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On N-Pure-High Subgroups of Abelian Torsion Groups

by

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All groups considered in this paper are abelian. For the general notation, we refer to Fuchs [4].

Let N be a subgroup of a group G. We say that a subgroup H of G is N-pure-high if it is maximal among the pure subgroups disjoint from N. Zorn's lemma guarantees the existence of N-pure-high subgroups.

Fuchs, in [4], Problem 14, proposes the study of pure-high subgroups of a group.

Benabdallah [1] proved that N-pure-high subgroups are N-high subgroups in torsion groups. Krivonos [6] characterized the group G which has a finite number of N-high subgroups of G.

In the case of torsion groups, the group G with the above property has a finite number of N-pure-high subgroups of G by [1]. But the converse is not trivial.

In this paper, we will characterize the torsion group G which has a finite number of N-pure-high subgroups of G and will prove that the converse is true.

Our groups are all the same type with Krivonos' groups. But the proof is evidently different from his.

§1. Preliminaries

We first quote some results which will be frequently used afterwards.

(1.1) (K. Benabdallah [1]). Every pure subgroup disjoint from a subgroup N of a torsion group G can be extended to a pure N-high subgroup of G.

(1.2) (K. Benabdallah [1]). Let N be a subgroup of a torsion group G. Then N-pure-high subgroups of G are N-high subgroups of G.

(1.3) (F. V. Krivonos [6]). Let N be a nonzero subgroup of a torsion group G. A subgroup A of G is the unique N-high subgroup of G if and only if $A = \bigoplus_{p\in\pi} G_p$, where π is a set of all prime numbers.

(1.4) (F. V. Krivonos [6]). A group G has a finite number of N-high subgroups of G for a subgroup N of G if and only if either of the following two holds:

(i) $N=0$, or

(ii) $N \neq 0$ and (a) G/N is torsion (b) for almost all primes p, $N[p]=G[p]$ or $N[p]=0$. (c) for any other primes p, $G_p=D\oplus R$, where D is the direct sum of a finite number of Prüfer groups and R is the direct sum of a finite number of finite cyclic groups, and $D[p]\subseteqq N[p]$.

(1.5) (R. S. Pierce [7]). Let G be a p-group. For each integers $k\geqq 0$, define $P_k=G[p]\cap p^kG$, $P_\infty=G[p]\cap G^1$, and $P_{\infty+1}=P_{\infty+2}=0$. Let H be a subgroup of G. Then H is a center of purity in G (that is, every H-high subgroup of G is pure) if and only if there exists k with $0\leqq k\leqq\infty$ such that $P_k\supseteqq H[p]\supseteqq P_{k+2}$.

(1.6) (K. Benabdallah [2]). Let S be a subsocle of a p-group G. If S is a center of boundedness, then S is strongly purifiable.

(S is said to be a center of boundedness if every S-high subgroup of G is bounded. S is said to be strongly purifiable if every pure subgroup H such that $H[p]<S$ can be extended to be a pure subgroup K supported by S.)

(1.7) (J. M. Irwin [5]). Let G be a p-group, and let H be a high subgroup of G. Then H contains a basic subgroup of G.

(1.8) (K. Benabdallah and J. M. Irwin [3]). Let $G=B\oplus D$ be a p-group, where B is bounded and D is divisible. Then every pure subgroup K of G is the direct sum of a bounded and a divisible subgroups. In particularly, the divisible part of K is equal to $K\cap D$.

§ 2. Uniqueness of the N-pure-high subgroups

In this section, we will characterize the torsion group G which has the unique N-high subgroup of G.

LEMMA 2.1. *Let N be a subgroup of a p-group G and A be an N-pure-high subgroup of G. If there exists an element b of $G[p]$ such that $h(b)<\infty$, $b\notin A$, and $b\notin N$, then G has an N-pure-high subgroup of G which is different from A and contains $\langle b\rangle$.*

Proof. Put $h(b)=n$. Then we write $b=p^nc$ for some $c\in G$. Since $\langle c\rangle$ is pure in G and $\langle c\rangle\cap N=0$, there exists an N-pure-high subgroup L of G containing $\langle c\rangle$ by (1.1). Suppose that $L=A$. Then we have $b\in A$, a contradiction. Hence $L\neq A$.

PROPOSITION 2.2. *Let N be a subgroup of a p-group G. Then G has the unique N-pure-high subgroup of G if and only if either $N=0$ or $N[p]=G[p]$.*

Proof. Suppose that G has the unique N-pure-high subgroup A of G and $A\neq 0$ and $N\neq 0$. We divide the following two cases and will lead to a contradiction in each case.

$\langle$*Case* I$\rangle$ A is nondivisible.

In this case, there exists a nonzero element a of $A[p]$ such that $h(a)<\infty$. By hypothesis, there exists a nonzero element g of $N[p]$. Now we will prove that there exists an element b of $G[p]$ such that $h(b)<\infty$, $b\notin A$, and $b\notin N$. Then it contradicts

(2.1) and hence the proof in this case is finished.

First we may assume that $h(a+g)=\infty$ for each $g\ (\neq 0)\in N[p]$. Indeed, if there exists a nonzero element g of $N[p]$ such that $h(a+g)<\infty$, then put $b=a+g$, and so we have $h(b)<\infty$, $b\notin A$, and $b\notin N$.

We can write $A[p]=\langle a\rangle\oplus A_1[p]$ for some subgroup A_1 of A. Suppose that $A_1[p]\neq 0$. If there exists an element a_1 of $A_1[p]$ such that $h(a_1)<\infty$, then, for some $g\ (\neq 0)\in N[p]$, put $b=a_1+(a+g)$, and so we have $h(b)<\infty$ and $b\notin A$. If $b\in N$, then we have $a_1+a\in A\cap N=0$ and $a_1=0$, a contradiction. Hence each nonzero element of $A_1[p]$ has an infinite height. On the other hand, since $h(a+g)=\infty$ for each $0\neq g\in N[p]$, it follows that $h(g)=h(a)<\infty$ for each $0\neq g\in N[p]$. Hence, for some $0\neq a'\in A_1[p]$, put $b=a'+g$, and so we have $h(b)<\infty$, $b\notin A$, and $b\notin N$. Hence we may assume that $A_1[p]=0$.

By (1.2), we have $G[p]=\langle a\rangle\oplus N[p]$. Since $h(g)=h(a)<\infty$ for each $0\neq g\in N[p]$, it follows that $h(a)=h(g)$ for each $g\ (\neq 0)\in N[p]$, by ([4], Cor. 27.8), there exists a bounded direct summand B of G containing $N[p]$. If $B[p]=G[p]$, then $B=G$, and hence G is bounded. This contradicts that $h(a+g)=\infty$. Hence we have $G[p]\neq B[p]=N[p]$ and write $G=A_0\oplus B$, where A_0 is a subgroup of G with $|A_0[p]|=p$. Then A_0 is an N-pure-high subgroup of G, and $A_0=A$. Since $h(a)<\infty$, A is bounded, and hence G is bounded, a contradiction.

⟨*Case* II⟩ A is divisible.

Since A is an absolute direct summand of G, we can write $G=A\oplus B$, where B is a subgroup of G containing N. By (1.2), we have $G[p]=A[p]\oplus B[p]$ and $B[p]=N[p]$.

First suppose that the height of each element of $B[p]$ is infinite. Then B is divisible. Hence B is a center of purity in G. Since all N-high subgroups of G are B-high in G, N is a center of purity. Thus G has the unique N-high subgroup A of G. By (1.3), either $A=G$ or $A=0$. If $A=G$, then $N=0$, a contradiction. Hence we may assume that there is a nonzero element x of $N[p]$ such that $h(x)<\infty$. Let $0\neq a\in A[p]$. Since $h(a)=\infty$, it follows that $h(a+x)<\infty$, $a+x\notin A$, and $a+x\notin N$. This is a contradiction by (2.1).

Conversely, if $N=0$, then G is the unique N-pure-high subgroup of G. If $N=G$, then 0 is the unique N-pure-high subgroup of G. Hence the proof is completed.

Next we will characterize the torsion group G which has the unique N-pure-high subgroup of G. Before we do it, we introduce two lemmas for ready reference.

LEMMA 2.3. *Let G be a torsion group and N_p be a subgroup of G_p for each prime p. Let H_p be an N_p-pure-high subgroup of G_p for each prime p, and $N=\bigoplus_p N_p$. Then $H=\bigoplus_p H_p$ is an N-pure-high subgroup of G.*

Proof. For integers $n\geqq 1$, we have

$$p^nH=\left(\bigoplus_{q\neq p} H_q\right)\oplus p^nH_p=\left(\bigoplus_{q\neq p} H_q\right)\oplus(H_p\cap p^nG)=\left(\bigoplus_{q} H_q\right)\cap p^nG=H\cap p^nG\,.$$

Hence H is pure in G. Furthermore $H\cap N=0$. By (1.1), there exists an N-pure-high subgroup L of G containing H. Let $L=\bigoplus_p L_p$. Since L_p is pure in G_p and contains H_p for each prime p, it follows that $L_p=H_p$. Hence $L=H$.

LEMMA 2.4. *Let G be a torsion group and N be a subgroup of G. If H is an N-pure-high subgroup of G, then H_p is an N_p-pure-high subgroup of G_p for each prime p.*

Proof. By (1.2), it is immediate.

THEOREM 2.5. *Let N be a subgroup of a torsion group G. Then G has the unique N-pure-high subgroup of G if and only if, for each prime p, either $N[p]=0$ or $N[p]=G[p]$.*

Proof. Suppose that G has the unique N-pure-high subgroup of G. (2.4) and (2.5), for each prime p, G_p has the unique N_p-pure-high subgroup of G_p. Hence, by (2.2), for each prime p, either $N[p]=0$ or $N[p]=G[p]$.

Conversely, suppose that, for each prime p, either $N[p]=0$ or $N[p]=G[p]$. Let A be an N-pure-high subgroup of G. By hypothesis, it follows that $A=\bigoplus_{p\in\pi} G_p$ for some set π of primes. By (1.2) and (1.3), G has a unique N-pure-high subgroup of G. Hence the proof is completed.

§ 3. Finiteness of N-pure-high subgroups

In this section, we will characterize the torsion group G which has a finite number of N-pure-high subgroups of G.

LEMMA 3.1. *Let N be a nonzero subgroup of a p-group G and A be an N-pure-high subgroup of G. If G has a finite number of N-pure-high subgroups of G and $A\neq 0$, then $N[p]$ is finite.*

Proof. We will prove it in the following two cases.

⟨*Case* I⟩ A is nondivisible.

Suppose that $|N[p]|\geqq\aleph_0$. Then there exist an independent set $\{g_i\,|\,g_i\in N[p],\ i=1, 2, \cdots\}$. Furthermore, since $A\neq 0$, there exists a nonzero element a of $A[p]$ such that $h(a)<\infty$.

Let $I=\{i\,|\,h(a+g_i)=\infty\}$ and $J=\{j\,|\,h(a+g_j)<\infty\}$. If $|I|$ is finite, then $|J|\geqq\aleph_0$ and so we may assume that $h(a+g_j)<\infty, j=1, 2, \cdots$. By (2.1), there exist N-pure-high subgroups A_j containing $\langle a+g_i\rangle$ different from A for $j=1, 2, \cdots$. Then for distinct indices i, j, we have $A_i\neq A_j$. Indeed, if $A_i=A_j$, then we have

$$g_i-g_j(a+g_i)-(a+g_j)\in A_i\cap N=0\ ,$$

a contradiction. Hence G has a countable number of N-pure-high subgroups of G, and hence this contradicts the hypothesis.

Suppose that $|I|\geqq\aleph_0$. Then we may assume that $h(a+g_i)=\infty$, $i=1, 2, \cdots$. We can write $A[p]=\langle a\rangle\oplus A_1$ for some subgroup A_1 of A. First suppose that $A_1[p]\neq 0$. If there exists an element a_1 of $A_1[p]$ such that $h(a_1)<\infty$, then put $b_i=a_1+(a+g_i)$, and so we have $h(b_i)<\infty$, $b_i \notin N$, and $b_i \notin A$, $i=1, 2, \cdots$. In analogy with above, G has a countable number of N-pure-high subgroups of G, a contradiction. Hence each element of $A_1[p]$ has a infinite height. On the other hand, since $h(a+g_i)=\infty$ for $i=1, 2, \cdots$, it follows that $h(g_i)=h(a)<\infty$ for $i=1, 2, \cdots$. Hence, for some $0 \neq a' \in A_1[p]$, put $b_1=a'+g_i$, $i=1, 2, \cdots$, and so we have $h(b_i)<\infty$, $b_i \notin A$, and $b_i \notin N$, $i=1, 2, \cdots$. In analogy with above, it contradicts the hypothesis.

Suppose that $A_1[p]=0$. Then $A[p]=\langle a \rangle$. Since $h(a)<\infty$, $h(g_i)<\infty$, and $h(a+g_i)=\infty$, we have $h(a)=h(g_i)$ for $i=1, 2, \cdots$. Put $b_i=(a+g_1)+g_i$, $i=2, 3, \cdots$. Then we have $b_i \notin A$, $b_i \notin N$, and $h(b_i)<\infty$, for $i=2, 3, \cdots$. By (2.1), there exist N-pure-high subgroups B_i containing b_i for $i=2, 3, \cdots$. In analogy with above, it contradicts to the hypothesis.

⟨*Case* II⟩ A is divisible.

We may assume that all N-pure-high subgroups of G are divisible. By (1.2), we can write $G=A \oplus B$, where B is a subgroup of G containing N, and $G[p]=A[p]\oplus B[p]$ and $B[p]=N[p]$. If each element of $B[p]$ has an infinite height, then, in analogy with Case II of (2.2), N is a center of purity in G. Thus G has a finite number of N-high subgroups of G. By (1.4), $|N[p]|$ is finite.

If there exists and element b of $B[p]$ such that $h(b)<\infty$, then we have $h(a+b)<\infty$, $a+b \notin N$, and $a+b \notin A$ for some $0 \neq a \in A[p]$. By (2.1), there exists an N-pure-high subgroup L of G containing $\langle a+b \rangle$. Then L is divisible and so $h(a+b)=\infty$, a contradiction. Hence the proof is completed.

THEOREM 3.2. *Let N be a nonzero subgroup of a p-group G with $N[p] \neq G[p]$. Then G has a finite number of N-pure-high subgroups of G if and only if the following two hold*:

(a) *$G=D\oplus B$, where D is a divisible subgroup and B is a bounded subgroup with $|B[p]|<\aleph_0$, and*

(b) *Either $N[p]\supseteqq D[p]$ or the following two hold: $N[p]\cap D[p]=0$ and $r(D[p])\leqq 1$.*

Proof. Suppose that G has a finite number of N-pure-high subgroups of G. By (3.1), it follows that $N[p]$ is finite. Put $N_2=N[p]\cap G^1$ and so we have $N[p]=N_1\oplus N_2$ for some subgroup N_1 of $N[p]$. If $N_1=0$, then N is a center of purity in G by (1.5). By (1.4), (a) and (b) are satisfied. Hence we may assume that $N_1 \neq 0$.

By (1.5), there exists a pure N_2-high subgroup L of G containing N_1. Since the height of each nonzero element of N_1 is bounded, by ([4], 27.8) and ([4], 27, Ex. 5), we have

$$L=H\oplus K,$$

where K is a bounded subgroup of L with $K[p]=N_1$ and H is some subgroup of L.

Now we have $G^1 \cap L=L^1=H^1$. Then we can express

$$H[p]=H^1[p]\oplus H'$$

for some subgroup H' of $H[p]$. Put

$$H' = \bigoplus_{\alpha \in I} \langle x_\alpha \rangle .$$

First we will prove that $|I|$ is finite. Suppose that $|I|$ is infinite. Since $N_1 \neq 0$, there exists a nonzero element g of N_1 such that $h(g) < \infty$. Put

$$H'_\alpha = (\bigoplus_{\beta \in I - \{\alpha\}} \langle x_\beta \rangle) \oplus \langle x_\alpha + g \rangle$$

and

$$H_\alpha = H^1[p] \oplus H'_\alpha .$$

Let K_α be a H_α-high subgroup of L. Then $L[p] = H_\alpha \oplus K_\alpha[p]$, $K_\alpha[p]$ is finite, and the height of each nonzero element of $K_\alpha[p]$ is finite. Hence all H_α-high subgroups of L are boundary and so H_α is strongly purifiable by (1.6). Since there exists an element x of H_α such that $h(x) < \infty$, there exists a pure subgroup $H(\alpha)$ such that $H(\alpha)[p] = H_\alpha$. Now α ranges over all elements of I. If $H(\alpha) = H(\beta)$, $\alpha, \beta \in I$, $\alpha \neq \beta$, then we have

$$g = (g + x_\alpha) - x_\alpha \in H(\alpha) \cap N = 0 ,$$

a contradiction. Hence, by the hypothesis, $|I|$ is finite.

A socle of a high subgroup of H is finite. By (1.7), we can express

$$H = D_1 \oplus H_1 ,$$

where D_1 is a divisible subgroup of H and H_1 is a finite subgroup of H. Hence we have

$$G = D_1 \oplus A ,$$

where A is a subgroup of G and A contains $H_1 \oplus K$ and N_2. Since $A[p] = H_1[p] \oplus K[p] \oplus N_2$, we have

$$A = D_2 \oplus A' ,$$

where D_2 is a divisible subgroup of A, A' is a finite subgroup of A, and A' contains $H_1 \oplus K$. Since $N_2 \leqq A \cap G^1 = A^1 = D_2$, we have $N_2 = D_2[p]$. Hence we have

$$G = D_1 \oplus D_2 \oplus H_1 \oplus H_2$$

for some subgroup H_2 of A'.

Suppose that $D_2 \neq 0$. Put $G_0 = D_1 \oplus D_2 \oplus H_1$. Then every D_2-high subgroup of G_0 is a pure N-high subgroup of G. By (1.4), it follows that $D_1 = 0$. Hence put $D = D_2$ and $B = H_1 \oplus H_2$, and so (a) and (b) are satisfied.

We may assume that $D_2 = 0$. Put $D = D_1$. Since $G = D \oplus A$, A is bounded, and K is bounded pure in A, it follows that

$$G = D \oplus H_0 \oplus K$$

for some subgroup H_0 of A. We will prove that $r(D[p]) \leqq 1$. Suppose that $r(D[p]) > 1$. Let $0 \neq g \in D[p]$ and $0 \neq x \in N[p]$. Since $K[p] = N[p]$, we have

$$G = D' \oplus D'' \oplus K' \oplus K'' \oplus H_0 ,$$

where D' and D' are subgroups of D with $D''[p] = \langle g \rangle$ and K' and K'' are subgroups of K with $K'[p] = \langle x \rangle$. Since $h(g+x) < \infty$ and $g+x \in D'' \oplus K'$, we have

$$D'' \oplus K' = K_0 \oplus K_0'$$

where K_0 and K_0' are subgroups of $D'' \oplus K'$, K_0 is bounded pure in $D'' \oplus K'$, and $K_0[p] = \langle g+x \rangle$. Put $M = D' \oplus K_0 \oplus H_0$, and so M is pure in G, $M \cap N = 0$, and $G[p] = M[p] + N[p]$. Hence M is an N-pure-high subgroup of G. Since D' is a $\langle g \rangle$-high subgroup of D and D has at least countable $\langle g \rangle$-high subgroups of D by (1.4), it contradicts the hypothesis. Hence $r(D[p]) \leqq 1$ and so (a) and (b) are satisfied.

Conversely, suppose that (a) and (b) are satisfied. If either $D[p] = 0$ or $D[p] \leqq N[p]$, then it is immediate by (1.2) and (1.4). Hence we may assume that $N[p] \cap D[p] = 0$ and $r(D[p]) = 1$. Let A be an N-pure-high subgroup of G, and then, by (1.8), it follows that $A = D_1 \oplus R_1$, where D_1 is a divisible subgroup and R_1 is a bounded subgroup. If $D_1 \neq 0$, then we have $D_1 = A \cap D$ by (1.8). Since D is pure-simple, we have $D_1 = D$. By (1.4), G has a finite number of D-high subgroups of G. Every D-high subgroup of G is finite, and R_1 is a subgroup of some D-high subgroup of G. Hence G has a finite number of N-pure-high subgroups of G.

If $D_1 = 0$, then, in analogy with above, we has a finite number of N-pure-high subgroups of G. Hence the proof is completed.

THEOREM 3.3. *Let N be a subgroup of a torsion group G. G has a finite number of N-pure-high subgroups of G if and only if the following two hold*:

(i) *for almost all primes p, either $N[p] = 0$ or $N[p] = G[p]$, and*

(ii) *for any other primes p, G_p is the group mentioned in Theorem* 3.2.

Proof. Suppose that G has a finite number of N-pure-high subgroups of G. By (2.3) and (2.4), for almost all primes p, G_p has the unique N_p-pure-high subgroup of G_p, and for any other primes p, G_p has a finite number of N-pure-high subgroups of G_p. Hence, by (3.3), (i) and (ii) are satisfied.

Conversely, assume (i) and (ii). By (1.2) and (3.2), G has a finite number of N-pure-high subgroups of G.

(1.4) and the preceding result yield the next Corollary.

Example. Let G be a p-group and $G = D \oplus B$ where D is a divisible subgroup and B is a bounded subgroup and $B[p]$ is finite. Let N be a nonzero subgroup of G with $N[p] \subseteqq B[p]$. Then G has a finite number of N-pure-high subgroups of G by (3.2). But G has at least countable number of N-high subgroups of G.

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