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A New Class of Analytic Functions with Negative Coefficients

by

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A systematic investigation of a new class of analytic functions, which is defined in terms of fractional derivatives, is presented. Apart from various coefficient bounds, and a number of characterization and distortion theorems, many interesting and useful properties of this class of functions are given; some of these properties involve, for example, linear combinations and modified Hadamard products of several functions belonging to the class introduced and studied here. Relevance of one of the results obtained here to the celebrated Bieberbach conjecture (now de Branges's theorem) is also indicated.

1. Introduction and definitions

Let $\mathscr{A}$ denote the class of functions $f(z)$ defined by

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0) \tag{1.1}$$

which are analytic in the unit disk $\mathscr{U}=\{z: |z|<1\}$. Also let $\mathscr{T}$ be the subclass of $\mathscr{A}$ consisting of analytic and univalent functions $f(z)$ of the form (1.1). Schild [10] studied a subclass of $\mathscr{T}$ consisting of polynomials having $|z|=1$ as radius of schlichtness (univalence). Subsequently, Silverman [11] proved a number of useful results for the subclasses $\mathscr{T}^*(\alpha)$ and $\mathscr{C}(\alpha)$ of $\mathscr{T}$, where $\mathscr{T}^*(\alpha)$ and $\mathscr{C}(\alpha)$ denote, respectively, the classes of starlike functions of order α and convex functions of order α, $0 \leqq \alpha < 1$.

A function $f(z) \in \mathscr{T}$ is said to be in the class $\mathscr{P}^*(\alpha, \beta)$, which was studied recently by Gupta and Jain [5], if and only if

$$\left|\frac{f'(z)-1}{f'(z)+(1-2\alpha)}\right|<\beta \qquad (z \in \mathscr{U}) \tag{1.2}$$

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for $0 \leqq \alpha < 1$ and $0 < \beta \leqq 1$. The object of the present paper is to investigate systematically a new class $\mathscr{P}_{\lambda}^{*}(\alpha, \beta)$ of analytic functions $f(z)$ belonging to the class $\mathscr{A}$ and satisfying the condition

$$\left| \frac{\Gamma(2-\lambda) z^{\lambda-1} D_z^{\lambda} f(z) - 1}{\Gamma(2-\lambda) z^{\lambda-1} D_z^{\lambda} f(z) + (1-2\alpha)} \right| < \beta \qquad (z \in \mathscr{U}) \tag{1.3}$$

for $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha < 1$, and $0 < \beta \leqq 1$; here $D_z^{\lambda} f(z)$ denotes the fractional derivative of $f(z)$ of order λ, as defined below, with

$$D_z^0 f(z) = f(z) \quad \text{and} \quad D_z^1 f(z) = f'(z) . \tag{1.4}$$

Since the condition (1.3) reduces, when $\lambda = 1$, to (1.2), the subclass of $\mathscr{P}_1^{*}(\alpha, \beta)$ consisting of functions $f(z) \in \mathscr{T}$ is precisely the class $\mathscr{P}^{*}(\alpha, \beta)$ studied by Gupta and Jain [5]. Furthermore, in the special case when $\lambda = 0$, the condition (1.3) assumes the elegant form:

$$\left| \frac{z^{-1} f(z) - 1}{z^{-1} f(z) + (1-2\alpha)} \right| < \beta \qquad (z \in \mathscr{U}) \tag{1.5}$$

where, as before, $0 \leqq \alpha < 1$ and $0 < \beta \leqq 1$. Thus $\mathscr{P}_0^{*}(\alpha, \beta)$ represents the class of functions $f(z) \in \mathscr{A}$ satisfying the inequality (1.5).

Several essentially equivalent definitions of fractional derivatives and fractional integrals have been given in the literature (*cf.*, *e.g.*, [2, Chapter 13], [6], [8], [9], and [12, p. 28 *et seq.*]). We find it to be convenient to restrict ourselves to the following definitions used recently by Owa [7] (and by Srivastava and Owa [13]):

DEFINITION 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta , \tag{1.6}$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

DEFINITION 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta , \tag{1.7}$$

where $0 \leqq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1 above.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by

(1.8) $$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z),$$

where $0 \leqq \lambda < 1$ and $n \in \mathcal{N} \cup \{0\}$, $\mathcal{N}$ being, as usual, the set of natural numbers.

For the class of functions belonging to $\mathcal{P}_\lambda^*(\alpha, \beta)$ we prove a number of sharp results including, for example, coefficient and distortion theorems, and theorems involving modified Hadamard products.

2. A theorem on coefficient bounds

THEOREM 1. *A function $f(z)$ defined by* (1.1) *is in the class $\mathcal{P}_\lambda^*(\alpha, \beta)$ if and only if*

(2.1) $$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (1+\beta) a_n \leqq 2\beta(1-\alpha).$$

The result (2.1) *is sharp.*

Proof. Assume that the inequality (2.1) holds true and let $|z|=1$. Then we obtain

$$\left| \frac{\Gamma(2-\lambda) D_z^{\lambda} f(z)}{z^{1-\lambda}} - 1 \right| - \beta \left| \frac{\Gamma(2-\lambda) D_z^{\lambda} f(z)}{z^{1-\lambda}} + (1-2\alpha) \right|$$

$$= \left| -\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^{n-1} \right| - \beta \left| 2(1-\alpha) - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^{n-1} \right|$$

$$\leqq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} (1+\beta) a_n - 2\beta(1-\alpha)$$

$$\leqq 0, \quad \text{by hypothesis.}$$

Hence, by the maximum modulus theorem, we have $f(z) \in \mathcal{P}_\lambda^*(\alpha, \beta)$.

To prove the converse, assume that $f(z)$ is defined by (1.1) and is in the class $\mathcal{P}_\lambda^*(\alpha, \beta)$, so that the condition (1.3) readily yields

(2.2) $$\left| \frac{\Gamma(2-\lambda) z^{\lambda-1} D_z^{\lambda} f(z) - 1}{\Gamma(2-\lambda) z^{\lambda-1} D_z^{\lambda} f(z) + (1-2\alpha)} \right|$$

$$= \left| \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^{n-1} \right| \bigg/ \left| 2(1-\alpha) - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^{n-1} \right|$$

$$< \beta, \qquad z \in \mathcal{U}.$$

Since $|\mathrm{Re}(z)| \leqq |z|$ for any z, we find from (2.2) that

(2.3) $$\mathrm{Re}\left\{ \left[\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^{n-1} \right] \bigg/ \left[2(1-\alpha) - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^{n-1} \right] \right\} < \beta.$$

Choose values of z on the real axis so that $\Gamma(2-\lambda) z^{\lambda-1} D_z^{\lambda} f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we have

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n \leqq 2\beta(1-\alpha) - \beta \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n , \tag{2.4}$$

which gives the required assertion (2.1).

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z) = z - \frac{2\beta(1-\alpha)\Gamma(n+1-\lambda)}{(1+\beta)\Gamma(n+1)\Gamma(2-\lambda)} z^n . \tag{2.5}$$

Remark 1. From the work of Silverman ([11], p. 110, Theorem 2) it is known that a function $f(z)$ defined by (1.1) is in the class $\mathscr{T}^*(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha}\right) a_n \leqq 1 . \tag{2.6}$$

Thus we have

$$\mathscr{P}_{\lambda}^*(\alpha, \beta) \subset \mathscr{T}^*(\alpha) , \tag{2.7}$$

provided that $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha < 1$, and

$$0 < \beta \leqq \min_{n \geqq 2} \left\{ \frac{1}{\Phi(n) - 1} \right\} , \tag{2.8}$$

where, for convenience,

$$\Phi(n) = \frac{2(n-\alpha)\Gamma(n+1-\lambda)}{\Gamma(n+1)\Gamma(2-\lambda)} , \qquad 0 \leqq \alpha < 1 , \quad 0 \leqq \lambda \leqq 1 . \tag{2.9}$$

The function $\Phi(n)$ defined by (2.9) is positive and non-decreasing for $n \geqq 2$, and

$$\lim_{n \to \infty} \Phi(n) = \begin{cases} 2 , & \text{if } \lambda = 1 , \\ \infty , & \text{if } \lambda < 1 . \end{cases} \tag{2.10}$$

Consequently, (2.7) holds true for $0 \leqq \alpha < 1$ and $0 < \beta \leqq 1$ if and only if $\lambda = 1$.

Remark 2. When $\lambda = 1$, Theorem 1 reduces to the corresponding result due to Gupta and Jain ([5], p. 469, Theorem 1). It follows immediately that

$$\mathscr{P}_1^*(\alpha, \beta) = \mathscr{P}^*(\alpha, \beta) . \tag{2.11}$$

We record in passing the following interesting consequence of Theorem 1.

COROLLARY 1. *Let the function* $f(z)$ *defined by* (1.1) *belong to the class* $\mathscr{P}_{\lambda}^*(\alpha, \beta)$.
Then

$$a_n \leqq \frac{2\beta(1-\alpha)\Gamma(n+1-\lambda)}{(1+\beta)\Gamma(n+1)\Gamma(2-\lambda)} \tag{2.12}$$

for every integer $n \geqq 2$.

Remark 3. The assertion (2.12) of Corollary 1 may be rewritten as

$$a_n \leqq \frac{2\beta(1-\alpha)\Gamma(n+1-\lambda)}{(1+\beta)\Gamma(n+1)\Gamma(2-\lambda)} \leqq n \qquad (n \geqq 2) \tag{2.13}$$

for $0 \leqq \alpha < 1$, $0 < \beta \leqq 1$, and $0 \leqq \lambda \leqq 1$. Thus, if $\mathscr{S}$ denotes the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n \qquad (z \in \mathscr{U}) \tag{2.14}$$

that are analytic and univalent in $\mathscr{U}$, there do exist functions

$$f(z) \in \mathscr{P}_\lambda^*(\alpha, \beta)\,, \qquad 0 \leqq \alpha < 1\,, \quad 0 < \beta \leqq 1\,, \quad 0 \leqq \lambda < 1\,,$$

not necessarily in the class $\mathscr{S}$, for which the celebrated Bieberbach conjecture (now de Branges's theorem [1]):

$$|c_n| \leqq n \qquad (n \geqq 2) \tag{2.15}$$

holds true (*cf.* [3] and [4]).

3. A distortion theorem

THEOREM 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathscr{P}_\lambda^*(\alpha, \beta)$. *Then*

$$|f(z)| \geqq |z| - \frac{\beta(1-\alpha)(2-\lambda)}{1+\beta}|z|^2 \tag{3.1}$$

and

$$|f(z)| \leqq |z| + \frac{\beta(1-\alpha)(2-\lambda)}{1+\beta}|z|^2 \tag{3.2}$$

for $z \in \mathscr{U}$

Furthermore

$$|D_z^\lambda f(z)| \geqq \frac{1}{\Gamma(2-\lambda)}|z|^{1-\lambda} - \frac{2\beta(1-\alpha)}{(1+\beta)\Gamma(2-\lambda)}|z|^{2-\lambda} \tag{3.3}$$

and

$$|D_z^\lambda f(z)| \leqq \frac{1}{\Gamma(2-\lambda)}|z|^{1-\lambda} + \frac{2\beta(1-\alpha)}{(1+\beta)\Gamma(2-\lambda)}|z|^{2-\lambda} \tag{3.4}$$

whenever $z \in \mathscr{U}$.

Proof. Since $f(z) \in \mathscr{P}_\lambda^*(\alpha, \beta)$, in view of Theorem 1, we have

$$\frac{2(1+\beta)}{2-\lambda}\sum_{n=2}^{\infty} a_n \leqq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(1+\beta)a_n \leqq 2\beta(1-\alpha), \tag{3.5}$$

which evidently yields

$$\sum_{n=2}^{\infty} a_n \leqq \frac{\beta(1-\alpha)(2-\lambda)}{1+\beta}. \tag{3.6}$$

Consequently, we obtain

$$|f(z)| \geqq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geqq |z| - \frac{\beta(1-\alpha)(2-\lambda)}{1+\beta}|z|^2 \tag{3.7}$$

and

$$|f(z)| \leqq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leqq |z| + \frac{\beta(1-\alpha)(2-\lambda)}{1+\beta}|z|^2, \tag{3.8}$$

which prove the assertions (3.1) and (3.2).

Next, by using the second inequality in (3.5), we observe that

$$\begin{aligned} |\Gamma(2-\lambda)z^\lambda D_z^\lambda f(z)| &\geqq |z| - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n |z|^n \\ &\geqq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n \\ &\geqq |z| - \frac{2\beta(1-\alpha)}{1+\beta}|z|^2 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} |\Gamma(2-\lambda)z^\lambda D_z^\lambda f(z)| &\leqq |z| + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n |z|^n \\ &\leqq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n \\ &\leqq |z| + \frac{2\beta(1-\alpha)}{1+\beta}|z|^2, \end{aligned} \tag{3.10}$$

which prove the assertions (3.3) and (3.4).

Remark 4. Putting $\lambda=1$ in Theorem 2, we obtain the corresponding result given by Gupta and Jain ([5], p. 470, Theorem 2).

The following consequence of Theorem 2 is worthy of note:

COROLLARY 2. *Under the hypotheses of Theorem* 2, $f(z)$ *is included in a disk with its center at the origin and radius* r *given by*

$$r=1+\frac{\beta(1-\alpha)(2-\lambda)}{1+\beta}; \tag{3.11}$$

and $D_z^\lambda f(z)$ *is included in a disk with its center at the origin and radius R given by*

$$R=\frac{1}{\Gamma(2-\lambda)}\left\{1+\frac{2\beta(1-\alpha)}{1+\beta}\right\}. \tag{3.12}$$

4. Further properties of the class $\mathcal{P}_\lambda^*(\alpha, \beta)$

We begin by recalling the following useful result (see also Remark 2):

LEMMA 1 (Gupta and Jain [5, p. 469, Theorem 1]). *A function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{P}^*(\alpha, \beta)$ *if and only if*

$$\sum_{n=2}^{\infty} n(1+\beta)a_n \leqq 2\beta(1-\alpha). \tag{4.1}$$

This result is sharp, the extremal function being

$$f(z)=z-\frac{2\beta(1-\alpha)}{n(1+\beta)}z^n, \qquad n\in\mathcal{N}. \tag{4.2}$$

THEOREM 3. *Let* $0\leqq\lambda\leqq 1$, $0\leqq\alpha<1$, *and* $0<\beta\leqq 1$.
Then

$$\mathcal{P}_\lambda^*(\alpha, \beta)=\mathcal{P}_\lambda^*\left(\frac{1-\beta+2\alpha\beta}{1+\beta}, 1\right). \tag{4.3}$$

More generally, if $0\leqq\alpha'<1$ *and* $0<\beta'\leqq 1$, *then*

$$\mathcal{P}_\lambda^*(\alpha, \beta)=\mathcal{P}_\lambda^*(\alpha', \beta') \tag{4.4}$$

if and only if

$$\frac{\beta(1-\alpha)}{1+\beta}=\frac{\beta'(1-\alpha')}{1+\beta'}. \tag{4.5}$$

Proof. First assume that the function $f(z)$ is in the class $\mathcal{P}_\lambda^*(\alpha, \beta)$, and let the condition (4.5) hold true. Then, by using the assertion (2.1) of Theorem 1, we readily have

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n\leqq\frac{2\beta(1-\alpha)}{1+\beta}=\frac{2\beta'(1-\alpha')}{1+\beta'},$$

which shows that $f(z)\in\mathcal{P}_\lambda^*(\alpha', \beta')$, again with the aid of Theorem 1.

Reversing the above steps, we can similarly prove the other part of the equivalence (4.4) which, for $\beta'=1$, immediately yields the special case (4.3).

Conversely, the assertion (4.4) can easily be shown to imply the condition (4.5), and the proof of Theorem 3 is thus completed.

Next we state

THEOREM 4. *Let* $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha_1 \leqq \alpha_2 < 1$, *and* $0 < \beta \leqq 1$. *Then*

$$\mathscr{P}_\lambda^*(\alpha_1, \beta) \supset \mathscr{P}_\lambda^*(\alpha_2, \beta). \tag{4.6}$$

The proof of Theorem 4 uses Theorem 1 in a straightforward manner. The details may be omitted.

THEOREM 5. *Let* $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha < 1$, *and* $0 < \beta_1 \leqq \beta_2 \leqq 1$. *Then*

$$\mathscr{P}_\lambda^*(\alpha, \beta_1) \subset \mathscr{P}_\lambda^*(\alpha, \beta_2). \tag{4.7}$$

Proof. By using Theorem 3, we have

$$\mathscr{P}_\lambda^*(\alpha, \beta_1) = \mathscr{P}_\lambda^*\left(\frac{1-\beta_1+2\alpha\beta_1}{1+\beta_1}, 1\right) \tag{4.8}$$

and

$$\mathscr{P}_\lambda^*(\alpha, \beta_2) = \mathscr{P}_\lambda^*\left(\frac{1-\beta_2+2\alpha\beta_2}{1+\beta_2}, 1\right). \tag{4.9}$$

Furthermore

$$0 \leqq \frac{1-\beta_2+2\alpha\beta_2}{1+\beta_2} \leqq \frac{1-\beta_1+2\alpha\beta_1}{1+\beta_1} < 1 \tag{4.10}$$

for $0 \leqq \alpha < 1$ and $0 < \beta_1 \leqq \beta_2 \leqq 1$.

Consequently, by using Theorem 4, we arrive at our assertion (4.7).

COROLLARY 3. *Let* $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha_1 \leqq \alpha_2 < 1$, *and* $0 < \beta_1 \leqq \beta_2 \leqq 1$. *Then*

$$\mathscr{P}_\lambda^*(\alpha_2, \beta_1) \subset \mathscr{P}_\lambda^*(\alpha_1, \beta_1) \subset \mathscr{P}_\lambda^*(\alpha_1, \beta_2). \tag{4.11}$$

THEOREM 6. *Let* $0 \leqq \lambda \leqq \mu \leqq 1$, $0 \leqq \alpha < 1$, *and* $0 < \beta \leqq 1$. *Then*

$$\mathscr{P}_\lambda^*(\alpha, \beta) \supset \mathscr{P}_\mu^*(\alpha, \beta). \tag{4.12}$$

Proof. Let the function $f(z)$ defined by (1.1) be in the class $\mathscr{P}_\mu^*(\alpha, \beta)$. Then, by using Theorem 1, we have

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(1+\beta)a_n \leqq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\mu)}{\Gamma(n+1-\mu)}(1+\beta)a_n \leqq 2\beta(1-\alpha), \tag{4.13}$$

because

(4.14) $$1 \leqq \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \leqq \frac{\Gamma(n+1)\Gamma(2-\mu)}{\Gamma(n+1-\mu)} \leqq n$$

for $0 \leqq \lambda \leqq \mu \leqq 1$ and $n \geqq 2$. It follows from (4.13) that

(4.15) $$f(z) \in \mathscr{P}_{\lambda}^{*}(\alpha, \beta),$$

in view of Theorem 1, and the assertion (4.12) is thus proved.

THEOREM 7. *Let* $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha < 1$, *and* $0 < \beta \leqq 1$.
Then

(4.16) $$\mathscr{P}^{*}(\alpha, \beta) \subset \mathscr{P}_{\lambda}^{*}(\alpha, \beta).$$

Proof. Let the function $f(z)$ defined by (1.1) belong to the class $\mathscr{P}^{*}(\alpha, \beta)$. Then, by using Lemma 1, we have

(4.17) $$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(1+\beta)a_n \leqq \sum_{n=2}^{\infty} n(1+\beta)a_n \leqq 2\beta(1-\alpha),$$

because

(4.18) $$1 \leqq \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \leqq n$$

for $0 \leqq \lambda \leqq 1$ and $n \geqq 2$.

Equation (4.17), in conjunction with Theorem 1, completes the proof of the assertion (4.16).

5. Theorems involving modified Hadamard products

Let $f(z)$ be defined by (1.1), and let

(5.1) $$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geqq 0).$$

The modified Hadamard product of $f(z)$ and $g(z)$ is defined here by

(5.2) $$f * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

The following result depicts an interesting property of the modified Hadamard product of several functions.

THEOREM 8. *Let the functions* $f_1(z), f_2(z), \cdots, f_m(z)$ *defined by*

(5.3) $$f_j(z) = z - \sum_{n=2}^{\infty} c_{n,j} z^n \qquad (c_{n,j} \geqq 0)$$

be in the classes $\mathscr{P}_{\lambda}^{*}(\alpha_j, \beta_j)$, $j = 1, 2, \cdots, m$, *respectively. Also let*

(5.4) $$\lambda+\min_{1\leqq j\leqq m}\{\beta_j\}\geqq 1\,.$$

Then

(5.5) $$f_1*f_2*\cdots*f_m(z)\in\mathscr{P}_\lambda^*\left(\prod_{j=1}^m\alpha_j,\ \prod_{j=1}^m\beta_j\right).$$

Proof. Since $f_j(z)\in\mathscr{P}_\lambda^*(\alpha_j,\beta_j)$, $j=1,2,\cdots,m$, by using Theorem 1, we have

(5.6) $$\sum_{n=2}^\infty\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(1+\beta_j)c_{n,j}\leqq 2\beta_j(1-\alpha_j)$$

and

(5.7) $$\sum_{n=2}^\infty c_{n,j}\leqq\frac{\beta_j(1-\alpha_j)(2-\lambda)}{1+\beta_j}$$

for each $j=1,2,\cdots,m$.

Using (5.6) for any j_0 and (5.7) for the rest, we obtain

$$\sum_{n=2}^\infty\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\left[1+\prod_{j=1}^m\beta_j\right]\prod_{j=1}^m c_{n,j}\leqq\frac{2(2-\lambda)^{m-1}\prod_{j=1}^m\beta_j(1-\alpha_j)}{\sum_{\substack{j=1\\ j\neq j_0}}^m(1+\beta_j)}$$

$$\leqq\frac{2(2-\lambda)^{m-1}\prod_{j=1}^m\beta_j\left[1-\prod_{j=1}^m\alpha_j\right]}{\left[1+\min_{1\leqq j\leqq m}\{\beta_j\}\right]^{m-1}}$$

$$\leqq 2\prod_{j=1}^m\beta_j\left[1-\prod_{j=1}^m\alpha_j\right],$$

since

(5.8) $$0<\frac{2-\lambda}{1+\min_{1\leqq j\leqq m}\{\beta_j\}}\leqq 1\,.$$

Consequently, we have the assertion (5.5) with the aid of Theorem 1.

For $\alpha_j=\alpha$ and $\beta_j=\beta$, $j=1,2,\cdots,m$, Theorem 8 yields

COROLLARY 4. *Let each of the functions* $f_1(z), f_2(z),\cdots,f_m(z)$ *defined by* (5.3) *be in the same class* $\mathscr{P}_\lambda^*(\alpha,\beta)$. *Also let* $\lambda+\beta\geqq 1$.

Then

(5.9) $$f_1*f_2*\cdots*f_m(z)\in\mathscr{P}_\lambda^*(\alpha^m,\beta^m)\,.$$

Next we prove

THEOREM 9. *Let the functions $f(z)$ defined by* (1.1) *and $g(z)$ defined by* (5.1) *be in the classes $\mathscr{P}_{\lambda}^{*}(\alpha_1, \beta_1)$ and $\mathscr{P}_{\lambda}^{*}(\alpha_2, \beta_2)$, respectively.*

Then the modified Hadamard product $f*g(z)$ belongs to the class $\mathscr{P}_{\lambda}^{*}(2\alpha-\alpha^2, \beta)$, where

$$\alpha=\min\{\alpha_1, \alpha_2\} \quad \text{and} \quad \beta=\max\{\beta_1, \beta_2\}. \tag{5.10}$$

Proof. Since $f(z)\in\mathscr{P}_{\lambda}^{*}(\alpha_1, \beta_1)$ and $g(z)\in\mathscr{P}_{\lambda}^{*}(\alpha_2, \beta_2)$, in view of Theorem 1, we have

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(1+\beta)a_n \leqq 2\beta(1-\alpha)\frac{\beta_0(1-\alpha)(2-\lambda)}{1+\beta_0} \leqq 2\beta\{1-\alpha(2-\alpha)\}, \tag{5.11}$$

where $\beta_0=\min\{\beta_1, \beta_2\}$. Moreover, $0\leqq\alpha(2-\alpha)<1$ for $0\leqq\alpha<1$. Hence, by Theorem 1, the modified Hadamard product $f*g(z)$ is in the class $\mathscr{P}_{\lambda}^{*}(2\alpha-\alpha^2, \beta)$, with α and β given by (5.10).

COROLLARY 5. *Under the hypotheses of Theorem* 9, *the modified Hadamard product $f*g(z)$ belongs to the class $\mathscr{P}_{\lambda}^{*}(\alpha, \beta)$.*

Proof. In view of Theorem 4, we have

$$\mathscr{P}_{\lambda}^{*}(\alpha, \beta)\supset\mathscr{P}_{\lambda}^{*}(2\alpha-\alpha^2, \beta), \tag{5.12}$$

which, in conjunction with Theorem 9, shows that $f*g(z)\in\mathscr{P}_{\lambda}^{*}(\alpha, \beta)$, where α and β are given by (5.10).

Finally, we prove an interesting theorem on the modified Hadamard product (5.2) with extremal functions.

THEOREM 10. *Let the functions $f_i(z)$ $(i=1, 2)$ defined by* (5.3) *be in the class $\mathscr{P}_{\lambda}^{*}(\alpha, \beta)$.*

Then

$$f_1*f_2(z)\in\mathscr{P}_{\lambda}^{*}(\gamma(\alpha, \beta, \lambda), \beta), \tag{5.13}$$

where

$$\gamma(\alpha, \beta, \lambda)=1-\frac{\beta(1-\alpha)^2(2-\lambda)}{1+\beta}. \tag{5.14}$$

The result is sharp.

Proof. It suffices to prove that

$$\sum_{n=2}^{\infty}\frac{(1+\beta)\Gamma(n+1)\Gamma(2-\lambda)}{2\beta(1-\gamma)\Gamma(n+1-\lambda)}c_{n,1}c_{n,2}\leqq 1 \tag{5.15}$$

for $\gamma\leqq\gamma(\alpha, \beta, \lambda)$. By virtue of the Cauchy-Schwarz inequality, it follows from (2.1) that

$$\sum_{n=2}^{\infty} \frac{(1+\beta)\Gamma(n+1)\Gamma(2-\lambda)}{2\beta(1-\alpha)\Gamma(n+1-\lambda)} \sqrt{c_{n,1}c_{n,2}} \leqq 1 . \tag{5.16}$$

Hence we need to find the largest γ such that

$$\sum_{n=2}^{\infty} \frac{(1+\beta)\Gamma(n+1)\Gamma(2-\lambda)}{2\beta(1-\gamma)\Gamma(n+1-\lambda)} c_{n,1}c_{n,2} \leqq \sum_{n=2}^{\infty} \frac{(1+\beta)\Gamma(n+1)\Gamma(2-\lambda)}{2\beta(1-\alpha)\Gamma(n+1-\lambda)} \sqrt{c_{n,1}c_{n,2}} , \tag{5.17}$$

or, equivalently,

$$\sqrt{c_{n,1}c_{n,2}} \leqq \frac{1-\gamma}{1-\alpha} \qquad (n \geqq 2) . \tag{5.18}$$

In view of (5.16), it is sufficient to find the largest γ such that

$$\frac{2\beta(1-\alpha)\Gamma(n+1-\lambda)}{(1+\beta)\Gamma(n+1)\Gamma(2-\lambda)} \leqq \frac{1-\gamma}{1-\alpha} . \tag{5.19}$$

The inequality (5.19) yields

$$\gamma \leqq 1 - \frac{2\beta(1-\alpha)^2}{1+\beta} \Psi(n) \qquad (n \geqq 2) , \tag{5.20}$$

where

$$\Psi(n) = \frac{\Gamma(n+1-\lambda)}{\Gamma(n+1)\Gamma(2-\lambda)} . \tag{5.21}$$

Since $\Psi(n)$ defined by (5.21) is a decreasing function of n $(n \geqq 2)$ for fixed λ, we have

$$\gamma \leqq \gamma(\alpha, \beta, \lambda) = 1 - \frac{2\beta(1-\alpha)^2\Gamma(3-\lambda)}{(1+\beta)\Gamma(3)\Gamma(2-\lambda)} , \tag{5.22}$$

that is,

$$\gamma \leqq \gamma(\alpha, \beta, \lambda) = 1 - \frac{\beta(1-\alpha)^2(2-\lambda)}{1+\beta} , \tag{5.23}$$

which proves the assertion (5.13) under the constraint (5.14).

Finally, by taking the functions

$$f_i(z) = z - \frac{\beta(1-\alpha)(2-\lambda)}{1+\beta} z^2 \qquad (i = 1, 2) , \tag{5.24}$$

we can prove that the result is sharp.

COROLLARY 6. *Let the functions* $f_i(z)$ $(i=1, 2)$ *defined by* (5.3) *be in the class* $\mathscr{P}_1^*(\alpha, \beta)$.

Then

$$f_1 * f_2(z) \in \mathscr{P}_1^*(\gamma(\alpha, \beta), \beta) , \tag{5.25}$$

where

$$\gamma(\alpha, \beta)=1-\frac{\beta(1-\alpha)^2}{1+\beta}. \tag{5.26}$$

The result is sharp for functions

$$f_i(z)=z-\frac{\beta(1-\alpha)}{1+\beta}z^2 \qquad (i=1, 2). \tag{5.27}$$

Putting $\lambda=0$ in Theorem 10, we have

COROLLARY 7. *Let the functions* $f_i(z)$ $(i=1, 2)$ *defined by* (5.3) *be in the class* $\mathscr{P}_0^*(\alpha, \beta)$.

Then

$$f_1 * f_2(z) \in \mathscr{P}_0^*(\delta(\alpha, \beta), \beta), \tag{5.28}$$

where

$$\delta(\alpha, \beta)=1-\frac{2\beta(1-\alpha)^2}{1+\beta}. \tag{5.29}$$

The result is sharp for the functions

$$f_i(z)=z-\frac{2\beta(1-\alpha)}{1+\beta}z^2 \qquad (i=1, 2). \tag{5.30}$$

6. Linear combination of functions in the class $\mathscr{P}_\lambda^*(\alpha, \beta)$

THEOREM 11. *Let each of the functions* $f_1(z), f_2(z), \cdots, f_m(z)$ *defined by* (5.3) *be in the same class* $\mathscr{P}_\lambda^*(\alpha, \beta)$.

Then the function $h(z)$ *given by*

$$h(z)=\frac{1}{m}\sum_{j=1}^{m} f_j(z) \tag{6.1}$$

is also in the class $\mathscr{P}_\lambda^*(\alpha, \beta)$.

Proof. By the definition (6.1) of $h(z)$, we have the expansion

$$h(z)=z-\sum_{n=2}^{\infty}\left[\frac{1}{m}\sum_{j=1}^{m} c_{n,j}\right]z^n. \tag{6.2}$$

Since $f_j(z) \in \mathscr{P}_\lambda^*(\alpha, \beta)$ for every $j=1, 2, \cdots, m$, by using Theorem 1, we obtain

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(1+\beta)\left[\frac{1}{m}\sum_{j=1}^{m} c_{n,j}\right] \leqq 2\beta(1-\alpha), \tag{6.3}$$

which, in view of Theorem 1, yields Theorem 11.

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