p_2, p_4, p_5, p_6, p_8). \]

(II) The discriminant is given as follows:
\[ \Delta = p_2^3(p_6^2 - p_5^2 p_6)t^4 + \cdots + 432t^{10}. \]
It follows that \(\text{ord}_e = 0(\Delta) = 4\) if \(p_5 \neq 0\) and \(J_1 := p_6^2 - p_5^2 p_6 \neq 0\), in which case the singular fibre at \(t = 0\) is of type \(I_4\):
\[ f^{-1}(0) = \Theta_0 + \Theta_4 + \Theta_2 + \Theta_3, \]
where 4 smooth rational curves \(\Theta_i\) meet in a cyclic way like "4" (\(\Theta_0\) is the identity component).

At \(t = \infty\), there is a singular fibre of type \(II\), as in (No. 1, II). For general \(\lambda\), there are no other reducible fibres, and hence the trivial lattice is \(T = A_3\), and we have \(L = D_5, M = D_5^*\) by [OS].
(III) The lattice \( D_5^\pm \) has 10 minimal vectors of minimal norm 1 and 32 vectors with norm 5/4. In view of (3.1), for \( P \in M = E(K) \), \( \langle P, P \rangle = 1 \) holds if and only if \( (P)O = 0 \) and \( (P) \) meets \( \Theta_2 \). The minimal vectors \( P \) have the following form:

\[
P = \left( \frac{t^2}{u^2}, \frac{t^3}{u^3} + ct^2 \right), \quad c = \frac{u^5 + p_2u^3 + p_4u - p_5}{2u^2},
\]

where \( u = sp_{\alpha}(P) \) should satisfy the equation in (IV).

(IV) The fundamental algebraic equation is a polynomial of degree 10 in \( u \):

\[
\Phi(u, \lambda) = u^{10} + (2p_2)u^8 + (2p_4 + p_2^2)u^6 + (-4p_6 + 2p_2p_4)u^4 + (-4p_8 + p_2^2)u^2 - p_2^2 = 0.
\]

(V) Let \( \pm u_i (i = 1, \cdots, 5) \) be the roots of this equation. By the relation of roots and coefficients, we have:

\[
\begin{align*}
p_2 &= -\varepsilon_2/2 \\
p_4 &= (\varepsilon_4 - p_2)/2 \\
p_5 &= u_1 \cdots u_5 \\
p_6 &= (\varepsilon_6 + 2p_2p_4)/4 \\
p_8 &= (p_2^2 - \varepsilon_8)/4.
\end{align*}
\]

Here \( \varepsilon_2 \) denotes the \( d \)-th elementary symmetric function of \( u_1^2, \cdots, u_5^2 \). This gives an explicit formula of the fundamental invariants of \( W(D_5^\pm) \), and we deduce that the family in question is an excellent family for No. 5 by the same argument as (No. 4, V).

(VI) The 5 rational points

\[
Q_i = \left( \frac{t^2}{u_i^2}, \frac{t^3}{u_i^3} + ct^2 \right) \quad (i = 1, \cdots, 5)
\]

(\( c_i \) is determined as in (III)) generate a sublattice of \( E(K) \approx D_5^\pm \) of index 2. The full MWL is generated by \( Q_i \) and one more rational point of the form

\[
Q_0 = \left( \frac{t^2}{u_0^2} + at, \frac{t^3}{u_0^3} + ct^2 + dt \right) \quad \text{with} \quad u_0 = \frac{1}{2}(u_1 + \cdots + u_5).
\]

(VII) By (II), the frame invariant is given by \( J(\lambda) = p_5J_1 \). Both \( p_5 \) and \( J_1 \) are expressed as the product of linear forms in \( u_i \) which correspond to the short vectors of norm 1 or 5/4 in \( D_5^\pm \), taken modulo \( \{\pm 1\} \). Thus

\[
J_1 = p_8^2 - p_2^2p_6 = \pm \prod \frac{1}{2}(u_1 + \cdots + u_5),
\]

where the product runs over 16 transforms of \( 1/2(u_1 + \cdots + u_5) \) under \( W(D_5^\pm) \) modulo \( \{\pm 1\} \).

(VIII) For an explicit example, see [U].
6. Two families for No. 6 \((L = A_5)\)

6.1. No. 6 (family of type \(E_7\)).

(I) No. 6: \(T=A_2 \oplus A_1, L=A_5, M=A^*_5\)

\[ y^2 + p_3xy + p_5t = x^3 + x^2(p_2t) + x(p_4t^2 + t^3) + (p_6t^3) \]
\[ \lambda = (p_2, p_3, p_4, p_5, p_6) \]

Observe that the weight-set of \(A_5\) is \(\{2, 3, 4, 5, 6\}\) and that this equation is obtained by magic of weights from the weight-table of type \(E_7\) (cf. §3, 3.2).

(II) The discriminant is as follows:

\[ \Delta = -p_3^3(-p_3^3p_4^5 + p_2p_3p_5^3 - p_3^3 + p_3^3p_6)t^3 + (-27p_5^4 + p_3(\cdots))t^4 + \cdots - 64t^9. \]

The singular fibre at \(t=\infty\) is of type \(III\) for any \(\lambda\), as in No. 2. The singular fibre at \(t=0\) is reducible with 3 irreducible components:

\[ f^{-1}(0) = \Theta_{0,0} + \Theta_{0,1} + \Theta_{0,2} \]

under the condition

\[ J(\lambda) = -p_3^2p_4p_5 + p_2p_3p_5^2 - p_3^3 + p_3^3p_6 \neq 0. \]

Indeed, it is of type either \(I_3\) or \(IV\) according to whether \(p_3 \neq 0\) or =0 under the said condition. For, if \(p_3 \neq 0\), then \(\text{ord}_{t=0}(\Delta) = 3\) and the defining equation reduces to a nodal cubic \(y^2 + p_3xy = x^3\) (with the node at \((0, 0)\)) as \(t \to 0\), which implies ([N], [T]) that the singular fibre is of type \(I_3\) (3 smooth rational curves forming a triangle). Next, if \(p_3 = 0\) (but \(J \neq 0\)), then \(\text{ord}_{t=0}(\Delta) = 4\) and the defining equation reduces to the cuspidal cubic \(y^2 = x^3\) as \(t \to 0\). This shows that the singular fibre is of type \(IV\).

For \(\lambda\) generic, there are no other reducible fibres and so the trivial lattice is \(T = A_2 \oplus A_1\), which implies that \(L = A_5\), \(M = A^*_5\) ([OS]).

(III) The lattice \(A^*_5\) has 12 minimal vectors of minimal norm 5/6 which are divided into 2 orbits under the Weyl group \(W(A_5) = S_6\). The sum of 6 vectors in each orbit is zero, and any 5 vectors among them give a set of free generators of \(A^*_5\).

By (3.1), \(P \in M = E(K)\) satisfies \(\langle P, P \rangle = 5/6\) if and only \((PO) = 0\) and \((P)\) meets the non-identity components at \(t = 0\) and at \(t = \infty\); the two components \(\Theta_{0,i}(i = 1, 2)\) correspond to the 2 orbits. The minimal vectors \(P\) have the following form:

\[ P = (at, ct^2 + dt). \]

By the defining equation, \(a, c, d\) should satisfy

\[ a = c^2, \quad d = 0 \quad \text{or} \quad d = -ap_3 - p_5, \quad -a^2 + 2cd - a^2p_2 + ap_3 - ap_4 + cp_5 - p_6 = 0. \]

(IV) Substituting \(a = c^2, d = 0\) into the third relation, we obtain the fundamental algebraic equation:
\( \Phi(c, \lambda) = c^6 + c^4 p_2 - c^3 p_3 + c^2 p_4 - cp_5 + p_6 = 0 \),

which is nothing but the generic algebraic equation of degree 6 (cf. (1.6)).

(V) Let \( c = u_i \ (i = 1, \cdots, 6) \) be the 6 roots of this equation. By the relation of roots and coefficients (in the standard sense), \( p_d \) is equal to the \( d \)-th elementary symmetric function of \( u_i \ (i = 1, \cdots, 6) \) for \( d = 2, \cdots, 6 \) and \( u_1 + \cdots + u_6 = 0 \). In other words, we have:

\[
p_2 = \sum_{i < j} u_i u_j, \quad \cdots, \quad p_6 = \prod_{i} u_i.
\]

This represents an explicit formula of the fundamental invariants of \( W(A_5) = \mathcal{S}_6 \), and we conclude that the family in question is an excellent family for No. 6.

(VI) The 6 rational points

\[
Q_i = (u_i^2 t, u_i t^2) \quad (i = 1, \cdots, 6)
\]

form a \( W(A_5) \)-orbit of minimal vectors, and they generate \( E(K) \cong \mathbb{A}^6_5 \). Note that

\[
Q_1 + \cdots + Q_6 = 0 \quad \text{(in} \ E(K), \ \text{of course)}
\]

corresponding to \( u_1 + \cdots + u_6 = 0 \), since the specialization homomorphism \( sp^*: E(K) \rightarrow \sum \mathbb{Z}u_i \) is an injective map for \( \lambda \) generic (cf. [S5], Th. 9.5).

(VII) The nondegeneracy condition of MWL is stated in Theorem 2, where the frame invariant is now given by \( J(\lambda) \) in (II). In terms of \( u_6 \), it is expressed as a product of 15 linear forms (corresponding to short vectors of norm 4/3 in a \( W(A_5) \)-orbit):

\[
J = \prod_{i < j} (u_i + u_j),
\]

while \( \delta(\lambda) \) is equal to the (ordinary) discriminant of the equation \( \Phi(c, \lambda) = 0 \), i.e.

\[
\delta = \left\{ \prod_{i < j} (u_i - u_j) \right\}^2.
\]

Therefore the MWL is nondegenerate if and only if \( u_i \neq \pm u_j \) for any \( i < j \).

(VIII) Example. Take \( u_1 = i - 1 \ (i = 1, \cdots, 5) \) and \( u_6 = -10 \), which satisfies the above nondegeneracy condition. Then

\[
p_2 = -65, \quad p_3 = -300, \quad p_4 = -476, \quad p_5 = -240, \quad p_6 = 0,
\]

so the elliptic curve \( E/Q(t) \)

\[
y^2 - 300xy - 240ty = x^3 - 65tx^2 + (-476t^2 + t^3)x
\]
gives a \( Q \)-split example for No. 6. Free generators of \( E(Q(t)) \cong \mathbb{A}^6_5 \) are given by any 5 points among

\[
\left\{ Q_i \right\} = \{ (0, 0), (t, t^2), (4t, 2t^2), (9t, 3t^2), (16t, 4t^2), (100t, -10t^2) \}.
\]
The 30 minimal vectors (roots) in the narrow MWL $E(\mathcal{O}(t))^0 \simeq A_5$ are given by \{\(Q_i - Q_j\) \((i < j)\)\}.

6.2. **No. 6 (family of type \(E_6\)).**

(I) **No. 6':** \(T = A_2 \oplus A_1, L = A_5, M = A_8^5\)

\[
y^2 = x^3 + p_4x^2 + (p_5t + p_2t^2)x + (p_6t^2 + p_3t^3 + t^4)
\]

\[
\lambda = (p_2, p_3, p_4, p_5, p_6).
\]

This equation is obtained by magic of weights from the weight-table of type \(E_6\) (cf. §3, 3.2).

(II) The discriminant is as follows:

\[
\Delta = -16p_2^2(-p_2^2 + 4p_4p_6)t^2 - 32(2p_3^2 + p_4(p\cdots))t^3 + \cdots + -432t^8.
\]

The singular fibre at \(t = \infty\) is of type \(IV\) for any \(\lambda\), as in No. 3. The singular fibres at \(t = 0\) is either \(I_2\) or \(III\) (according to whether \(p_4 \neq 0\) or \(= 0\)) under the condition

\[
J(\lambda) = -p_2^2 + 4p_4p_6 \neq 0.
\]

This can be checked in the same way as before, so it will be omitted. For \(\lambda\) generic, we have again \(T = A_2 \oplus A_1\) so that \(L = A_5, M = A_8^5\).

(III) By the same argument as above, the 12 minimal vectors \(P = (x, y) \in E(K) \simeq A_8^5\) have the x-coordinate of the form \(x = at\). Then the right hand side of the defining equation in (I) becomes

\[
t^2\{t^2 + (a^3 + ap_2 + p_3)t + (a^2p_4 + ap_5 + p_6)\},
\]

which should be a square in \(k[t]\).

(IV) Thus we obtain the fundamental algebraic equation:
\[
\Phi(a, \lambda) = (a^3 + ap_2 + p_3)^2 - 4(a^2 p_4 + ap_5 + p_6) \\
= a^6 + 2a^4 p_2 + 2a^3 p_3 + a^2(p_2^2 - 4p_4) + a(2p_2 p_3 - 4p_5) + p_3^2 - 4p_6 \\
= 0.
\]

(V) Let \( a = u_i \ (i = 1, \ldots, 6) \) be the 6 roots of this equation. Then the relation of roots and coefficients can be rewritten as follows:

\[
\begin{aligned}
p_2 &= \varepsilon_2/2 \\
p_3 &= -\varepsilon_3/2 \\
p_4 &= -(\varepsilon_4 - p_2^2)/4 \\
p_5 &= (\varepsilon_5 + 2p_2 p_3)/4 \\
p_6 &= -(\varepsilon_6 - p_3^2)/4,
\end{aligned}
\]

where \( \varepsilon_d \) denotes the \( d \)-th elementary symmetric function of \( u_i \ (i = 1, \ldots, 6) \) \((u_1 + \cdots + u_6 = 0)\). In this way, we have obtained another excellent family for No. 6.

(VI) The 6 rational points

\[
Q_i = \left( u_i t, t \left( t + \frac{1}{2}(u_i^3 + u_i p_2 + p_3) \right) \right) \quad (i = 1, \ldots, 6)
\]

generate \( E(K) \cong A_6^* \), where we have \( Q_1 + \cdots + Q_6 = 0 \) as before. Note that, in this case, we have \( u_i = -2sp_{e_1}(Q_i) \) as in (No. 3, IV).

(VII) The frame invariant \( J(\lambda) \) in (II) now is an invariant of degree 10 with the following expression:

\[
J = \frac{1}{2^4} \prod_{i < j < k} (u_i + u_j + u_k),
\]

the factors of which correspond to a half of 20 vectors of norm 3/2 in \( A_6^* \). Thus the MWL is nondegenerate if and only if \( u_i \neq u_j \) and \( u_i + u_j + u_k \neq 0 \) for \( i < j < k \).

(VIII) Example. Take \( u_i = 2(i-1) \ (i = 1, \cdots, 5) \) and \( u_6 = -20 \), which satisfies the above nondegeneracy condition. Then

\[
p_2 = -130, \quad p_3 = 1200, \quad p_4 = 6129, \quad p_5 = -79920, \quad p_6 = 360000,
\]

so we get the elliptic curve \( E/\mathbb{Q}(t) \)

\[
y^2 = x^3 - 6129x^2 + (-79920t - 130t^2)x + (360000t^2 + 1200t^3 + t^4),
\]

which is a \( \mathbb{Q} \)-split example for No. 6 using the new family. Free generators of \( E(\mathbb{Q}(t)) \cong A_6^* \) are given by the 5 points:

\[
(0, t(600 + t)), \quad (2t, t(474 + t)), \quad (4t, t(372 + t)), \quad (6t, t(318 + t)), \quad (8t, t(336 + t)).
\]
7. No. 7 \( (L = D_4 \oplus A_1) \)

(I) **No. 7:** \( T = A_1^{\oplus 3}, L = D_4 \oplus A_1, M = D_4^* \oplus A_1^* \)

\[
y^2 = x^3 + x^2(p_6 + p_2t) + xt(t - p_4)(t - q_4) + q_2t^2(t - p_4)^2
\]

\[
\lambda = (p_2, p_4, q_4, p_6, q_2).
\]

N.B. This is the first case in the families of Theorem 1 where the elliptic surface has 3 reducible fibres. In addition to the 2 reducible fibres over \( t = 0, \infty \) as in No. 4, we need one more, say over \( t = v \). This \( v = p_4 \) should serve as an invariant of weight 4 when we employ the weight-table of type \( E_7 \). This is a rough idea for getting the above equation.

(II) The discriminant:

\[
A = -16t^2(t - p_4)^2\{p_2^2(4p_6q_2 - q_4^2) + \cdots + 4t^2\}.
\]

The singular fibre at \( t = \infty \) is of type \( III \) for any \( \lambda \), as in No. 2. Assume \( p_4 \neq 0 \). Then at \( t = 0 \) (or \( t = p_4 \), the Weierstrass equation reduces to the cubic curve

\[
y^2 = x^3 + p_6x^2 \quad \text{(or } y^2 = x^3 + x^2(p_6 + p_2p_4))
\]

with a singular point at \((0, 0)\) (node or cusp as the case may be). Thus the singular fibre at \( t = 0 \) is either \( I_2 \) or \( III \) (according to whether \( p_6 \neq 0 \) or \( = 0 \)) provided that

\[
J_1 = q_4^2 - 4q_2p_6 \neq 0.
\]

Similarly, the singular fibre at \( t = p_4 \) is either \( I_2 \) or \( III \) (according to whether \( p_6 + p_2p_4 \neq 0 \) or \( = 0 \)) provided that

\[
J_2 = (p_4 - q_4)^2 - 4q_2(p_6 + p_2p_4) \neq 0.
\]

For \( \lambda \) generic, there are no other reducible fibres than those three, and so the trivial lattice is \( T = A_1^{\oplus 3} \), which implies that \( L = D_4 \oplus A_1, M = D_4^* \oplus A_1^* \) ([OS]). (We note that the orthogonal complement of a root (i.e. of \( A_1 \)) in \( D_6 \) is \( D_4 \oplus A_1 \).)

(III) The lattice \( M = D_4^* \oplus A_1^* \) has 2 minimal vectors of minimal norm 1/2 coming from \( A_1^* \), and 24 vectors of norm 1 coming from \( D_4^* \); the latter divides into 3 orbits under \( W(D_4) \). The number of roots in \( L \) is 26 (= 24 + 2).

On the other hand, we have by the formula (3.1)

\[
\langle P, P \rangle = 2 + 2(PO) - \begin{pmatrix} \frac{1}{2} \\ - \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ - \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

for \( P \in M = E(K) \); the 3 terms correspond to the reducible fibres at \( t = 0, p_4, \infty \), with the value 1/2 if \( (P) \) passes through the non-identity component there. In particular, we have \( \langle P, P \rangle = 1/2 \) if and only \( (PO) = 0 \) and \( (P) \) meets all the non-identity components at \( t = 0, p_4, \infty \). Hence we see that the rational points
\[ \pm Q_0 = (0, \pm u_0(t-p_4)) \in E(k) \quad (u_0^2 = q_2) \]

give the minimal vectors, since they reduce to the singular points \((0, 0)\) at \(t = 0, p_4, \infty\).

More generally, the short vectors \(P\) meeting the non-identity components at \(t = 0, \infty\) have the following form:

\[ P = (at, ct^2 + dt). \]

The condition for such a \(P\) to satisfy the defining equation is:

\[
\begin{align*}
c^2 &= a + q_2 \\
2cd &= a^3 + a^2p_2 - ap_4 + q_4 - 2p_4q_2 \\
d^2 &= a^2p_6 + p_2^2q_2 + ap_4q_4.
\end{align*}
\]

(IV) Eliminating \(a, d\), we obtain the fundamental algebraic equation, which factorizes as follows:

\[ \Phi(c, \lambda) = (c^2 - q_2)^2 \Psi(c, \lambda) = 0, \]

where

\[
\Psi(c, \lambda) = c^8 + c^6(2p_2 - 4q_2) + c^4(p_2^2 - 2p_4 - 6p_2q_2 + 6q_2^2 - 2q_4) \\
+ c^2(-2p_2p_4 - 4p_6 - 2p_2^2q_2 + 6p_2q_2^2 - 4q_2^3 - 2p_2q_4 + 4q_2q_4) \\
+ (p_4 - p_2q_2 + q_2^2 - q_4)^2.
\]

The first factor corresponds to \(\pm Q_0\) mentioned above: \(c^2 = q_2, a = 0\). The 8 roots of \(\Psi\) define 8 rational points \(\pm Q_i (i = 1, \cdots, 4)\), corresponding to 8 vectors of norm 1 in \(D_4\) forming one orbit under \(W(D_4)\), which generate an index 2 subgroup (cf. No. 4 or No. 5).

(V) Let \(\pm u_i (i = 1, \cdots, 4)\) be the 8 roots of this equation, and write down the relation of roots and coefficients. Then, letting \(\epsilon_2, \epsilon_4\) denote the \(d\)-th elementary symmetric function of \(u_1^2, \cdots, u_4^2\), we obtain the following:

\[
\begin{align*}
q_2 &= u_0^2 \\
p_2 &= -(-4q_2 + \epsilon_2)/2 \\
p_4 &= (p_2^2 - 4p_2q_2 + 4q_2^2 - \epsilon_4 + 2\epsilon_4)/4 \\
q_4 &= (p_2^2 - 8p_2q_2 + 8q_2^2 - \epsilon_4 - 2\epsilon_4)/4 \\
p_6 &= (-2p_2p_4 - 2\epsilon_2q_2 + 6p_2q_2^2 - 4q_2^3 - 2p_2q_4 + 4q_2q_4 + \epsilon_4)/4,
\end{align*}
\]

where we set \(\epsilon_4 = u_1 \cdots u_4\). This gives an explicit formula of the fundamental invariants of \(W(D_4 \oplus A_1) = W(D_4) \times W(A_1)\) in terms of the splitting variables \(u_0, u_1, \cdots, u_4\), and we conclude that the family in question is an excellent family for No. 7.

(VI) Generators. The 5 rational points

\[ Q_i = (u_i^2 - q_2)t, u_i t^2 + dt (i = 1, \cdots, 4) \]
and \( Q_0 \) generate an index 2 subgroup of \( E(K) \cong D_4^* \oplus A_4^* \). Taking one more point

\[
Q_5 = (u_5^2 - q_2)(t - p_4), \quad u_5(t - p_4)(t - \cdots), \quad u_5 = \frac{1}{2}(u_1 + \cdots + u_4),
\]

we get generators of the full MWL \( E(K) \).

(VII) The frame invariant is given by

\[
J(\lambda) = p_4 J_1 J_2 \quad \text{(cf. II).}
\]

In terms of \( u_i, p_4, J_1, J_2 \) are expressed as follows:

\[
p_4 = \prod \frac{1}{2}(u_1 + u_2 + u_3 + u_4),
\]

\[
J_1 = \prod \left( u_0 + \frac{1}{2}(u_1 - u_2 - u_3 - u_4) \right) \left( u_0 - \frac{1}{2}(u_1 - u_2 - u_3 - u_4) \right),
\]

\[
J_2 = \prod_{i=1}^{4} (u_0 + u_i)(u_0 - u_i),
\]

where the product for \( p_4, J_1 \) is taken over 4 choices of changing even number of signs of \( u_2, u_3, u_4 \) of the given linear form. The other invariant \( \delta(\lambda) \) in Theorem 2 is equal to

\[
\delta = \left\{ u_0, \prod_{1 \leq i < j \leq 4} (u_i + u_j)(u_i - u_j) \right\}^2.
\]

Therefore the MWL is nondegenerate if and only if \( u_0 \neq 0, u_i \neq \pm u_j \) for any \( 1 \leq i < j \leq 4 \) and none of the linear factors of \( J \) vanish.

(VIII) Example. Let \( u_0 = 6, u_1 = 2, u_2 = 4, u_3 = 8, u_4 = 16 \). Then

\[
q_2 = 36, \quad p_2 = -98, \quad p_4 = 2025, \quad q_4 = 5825, \quad p_6 = 271350.
\]

So the equation of the elliptic curve \( E(\mathbb{Q}(t)) \) is:

\[
y^2 = x^3 + (271350 - 98t)x^2 + (t - 5825)(t - 2025)x + 36t^2(t - 2025)^2
\]

and the generators of \( E(\mathbb{Q}(t)) \) are given by

\[
Q_0 = (0, 6t^2 - 12150t), \quad Q_1 = (-32t, 2t^2 - 6930t), \quad Q_2 = (-20t, 4t^2 - 4500t),
\]

\[
Q_3 = (28t, 8t^2 - 26280t), \quad Q_4 = (220t, 16t^2 + 126000t), \quad Q_5 = (-35(t - 2025), (t - 2025)(t + 20475)).
\]
8. No. 8, No. 16, No. 27, No. 43 ($L = A_4, \cdots, A_1$)

8.1. No. 8 ($L = A_4$).

(I) No. 8: $T = A_4$, $L = A_4$, $M = A_4^*$

$$y^2 + p_5 xy + p_3 t^2 y = x^3 + x^2 (p_4 t) + x (p_2 t^3) + t^5, \quad \lambda = (p_2, p_3, p_4, p_5)$$

Magic of weights works here using the weight-table of type $E_8$ (cf. 3.2).

(II) The discriminant:

$$\Delta = -t^5 \{ p_5^4 (p_3 p_4 - p_2 p_3 p_5 + p_2^3) + \cdots + 432 t^5 \}$$

The singular fibre at $t = \infty$ is of type II, as in No. 1. The singular fibre at $t = 0$ is of type $I_5$ under the condition:

$$p_5 \neq 0, \quad J_4 = p_3^2 p_4 - p_2 p_3 p_5 + p_2^2 \neq 0.$$  

This is because then the discriminant has order 5 at $t = 0$ and the Weierstrass equation reduces to the cubic curve $y^2 + p_5 xy = x^3$ with a node at $(0, 0)$. Thus we can write

$$f^{-1}(0) = \Theta_0 + \cdots + \Theta_4,$$

where the 5 irreducible components $\Theta_j$ are smooth rational curves forming a pentagon, numbered in a cyclic way. (We choose $\Theta_0$ as the identity component as usual.)

For $\lambda$ generic, there are no other reducible fibres, and the trivial lattice is $T = A_4$ and $L = A_4$, $M = A_4^*$ ([OS]).

(III) The lattice $M = A_4^*$ has 10 minimal vectors of minimal norm 4/5, which divide into 2 orbits under $W(A_4) = S_5$, and $M$ is generated by any 4 of the 5 vectors in an orbit (cf. (No. 6, III)). The number of roots in $L = A_4$ is 20.

On the other hand, for $P \in E(K) \approx A_4^*$, suppose that ($P$) meets $\Theta_j$, then the formula (3.1) gives
Thus the minimal vectors are exactly those $P$ for which $(PO)=0$ and $(P)$ meets $\Theta_j$ for $j=2$ or 3; the latter distinguishes the 2 orbits.

(IV) There are 5 rational points of the form

$$P = \left( \frac{t^2}{u^2}, \frac{t^3}{u^3} \right) \quad (u = sp_{\infty} (P)).$$

Indeed, substituting the coordinates into the defining equation, we get the fundamental algebraic equation in this case:

$$\Phi(\mu, \lambda) = u^5 + p_3 u^3 - p_3 u^2 + p_4 u - p_5 = 0$$

which is the generic algebraic equation of degree 5.

(V) Let $u_1, \cdots, u_5$ be the 5 roots; we have $u_1 + \cdots + u_5 = 0$. Then $p_d$ ($d=2, 3, 4, 5$) is equal to the $d$-th elementary symmetric function of $u_1, \cdots, u_5$. Thus we see that the family in question is an excellent one for No. 8.

(VI) Let

$$Q_i = \left( \frac{t^2}{u_i^2}, \frac{t^3}{u_i^3} \right) \quad (i = 1, \cdots, 5).$$

Obviously, each $P = Q_i$ is a short vector (i.e. $(PO) = 0$) such that $(P)$ passes through a non-identity component $\Theta_j$ ($j>0$) (note that $P$ reduces to the node $(0,0)$ at $t = 0$). Observe that these 5 points are conjugate under the Galois group over $Q(\lambda)$, i.e. they form a single orbit under $W(A_4)$. Hence $\{Q_i\}$ give 5 minimal vectors in $M = A_4$, which generate $M$. Incidentally, notice that they pass through one and the same component, say $\Theta_2$. For the corresponding sections $(Q_i)$ (in the elliptic surface) are disjoint everywhere, since $Q_i - Q_k$ is a root in $L = A_4$ for any $i \neq k$.

(VII) The frame invariant is given by

$$J(\lambda) = p_5 J_1 \quad \text{(cf. II)},$$

which is expressed as follows in terms of $u_i$:

$$p_5 = u_1 \cdots u_5,$$

$$J_1 = - \prod_{i<j} (u_i + u_j).$$

So far we have omitted the proof of such identities as above. As an illustration, let us check the last identity. Let $F(u) = \Phi(u, \lambda)$; it has 5 roots $u_i$. Assume for a moment
that \( u_i + u_j = 0 \). Then \( F(u) \) and \( F(-u) \) have a common zero \( u = u_i \). By eliminating \( u \), we obtain a nontrivial relation among \( p_a \), which turns out to be \( J_1 = 0 \). Thus \( J_1 \) is divisible by the product of \( u_i + u_j \). Comparing the degree, we see that \( J_1 \) is a constant multiple of this product. A special case determines the constant to be \(-1\).

The MWL is nondegenerate if and only if \( u_i \neq 0 \) and \( u_i \neq \pm u_j \).

(VIII) Example. Let \( u_1 = 1, u_2 = 2, u_3 = 3, u_4 = 4, u_5 = -10 \). Then it is easy to see that the above nondegeneracy condition holds. Since we have

\[
p_2 = -65, \quad p_3 = -300, \quad p_4 = -476, \quad p_5 = -240,
\]

the defining equation of the elliptic curve \( E/\mathbb{Q}(t) \) is:

\[
y^2 - 240xy - 300t^2 y = x^3 - 476tx^2 - 65t^3 x + t^5.
\]

(A direct calculation shows that the discriminant has 5 simple roots (conjugate over \( \mathbb{Q} \)) at \( t \neq 0 \) and order 5 zero at \( t = 0 \), which checks also that the MWL is nondegenerate for this example.)

The minimal vectors \( \{Q_i\} \in E(\mathbb{Q}(t)) = A_4^\times \) are as follows:

\[
(t^2, t^3), \quad \left( \frac{t^2}{4}, \frac{t^3}{8} \right), \quad \left( \frac{t^2}{9}, \frac{t^3}{27} \right), \quad \left( \frac{t^2}{16}, \frac{t^3}{64} \right), \quad \left( \frac{t^2}{100}, \frac{-t^3}{1000} \right).
\]

and the remaining minimal vectors \( \{-Q_i\} \) are given by

\[
(t^2, -t^3 + 540t^2), \quad \left( \frac{t^2}{4}, \frac{t^3}{8} + 360t^2 \right), \quad \text{etc.}
\]

(Note that \(-P\) is not equal to \((x, -y)\) for \( P = (x, y) \) in general. Of course, we can rewrite the curve by a simple coordinate change in a more familiar style:

\[
y^2 = x^3 + (14400 - 476t)x^2 + (36000t^2 - 65t^3)x + 22500t^4 + t^5,
\]

but then the generators \( \{Q_i\} \) lose its simplest expression.) Other short vectors such as \( Q_i + Q_j \) (of norm 6/5) or \( Q_i - Q_j \) (norm 2: roots of \( A_4 \)) can be computed via the addition theorem. For instance,

\[
Q_1 + Q_2 = \left( \frac{t^2}{9} + 196t, \frac{t^3}{27} + 98t^2 + 47040t \right)
\]

\[
Q_2 - Q_1 = (345600 - 1324t + t^2, -165888000 + 1153920t - 1986t^2 + t^3)
\]

Observe that \( sp_{\infty}(Q_1 + Q_2) = u_1 + u_2 = 3 \) or \( sp_{\infty}(Q_2 - Q_1) = 2 - 1 = 1 \) is reflected in the highest coefficients. Also we see that \( (Q_2 - Q_1) \) is disjoint from \( (O) \) everywhere ([S2], Lemma 10.9), i.e. \( (Q_1) \) and \( (Q_2) \) are disjoint and pass the same irreducible component at \( t = 0 \). Let us check directly that this component must be \( \Theta_j \) with \( j = 2 \) or 3. First we have \( \langle Q_i, Q_i \rangle = 2 - c \), where \( c = contr_0(Q_i, Q_i) = 4/5 \) or 6/5 according as \( j = 1, 4 \) or \( j = 2, 3 \). Then (3.2) implies

\[
\langle Q_1, Q_2 \rangle = 1 + (Q_1O) + (Q_2O) - (Q_1Q_2) - c = 1 + 0 + 0 - 0 - c = 1 - c.
\]
Thus

\[ \langle Q_1 + Q_2, Q_1 + Q_2 \rangle = \langle Q_1, Q_1 \rangle + \langle Q_2, Q_2 \rangle + 2 \langle Q_1, Q_2 \rangle = (2 - c) + (2 - c) + 2(1 - c) = 6 - 4c. \]

On the other hand, the expression for \(Q_1 + Q_2\) shows that it has a norm smaller than 2, since it reduces to \((0, 0)\) at \(t = 0\). This implies that \(c > 1\), hence \(c = 6/5\), proving that \(j = 2\) or 3 and hence that \(Q_i\) are minimal vectors with norm 4/5. (This argument works for arbitrary \(\lambda\) with nondegenerate MWL, while the proof in (VI) is for generic \(\lambda\). N.B. The generic case, combined with a specialization argument, also proves the general case.)

Figure No. 8

![Image](image1)

Figure No. 16

![Image](image2)

8.2. **No. 16 \((L = A_3)\).**

(I) **No. 16:** \(T = D_5\), \(L = A_3\), \(M = A_{\frac{5}{2}}\)

\[ y^2 + p_3 t^2 y = x^3 + x (p_4 t) + x (p_2 t^3) + t^5, \quad \lambda = (p_2, p_3, p_4). \]

This is obtained from No. 8 by letting \(p_5 = 0\). It turns out that magic of weights works in this case too.

(II) The discriminant:

\[ A = -t^7 \{ 16 p_3^2 p_4^3 + \cdots + 432 t^3 \}. \]

The singular fibre at \(t = \infty\) is of type \(II\), as before. The singular fibre at \(t = 0\) is of type \(I_1^*\) under the condition:

\[ p_3 \neq 0, \quad p_4 \neq 0. \]

This is because then the discriminant has order 7 at \(t = 0\) and the Weierstrass equation reduces to the cuspidal cubic \(y^2 = x^3\) (cf. [K], [N], [T]). Thus we have

\[ f^{-1}(0) = \Theta_0 + \cdots + \Theta_3 + 2 \Theta_4 + 2 \Theta_5, \]

where the 6 irreducible components \(\Theta_j\) intersect as follows:
\( (\Theta_0 \Theta_4) = (\Theta_2 \Theta_4) = 1, \quad (\Theta_1 \Theta_3) = (\Theta_3 \Theta_3) = 1, \quad (\Theta_4 \Theta_3) = 1 \)

with all other \((\Theta_i \Theta_j) = 0\). In this case, the group structure on \(f^{-1}(0)/\Theta_i^j\) is a cyclic group of order 4, which is indexed by \(j = 0, 1, 2, 3\) as \(j \in \mathbb{Z}/4\mathbb{Z}\).

For \(\lambda\) generic, there are no other reducible fibres, and the trivial lattice is \(T = D_5\) and \(L = A_3, M = A_3^*\) ([OS]).

(III) The lattice \(M = A_3^*\) has 8 minimal vectors of minimal norm \(3/4\), forming 2 orbits under \(W(A_3) = S_4\), and \(M\) is generated by any 3 of the 4 vectors in an orbit. The number of roots in \(L = A_3\) is 12.

On the other hand, for \(P \in E(K) \simeq A_3^*\), suppose that \((P)\) meets \(\Theta_j\), then by (3.1) (cf. [S2], p. 229), we have

\[
\langle P, P \rangle = 2 + 2(PO) = \begin{cases} 
5 & (j = 1, 3) \\
4 & (j = 2) \\
1 & (j = 0) \\
0 & (j = 0) 
\end{cases}
\]

Thus the minimal vectors are exactly those \(P\) for which \((PO) = 0\) and \((P)\) meets \(\Theta_j\) for \(j = 1\) or 3; the latter distinguishes the 2 orbits.

(IV) Arguing as in (No. 8, III), we obtain the fundamental algebraic equation for No. 16:

\[
\Phi(u, \lambda) = u^4 + p_2 u^2 - p_3 u + p_4 = 0.
\]

Observe that this is deduced from the previous one by setting \(p_5 = 0\) (and dividing by \(u \neq 0\)).

(V) Let \(u_1, \ldots, u_4\) be the roots of the above equation. This is compatible with the notation of (No. 8, III) by letting \(u_5 = 0\) (recall that \(p_3\) is the product of \(u_1, \ldots, u_4\)). Hence the formula of the fundamental invariants for \(W(A_3)\) is the same as before (but with \(u_5 = 0\)), and it follows that our family is an excellent one for No. 16.

(VI) The 4 rational points \(Q_i = (t^2/|u_i^2|, t^3/|u_i^3|) \quad (i = 1, \ldots, 4)\) defined as in (No. 8, VI), give a half of minimal vectors, and any 3 among them give free generators of the MWL \(E(K) \simeq A_3^*\). (Note that \(Q_5\) "degenerates" to \(O\) under the specialization \(u_5 \to 0\)).

(VII) The frame invariant is given by

\[
J = p_3 p_4, \quad p_3 = \prod_{1 \leq i < j \leq 3} (u_i + u_j), \quad p_4 = u_1 \cdots u_4
\]

Hence the MWL is nondegenerate \(\simeq A_3^*\) if and only if we have \(u_i \neq 0\) and \(u_i \neq \pm u_j\) for all \(i, j\).

(VIII) Example. Take \(u_1 = 1, u_2 = 2, u_3 = 3, u_4 = -6\) (and \(u_5 = 0\)). Then

\[
y^2 - 60t^2y = x^3 - 36tx^2 - 25t^3x + t^5
\]

has the MWL \(E(Q(t)) \simeq A_3^*\), which is generated by
\[ Q_1 = (t^2, t^3), \quad Q_2 = \left( t^2, \frac{t^3}{4}, \frac{t^3}{8} \right), \quad Q_3 = \left( t^2, \frac{t^3}{9}, \frac{t^3}{27} \right). \]

8.3. No. 27 \((L = A_2)\).

(I) No. 27: \(T = E_6, \ L = A_2, \ M = A_2^*\)

\[ y^2 + p_3 t^2 y = x^3 + p_2 t^3 x + t^5, \quad \lambda = (p_2, p_3). \]

This is obtained from No. 16 by letting \(p_4 = 0\). Again magic of weights works here.

(II) The discriminant:

\[ \Delta = -t^8 \{ 27 p_2^2 + (64 p_3^3 + 216 p_3 t^3 + 432 t^2) \}. \]

The singular fibre at \(t = \infty\) is of type \(II\), as before. The singular fibre at \(t = 0\) is of type \(IV^*\) under the condition:

\[ p_3 \neq 0. \]

For the discriminant has order 8 at \(t = 0\), while the \(j\)-invariant has no pole there (cf. [K], [N], [T]). There are 7 irreducible components of which 3 have multiplicity 1:

\[ f^{-1}(0) = \Theta_0 + \Theta_1 + \Theta_2 + \cdots, \]

and the algebraic group \(f^{-1}(0)\) is the product of \(\Theta_5^* \simeq G_2\) and a cyclic group of order 3, indexed by \(j = 0, 1, 2\).

For \(\lambda\) generic, the trivial lattice is \(T = E_6\) and \(L = A_2, \ M = A_2^*\).

(III) The root lattice \(L = A_2\) is one of the most famous lattice: the hexagonal lattice (cf. [CS]). It has 6 roots (norm 2), and its dual lattice \(M = A_2^*\) has also 6 minimal vectors (of minimal norm \(2/3\)), but they divide into 2 orbits under \(W(A_2) = S_3\). Any 2 vectors in an orbit give free generators of \(M\).

On the other hand, we have for \(P \in E(K) \simeq A_2^*\)

\[ \langle P, P \rangle = 2 + (2P) - \begin{cases} 4 & (j = 1, 2) \\ 3 & (j = 0) \end{cases}, \]

if \((P)\) meets \(\Theta_j\) (cf. (3.1), [S2], p. 229). Thus the minimal vectors are the short vectors meeting \(\Theta_j\) for \(j = 1\) or 2, and the latter distinguishes the 2 orbits.

(IV) The fundamental algebraic equation for No. 27:

\[ \Phi(u, \lambda) = u^3 + p_2 u - p_3 = 0 \]

is obtained from (No. 16, IV) by setting \(p_4 = 0\).

(V) Let \(u_1, u_2, u_3\) be the roots of the above cubic equation \((u_1 + u_2 + u_3 = 0)\). Again this is compatible with the notation of No. 8 and No. 16 by letting \(u_4 = u_5 = 0\) (recall that \(p_4\) is the product of \(u_1, \cdots, u_4\)). It will now be evident that the family in question is excellent for No. 27.

(IV) The 3 rational points \(Q_i = (t^2/u_i^2, t^3/u_i^3)\) \((i = 1, 2, 3)\) defined as in (No. 8,
VI), give a half of 6 minimal vectors, and any 2 among them give free generators of the MWL \( E(K) \cong A_2^* \).

(VII) The frame invariant is given by

\[ J = p_3 = u_1 u_2 u_3. \]

Thus the MWL is nondegenerate \((\cong A_2^*)\) if and only if we have \( u_i \neq 0 \) and \( u_i \neq u_j \) for any \( i < j \), or \( p_3 \neq 0, 4p_2^2 + 27p_3^3 \neq 0 \) in terms of \( \lambda = (p_2, p_3) \).

(VIII) Example. Take \( u_1 = 1, u_2 = 2, u_3 = -3 \). Then

\[ y^2 - 6t^2 y = x^3 - 7t^3 x + t^5 \]

has the MWL \( E(Q(t)) \cong A_2^* \), which is generated by

\[ Q_1 = (t^2, t^3), \quad Q_2 = \left( \frac{t^2}{4}, \frac{t^3}{8} \right), \quad Q_3 = \left( \frac{t^2}{9}, \frac{-t^3}{27} \right) \quad (= -Q_1 - Q_2). \]

---

8.4. No. 43 \((L = A_1)\).

(I) No. 43: \( T = E_7, L = A_1, M = A_1^* \)

\[ y^2 = x^3 + p_2 t^3 x + t^5, \quad \lambda = p_2. \]

This is obtained from No. 27 by letting \( p_3 = 0 \). Again magic of weights works here.

(II) The discriminant is:

\[ \Delta = -16t^9(4p_2^3 + 27t). \]

The singular fibre at \( t = \infty \) is of type \( II \), as before. The singular fibre at \( t = 0 \) is of type \( III^* \) under the condition:

\[ p_2 \neq 0, \]

since the discriminant has order 9 at \( t = 0 \), while the \( j \)-invariant has no pole there (cf. [K], [N], [T]). There are 8 irreducible components of which 2 have multiplicity 1:
\[ f^{-1}(0) = \Theta_0 + \Theta_1 + \cdots, \]

and the algebraic group \( f^{-1}(0) \) is the product of \( \Theta_0 \approx G_a \) and a cyclic group of order 2, indexed by \( j=0, 1 \).

If \( p_2 \neq 0 \), the trivial lattice is \( T=E_7 \) and \( L=A_1, M=A_1^2 \).

(III) The lattice \( M = A_1^2 \) has 2 minimal vectors of minimal norm 1/2, which are interchanged by \( W(A_1^{2}) = \{ \pm 1 \} \), and \( M \) is generated by them. On the other hand, for \( P \in E(K) \approx A_1^2 \), we have

\[ \langle P, P \rangle = 2 + 2(PO) - \begin{cases} 
3 & (j=1) \\
2 & (j=0)
\end{cases}, \]

if \( (P) \) meets \( \Theta_j \) (cf. [S2], p. 229). Thus \( P \) is a minimal vector if and only if \( (PO)=0 \) and \( (P) \) meets \( \Theta_1 \).

(IV) The fundamental algebraic equation for No. 43 reads:

\[ \Phi(u, \lambda) = u^2 + p_2 = 0. \]

(V) Let \( u_1, u_2 \) be the 2 roots of the above equation \( (u_1 + u_2 = 0) \). This is compatible with the previous notation of No. 8, No. 16 and No. 27. Obviously the family defines an excellent family for No. 43.

(VI) The rational point \( Q_1 = (t^2/u_1^2, t^3/u_1^3) \) is a generator of \( E(K) \approx A_1^2 \).

(VII) The frame invariant is given by

\[ J = p_2 = u_1 u_2 = -u_1^3. \]

Thus the MWL is nondegenerate \( (\approx A_1^2) \) if and only if we have \( u_1 \neq 0 \), or equivalently, \( p_2 \neq 0 \).

(VIII) Example. Take \( u_1 = 1, u_2 = -1 \). Then

\[ y^2 = x^3 - t^3 x + t^5 \]

has the MWL \( E(Q(t)) \approx A_1^4 \), which is generated by \( Q_1 = (t^2, t^3) \). We have \( Q_2 = -Q_1 = (t^2, -t^3) \) and

\[ 2Q_1 = \left( \frac{t^2 - 6t + 1}{4}, \frac{t^3 + 15t^2 - 9t + 1}{8} \right), \]

the latter being a root of the narrow MWL \( E(Q(t)) \approx A_1 \).

Incidentally the familiar elliptic curve defined over \( Q(j) \) with the \( j \)-invariant \( j \)

\[ y^2 = x^3 - sx + s, \quad s = \frac{27j}{4(j - 12^3)} \]

can be transformed to the above example by a simple coordinate change. The obvious rational point \( (x, y) = (1, 1) \) corresponds to \( Q_1 \) and is a generator of \( E(Q(j)) \), if \( j \) is a variable over \( Q \).
Remark. As shown in the above discussion, the 4 families treated in this section are related by specializing the parameters: \( p_5 \to 0, p_4 \to 0 \) and \( p_3 \to 0 \), or in terms of \( u_i, u_5 \to 0, u_4 \to 0 \) and \( u_3 \to 0 \). While the variables "downstairs" are sufficient for describing how the singular fibre at \( t = 0 \) changes, i.e. how the frame is broken (in the terminology of 3.2), the degeneration of MWL can be best described in terms of the variables "upstairs" or the splitting variables \( u_i \).

9. No. 9, No. 10, No. 11 (\( L = D_4, A_3 \oplus A_1, A_2^{\oplus 2} \))

9.1. No. 9 (\( L = D_4 \)).

(I) No. 9: \( T = D_4, L = D_4, M = D_4^* \) (cf. [SS5], §6)

\[ y^2 = x^3 + x(p_4 - t^2) + (q_6 + q_4 t + q_2 t^2), \quad \lambda = (q_2, q_4, p_4, q_6). \]

This case has been treated in [SS5], which we review here to compare with other cases and also to make some supplements. The equation is weighted homogeneous of total weight 6; the relevant weight-table (cf. §3, 3.2) should be called of type \( D_4 \).

(II) The discriminant is a polynomial of degree 6:

\[ \Delta = -2^4(4(p_4 - t^2)^3 + 27(q_6 + q_4 t + q_2 t^2)^2), \]

and we have a singular fibre of type \( I_5^0 \) at \( t = \infty \):

\[ f^{-1}(\infty) = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + 2 \Theta_4. \]

There are no other reducible fibres in general, so for \( \lambda \) generic, the trivial lattice is \( T = D_4 \) and \( L = D_4, M = D_4^* \) (cf. [OS]).

(III) In a standard realization (cf. (No. 4, III)), we have

\[ D_4 \subset \mathbb{Z}^4 \subset D_4^*. \]

The Weyl group \( W(D_4) \) is generated by \( S_4 \) (permutations of 4 coordinates) and sign-change at 2 coordinates, and has order \( 2^3 \cdot 4! \). The lattice \( M = D_4^* \) has 24 minimal vectors of minimal norm 1, which divide into 3 orbits under \( W(D_4) \). If we denote by \( \{ e_i \} \) the unit vectors in \( \mathbb{Z}^4 \), then they are: (i) \( \{ \pm e_i \} \), (ii) \( \{ \pm f, f - e_i - e_j \} \) \( i < j \), (iii) \( \{ \pm (f - e_i) \} \), where we set \( f = \frac{1}{2}(e_1 + \cdots + e_4) \). The quotient group \( D_4^*/D_4 \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), with representatives \( \{ 0, e_1, f, f - e_1 \} \). In particular, \( M \) is generated by \( \{ e_1, e_2, e_3, f \} \). The roots in \( L = D_4 \) are \( \{ \pm e_i \pm e_j \) \( i \neq j \}\).

The minimal vectors \( P \in E(K) \approx D_4^* \) are precisely the short vectors (\( P \)) which meet \( \Theta_j \) \((j > 0)\) (use (3.1)); the 3 cases \( j = 1, 2, 3 \) correspond to the 3 orbits under \( W(D_4) \). In terms of Weierstrass coordinates, they are given by

\[ P = (at + b, dt + e). \]

(IV) The condition for \( a, b, d, e \) is easily written down. In particular, we see \( a = 0, 1 \) or \(-1\), which correspond to the 3 cases as above, say \( j = 1, 2 \) or 3 (by renaming \( \Theta_j \) if necessary). Let \( a = 0 \); then \( b = q_2 - d^2, e = q_4/(2d) \) (if \( d \neq 0 \), and elimination yields the fundamental algebraic equation for \( d \):
\[ \Phi(d, \lambda) = d^8 - 3q_2 d^6 + (p_4 + 3q_2^3) d^4 - (q_6 + q_2 p_4 + q_2^3) d^2 + \left( \frac{1}{2} q_4 \right)^2 = 0 \, . \]

(V) Let \( \pm u_i \) \((i = 1, \ldots, 4)\) be the roots of this equation. Then the relation of roots and coefficients gives the formula (cf. [S5], Th. (D_4), p. 678):

\[
\begin{align*}
q_2 &= \varepsilon_2^d / 3 \\
p_4 &= \varepsilon_4 - 3q_2^2 \\
q_4 &= u_1 \cdots u_4 \\
q_6 &= \varepsilon_6 - q_2 p_4 - q_2^2.
\end{align*}
\]

Here \( \varepsilon_2^d \) denotes the \( d \)-th elementary symmetric function of \( u_1^2, \ldots, u_4^2 \). This gives an explicit formula of the fundamental invariants of \( W(D_4) \), and we see that the family in question is an excellent family for No. 9 (cf. No. 4, No. 5).

(IV) The 4 rational points

\[ Q_i = (q_2 - u_i^2, u_i t + u_i u_4 u_t) \quad (i = 1, \ldots, 4), \quad \{i, j, k, l\} = \{1, 2, 3, 4\} \]

generate the sublattice \( \approx \mathbb{Z}^4 \) of \( E(K) \approx D_4^* \). The full MWL is generated by \( Q_i \) and one more rational point corresponding to \(-f\), which has the form

\[ Q_0 = (x, y), \quad \begin{cases} 
x = - t + q_2 + \sum_{i < j} u_i u_j \\
y = \left( \sum_{i} u_i \right) t - \left( \sum_{i < j} \left( u_i^2 u_j + u_i u_j^2 \right) + 2 \sum_{i < j < k} u_i u_j u_k \right) \, .
\end{cases} \]

(VII) By (II), the frame invariant can be taken as the constant 1. The MWL is nondegenerate if and only if \( u_i \neq \pm u_j \) \((i < j)\).

(VIII) For an example, see [S5], p. 683.

9.2. No. 10 \((L = A_3 \oplus A_1)\).

(I) No. 10: \( T = A_3 \oplus A_1, \, L = A_3 \oplus A_1, \, M = A_3^* \oplus A_3^* \)

\[ y^2 + p_3 x y = x^3 + x^2 (p_2 t) + x(p_4 t^2 + t^3) + q_2 t^4, \quad \lambda = (p_2, p_3, p_4, q_2). \]

The equation is obtained by magic of weights using the weight-table of type \( E_7 \).

(II) The discriminant is:

\[ A = - t^4 \{ p_3^2 (-p_2^2 + p_3^2 q_2) + \cdots + 64 t^5 \} . \]

We have a singular fibre of type \( III \) at \( t = \infty \), as in No. 2. The singular fibre at \( t = 0 \) is of type \( I_4 \) under the condition:

\[ p_3 \neq 0, \quad J_1 = - p_2^2 + p_3^2 q_2 \neq 0. \]

This is verified as in (No. 5, II), noting that the reduced curve at \( t = 0 \) has a node at \((0, 0)\). Let
\[ f^{-1}(0) = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3, \quad f^{-1}(\infty) = \Theta'_0 + \Theta'_1. \]

There are no other reducible fibres in general. Hence, for \( \lambda \) generic, the trivial lattice is \( T = A_3 \oplus A_1 \) and \( L = A_3 \oplus A_1, M = A^*_3 \oplus A^*_1 \) (cf. [OS]).

(III) The lattice \( M = A^*_3 \oplus A^*_1 \) has 2 minimal vectors of minimal norm 1/2 (coming from the factor \( A^*_1 \)), say \( \pm \tilde{u}_0 \), and 8 vectors of norm 3/4 (coming from the other factor) which divide into 2 orbits under \( W(A_3) \). If we denote by \( \tilde{u}_i (i=1, \ldots, 4) \) an orbit, then \( \{ \tilde{u}_i (i=0, \ldots, 3) \} \) forms a basis of \( M \) (note that \( \sum_{i=1}^4 \tilde{u}_i = 0 \)).

The rational points \( Q_i \in E(K) \) corresponding to \( \tilde{u}_i \in M \) can be described as follows. They are short vectors, i.e. \( (Q_i, O) = 0 \), and the minimal vectors \( \pm Q_0 \) meet the components \( \Theta_2 \) and \( \Theta'_1 \); the formula (3.1) says \( 2 - 1/2 = 1/2 \). The 4 points \( Q_i \) pass through \( \Theta'_1 \) and either one of \( \Theta_j (j=1, 3) \), and (3.1) reads \( 2 - 3/4 - 1/2 = 3/4 \). In terms of Weierstrass coordinates, they are given by

\[ P = (at, ct^2 + dt). \]

(IV) Substituting it into the defining equation, we get the condition for \( a, c, d \):

\[ a = c^2 - q_2, \quad d = 0 \quad \text{or} \quad d = -ap_3, \quad a^3 + a^2p_2 - ap_3 + ap_4 = 2cd. \]

Consider the case \( d = 0 \) (\( d = -ap_3 \) corresponds to changing \( P \to -P \)). Then eliminating \( a \), we get the fundamental algebraic equation:

\[ \Phi(c, \lambda) = a^3 + a^2p_2 - ap_3 + ap_4 = (c^2 - q_2)(c^4 + c^2(p_2 - 2q_2) - cp_3 + p_4 - p_2q_2 + q_2^2) = 0. \]

(V) Let \( c = \pm u_0 \) be the roots of the first factor \( c^2 - q_2 \); then we have \( a = 0 \) and \( Q_0 = (0, u_0 t^2), \quad q_2 = u_0^2 \).

Next let \( \pm u_i (i=1, \ldots, 4) \) be the roots of the second factor of \( \Phi(c, \lambda) \). By the relation of roots and coefficients, we have \( u_1 + u_2 + u_3 + u_4 = 0 \) and

\[
\begin{align*}
\varepsilon_2 &= p_2 - 2q_2 \\
\varepsilon_3 &= p_3 \\
\varepsilon_4 &= p_4 - p_2q_2 + q_2^2 \\
\end{align*}
\]

i.e.

\[
\begin{align*}
q_2 &= u_0^2 \\
p_2 &= \varepsilon_2 + 2q_2 \\
p_3 &= \varepsilon_3 \\
p_4 &= \varepsilon_4 + p_2q_2 - q_2^2.
\end{align*}
\]

Here \( \varepsilon_d \) is the \( d \)-th elementary symmetric function of \( u_1, \ldots, u_4 \). This gives an explicit formula of the fundamental invariants of the Weyl group \( W(A_3 \oplus A_1) = W(A_3) \times W(A_1) \), and we see that the family in question is an excellent family for No. 10.

(VI) Generators. Let

\[ Q_i = ((u_i^2 - q_2)t, u_it^2) \quad (i = 1, \ldots, 4). \]

Then \( Q_4 + \cdots + Q_4 = 0 \) and \( \{ Q_0, Q_1, Q_2, Q_3 \} \) generate the full MWL \( E(K) \cong M = A^*_3 \oplus A^*_1 \).

(VII) By (II), the frame invariant is given by
\[ J(\lambda) = p_3 J_1 = p_3 (-p_3^2 + p_3^2 q_2). \]

In terms of splitting variables \( u_i \), we have
\[
p_3 = -(u_1 + u_2)(u_1 + u_3)(u_2 + u_3) \\
J_1 = -\prod_{i=1}^{4} (u_0 + u_i)(u_0 - u_i).
\]

The invariant \( \delta(\lambda) \) is given by the product of all roots of \( L = A_3 \oplus A_1 \), expressed in terms of \( u_i \):
\[
\delta = (2u_0)^2 \prod_{1 \leq i < j \leq 4} (u_i - u_j)^2.
\]

Therefore the MWL is nondegenerate if and only if \( u_0 \neq 0 \) and \( u_i \neq \pm u_j (0 \leq i < j \leq 4) \).

(VIII) Example. Take \( u_0 = 4, u_1 = 0, u_2 = 1, u_3 = 2, u_4 = -3 \), which satisfy the above condition. Then \( q_2 = 16, p_2 = 25, p_3 = -6, p_4 = 144 \). So the elliptic curve \( E(\mathbb{Q}(t)) \)
\[
y^2 - 6xy = x^3 + 25tx^2 + (144t^2 + t^3)x + 16t^4
\]
has the MWL \( E(\mathbb{Q}(t)) \cong M = A_3^1 \oplus A_1^1 \). The rational points \( Q_0, \ldots, Q_4 \) are given as follows:
\[
(0, 4t^2), \quad (-16t, 0), \quad (-15t, t^2), \quad (-12t, 2t^2), \quad (-7t, -3t^2),
\]
of which the first 4 elements form free generators. The duplication formula on \( E \) gives
\[
2Q_0 = \left( \frac{1}{64} (20160 - 1312t + t^2), \frac{1}{512} (-2419200 + 137280t - 856t^2 - t^3) \right),
\]
which confirms that the point \( 2Q_0 \) has norm \( 2 \cdot 1/2 = 2 \), i.e., it is a root in \( A_1 \in L \). (This also checks that \( (Q_0) \) passes through \( \Theta_2 \).) For other \( P = 2Q_i \), the \( x \)-coordinate is a rational function of degree 4 in \( t \), reflecting the fact that their norm \( 2^2 \cdot 3/4 = 3 \) shows \( (PO) = 1 \).
9.3. No. 11 ($L = A_2^{\oplus 2}$).

(I) No. 11: $T = A_2^{\oplus 2}$, $L = A_2^{\oplus 2}$, $M = A_2^{\oplus 2}$

\[ y^2 + p_2xy + p_3y = x^3 + x(q_2t^2) + (q_3t^3 + t^4), \quad \lambda = (p_2, p_3, q_2, q_3). \]

This equation is obtained by magic of weights using the weight-table of type $E_6$.

(II) The discriminant is:

\[ \Delta = t^3 \{-p_2^2(-p_3^3 - p_2^2p_3q_2 + p_3^2q_3) + (-27p_3^4 + p_2(\cdots)t + \cdots - 432t^5) \}. \]

We have a singular fibre of type $IV$ at $t = \infty$, as in No. 3. The singular fibre at $t = 0$ is of type $I_3$ or $IV$ under the condition:

\[ J = -p_3^3 - p_2^2p_3q_2 + p_3^2q_3 \neq 0. \]

This is verified as in No. 6, noting that the reduced curve at $t = 0$ has a node or a cusp at $(0, 0)$ according as $p_2$ is zero or not. Let

\[ f^{-1}(0) = \Theta_0 + \Theta_1 + \Theta_2, \quad f^{-1}(\infty) = \Theta'_0 + \Theta'_1 + \Theta'_2. \]

For $\lambda$ generic, there are no other reducible fibres, so the trivial lattice is $T = A_2^{\oplus 2}$, which implies $L = A_2^{\oplus 2}$, $M = A_2^{\oplus 2}$ (cf. [OS]).

(III) The lattice $A_2^*$ has 6 minimal vectors of norm 2/3, which form 2 orbits under $W(A_2) = \mathcal{S}_3$; 3 vectors in each orbit sum up to 0 and any 2 among them generate this lattice. Hence the lattice $M = A_2^{\oplus 2}$ has 12 minimal vectors of minimal norm 2/3, which form 4 orbits under $W(L) = W(A_2) \times W(A_2)$ of the form:

\[ \{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}, \quad \{-\tilde{u}_1, -\tilde{u}_2, -\tilde{u}_3\}, \quad \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}, \quad \{-\tilde{v}_1, -\tilde{v}_2, -\tilde{v}_3\}, \]

where $\tilde{u}_1 + \tilde{u}_2 + \tilde{u}_3 = 0$, $\tilde{v}_1 + \tilde{v}_2 + \tilde{v}_3 = 0$.

By (3.1), $P \in E(K)$ is a minimal vector if and only if $(PO) = 0$ and $(P)$ passes through a non-identity component both at $t = 0$ and $t = \infty$. In terms of Weierstrass coordinates, we have

\[ P = (at, ct^2 + dt). \]

As in No. 3, we have $c = \pm 1$ and $d = \pm 2sp\alpha(P)$.

(IV) It follows from the defining equation that $a$, $c$, $d$ should satisfy the condition:

\[ c = 1 \text{ or } -1, \quad d = 0 \text{ or } d = -ap_2 - p_3, \quad a^3 + a(q_2 - cp_3) - 2cd - cp_3 + q_3 = 0. \]

The 4 choices of $c$, $d$ correspond to the 4 orbits mentioned above. The choice $c = 1$, $d = 0$ (or $c = -1$, $d = -ap_2 - p_3$) leads to one of the fundamental equations:

\[ \Phi_1(a, \lambda) = a^3 + a(-p_2 + q_2) - p_3 + q_3 = 0, \]

and the choice $c = -1$, $d = 0$ (or $c = 1$, $d = -ap_2 - p_3$) to another:

\[ \Phi_2(a, \lambda) = a^3 + a(p_2 + q_2) + p_3 + q_3 = 0. \]

(V) Let $a = u_i$ ($i = 1, 2, 3$) (resp. $v_i$ ($i = 1, 2, 3$)) be the roots of the first (resp.
second) equation. By the relation of roots and coefficients, we have \( u_1 + u_2 + u_3 = 0, \)
\( v_1 + v_2 + v_3 = 0 \) and
\[
\begin{align*}
p_2 &= \frac{1}{2} (-e_2 + \eta_2) \\
q_2 &= \frac{1}{2} (e_2 + \eta_2) \\
p_3 &= \frac{1}{2} (e_3 - \eta_3) \\
q_3 &= \frac{1}{2} (-e_3 - \eta_3).
\end{align*}
\]

Here \( e_d \) is the \( d \)-th elementary symmetric function of \( u_1, u_2, u_3, \) and \( \eta_d \) is that of \( v_1, v_2, v_3. \)

Therefore the two extensions of polynomial rings with Galois group \( W(A_2) = S_3 \)
\[
\mathcal{O}[u_1, u_2] \supseteq \mathcal{O}[e_2, e_3], \quad \mathcal{O}[v_1, v_2] \supseteq \mathcal{O}[\eta_2, \eta_3]
\]
glue into a single extension with Galois group \( W(A_2) \times W(A_2) \)
\[
\mathcal{O}[u_1, u_2, v_1, v_2] \supseteq \mathcal{O}[e_2, e_3, \eta_2, \eta_3] = \mathcal{O}[p_2, q_2, p_3, q_3].
\]
This proves that the family in question is an excellent family for No. 11.

(VI) Generators. Let
\[
P_i = (u_it, t^2), \quad Q_i = (v_it, -t^2) \quad (i = 1, \hdots, 3).
\]
Then \( \{P_1, P_2, Q_1, Q_2\} \) generate the full MWL \( E(K) \cong M = A_2^2 \oplus A_2^2 \). Note that
\( P_1 + P_2 + P_3 = 0 \), since the 3 points are collinear, lying on the line \( y = t^2 \), and similarly
for \( Q_i \).

(VII) The frame invariant is given by \( J \) in (II), which is expressed as
\[
J(\lambda) = \frac{1}{8} \prod_{i,j=1}^{3} (u_i - v_j)
\]
in terms of splitting variables \( u_i, v_i \). The invariant \( \delta(\lambda) \) is equal to the product of the
discriminants of \( \Phi_1 \) and \( \Phi_2 \). Therefore the MWL is nondegenerate if and only if \( u_1, u_2, u_3, v_1, v_2, v_3 \) are mutually distinct.

(VIII) Example. Take \( u_1 = 0, u_2 = 1, v_1 = 2, v_2 = 3 \), which satisfy the above
condition. Then we get an elliptic curve \( E/\mathcal{O}(t) \)
\[
y^2 - 9xy + 15ty = x^3 - 10t^2x + 15t^3 + t^4,
\]
having the MWL \( E(\mathcal{O}(t)) \cong M = A_2^2 \oplus A_2^2 \). The generators \( P_1, P_2, Q_1, Q_2 \) are given by
\( (0, t^2), \quad (t, t^2), \quad (2t, -t^2), \quad (3t, -t^2). \)

10. No. 15 and No. 26 (\( L = A_2 \oplus A_1, A_2^{\oplus 2} \))

10.1. No. 15 (\( L = A_2 \oplus A_1 \)).

(I) No. 15: \( T = A_5, \quad L = A_2 \oplus A_1, \quad M = A_2^2 \oplus A_2^2 \)
\[
y^2 + q_2p_3xy + p_3t^2y = x^3 + p_2q_2tx^2 + (p_2 + q_2)t^3x + t^5, \quad \lambda = (p_2, p_3, q_2).
\]
Note that this equation is obtained from No. 8 by specializing the parameters in a way less obvious than the case No. 16. It will be seen below that this arises quite naturally when we use the splitting variables.

(II) The discriminant is:

$$\Delta = t^6 \left( q_2^4 p_3^4 (q_2^2 + p_3^2 - 2q_2^2 p_2 + q_2 p_2^2) + \cdots - 432t^4 \right).$$

The singular fibre at \( t = \infty \) is the same type II as in No. 8. The singular fibre at \( t = 0 \) is of type \( I_6 \) under the condition:

$$q_2 \neq 0, \quad p_3 \neq 0, \quad J_1 = q_2^2 + p_3^2 - 2q_2^2 p_2 + q_2 p_2^2 \neq 0,$$

since \( \Delta \) has order 6 at \( t = 0 \) and the reduced equation gives a nodal cubic \( y^2 + q_2 p_3 xy = x^3 \) (with the node \((0, 0))\). It has 6 irreducible components \( \Theta_j \) \((j = 0, \cdots, 5)\) forming a hexagonal cycle.

For \( \lambda \) generic, the trivial lattice is \( T = A_5 \), and we have \( L = A_2 \oplus A_1, M = A_2^* \oplus A_1^* \) (cf. [OS]).

(III) Given \( P \in E(K) \cong M \), suppose that \( (P) \) meets \( \Theta_j \). By the formula (3.1), we have

$$\langle P, P \rangle = 2 + 2(PO) - \begin{cases} 5 & (j = 1, 5) \\ 6 \\ 4 & (j = 2, 4) \\ 3 \\ 2 & (j = 3) \\ 0 & (j = 0). \end{cases}$$

Hence \( P \) has norm 1/2 (resp. 2/3 or 7/6) if and only if \( (PO) = 0 \) and \( (P) \) passes through \( \Theta_3 \) (resp. \( \Theta_2, \Theta_4 \) or \( \Theta_1, \Theta_5 \)).

(IV) As in No. 8, there are 5 rational points:

$$P = \left( \frac{t^2}{u^2}, \frac{t^3}{u^3} \right) \quad (u = sp_{\infty}(P)),$$

where \( u \) satisfies the fundamental algebraic equation:

$$\Phi(u, \lambda) = u^5 + (p_2 + q_2)u^3 - p_3 u^2 + p_2 q_2 u - q_2 p_3 = (u^2 + q_2)(u^3 + p_2 u - p_3) = 0.$$

(V) Let \( \pm u_0 \) be the roots of \( u^2 + q_2 = 0 \) and let \( u_1, u_2, u_3 \) be the roots of \( u^3 + p_2 u - p_3 = 0 \) \((u_1 + u_2 + u_3 = 0)\). The relation of roots and coefficients gives

$$\begin{align*}
q_2 &= -u_0^2 \\
p_2 &= u_1 u_2 + u_1 u_3 + u_2 u_3 \\
p_3 &= u_1 u_2 u_3.
\end{align*}$$
This represents the fundamental invariants of the Weyl group $W(L) = W(A_2) \times W(A_1)$, and we can conclude that our family is an excellent family for No. 15.

**Remark.** At this point, we can explain the degeneration of MWL from No. 8 to No. 15. In the notation of (No. 8, II), we should have $p_5 \neq 0$ and $J_1 = 0$ in order to make the singular fibre at $t = 0$ to be of type $I_6$. In terms of the splitting variables $u_1, \cdots, u_5$ ($u_1 + \cdots + u_5 = 0$) in (No. 8, V), the invariants $p_5, J_1$ are expressed as in (No. 8, VII) and so we should have $u_i \neq 0$ for all $i$ and $u_i + u_j = 0$ for some $i < j$.

Now let $u_4 = u_0$ and $u_5 = -u_0$ so that $u_1 + u_2 + u_3 = 0$. Then the elementary symmetric functions $p'_2$ of $u_1$, $u_2$, $u_3$, which were denoted by $p_2$ in No. 8, are rewritten as follows:

$$p'_2 = p_2 + q_2, \quad p'_3 = p_3, \quad p'_4 = p_2 q_2, \quad p'_5 = q_2 p_3$$

This defines the family for No. 15 under consideration.

(VI) Let

$$Q_i = \begin{pmatrix} t^2 \\ u_i^2 \\ t^3 \\ u_i^3 \end{pmatrix} \quad (i = 0, \cdots, 3).$$

Observe that $\{ \pm Q_0 \}$ and $\{ Q_1, Q_2, Q_3 \}$ are stable under the Galois group $W(L)$ (for $\lambda$ generic). It follows that $\pm Q_0$ are the minimal vectors of norm $1/2$ and $Q_1, Q_2, Q_3$ form one orbit of vectors of norm $2/3$, and that $Q_1, Q_2$ and $Q_0$ generate the MWL $E(K) \cong M = A^*_2 \oplus A^*_1$. In view of (III), $Q_0$ meets $\Theta_3$, while $(Q_i) (i = 1, 2, 3)$ meet either $\Theta_2$ or $\Theta_4$.

(VII) The frame invariant is given by

$$J(\lambda) = q_2 p_3 J_1 \quad (cf. \ II),$$

where $J_1$ is expressed as follows in terms of $u_i$:

$$J_1 = \prod_{i=1}^{3} (u_0 - u_i)(u_0 + u_i).$$

Thus the MWL is nondegenerate if and only if all $u_0, \cdots, u_3$ are different from $0$ and from each other and $u_0 \neq -u_i$ for any $i = 1, 2, 3$.

(VIII) **Example.** Let $u_0 = 4, u_1 = 1, u_2 = 2, u_3 = -3$. Then the elliptic curve

$$y^2 + 96xy - 6t^5 y = x^3 + 112tx^2 - 23t^3 x + t^5$$

has a rank 3 MWL $E(Q(t)) \cong M = A^*_2 \oplus A^*_1$, which is generated by

$$Q_1 = (t^2, t^3), \quad Q_2 = \left( t^2, \frac{t^3}{8} \right), \quad Q_0 = \left( t^2, \frac{t^3}{16}, \frac{t^3}{64} \right).$$
10.2. No. 26 ($L = A_1^{\oplus 2}$)

(I) No. 26: $T = D_6$, $L = A_1^{\oplus 2}$, $M = A_1^{\oplus 2}$

$$y^2 = x^3 + p_2q_2tx^2 - (p_2 + q_2)t^3x + t^5, \quad \lambda = (p_2, q_2).$$

This is obtained from No. 15 by letting $p_3 = 0$ and changing the sign of $p_2, q_2$.

(II) The discriminant is:

$$\Delta = 16t^8\{p_2^2q_2^2(p_2 - q_2)^2 + \cdots - 27t^2\}.$$

The singular fibre at $t = \infty$ is the same type II as in No. 8. The singular fibre at $t = 0$ is of type $I_2^*$ provided that

$$p_2 \neq 0, \quad q_2 \neq 0, \quad p_2 \neq q_2.$$

For $\Delta$ has order 8 and the $j$-invariant has a pole of order 2 at $t = 0$. It has 7 irreducible components $\Theta_j$ ($j = 0, \cdots, 6$), which we arrange in a similar way to (No. 16, II); in particular, $\Theta_j$ ($j = 0, \cdots, 3$) have multiplicity 1.

For $\lambda$ generic, the trivial lattice is $T = D_6$, and $L = A_1^{\oplus 2}$, $M = A_1^{\oplus 2}$ (cf. [OS]).

(III) The lattice $M$ is generated by 2 minimal vectors (of norm 1/2) which are orthogonal to each other. There are 4 minimal vectors and 4 vectors of norm 1. By (3.1), if $(P)$ meets $\Theta_j$, then

$$\langle P, P \rangle = 2 + 2(PO) - \begin{cases} 3 & (j = 1, 3) \\ 2 & (j = 2) \\ 1 & (j = 0) \end{cases}.$$

Hence $P$ has norm 1/2 (resp. 1) if and only if $(PO) = 0$ and $(P)$ passes through $\Theta_1$ or $\Theta_3$ (resp. $\Theta_2$).

(IV) In this case, there are 4 rational points of the form:
\[ P = \left( \frac{t^2}{u^2}, \frac{t^3}{u^3} \right), \]

and the fundamental algebraic equation is given by
\[ \Phi(u, \lambda) = (u^2 - p_2)(u^2 - q_2) = 0. \]

(V) Let \( u_1, u_2 \) be the splitting variables such that
\[ p_2 = u_1^2, \quad q_2 = u_2^2. \]

For generic \((p_2, q_2)\), the extension \( Q(u_1, u_2)/Q(p_2, q_2) \) is a Galois extension whose Galois group is the Weyl group \( W(L) = W(A_1) \times W(A_1) = \{ \pm 1 \} \times \{ \pm 1 \} \). It is easy to see that our family is an excellent family for No. 26.

(VI) Let
\[ Q_i = \left( \frac{t^2}{u_i^2}, \frac{t^3}{u_i^3} \right) \quad (i = 1, 2). \]

We claim that \( Q_1, Q_2 \) are minimal vectors such that \( \langle Q_1, Q_2 \rangle = 0 \). Indeed, let \( \sigma \) be the automorphism of \( Q(u_1, u_2) \) sending \( u_1 \) to \(-u_1\) and leaving \( u_2 \) fixed. Then we have \( Q_1 = -Q_1, \quad Q_2 = Q_2 \). On the other hand, we have \( \langle Q_1, Q_2 \rangle = \langle Q_1, Q_2 \rangle \) in general (see [S2], Prop. 8.13). This proves \( \langle Q_1, Q_2 \rangle = 0 \). Now both \( Q_1, Q_2 \) have norm < 2 by (3.1) mentioned above. Therefore the only possibility is that they are minimal vectors in \( M = A_1^{\oplus 2} \). (The claim can be also proven by other argument which works even if the extension is trivial; cf. example below).

(VII) The frame invariant is given by
\[ J(\lambda) = p_2q_2(p_2 - q_2), \]
while \( \delta = (2u_1)(2u_2))^2 = 2^4p_2q_2 \). Hence the MWL is nondegenerate if and only if \( u_1 \neq 0, u_2 \neq 0, u_1 \pm u_2 \neq 0 \).

(VIII) Example. Let \( u_1 = 1, u_2 = 2 \). Then we get the elliptic curve
\[ y^2 = x^3 + 4tx^2 - 5t^3x + t^5, \]
and the rational points
\[ Q_1 = (t^2, t^3), \quad Q_2 = \left( \frac{t^2}{4}, \frac{t^3}{8} \right) \]
forming an orthogonal basis of \( E(Q(t)) \cong M = A_1^{\oplus 2} \). By the addition theorem, we have
\[ Q_1 - Q_2 = (-4t + t^2, 6t^2 - t^3), \quad Q_1 + Q_2 = \left( \frac{1}{9}(-36t + t^2), \frac{1}{27}(126t^2 + t^3) \right); \]
these are also short vectors and, in fact, \( \{ \pm Q_1 \pm Q_2 \} \) are all the vectors of norm 1. Note also that the above expression implies that \( (Q_1) \) and \( (Q_2) \) pass through different irreducible components at \( t = 0 \); by renumbering if necessary, \( (Q_1) \) meets \( \Theta_1 \) and \( (Q_2) \).
meets $\Theta_3$, and $\{\pm Q_1 \pm Q_2\}$ meet $\Theta_2$. The 4 roots in $L = A_1^{\oplus 2}$ are given by $\pm 2Q_1$, $\pm 2Q_2$; for instance,

$$2Q_1 = \left(\frac{1}{4}(9 + 2t + t^2), \frac{1}{8}(-27 - 33t + 3t^2 + t^3)\right).$$

This shows directly that $Q_1$ is a minimal vector, since $2Q_1$ has norm $\leq 2$ as a short vector so that $\langle Q_1, Q_1 \rangle \leq 2/2 = 1/2$. Combined with the above formula for $Q_1 \pm Q_2$ showing that their norm is less than 2, we deduce that $|\langle Q_1, Q_2 \rangle| < 1/2$, which implies that $\langle Q_1, Q_2 \rangle = 0$ by (3.2) and [S2], p. 229.

References


Department of Mathematics,
Rikkyo University
Nishi-Ikebukuro, Toshimaku,
Tokyo 171, Japan