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Representations of the Fundamental Groups of 3-Manifolds, I

by

Moto-o TAKAHASHI

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In [4], we have considered the representations of the fundamental groups of 3-manifolds obtained by Dehn surgeries along 2-bridge knots. In this paper, we shall show that this method can be applied also to manifolds which are not obtained by Dehn surgery along a knot.

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§ 1. Lens space conjecture

Unless otherwise stated, we denote by M a closed orientable connected 3-manifold and by $\pi_1(M)$ its fundamental group. By a lens space we mean a closed 3-manifold obtained by glueing the boundaries of two solid tori. Thus we include S^3 and $S^2 \times S^1$ in lens spaces. Let Z_n be the finite cyclic group of order n.

First we consider the following conjecture:

Conjecture 1 (Haken). If $\pi_1(M) \cong Z_n$, then M is a lens space.

We call this conjecture the lens space conjecture. Obviously this conjecture for $n=1$ is just the Poincaré conjecture. We first derive some consequences from this conjecture.

THEOREM 1. *Suppose that the lens space conjecture is true. Then, if $\pi_1(M)$ is abelian, then either M is a lens space or M is homeomorphic to $S^1 \times S^1 \times S^1$.*

Proof. Suppose that the lens space conjecture is true and that $\pi_1(M)$ is abelian. Then by Epstein [1], $\pi_1(M)$ is isomorphic to one of the following groups:

$$Z_n, \qquad Z, \qquad Z_n \times Z, \qquad Z \times Z, \qquad Z \times Z \times Z.$$

If $\pi_1(M)$ is finite, then it must be isomorphic to Z_n, and hence by the lens space conjecture M is a lens space. If $\pi_1(M)$ is infinite, then $H_1(M)$ ($\cong \pi_1(M)$) is infinite and hence by Waldhausen [6], M is sufficiently large. So M contains an incompressible surface F. Since $\pi_1(M)$ is abelian, the genus of F must be 0 or 1. First suppose that the genus of F is 0, that is, F is a 2-sphere. By a standard argument we can assume that F is separating in M, unless M is homeomorphic to $S^2 \times S^1$. Then M is the connected sum of two closed 3-manifolds M_1 and M_2 which are not homeomorphic to S^3, and $\pi_1(M)=\pi_1(M_1)*\pi_1(M_2)$, (a free product). Since the lens space conjecture implies Poincaré conjecture, $\pi_1(M_1)$ and $\pi_1(M_2)$ are non-trivial. But then $\pi_1(M)$ cannot be abelian. Next suppose that the genus of F is 1, that is, F is a torus. If F is separating, then $\pi_1(M)$ is an amalgamated free product

$$\pi_1(M)=\pi_1(M_1)*_{\pi_1(F)}\pi_1(M_2)$$

where $\partial M_1=\partial M_2=F$. Since $\pi_1(M)$ is abelian, this amalgamated free product must be trivial:

$$\pi_1(M_1)\ (\text{or } \pi_1(M_2))\cong\pi_1(F)\cong Z\times Z\,.$$

But there does not exist a 3-manifold M_1 such that ∂M_1 is a torus and $\pi_1(M_1)=Z\times Z$. For, by Waldhausen [7], there is an incompressible surface G in M_1 such that $0\neq[\partial G]\in H_1(\partial M_1)$. G must be a 2-disk or an annulus. In either case, a contradiction arises. Finally suppose that F is a torus and non-separating in M. We choose a base point P on F. Let a and b be loops on F which represent independent generators of $\pi_1(F)=Z\times Z$. Let c be a loop in M such that c intersects F transversely only at one point P. Since $\pi_1(M)$ is abelian, we have $ab=ba$, $ac=ca$, $bc=cb$. Moreover $a^lb^mc^n=1$ implies $n=0$ (since the intersection number of $a^lb^mc^n$ with F must be 0) and hence $l=0$ and $m=0$ (since $\pi_1(F)\to\pi_1(M)$ is injective). Thus $\pi_1(M)$ contains a subgroup isomorphic to $Z\times Z\times Z$. Hence by Epstein's result mentioned above, $\pi_1(M)$ must be isomorphic to $Z\times Z\times Z$. Since we are assuming the lens space conjecture and hence Poincaré conjecture, M must be irreducible. And M is also sufficiently large. So it is determined by $\pi_1(M)$ (Waldehausen [7]). Hence M must be homeomorphic to $S^1\times S^1\times S^1$. This completes the proof of the theorem.

COROLLARY 2. *Suppose that the lens space conjecture is true. Then, if M has a Heegaard splitting of genus* 2 *and $\pi_1(M)$ is abelian, then M is a lens space. In other words, if M is of Heegaard genus* 2, *then $\pi_1(M)$ is non-abelian.*

Proof. This follows immediately from the Theorem 1, since $S^1\times S^1\times S^1$ does not have Heegaard splittings of genus 2.

COROLLARY 3. *Suppose that the lens space conjecture is true. Then,* (i) *if $\pi_1(M)\cong Z$, then M is homeomorphic to $S^2\times S^1$, and* (ii) *if $\pi_1(M)\cong Z\times Z\times Z$, then M is homeomorphic to $S^1\times S^1\times S^1$.*

The following conjecture is well-known:

Conjecture 2. If Z_n acts freely on S^3, then the quotient space is a lens space.

THEOREM 4 (Haken). *The lens space conjecture is equivalent to the conjunction of Poincaré conjecture and Conjecture* 2.

Proof. Clearly the lens space conjecture implies Poincaré conjecture and Conjecture 2. Conversely suppose that Poincaré conjecture and Conjecture 2 are true but the lens space conjecture is false. Then there exists a 3-manifold M with $\pi_1(M) \cong Z_n$ which is not a lens space. Consider the universal cover $\tilde{M}$ of M. By Poincaré conjecture $\tilde{M}$ is homeomorphic to S^3 since $\tilde{M}$ is compact. The covering translations constitute a group isomorphic to Z_n and this group acts freely on $\tilde{M}$ and the quaotient space is M. This contradicts Conjecture 2.

§ 2. Representations of $\pi_1(M)$

First we define the following four groups:

$$PGL(2, C) = GL(2, C)/\{\lambda E\}\ ,^{*)}$$

$$PSL(2, C) = SL(2, C)/\{\pm E\}\ ,$$

$\mathfrak{M}$ = "the group of all Möbius transformations

$$w = (az+b)/(cz+d)\ ,$$

where a, b, c, $d \in C$ and $ad - bc \neq 0$."

$I^+(H^3)$ = "the group of all orientation-preserving isometries of the hyperbolic 3-space H^3."

Then it is known that these four groups are all isomorphic:

$$PGL(2, C) \cong PSL(2, C) \cong \mathfrak{M} \cong I^+(H^3)\ .$$

Hereafter, by a representation of $\pi_1(M)$ we shall mean a representation of $\pi_1(M)$ into $PGL(2, C)$. Let h and h' be two representations of $\pi_1(M)$. h and h' are said to be equivalent if there exists an $A \in PGL(2, C)$ such that for all $x \in \pi_1(M)$, $h'(x) = Ah(x)A^{-1}$.

In many cases, the number of the equivalence classes of representations of $\pi_1(M)$ is finite with the exception of the connected sums of lens spaces, some of sufficiently large manifolds, etc. Let $\delta(M)$ be the number of the equivalence classes of representations of $\pi_1(M)$. Then $\delta(M)$ is a (computable) invariant of M. We conjecture the following:

Conjecture 3. If M is irreducible but not sufficiently large, then $\delta(M)$ is finite.

A representation is said to be abelian, cyclic, trivial, etc., if so is its image.

Conjecture 4. If M is not homeomorphic to S^3, then there exists a non-trivial representation of $\pi_1(M)$.

Obviously this conjecture implies Poincaré conjecture.

Conjecture 5. If M is irreducible but not sufficiently large, and not a lens space, then there exists a non-abelian representation of $\pi_1(M)$.

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*) E is the identity matrix.

This conjecture implies the lens space conjecture.

Example. There exists an irreducible, sufficiently large, closed 3-manifold M such that

$$\pi_1(M) \cong \langle a, b \mid a^3b^2a^3b^{-1} = b^3a^2b^3a^{-1} = 1 \rangle .$$

This $\pi_1(M)$ is non-abelian (in fact, a non-trivial amalgamated free product, see [2]), but it can be shown that all the representations of $\pi_1(M)$ are abelian.

§ 3. A class of 3-manifolds

For 3-manifolds obtained by Dehn surgeries along 2-bridge knots, the computation of the representations of $\pi_1(M)$ is carried out in [4]. The remainder of this paper is devoted to computing all the representations of $\pi_1(M)$ for a certain class of 3-manifolds. The class of 3-manifolds we will consider appears in [3], and each manifold in this class has a Heegaard splitting of genus 2 and has the corresponding presentation of the fundamental group in which one of the relators is of length 10.

In order to describe the class of 3-manifolds, first we consider a solid torus V of genus 2. V can be viewed as obtained from a 3-disk D^3 by glueing α^+ to α^- and β^+ to β^-, where $\alpha^+, \alpha^-, \beta^+, \beta^-$ are disjoint 2-disks on ∂D^3. Then, $\alpha = \alpha^+ = \alpha^-$ and $\beta = \beta^+ = \beta^-$ (in V) constitute a system of meridian disks of V. Let c be the loop on V as shown in Fig. 1. (We glue α^+ to α^- and β^+ to β^- so that the points with the same number coincide.) We attach a 2-handle $D^2 \times D^1$ to V along c, that is, we glue $\partial D^2 \times D^1$ to

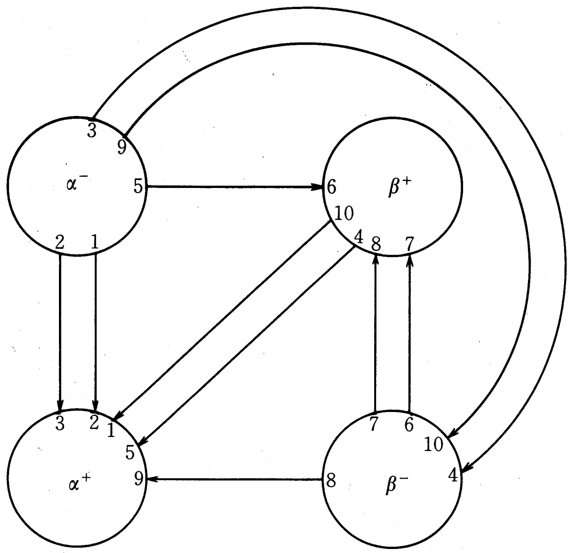

Fig. 1

$\bar{N}(c)$, where D^2 is a 2-disk, D^1 is $[0, 1]$ and $\bar{N}(c)$ is the closure of a regular neighborhood of c in ∂V. Then we obtain a 3-manifold N with a torus as its boundary. $\pi_1(M)$ has the following presentation:

$$\pi_1(M) \cong \langle a, b \mid a^3b^{-1}ab^3ab^{-1} = 1 \rangle, \tag{1}$$

where a and b are generators corresponding to the meridian disk α and β respectively, and the relator corresponds to the loop c and is read from Figure 1. Let $\gamma^+ = D^2 \times \{0\}$ and $\gamma^- = D^2 \times \{1\}$. $\gamma^+ \cup \gamma^- \cup \partial\beta \subseteq \partial N$ is called the reverse graph of c. Since ∂N is a torus, its universal covering space P is a plane. The reverse graph of c induces an infinite graph on P, as shown in Fig. 2. This is called the reverse development of c. (Cf. [3].)

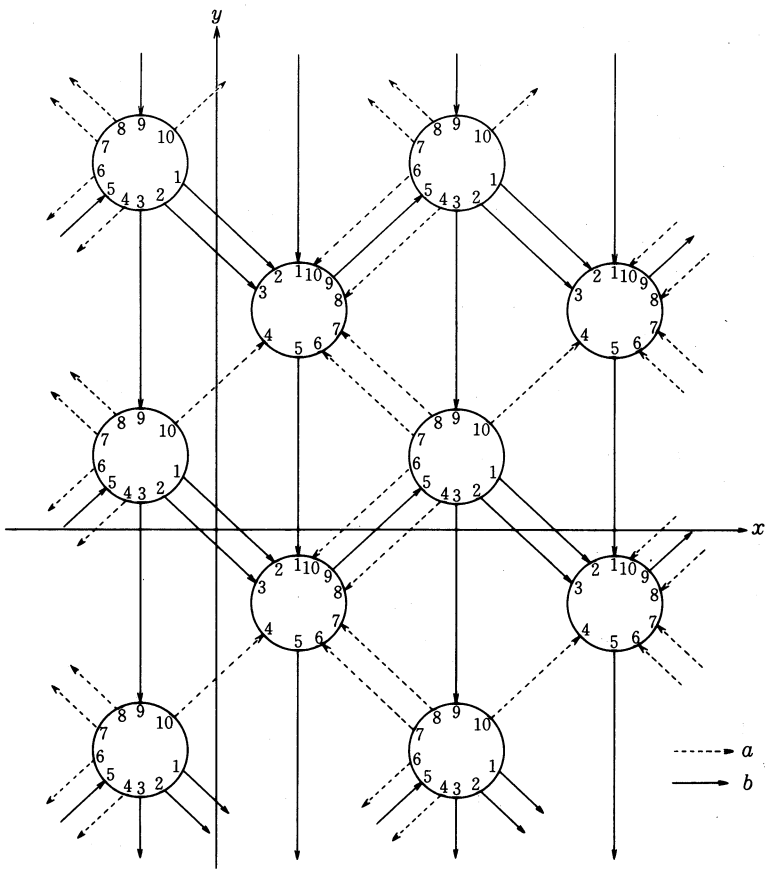

Fig. 2

Now let T be a solid torus. If we glue ∂T to ∂N in any way, then we obtain a closed orientable 3-manifold. It is determined by the homotopy type of a loop d on ∂N which is identified with a meridian of T by the glueing. This homotopy type is, in turn, determined by a pair (m, n) of relatively prime integers, where d is homotopic to $mx+ny$ and, x and y are loops on ∂N as shown in Figure 2. The closed manifold obtained is denoted by $M_{m,n}$. Obviously, $M_{-m,-n}=M_{m,n}$. Hence we can assume $m\geqq 0$. Since x and y correspond to the words $ab^{-1}a^2b^{-1}$ and ab^2 respectively and they commute in $\pi_1(N)$, we have presentations

$$\begin{aligned}\pi_1(M_{m,n}) &\cong \langle a, b \mid a^3b^{-1}ab^3ab^{-1}=(ab^{-1}a^2b^{-1})^m(ab^2)^n=1\rangle \\ &\cong \langle a, b \mid a^3b^{-1}ab^3ab^{-1}=(b^{-3}a^2b^{-1})^m(ab^2)^j=1\rangle ,\end{aligned} \tag{2}$$

where $j=m+n$.

§4. Computation of representations of $\pi_1(N)$

We shall find all the representations of $\pi_1(N)$ and of $\pi_1(M_{m,n})$. First we determine all the representations of $\pi_1(N)$.

LEMMA 4. *For any non-negative integer n, let*

$$\begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}^n .$$

Moreover we define polynomials $\rho_n=\rho_n(x, y)$, inductively as follows:

$$\rho_0=0 , \qquad \rho_1=1 , \qquad \rho_{n+2}=x\rho_{n+1}+y\rho_n .$$

Let $x=p+s$ and $y=qr-ps$. Then, we have

$$p_n=p\rho_n+y\rho_{n-1} , \qquad q_n=q\rho_n ,$$

$$r_n=r\rho_n , \qquad s_n=s\rho_n+y\rho_{n-1} .$$

Proof. By the induction on n.

Note that

$$\rho_2=x , \qquad \rho_3=x^2+y , \qquad \rho_4=x^3+2xy , \qquad \rho_5=x^4+3x^2y+y^2 , \cdots .$$

COROLLAY. $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)^n$ *is a scalar matrix λE, if and only if $\rho_n(x, y)=0$, where $x=p+s$ and $y=qr-ps$.*

Proof. $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)^n$ is a scalar matrix if and only if $p_n-s_n=q_n=r_n=0$. By Lemma 1, this condition is equivalent to $\rho_n=0$. q.e.d.

Here we note that any matrix A in $GL(2, C)$ has its Jordan normal form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} .$$

If the latter is the case, A is called parabolic.

Now let $A, B\in GL(2, C)$. We define

$$A \approx B \Leftrightarrow \exists \lambda \neq 0 , \qquad \lambda A=B .$$

That is, $A \approx B$ if and only if $\bar{A} = \bar{B}$, where $\bar{A}$ and $\bar{B}$ are elements of $PGL(2, C)$ corresponding to A and B, respectively.

Let h be a representation of $\pi_1(N)$, and let $h(a) = \bar{A}$ and $h(b) = \bar{B}$. Then, by (1),

$$\bar{A}^3 \bar{B}^{-1} \bar{A} \bar{B}^3 \bar{A} \bar{B}^{-1} = 1 , \tag{3}$$

that is, $A^3 B^{-1} AB^3 AB^{-1} \approx \bar{E}$. Conversely, if $\bar{A}$, $\bar{B} \in PGL(2, C)$ are such that (3) holds, then the equations $h(a) = \bar{A}$ and $h(b) = \bar{B}$ define a representation of $\pi_1(N)$.*) We denote these equations by

$$a \to A , \qquad b \to B . \tag{4}$$

THEOREM 6. (i) *Let A, $B \in GL(2, C)$ and $\bar{B} = \bar{A}^{-5}$, then* (4) *defines an abelian representation of $\pi_1(N)$. Every abelian representation is obtained in this way. Two such representations are equivalent if and only if the corresponding $\bar{A}$'s are conjugate in $PGL(2, C)$.*

(ii) *Let λ, $\mu \in C$ be such that $\lambda\mu \neq 0$ and $\lambda^3 \neq \mu^3$. Let*

$$p = \lambda^6 + 2\lambda^5\mu + 3\lambda^4\mu^2 + 2\lambda^3\mu^3 + 2\lambda^2\mu^4 + \lambda\mu^5 ,$$
$$s = -\lambda^5\mu - 2\lambda^4\mu^2 - 2\lambda^3\mu^3 - 3\lambda^2\mu^4 - 2\lambda\mu^5 - \mu^6 ,$$

and let q, $r \in C$ be such that both are not zero and $qr = ps - \lambda^3\mu^3(\lambda^3 - \mu^3)^2$. Then the correspondence

$$a \to A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad b \to B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

defines a non-abelian representation of $\pi_1(N)$.

(iii) *Also the correspondence*

$$a \to A = \begin{pmatrix} 10 & 1 \\ 11 & 2 \end{pmatrix}, \qquad b \to B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

defines a non-abelian representation of $\pi_1(N)$.

(iv) *Every non-abelian representation of $\pi_1(N)$ is equivalent to one of the representations defined in* (ii) *and* (iii).

(v) *A representation defined in* (ii) *and the one defined in* (iii) *are not equivalent.*

(iv) *Two representations*

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*) $\tilde{h}(a) = \bar{A}$ and $\tilde{h}(b) = \bar{B}$ clearly define a representation of the free group G generated by a and b. Let $\mathcal{N}$ be the least normal subgroup generated by $a^3b^{-1}ab^3ab^{-1}$. By (3), $\tilde{h}(\mathcal{N}) = \{\bar{E}\}$. So h is uniquely defined by the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\ \tilde{h}\ } & PGL(2, C) \\ {\scriptstyle j}\downarrow & \nearrow {\scriptstyle h} & \\ G/\mathcal{N} = \langle a, b \mid a^3b^{-1}ab^3ab^{-1} = 1 \rangle , & & \end{array}$$

where j is the natural homomorphism.

$$a \to \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad b \to \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

and

$$a \to \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}, \qquad b \to \begin{pmatrix} \lambda' & 0 \\ 0 & \mu' \end{pmatrix},$$

which are defined as in (ii) *are equivalent, if and only if one of the following is satisfied*:

(I) $(\lambda:\mu=\lambda':\mu'$ *or* $\lambda:\mu=\mu':\lambda')$ *and* $qr\neq 0$,

(II) $\lambda:\mu=\lambda':\mu'$ *and* $(q=q'=0$ *or* $r=r'=0)$,

(III) $\lambda:\mu=\mu':\lambda'$ *and* $(q=r'=0$ *or* $r=q'=0)$.

Proof. The abelian case (i) is obvious. Suppose that h is a non-abelian representation of $\pi_1(N)$ defined by

$$a \to A\,, \qquad b \to B\,.$$

First suppose that B is not parabolic. Then we can assume that B is in Jordan normal form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ with $\lambda\neq\mu$, $\lambda\mu\neq 0$. Let $A=\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Then we must have $A^3B^{-1}AB^3AB^{-1}\approx E$. Let $x=p+s$ and $y=qr-ps$. Then by Lemma 1 we have

$$p_3=p\rho_3+y\rho_2\,, \qquad q_3=q\rho_3\,, \qquad r_3=r\rho_3\,, \qquad s_3=s\rho_3+y\rho_2\,.$$

By computation we have

$$A^3B^{-1}AB^3AB^{-1}\approx\begin{pmatrix} p_3 & q_3 \\ r_3 & s_3 \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} \lambda^3 & 0 \\ 0 & \mu^3 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}$$

$$=\begin{pmatrix} q_3pr\lambda^4\mu+p_3p^2\lambda^3\mu^2+q_3rs\lambda\mu^4+p_3qr\mu^5, & q_3qr\lambda^5+p_3pq\lambda^4\mu+q_3s^2\lambda^2\mu^3+p_3qs\lambda\mu^4 \\ s_3pr\lambda^4\mu+r_3p^2\lambda^3\mu^2+s_3rs\lambda\mu^4+r_3qr\mu^5, & s_3qr\lambda^5+r_3pq\lambda^4\mu+s_3s^2\lambda^2\mu^3+r_3qs\lambda\mu^4 \end{pmatrix}.$$

Hence we must have

$$q_3pr\lambda^4\mu+p_3p^2\lambda^3\mu^2+q_3rs\lambda\mu^4+p_3qr\mu^5=s^3qr\lambda^5+r_3pq\lambda^4\mu+s_3s^2\lambda^2\mu^3+r_3qs\lambda\mu^4\,,$$

$$q_3qr\lambda^5+p_3pq\lambda^4\mu+q_3s^2\lambda^2\mu^3+p_3qs\lambda\mu^4=0\,,$$

$$s_3pr\lambda^4\mu+r_3p^2\lambda^3\mu^2+s_3rs\lambda\mu^4+r_3qr\mu^5=0\,;$$

or,

$$X\equiv s_3qr\lambda^5-p_3p^2\lambda^3\mu^2+s_3s^2\lambda^2\mu^3-p_3qr\mu^5=0\,, \tag{5}$$

$$q(\rho_3qr\lambda^4+p_3p\lambda^3\mu+\rho_3s^2\lambda\mu^3+p_3s\mu^4)=0\,, \tag{6}$$

$$r(s_3p\lambda^4+\rho_3p^2\lambda^3\mu+s_3s\lambda\mu^3+\rho_3qr\mu^4)=0\,. \tag{7}$$

Suppose that $q\neq 0$ and $r\neq 0$. From (6) and (7), it follows that

$$Y\equiv\rho_3qr\lambda^4+p_3p\lambda^3\mu+\rho_3s^2\lambda\mu^3+p_3s\mu^4=0\,, \tag{8}$$

$$Z\equiv s_3p\lambda^4+\rho_3p^2\lambda^3\mu+s_3s\lambda\mu^3+\rho_3qr\mu^4=0\,. \tag{9}$$

Note that

$$\rho_3 X \equiv s_3\lambda Y + p_3\mu Z . \tag{10}$$

Now

$$qr = y + ps , \qquad \rho_3 = x^2 + y , \qquad p_3 = px^2 + (p+x)y .$$

Substituting these into (8), we obtain

$$\begin{aligned} &y^2\mu^4 + y\{p(s+x)\lambda^4 + p^2\lambda^3\mu + s(s+x)\lambda\mu^3 + (x^2+ps)\mu^4\} \\ &\quad + \{psx^2\lambda^4 + p^2x^2\lambda^3\mu + s^2x^2\lambda\mu^3 + psx^2\mu^4\} = 0 . \end{aligned} \tag{11}$$

Similarly we obtain from (9)

$$\begin{aligned} &y^2\mu^4 + y\{p(s+x)\lambda^4 + p^2\lambda^3\mu + s(s+x)\lambda\mu^3 + (x^2+ps)\mu^4\} \\ &\quad + \{psx^2\lambda^4 + p^2x^2\lambda^3\mu + s^2x^2\lambda\mu^3 + psx^2\mu^4\} = 0 . \end{aligned} \tag{12}$$

Subtracting (12) from (11), we have

$$y^2(\lambda^4 - \mu^4) + y(sx\lambda^4 + px\lambda^3\mu - sx\lambda\mu^3 - px\mu^4) = 0 .$$

Since $y \neq 0$, we have

$$y(\lambda^4 - \mu^4) + (sx\lambda^4 + px\lambda^3\mu - \mathrm{s}x\lambda\mu^3 - px\mu^4) = 0 .$$

Hence, if $\lambda^4 \neq \mu^4$, we have

$$y = -\frac{(\lambda^2 + \lambda\mu + \mu^2)(s\lambda + p\mu)}{(\lambda+\mu)(\lambda^2+\mu^2)} x . \tag{13}$$

On the other hand we must also have

$$B^3AB^{-1}A^3B^{-1}A \approx E .$$

And

$$\begin{aligned} B^3AB^{-1}A^3B^{-1}A &\approx \begin{pmatrix} \lambda^3 & 0 \\ 0 & \mu^3 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} p_3 & q_3 \\ r_3 & s_3 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ &= \begin{pmatrix} \lambda^3(s_3qr\lambda^2 + 2pqr\rho_3\lambda\mu + p_3p^2\mu^2) & \lambda^3q(s_3s\lambda^2 + \rho_3(ps+qr)\lambda\mu + p_3p\mu^2) \\ \mu^3r(s_3s\lambda^2 + \rho_3(ps+qr)\lambda\mu + p_3p\mu^2) & \mu^3(s_3s^2\lambda^2 + 2qrs\rho_3\lambda\mu + p_3qr\mu^2) \end{pmatrix} . \end{aligned}$$

So we must have

$$s_3s\lambda^2 + \rho_3(ps+qr)\lambda\mu + p_3p\mu^2 = 0 , \tag{14}$$

or,

$$y^2\lambda\mu + y(s\lambda + p\mu)\{(s\lambda+p\mu) + (p+s)(\lambda+\mu)\} + x^2(s\lambda+p\mu)^2 = 0 . \tag{15}$$

Substituting (13) into (15), we easily obtain

$$\lambda^3\mu^3x = (\lambda+\mu)(\lambda^2+\mu^2)(\lambda^2+\lambda\mu+\mu^2)(s\lambda+p\mu) . \tag{16}$$

From this it follows that

$$p(\lambda^5\mu+2\lambda^4\mu^2+2\lambda^3\mu^3+3\lambda^2\mu^4+2\lambda\mu^5+\mu^6)$$
$$+s(\lambda^6+2\lambda^5\mu+3\lambda^4\mu^2+2\lambda^3\mu^3+2\lambda^2\mu^4+\lambda\mu^5)=0\,. \tag{17}$$

Since this holds also when some scalar is multiplied to the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, we may assume from (17) that

$$p=\lambda^6+2\lambda^5\mu+3\lambda^4\mu^2+2\lambda^3\mu^3+2\lambda^2\mu^4+\lambda\mu^5\,, \tag{18}$$

$$s=-(\lambda^5\mu+2\lambda^4\mu^2+2\lambda^3\mu^3+3\lambda^2\mu^4+2\lambda\mu^5+\mu^6)\,. \tag{19}$$

Then, we have

$$x=p+s=(\lambda+\mu)(\lambda^2+\mu^2)(\lambda^3-\mu^3)\,, \tag{20}$$

$$s\lambda+p\mu=\lambda^3\mu^3(\lambda-\mu)\,. \tag{21}$$

Hence by (13),

$$y=qr-ps=-\lambda^3\mu^3(\lambda^3-\mu^3)^2\,. \tag{22}$$

It follows that $\lambda^3\neq\mu^3$. In the above, the case $\lambda^4=\mu^4$ was excluded. In this case, we must have $s\lambda+p\mu=0$ or $x=0$. But, if $s\lambda+p\mu=0$, then by (15) we must have $y^2\lambda\mu=0$. This is impossible. If $x=0$, then (16) holds (since $\lambda\neq\mu$) and hence we obtain (18), (19) and (22) also in this case.

It remains the case $q=0$ or $r=0$. We shall show that (18), (19) and (22) hold also in this case. Since the case $q=0$ is treated similarly, we only treat the case $r=0$. Then we have $q\neq 0$ (otherwise we would have $AB=BA$) and

$$A=\begin{pmatrix} p & q \\ 0 & s \end{pmatrix}, \qquad B=\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

(5), (8) and (14) are available in this case. So, if we show that (9) is also available, then (18), (19) and (22) will follow. We show it by using (10). Since $y=-ps\neq 0$, we have

$$\rho_3=p(x^2+y)+xy=p^3\,.$$

So, we have $p_3\mu\neq 0$, $X=0$, $Y=0$. Hence by (10), we have $Z=0$. Thus (9) is available, as desired.

Thus, we have proved that if h is a non-abelian representation of $\pi_1(N)$ in which $h(b)$ is not parabolic, then h is equivalent to a representation defined in (ii) of Theorem 5.

Conversely, suppose that $\lambda\mu=0$, $\lambda^3\neq\mu^3$, ($q\neq 0$ or $r\neq 0$) and that (18), (19) and (22) hold. Direct computation shows that the correspondence

$$a\to A=\begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad b\to B=\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

defines a non-abelian representation of $\pi_1(N)$. Thus (ii) of Theorem 5 is proved.

Next suppose that h is a non-abelian representation of $\pi_1(N)$ defined by $a\to A$,

$b \to B$, and that B is parabolic. Then we may assume that $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Since $A^3 B^{-1} A B^3 A B^{-1} \approx E$, we have $BA^{-3}B \approx AB^3A$, that is,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} s_3 & -q_3 \\ -r_3 & p_3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

or,

$$\begin{pmatrix} s_3 - r_3 & s_3 + p_3 - r_3 - q_3 \\ -r_3 & p_3 - r_3 \end{pmatrix} \approx \begin{pmatrix} p^2 + 3pr + qr & pq + 3ps + qs \\ pr + 3r^2 + rs & qr + 3rs + s^2 \end{pmatrix}. \tag{23}$$

We show that $r \neq 0$. Suppose that $r = 0$. Then,

$$\begin{aligned} s_3 &= s\rho_3 + y\rho_2 \\ &= s(x^2 + y) + yx \\ &= s(x^2 - ps) - psx \\ &= s^3, \end{aligned}$$

and similarly we have $p_3 = p^3$. Hence (23) becomes

$$\begin{pmatrix} s^3 & s^3 + p^3 - q_3 \\ 0 & p^3 \end{pmatrix} \approx \begin{pmatrix} p^2 & pq + 3ps + qs \\ 0 & s^2 \end{pmatrix}.$$

So we must have

$$p^5 = s^5 \tag{24}$$

and

$$s^2(s^3 + p^3 - q_3) = p^3(pq + 3ps + qs). \tag{25}$$

Since $AB \approx BA$, we must have $p \neq s$. Hence it follows that

$$p^4 + p^3 s + p^2 s^2 + ps^3 + s^4 = 0. \tag{26}$$

Moreover from (25) we have

$$\begin{aligned} -3p^4 s + p^3 s^2 + s^5 &= s^2 q_3 + p^4 q + p^3 sq \\ &= s^2 q(x^2 + y) + p^4 q + p^3 sq \\ &= q(p^4 + p^3 s + p^2 s^2 + ps^3 + s^4) \\ &= 0. \end{aligned}$$

From this and (26), we must have $p = s = 0$, a contradiction. Thus $r \neq 0$.

Now from (23) we have

$$\begin{aligned} (s_3 - r_3) : (p^2 + 3pr + qr) &= (s_3 + p_3 - r_3 - q_3) : (pq + 3ps + qs) \\ &= (-\rho_3) : (p + 3r + s) \\ &= (p_3 - r_3) : (qr + 3rs + s^2), \end{aligned}$$

or,

$$(s_3-r_3)(p+3r+s)+(p^2+3pr+qr)\rho_3=0 \tag{27}$$

$$(p_3-r_3)(p+3r+s)+(s^2+3sr+qr)\rho_3=0 \tag{28}$$

$$(s_3+p_3-r_3-q_3)(p+3r+s)+(pq+3ps+qs)\rho_3=0\,. \tag{29}$$

But,

$$\begin{aligned}
\rho_3&=x^2+y\,,\\
p+3r+s&=x+3r\,,\\
s_3-r_3&=(s-r)(x^2+y)+xy\,,\\
p_3-r_3&=(p-r)(x^2+y)+xy\,,\\
s_3+p_3-r_3-q_3&=(x-r-q)(x^2+y)+2xy\,,\\
p^2+3pr+qr&=p(x+3r)+y\,,\\
a^2+3sr+qr&=s(x+3r)+y\,,\\
pq+3ps+qs&=q(x+3r)-3y\,.
\end{aligned}$$

Hence, (27) and (28) become the same equation

$$(x+3r)(x-r)(x^2+y)+xy(x+3r)+y(x^2+y)=0\,, \tag{30}$$

while (29) becomes

$$(x+3r)(x-r)(x^2+y)+2xy(x+3r)-3y(x^2+y)=0\,. \tag{31}$$

From (30) and (31), we have

$$x(x+3r)=4(x^2+y)\,, \tag{32}$$

since $y\neq 0$. Hence $x\neq 0$ and

$$r=x+\frac{4y}{3x}\,. \tag{33}$$

Substituting it in (30) we have

$$(x^2+y)(x^2+16y)=0\,.$$

But if $x^2+y=0$, then

$$\begin{pmatrix} s_3-r_3 & s_3+p_3-r_3-q_3 \\ -r_3 & p_3-r_3 \end{pmatrix}=\begin{pmatrix} xy & 2xy \\ 0 & xy \end{pmatrix},$$

$$\begin{pmatrix} p^2+3pr+qr & pq+3ps+qs \\ pr+3r^2+rs & qr+3rs+s^2 \end{pmatrix}=\begin{pmatrix} y & -3y \\ 0 & y \end{pmatrix},$$

and

$$\begin{pmatrix} xy & 2xy \\ 0 & xy \end{pmatrix} \approx \begin{pmatrix} y & -3y \\ 0 & y \end{pmatrix}.$$

This contradicts (23). Hence

$$x^2 + 16y = 0. \tag{34}$$

From this and (33) we have

$$r = \frac{11}{12}x. \tag{35}$$

Conversely, (34) and (35) together with $y \neq 0$, $p \neq s$ are sufficient for a non-abelian representation of $\pi_1(N)$. Now from (34) and (35), the values of p and s determine the values of q and r. We shall show the resulting representations are all equivalent, irrespective of the values of p and s. Now the correspondence

$$a \to \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p+tr & q-tp+ts-t^2r \\ r & s-tr \end{pmatrix},$$

$$b \to \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

gives an equivalent representation to the original one. That is, (p, s) and $(p+tr, s-tr)$ give equivalent representations. But, $(p+tr)/(s-tr)$ takes arbitrary values at t varies. Thus, all the representations considered in this case are equivalent. So, as a representative of these we can choose the one defined by

$$a \to \begin{pmatrix} 10 & 1 \\ 11 & 2 \end{pmatrix}, \qquad b \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus we have proved (iii) and (iv) of the theorem. Moreover (v) is obvious since $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are not conjugate.

Finally we shall show (vi). Let h and h' be two non-abelian representations defined respectively by

$$a \to \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad b \to \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix};$$

and

$$a \to \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}, \qquad b \to \begin{pmatrix} \lambda' & 0 \\ 0 & \mu' \end{pmatrix}.$$

Since they are non-abelian it follows that $\lambda \neq \mu$ and $\lambda' \neq \mu'$. And $\{\lambda, \mu\}$ and $\{\lambda', \mu'\}$ are eigenvalues of $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \lambda' & 0 \\ 0 & \mu' \end{smallmatrix}\right)$, respectively. The last two matrices are conjugate in $GL(2, C)$ if and only if $\{\lambda, \mu\} = \{\lambda', \mu'\}$. Hence these are conjugate in $PGL(2, C)$ if and only if $\lambda : \mu = \lambda' : \mu'$ or $\lambda : \mu = \mu' : \lambda'$. So, this condition is necessary for the equivalence of h and h'.

Suppose that h and h' are equivalent. Then, for some $A \in GL(2, C)$,

$$A\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}A^{-1} \approx \begin{pmatrix}\lambda' & 0\\ 0 & \mu'\end{pmatrix}, \tag{36}$$

$$A\begin{pmatrix}p & q\\ r & s\end{pmatrix}A^{-1} \approx \begin{pmatrix}p' & q'\\ r' & s'\end{pmatrix}. \tag{37}$$

If $\lambda : \mu = \lambda' : \mu' \neq -1$, then by (36), A must be of the form $\left(\begin{smallmatrix}\xi & 0\\ 0 & \eta\end{smallmatrix}\right)$. So, $q=0$ implies $q'=0$, and $r=0$ implies $r'=0$. If $\lambda : \mu = \mu' : \lambda' \neq -1$, then by (36), A must be of the form $\left(\begin{smallmatrix}0 & \xi\\ \eta & 0\end{smallmatrix}\right)$. So, $q=0$ implies $r'=0$, and $r=0$ implies $q'=0$. If $\lambda : \mu = -1$, then we may assume that $\lambda = 1$, $\mu = 1$. Then by (18), (19) and (22), we have $p=1$, $s=-1$, $qr-ps=4$. So, $qr \neq 0$.

These considerations show that if h and h' are equivalent, then one of the conditions (I), (II), (III) of Theorem 5, (vi), holds. Next suppose that (I) holds. Since

$$\begin{aligned}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\mu & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1} &= \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix},\\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1} &= \begin{pmatrix}s & r\\ q & p\end{pmatrix},\end{aligned} \tag{38}$$

we may assume that $\lambda : \mu = \lambda' : \mu'$. But then we may also assume that $\lambda = \lambda'$ and $\mu = \mu'$. Then we have by (ii) that $p = p'$, $s = s'$ and $qr = q'r' \neq 0$. So there exists an $\alpha \neq 0$ such that $\alpha^2 q = q'$. Now

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}^{-1} = \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} = \begin{pmatrix}\lambda' & 0\\ 0 & \mu'\end{pmatrix},$$

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}^{-1} = \begin{pmatrix}p & \alpha^2 q\\ \alpha^{-2} r & s\end{pmatrix} = \begin{pmatrix}p' & q'\\ r' & s'\end{pmatrix}.$$

Hence the two representations are equivalent.

Next we assume that (II) holds. We may assume that $\lambda = \lambda'$, $\mu = \mu'$. If $r = r' = 0$, then $q = 0$, $q' \neq 0$, for otherwise the representations become abelian. So by the same reason as above, the two representations are equivalent. Similarly for the case $q = q' = 0$. The case (III) is reduced to the case (II) by (38). This completes the proof of Theorem 5.

§5. Computation of representations of $\pi_1(M_{m,n})$

THEOREM 7. *If* $(m, n) \neq \pm(0, 1)$, $\pm(1, 0)$, *then* $\pi_1(M_{m,n})$ *has a non-abelian representation and hence it is non-abelian. Moreover* $M_{0,1}$ *is the lens space of type* (9, 2) *and* $M_{1,0}$ *is the lens space of type* (13, 3). *Therefore the lens space conjecture holds for the class of* 3*-manifolds* $\{M_{m,n}\}$.

The rest of this section is devoted to proving this theorem.

We consider the representations of

$$\pi_1(M_{m,n})=\langle a,\ b\,|\,a^3b^{-1}ab^3ab^{-1}=(ab^{-1}a^2b^{-1})^m(ab^2)^n=1\rangle$$
$$=\langle a,\ b\,|\,a^3b^{-1}ab^3ab^{-1}=(b^{-3}a^2b^{-1})^m(ab^2)^j=1\rangle\,,$$

where $j=m+n$. Let

$$a\to A=\begin{pmatrix} p & q\\ r & s\end{pmatrix},\qquad b\to B=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$$

be a representation of $\pi_1(N)$ defined in Theorem 6(ii). Then it becomes a representation of $\pi_1(M_{m,n})$ if and only if

$$(B^{-3}A^2B^{-1})^m(AB^2)^j\approx E\,.$$

Let

$$ab^2\to\begin{pmatrix} p & q\\ r & s\end{pmatrix}\begin{pmatrix}\lambda^2 & 0\\ 0 & \mu^2\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$
$$b^{-3}a^2b^{-1}\to\begin{pmatrix}\mu^3 & 0\\ 0 & \lambda^3\end{pmatrix}\begin{pmatrix} p_2 & q_2\\ r_2 & s_2\end{pmatrix}\begin{pmatrix}\mu & 0\\ 0 & \lambda\end{pmatrix}=\begin{pmatrix}\alpha'' & \beta''\\ \gamma'' & \delta''\end{pmatrix}.$$

Then,

$$\alpha=\lambda^2p=\lambda^3(\lambda^5+2\lambda^4\mu+3\lambda^3\mu^{-2}+2\lambda^2\mu^3+2\lambda\mu^4+\mu^5)$$
$$\text{(abbreviated by }\lambda^3\ (1,\ 2,\ 3,\ 2,\ 2,\ 1))\,,$$
$$\beta=\mu^2q\,,$$
$$\gamma=\lambda^2r\,,$$
$$\delta=\mu^2s=-\mu^3\ (1,\ 2,\ 2,\ 3,\ 2,\ 1)\,,$$
$$\alpha''=\mu^4p^2=\lambda\mu^4(\lambda^3-\mu^3)\ (1,\ 3,\ 6,\ 7,\ 9,\ 8,\ 6,\ 3,\ 1)\,,$$
$$\beta''=\lambda\mu^3xq=\lambda\mu^3(\lambda+\mu)(\lambda^2+\mu^2)(\lambda^3-\mu^3)q\,,$$
$$\gamma''=\lambda^3\mu xr=\lambda^3\mu(\lambda+\mu)(\lambda^2+\mu^2)(\lambda^3-\mu^3)r\,,$$
$$\delta''=\lambda^4s_2=-\lambda^4\mu(\lambda^3-\mu^3)\ (1,\ 3,\ 6,\ 8,\ 9,\ 7,\ 6,\ 3,\ 1)\,.$$

Let

$$\alpha'=\lambda\mu^4\ (1,\ 3,\ 6,\ 7,\ 9,\ 8,\ 6,\ 3,\ 1)\,,$$
$$\beta'=\lambda\mu^3(\lambda+\mu)(\lambda^2+\mu^2)q\,,$$
$$\gamma'=\lambda^3\mu(\lambda+\mu)(\lambda^2+\mu^2)r\,,$$
$$\delta'=-\lambda^4\mu\ (1,\ 3,\ 6,\ 8,\ 9,\ 7,\ 6,\ 3,\ 1)\,.$$

Then,

$$\begin{pmatrix}\alpha'' & \beta'' \\ \gamma'' & \delta''\end{pmatrix} \approx \begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}.$$

Note that ab^2 and $b^{-3}a^2b^{-1}$ commute in $\pi_1(N)$. Hence

$$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}$$

must commute in $PGL(2, C)$.

Let $\kappa\xi$, $\kappa\eta$ be the eigen-values of $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$, and $\theta\xi'$, $\theta\eta'$ be those of $\left(\begin{smallmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{smallmatrix}\right)$. Then,

$$\begin{aligned} &\kappa(\xi+\eta)=\alpha+\beta\,, \qquad \kappa^2\xi\eta=\alpha\delta-\beta\gamma\,, \\ &\theta(\xi'+\eta')=\alpha'+\delta'\,, \qquad \theta^2\xi'\eta'=\alpha'\delta'-\beta'\gamma'\,. \end{aligned} \tag{39}$$

Hence we have

$$\begin{aligned} &(\xi+\eta)^2(\alpha\delta-\beta\gamma)=\xi\eta(\alpha+\delta)^2\,, \\ &(\xi'+\eta')^2(\alpha'\delta'+\beta'\gamma')=\xi'\eta'(\alpha'+\delta')^2\,. \end{aligned} \tag{40}$$

We first assume that $\xi\neq\eta$, $\xi\neq-\eta$. Then, for some $P\in GL(2, C)$,

$$P\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}P^{-1}=\begin{pmatrix}\kappa\xi & 0 \\ 0 & \kappa\eta\end{pmatrix}.$$

Since $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{smallmatrix}\right)$ commute in $PGL(2, C)$, $P\left(\begin{smallmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{smallmatrix}\right)P^{-1}$ is also diagonal and equal to either $\left(\begin{smallmatrix}\theta\xi' & 0 \\ 0 & \theta\eta'\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}\theta\eta' & 0 \\ 0 & \theta\xi'\end{smallmatrix}\right)$. We may assume that $P\left(\begin{smallmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{smallmatrix}\right)P^{-1}=\left(\begin{smallmatrix}\theta\xi' & 0 \\ 0 & \theta\eta'\end{smallmatrix}\right)$. Then

$$P\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}P^{-1}=\begin{pmatrix}\kappa\theta\xi\xi' & 0 \\ 0 & \kappa\theta\eta\eta'\end{pmatrix}.$$

So the eigen-values of $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{smallmatrix}\right)$ are $\kappa\theta\xi\xi'$ and $\kappa\theta\eta\eta'$, and we have

$$\alpha\alpha'+\beta\gamma'+\gamma\beta'+\delta\delta'=\kappa\theta(\xi\xi'+\eta\eta')\,. \tag{41}$$

Also by (39) we have

$$\kappa\theta(\xi+\eta)(\xi'+\eta')=(\alpha+\delta)(\alpha'+\delta')\,. \tag{42}$$

From (41) and (42) we have

$$(\alpha\alpha'+\beta\gamma'+\gamma\beta'+\delta\delta')(\xi+\eta)(\xi'+\eta')=(\alpha+\delta)(\alpha'+\delta')(\xi\xi'+\eta\eta')\,,$$

or

$$P_0(\xi\eta'+\eta\xi')=Q_0(\xi\xi'+\eta\eta')\,,$$

where

$$P_0=\alpha\alpha'+\beta\gamma'+\gamma\beta'+\delta\delta' \quad \text{and} \quad Q_0=\alpha\delta'-\beta\gamma'-\gamma\beta'+\delta\alpha'\,.$$

From (43) we have

$$\xi':\eta'=(P_0\xi-Q_0\eta):(Q_0\xi-P_0\eta)\,.$$

So if θ is suitably chosen, we may assume

$$\xi'=P_0\xi-Q_0\eta\,,\qquad \eta'=Q_0\xi-P_0\eta\,,$$

unless they are both zero.

Now the second relator becomes

$$(b^{-3}a^2b^{-1})^m(ab^2)^j\to\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}^m\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^j\approx E\,.$$

From this, it is necessary that two eigen-values of

$$\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}^m\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^j$$

coincide, that is, $\xi'^m\xi^j=\eta'^m\eta^j$, or

$$(P_0\xi-Q_0\eta)^m\xi^j=(Q_0\xi-P_0\eta)^m\eta^j\,. \tag{43}$$

But this condition is also sufficient in this case. For, if $\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{smallmatrix}\right)^m\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)^j\approx E$ does not hold but (44) holds, then $\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{smallmatrix}\right)^m\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)^j$ must be parabolic and hence $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is also parabolic, since $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{smallmatrix}\right)$ commute. But it is impossible since we assume $\xi\neq\eta$.

In order to obtain representations of $\pi_1(M_{m,n})$, we solve the simultaneous homogeneous equations

$$(\xi+\eta)^2(\alpha\delta-\beta\gamma)=\xi\eta(\alpha+\delta)^2\,, \tag{40}$$

$$(P_0\xi-Q_0\eta)^m\xi^j=(Q_0\xi-P_0\eta)^m\eta^j\,. \tag{43}$$

As in [4], we consider the solutions of these equations as the intersection of two algebraic curves in $CP^1\times CP^1$ with the coordinate system $\{(\lambda,\mu;\xi,\eta)\}$, where λ, μ are not both zero, and ξ, η are not both zero, and $(\lambda,\mu;\xi,\eta)$ and $(\lambda',\mu';\xi',\eta')$ denote the same point iff $\lambda'=\sigma\lambda$, $\mu'=\sigma\mu$, $\xi'=\tau\xi$, $\eta'=\tau\eta$, for some $\sigma\neq 0$ and $\tau\neq 0$. These solutions give desired representations iff $\lambda\mu\neq 0$, $\lambda^3\neq\mu^3$, $\xi\eta\neq 0$, $\xi\neq\eta$, $\xi\neq-\eta$.

Now,

$$\alpha\delta-\beta\gamma=-\lambda^2\mu^2y=\lambda^5\mu^5(\lambda^3-\mu^3)^2\,,$$

$$\alpha+\delta=(\lambda^2-\mu^2)\,(1,2,4,3,4,2,1)\,.$$

So, (40) becomes

$$(\xi+\eta)^2\lambda^5\mu^5(\lambda^2+\lambda\mu+\mu^2)^2=\xi\eta(\lambda+\mu)^2\,(1,2,4,3,4,2,1)^2\,. \tag{44}$$

Moreover,

$$\begin{aligned}P_0&=\alpha\alpha'+\beta\gamma'+\gamma\beta'+\delta\delta'\\ &=-\lambda^6\mu^6(\lambda-\mu)^2(\lambda+\mu)\,(1,2,4,5,4,2,1)\,.\end{aligned}$$

$$Q_0 = -\lambda\mu(\lambda-\mu)^2(\lambda+\mu)\ (1,\ 6,\ 22,\ 56,\ 113,\ 185,\ 261,\ 316,\ 339,\ 316,\ 261,\ 185,\ 113,\ 56,\ 22,\ 6,\ 1)$$
$$= -\lambda\mu(\lambda-\mu)^2(\lambda+\mu)(\lambda^2+\lambda\mu+\mu^2)^2\ (1,\ 4,\ 11,\ 20,\ 31,\ 37,\ 43,\ 37,\ 31,\ 20,\ 11,\ 4,\ 1)\,.$$

In (44), if $\lambda+\mu=0$, then $\xi+\eta=0$. So it does not give representations. So we can devide (43) by $-\lambda\mu(\lambda-\mu)^2(\lambda+\mu)$. Then we have

$$(P\xi-Q\eta)^m\xi^j=(Q\xi-P\eta)^m\eta^j\,, \tag{45}$$

where $P=\lambda^5\mu^5$ (1, 2, 4, 5, 4, 2, 1) and

$$Q=(\lambda^2+\lambda\mu+\mu^2)^2\ (1,\ 4,\ 11,\ 20,\ 31,\ 37,\ 43,\ 37,\ 31,\ 20,\ 11,\ 4,\ 1)\,.$$

So we solve the simultaneous equations (44) and (45).

In (44),

(i) if $\lambda\mu=0$, then $\xi\eta=0$;
(ii) if $\lambda^2+\lambda\mu+\mu^2=0$, then $\xi\eta=0$;
(iii) if $\lambda=\mu$, then $9(\xi+\eta)^2=1156\xi\eta$;
(iv) if $\xi\eta=0$, then $\lambda\mu=0$ or $\lambda^2+\lambda\mu+\mu^2=0$;
(v) if $\xi=\eta$, then $4\lambda^5\mu^5(\lambda^2+\lambda\mu+\mu^2)^2=(\lambda+\mu)^2(1, 2, 4, 3, 4, 2, 1)$;
(vi) if $\xi=-\eta$, then $\mu=-\lambda$ or $(1, 2, 4, 3, 4, 2, 1)=0$.

By the way there are only finitely many such exceptional points.

First we remark that $\lambda=\mu$ does not occur in any solution of (44) and (45). For, if $\lambda=\mu$, then from (44) we have $9(\xi+\eta)^2=1156\xi\eta$, that is,

$$9\xi^2-1138\xi\eta+9\eta^2=0\,.$$

So if we put $x=\xi/\eta$, then x satisfies the equation

$$9x^2-1138x+9=0\,. \tag{46}$$

Moreover, by (45) we have

$$(Px-Q)^m x^j=(Qx-P)^m\,,$$
$$x^j=\left(\frac{Qx-P}{Px-Q}\right)^m=\left(\frac{2259x-19}{19x-2259}\right)^m.$$

So, if we put

$$y=\frac{2259x-19}{19x-2259}\,,$$

we have

$$9y^2-17938y+9=0\,, \tag{47}$$

and

$$x^j = y^m . \tag{48}$$

And each of x^n and y^n satisfies the equation of form

$$9^n t^2 - kt + 9^n = 0 ,$$

where k is an integer relatively prime to 9. So we must have $|j| = |m|$ and since j and m are relatively prime, we have $|j| = |m| = 1$ and $x = y$ or y^{-1}, by (48). But this is impossible by (46) and (47). So $\lambda \neq \mu$ in any solution of (44) and (45).

Now, let

$$f(\lambda, \mu; \xi, \eta) = (\xi + \eta)^2 \lambda^5 \mu^5 (\lambda^2 + \lambda\mu + \mu^2)^2 - \xi\eta(\lambda + \mu)^2 (1, 2, 4, 3, 4, 2, 1)^2 \tag{49}$$

and

$$g_{m,j}(\lambda, \mu; \xi, \eta) = (P\xi - Q\eta)^m \xi^j - (Q\xi - P\eta)^m \eta^j \tag{50}$$

and we consider the simultaneous equations

$$f(\lambda, \mu; \xi, \eta) = 0 , \qquad g_{m,j}(\lambda, \mu; \xi, \eta) = 0 , \tag{51}$$

in $CP^1 \times CP^1$. f is of degree (2, 14) and $g_{m,j}$ is of degree $(m + |j|, 16m)$. By Bezout's theorem for $CP^1 \times CP^1$ (cf. [4]), the total sum of the number of intersection is

$$2 \cdot 16m + 14(m + |j|) = 46m + 14|j| .$$

First we compute the numbers of interesections at

$$(\lambda, \mu : \xi, \eta) = (0, 1; 0, 1) , \quad (0, 1; 1, 0) , \quad (1, 0; 0, 1) , \quad (1, 0; 1, 0) .$$

We easily see that the numbers of intersections at these points are the same. So we only compute that of the point $F = (0, 1; 0, 1)$.

Now the only parametrization of (49) with center at F is given by

$$\lambda = t , \quad \mu = 1 ; \quad \xi = t^5 - 4t^6 + 6t^7 - t^9 + 34t^{10} + \cdots , \qquad \eta = 1 .$$

Then

$$(P\xi - Q\eta)^m \xi^j = \pm t^{5j} + \cdots ,$$

$$(Q\xi - P\eta)^m \eta^j = \pm t^{8m} + \cdots .$$

So, if $5j \neq 8m$, $j \geqq 0$, then

$$\text{ord}\,(g_{m,j}) = \min\,(5j, 8m) .$$

Moreover we can show that if $5j = 8m$, i.e. $(m, j) = (5, 8)$, then

$$\text{ord}\,(g_{m,j}) = 41 .$$

If $j < 0$, we easily obtain

$$\text{ord}\,(g_{m,j}) = 0 .$$

So, if we denote the number intersection at F by $\iota(F)$, then

$$\iota(0,\ 1;\ 0,\ 1)=\iota(0,\ 1;\ 1,\ 0)=\iota(1,\ 0;\ 0,\ 1)=\iota(1,\ 0;\ 1,\ 0)$$
$$=\begin{cases}\min(5j, 8m)\,, & \text{if } j\geqq 0,\ 5j\neq 8m\,,\\ 41\,, & \text{if } (m, j)=(5, 8)\,,\\ 0\,, & \text{if } j<0\,.\end{cases}$$

Next we consider the exceptional points with $\lambda^2+\lambda\mu+\mu^2=0$, $\xi\eta=0$. Let ω denote a root of $\omega^2+\omega+1=0$. Then there are four points under consideration:

$$(\lambda,\ \mu;\ \xi,\ \eta)=(\omega,\ 1;\ 0,\ 1)\,,\quad (\omega^2,\ 1;\ 0,\ 1)\,,\quad (\omega,\ 1;\ 1,\ 0)\,,\quad (\omega^2,\ 1;\ 1,\ 0)\,.$$

Of course the numbers of intersection at these points are equal. So we only treat the point $(\omega, 1; 0, 1)$. The place with center at this point is given by

$$\lambda=\omega+t\,,\qquad \mu=1\,;\qquad \xi=-3\omega t^2+\cdots\,,\qquad \eta=1\,.$$

If $j\geqq 0$, then ord $(g_{m,j})=0$. So the number of interesection $\iota(\omega,\ 1;\ 0,\ 1)=0$. If $j<0$, then

$$(P\xi-Q\eta)^m\eta^{-j}=3^m t^{2m}+\cdots\,,$$
$$(Q\xi-P\eta)^m\xi^{-j}=(-3\omega)^{-j}t^{-2j}+\cdots\,.$$

So if $2m\neq -2j$, then $\iota(\omega,\ 1;\ 0,\ 1)=\min(2m,\ -2j)$. If $2m=-2j$, i.e. $(m, j)=(1,\ -1)$, then $\iota(\omega,\ 1;\ 0,\ 1)=3$. Thus

$$\iota(\omega,\ 1;\ 0,\ 1)=\iota(\omega^2,\ 1;\ 0,\ 1)=\iota(\omega,\ 1;\ 1,\ 0)=\iota(\omega^2,\ 1;\ 1,\ 0)$$
$$=\begin{cases}0\,, & \text{if } j\geqq 0\,,\\ \min(2m,\ -2j) & \text{if } j<0,\ (m, j)\neq(1,\ -1)\,,\\ 3\,, & \text{if } (m, j)=(1,\ -1)\,.\end{cases}$$

Next we compute the number of intersection at $(-1,\ 1;\ -1,\ 1)$. Let $\lambda=-1+t$, $\mu=1$, $\eta=1$. Then, by (44), $\xi=-1+3t-3t^2\cdots$, or $\xi=-1-3t-6t^2+\cdots$. If $\xi=-1+3t-3t^2\cdots$, then

$$\begin{aligned}&(P\xi-Q\eta)^m\xi^j-(Q\xi-P\eta)^m\eta^j\\ &=\{(-1+t)^5(3-9t)(-1+3t)-(1-2t)(7-42t)\}^m(-1+3t)^j\\ &\quad-\{(1-2t)(7-42t)(-1+3t)-(-1+t)^5(3-9t)\}^m+\cdots\\ &=(-4+23t)^m(-1+3t)^j-(-4+101t)^m+\cdots\\ &=\{(-4)^m(-1)^j-(-4)^m\}+2(-4)^{m-1}\{39m-2(-3^j)\}t+\cdots\,.\end{aligned}$$

So

$$\operatorname{ord}(g_{m,j})=\begin{cases}0, & \text{if } j \text{ is odd,}\\ 1, & \text{if } j \text{ is even.}\end{cases}$$

The same holds when $\xi=-1-3t-6t^2+\cdots$. Thus

$$\iota(-1,1;-1,1)=\begin{cases}0, & \text{if } j \text{ is odd,}\\ 1\times 2=2, & \text{if } j \text{ is even.}\end{cases}$$

But if j is even, there exists a homomorphism

$$\pi_1(M_{m,n})\cong\langle a,\, b\,|\,a^3b^{-1}ab^3ab^{-1}=(b^{-3}a^2b^{-1})^m(ab^2)^j=1\rangle$$
$$\to\langle a,\, b\,|\,a^2=b^2=(ab)^3=1\rangle\,.$$

The latter group has the representations

$$a\to\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad b\to\begin{pmatrix}p & q\\ r & s\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}s & -q\\ -r & p\end{pmatrix},$$

where

$$ab\to\begin{pmatrix}ps+qr & -2pq\\ -2rs & ps+qr\end{pmatrix}$$

and $3(ps)^2+10(ps)(qr)+3(qr)^2=0$, so that $(ab)^3\to 1$. In some sense, these representations may be counted twice. So we do not subtract $\iota(-1,1;-1,1)$ from the total number of intersection.

Next we compute the number of intersection at $(\lambda,\mu;\xi,\eta)$ $(\alpha,1;-1,1)$, where α is a root of

$$\phi(t)=t^6+2t^5+4t^4+3t^3+4t^2+2t+1=0\,.$$

First we remark that $\phi(t)$ is an irreducible polynomial in $Z[t]$ and hence does not have double roots. So there exist six roots of it and they determine six points $(\alpha,1;-1,1)$'s in $CP^1\times CP^1$.

We shall show that

$$\iota(\alpha,1;-1,1)=2m\,,$$

for each α. Now there are two places with center at this point. These are parametrized by

$$\xi=-1+t\,,\qquad \lambda=\alpha+at+bt^2+\cdots\,, \tag{52}$$
$$\eta=1\,,\qquad \mu=1\,.$$

In order to find the values of a, we substitute (52) into $f(\lambda,\mu;\xi,\eta)$. Then

$$0=f(-1+t,\,1;\,\alpha+at+bt^2+\cdots,\,1)$$
$$\equiv t^2(\alpha+at)^5\{(\alpha+at)^2+(\alpha+at)+1\}^2$$
$$-(-1+t)\{(1+\alpha)+at\}^2\phi(\alpha+at)^2$$

$$\equiv\{\alpha^5(\alpha^2+\alpha+1)^2+(\alpha+1)^2\phi'(\alpha)a^2\}t^2 \qquad (\text{mod. } t^3).$$

Thus, we have

$$a=\pm\sqrt{\frac{-\alpha^5(\alpha^2+\alpha+1)^2}{(\alpha+1)^2}}\frac{1}{\phi'(\alpha)}=\frac{\pm\alpha^2(\alpha^2+\alpha+1)\sqrt{-\alpha}}{(\alpha+1)\phi'(\alpha)}.$$

Next we compute the order of $g_{m,j}(\lambda, \mu; \xi, \eta)$. We assume $j>0$. The case $j<0$ can be treated similarly.

$$\begin{aligned}&g_{m,j}(-1+t,\ 1,\ \alpha+at+bt^2+\cdots,\ 1)\\&=(P\cdot(-1+t)-Q)^m(-1+t)^j-(Q\cdot(-1+t)-P)^m\\&=(Pt-(P+Q))^m(-1+t)^j-(Qt-(P+Q))^m.\end{aligned}$$

Here we notice that

$$P(\lambda,\ 1)+Q(\lambda,\ 1)=\phi(\lambda)\chi(\lambda),$$

where

$$\chi(\lambda)=\lambda^{10}+4\lambda^9+10\lambda^8+17\lambda^7+23\lambda^6+24\lambda^5+23\lambda^4+17\lambda^3+10\lambda^2+4\lambda+1.$$

Thus,

$$\begin{aligned}&(Pt-(P+Q))^m(-1+t)^j-(Qt-(P+Q))^m\\&\equiv(P(\alpha,\ 1)t-a\phi'(\alpha)\chi(\alpha)t)^m(-1)^j-(Q(\alpha,\ 1)t-a\phi'(\alpha)\chi(\alpha)t)^m\\&\equiv\{(P(\alpha,\ 1)-a\phi'(\alpha)\chi(\alpha))^m(-1)^j-(Q(\alpha,\ 1)-a\phi'(\alpha)\chi(\alpha))^m\}t^m \quad (\text{mod. } t^{m+1}).\end{aligned}$$

We show ord $(g_{m,j})=m$ by checking that the coefficient

$$c=(P(\alpha,\ 1)-a\phi'(\alpha)\chi(\alpha))^m(-1)^j-(Q(\alpha,\ 1)-a\phi'(\alpha)\chi(\alpha))^m\neq 0.$$

Suppose that $c=0$. Then

$$d=\frac{P(\alpha, 1)-a\phi'(\alpha)\chi(\alpha)}{Q(\alpha, 1)-a\phi'(\alpha)\chi(\alpha)}$$

is a $2m$-th root of unity and hence d must satisfy a cyclotomic equation. Thus if we show that d satisfies an irreducible non-cyclotomic equation in Q, then we have a contradiction and it shows that $c\neq 0$. Now

$$\begin{aligned}P(\alpha,\ 1)&=2\alpha^8, &&(\text{since } P(\lambda,\ 1)\equiv 2\lambda^8 \quad (\text{mod. } \phi(\lambda)).\\Q(\alpha,\ 1)&=-2\alpha^8, &&(\text{by the same reason as above})\\\chi(\alpha)&=-2\alpha^5, &&(\text{by the same reason as above})\\a\phi'(\alpha)&=\frac{\pm\alpha^2(\alpha^2+\alpha+1)\sqrt{-\alpha}}{\alpha+1}.\end{aligned}$$

$$d=\frac{\pm\dfrac{\alpha^2(\alpha^2+\alpha+1)\sqrt{-\alpha}}{\alpha+1}(-2\alpha^5)-2\alpha^8}{\pm\dfrac{\alpha^2(\alpha^2+\alpha+1)\sqrt{-\alpha}}{\alpha+1}(-2\alpha^5)+2\alpha^8}$$

$$=\frac{(\alpha^2+\alpha+1)\sqrt{-\alpha}\mp\alpha(\alpha+1)}{(\alpha^2+\alpha+1)\sqrt{-\alpha}\pm\alpha(\alpha+1)}$$

A further (rather long) calculation shows that d satisfies the irreducible equation

$$d^6+14d^5+63d^4+36d^3+63d^2+14d+1=0\,,$$

which is not cyclotomic.

Thus we have shown that ord $(g_{m,j})=m$, for each place with center at $(\alpha, 1; -1, 1)$. Since there are exactly two places with center at this point, we have that

$$\imath(\alpha,\ 1;\ -1,\ 1)=2|m|\,.$$

Since there are 6 different roots of $\phi(\alpha)=0$, we have

$$\sum_{\phi(\alpha)=0}(-1, 1; \alpha, 1)=12m\,.$$

Next note that

$$f(\lambda,\ \mu;\ \xi,\ \eta)=(\phi-\eta)^2\lambda^5\mu^5(\lambda^2+\lambda\mu+\mu^2)^2-\xi\eta\cdot\psi(\lambda,\ \mu)\,,$$

where

$$\begin{aligned}\psi(\lambda,\ \mu)&=(1,\ 6,\ 21,\ 50,\ 92,\ 134,\ 167,\ 178,\ 167,\ 134,\ 92,\ 50,\ 21,\ 6,\ 1)\\&=(1,\ 4,\ 8,\ 9,\ 8,\ 4,\ 1)\ (1,\ 2,\ 5,\ 5,\ 6,\ 5,\ 5,\ 2,\ 1)\,.\end{aligned}$$

Since these factors are reciprocal we can easily compute all the roots of $\psi(\lambda, 1)$, (e.g. by the aid of programable electronic calculator) and see that $\psi(\lambda, 1)$ does not have multiple roots.

Let β be any solution of $\psi(\lambda)$. Then by Walker [8], there exists exactly one place with center at $(\beta, 1; 1, 1)$, which is parametrized by

$$\begin{aligned}&\xi=1+t\,, &&\lambda=\beta+at^2+bt^3+\cdots\,,\\&\eta=1\,, &&\mu=1\,.\end{aligned}$$

Substituting these in $f(\lambda,\ \mu;\ \xi,\ \eta)$, we obtain

$$\begin{aligned}0&=f(\beta+at^2+bt^3+\cdots,\ 1;\ 1+t,\ 1)\\&=t^2(\beta+at^2+bt^3+\cdots)^5((\beta+at^2+bt^3+\cdots)^2+(\beta+at^2+bt^3+\cdots)+1)^2\\&\quad-(1+t)\psi(\beta+at^2+bt^3+\cdots,\ 1)\\&\equiv\{\beta^5(\beta^2+\beta+1)^2-\psi'(\beta)a\}t^2\qquad(\text{mod. } t^3)\,.\end{aligned}$$

Thus

$$a=\frac{\beta^5(\beta^2+\beta+1)}{\psi'(\beta)}.$$

Note that

$$\begin{aligned}
Q-P&=(\lambda^2+\lambda\mu+\mu^2)\psi(\lambda,\mu)\\
&=((\beta+at^2+bt^3\cdots)^2+(\beta+at^2+bt^3+\cdots)+1)(\psi'(\beta)at^2+\cdots)\\
&\equiv(\beta^2+\beta+1)\psi'(\beta)at^2\\
&=\beta^5(\beta^2+\beta+1)^3t^2 \qquad (\mathrm{mod.}\ t^3)\,,\\
P&\equiv\beta^5(\beta^6+2\beta^5+4\beta^4+5\beta^3+4\beta^2+2\beta+1) \qquad (\mathrm{mod.}\ t^2)\,,\\
Q&\equiv\lambda^5\mu^5(\lambda^6+2\lambda^5\mu+4\lambda^4\mu^2+5\lambda^3\mu^3+4\lambda^2\mu^4+2\lambda\mu^5+\mu^6) \qquad (\mathrm{mod.}\ \psi)\\
&\equiv\beta^5(\beta^6+2\beta^5+4\beta^4+5\beta^3+4\beta^2+2\beta+1) \qquad (\mathrm{mod.}\ t^2)\,.
\end{aligned}$$

Thus (assuming $j>0$)

$$\begin{aligned}
&g_{m,j}(\beta+at^2+bt^3+\cdots,\ 1;\ 1+t,\ 1)\\
&=(P\cdot(1+t)-Q)^m(1+t)^j-(Q\cdot(1+t)-P)^m\\
&=(Pt-(Q-P))^m(1+t)^j-(Qt+(Q-P))^m\\
&\equiv(P^mt^m-mP^{m-1}\beta^5(\beta^2+\beta+1)^3t^{m-1})(1+jt)\\
&\quad-(\bar{P}^mt^m+m\bar{P}^{m-1}\beta^5(\beta^2+\beta+1)^3t^{m+1}) \qquad (\mathrm{mod.}\ t^{m+2})\\
&=\{-2m\bar{P}^{m-1}\beta^5(\beta^2+\beta+1)^3+j\bar{P}^m\}t^{m+1}\,,
\end{aligned}$$

where

$$\bar{P}=\beta^5(\beta^6+2\beta^5+4\beta^4+5\beta^3+4\beta^2+2\beta+1)\,.$$

But it is easily seen that

$$\frac{\beta^5(\beta^2+\beta+1)^3}{\bar{P}}=\frac{(\beta^2+\beta+1)^3}{\beta^6+2\beta^5+4\beta^4+5\beta^3+4\beta^2+2\beta+1}$$

is irrational. So the coefficient

$$-2m\bar{P}^{m-1}\beta^5(\beta^2+\beta+1)^3+j\bar{P}^m\neq0\,.$$

Thus $\iota(\beta,\ 1;\ 1,\ 1)=\mathrm{ord}\,(g_{m,j})=m+1$. There are 14 solution of the equation $\psi(\lambda)=0$. So

$$\sum_{\psi(\beta)=0}\iota(\beta,1;1,1)=14m+14\,.$$

Recall that $j=m+n$. Now

$$\begin{aligned}
d_{m,n} &= (\text{the total sum of the numbers of intersections}) \\
&\quad -\iota(0,1;0,1)-\iota(0,1;1,0)-\iota(1,0;0,1)-\iota(1,0;1,0) \\
&\quad -\iota(\omega,1;0,1)-\iota(\omega^2,1;0,1)-\iota(\omega,1;1,0)-\iota(\omega^2,1;1,0) \\
&\quad -\sum_{\phi(\alpha)=0}\iota(\alpha,1;-1,1)-\sum_{\psi(\beta)=0}\iota(\beta,1;1,1) \\
&= 46m+14|j|-4\max(0,\min(5j,8m))-4\max(0,\min(-2j,2m)) \\
&\quad -12m-14m-14 \\
&= \begin{cases} 14j-12m-14, & \text{if } 5j>8m, \\ -6j+20m-14, & \text{if } 8m>5j>0, \\ -6j+20m-14, & \text{if } 0>2j>-2m, \\ -14j+12m-14, & \text{if } -2m>2j. \end{cases} \\
&= 2|5j-8m|+4|j+m|-14 = 2|5n-3m|-4|n+2m|-14,
\end{aligned}$$

if $(m,j)\neq(5\ 8)$, $(1,-1)$, $(0,1)$, $(1,0)$. Moreover,

if $(m,j)=(5,8)$, then $d_{m,n}=32=2|5n-3m|+4|n+2m|-18$,

if $(m,j)=(1,-1)$, then $d_{m,n}=8=2|5n-3m|+4|n+2m|-18$,

if $(m,j)=(0,1)$, then $d_{m,n}=0=2|5n-3m|+4|n+2m|-14$,

if $(m,j)=(1,0)$, then $d_{m,n}=6=2|5n-3m|+4|n+2m|-14$,

Thus

$$d_{m,n}=2|5n-3m|+4|n+2m|-14-\delta_{m,n},$$

where

$$\delta_{m,n}=\begin{cases} 0, & \text{if } (m,n)\neq(5,3),\ (1,-2), \\ 4, & \text{if } (m,n)=(5,3),\ (1,-2). \end{cases}$$

Let

$$h(x,y)=2|5x-3y|+4|x+2y|-14,$$

where $x, y\in R$. Then $h(x, y)$ is a continuous, piecewise linear function of x, y. In (x, y)-plane, $\{(x,y)\,|\,h(x,y)=0\}$ is the parallelogram L illustrated in Figure 3. $h(x,y)>0$ outside L, and $h(x,y)<0$ inside L. The interior of L does not contain any lattice point other than the origin and on L there are four lattice points $\pm(0,1)$, $\pm(1,0)$. This means that $d_{m,n}>0$ except $(m,n)=(0,1)$, (1.0).

So if $(m,n)\neq(0,1)$, (1.0), then there exists a non-abelian representation of $\pi_1(M_{m,n})$ and hence $\pi_1(M_{m,n})$ is non-abelian and $M_{m,n}$ is not a lens space. Moreover, by examining Heegaard diagrams we can show that $M_{(1,0)}$ is the lens space of type

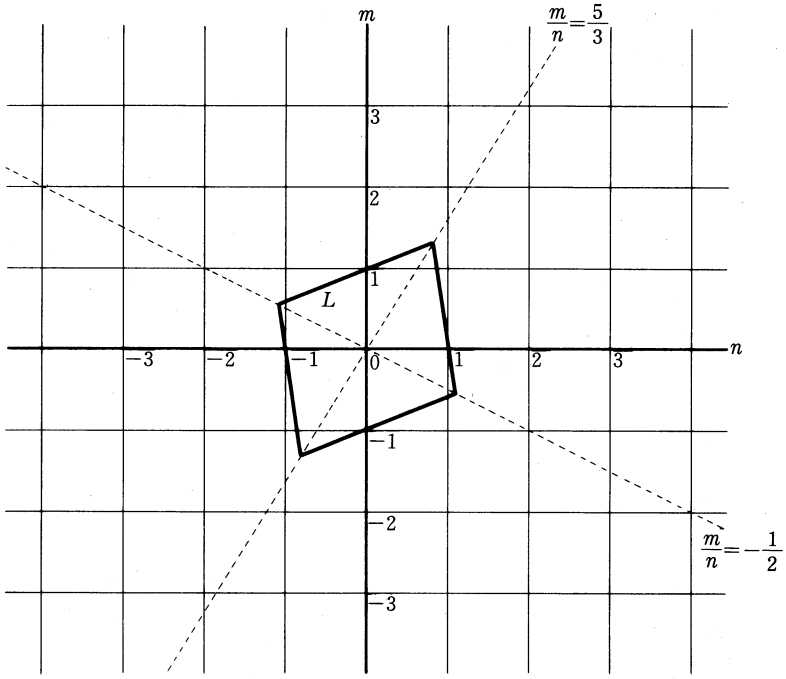

Fig. 3

(13, 3) and $M_{(0,1)}$ is the lens space of type (9, 2). Thus we conclude that for the class of manifolds $\{M_{m,n}\}$ the lens space conjecture holds.

§ 6. Remark

During the writing of this paper, we knew Thurston's theory [5]. If we used his theory with some device, the arguments of this paper could be fairly simplified. Moreover we can show that the interior of N admits a (complete) hyperbolic structure (with finite volume). This structure can be constructed by glueing together the faces of three ideal tetrahedra. We can also show that the critical cases $(m, n)=(5, 3), (1, -2)$, the manifold $M_{m,n}$ is sufficiently large. Indeed $M_{(1,-2)}$ contains an incompressible torus and $M_{(5,3)}$ contains an incompressible surface of genus 2. $M_{(9,13)}$ is also sufficiently large since $H_1(M_{(9,13)})$ is infinite. It seems likely that any other $M_{m,n}$ is not sufficiently large. There is no theoretical difficulty to check it but only a tedious effort would be necessary.

Also it can be shown that when $(m, j)=(1, -1), (1, 1), (2, 1)$, the manifold $M_{m,n}$ is a (special) Seifert fibered space and hence does not admit hyperbolic structure. $M_{(1,-2)}$ does not admit hyperbolic structure since it contains an incompressible torus. It seems likely that when $(m, n)\neq(0, 1), (1, 0), (1, -2), (1, -1), (1, 1), (2, 1)$, $M_{m,n}$ does admit hyperbolic structure. Thurston's hyperbolic Dehn surgery argument can apply. However it causes some difficulty when both positively oriented simplexes and negatively oriented simplexes occur.

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Institute of Mathematics
The University of Tsukuba
Ibaraki, Japan