COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 34, No. 1, 1985

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

Image Area and Functions of Uniformly Bounded Characteristic

by

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(Received December 24, 1983)

1. Introduction

Let f be a function meromorphic in $D=\{|z|<1\}$, and let $T(1,f)$ be the limit

$$T(1,f)=\lim_{r\to 1} T(r,f)$$

of the Shimizu-Ahlfors characteristic function

$$T(r,f)=\pi^{-1}\int_0^r t^{-1}\left[\iint_{|z|<t} f^{\#}(z)^2 dx\,dy\right]dt\,,\qquad 0<r<1\,,$$

where

$$f^{\#}=|f'|/(1+|f|^2)\,.$$

For $w\in D$ as a parameter we write the composed function

$$f_w(z)=f((z+w)/(1+\bar{w}z))\,,\qquad z\in D\,.$$

Then, f is said to be of uniformly bounded characteristic in D, $f\in UBC$ in notation, if

$$\|f\|_T\equiv\sup_{w\in D} T(1,f_w)<\infty\,,$$

while f is said to be of class UBC_0, $f\in UBC_0$ in notation, if

$$\lim_{|w|\to 1} T(1,f_w)=0\,.$$

It is known that $UBC_0\subset UBC$ [3, Lemma 2.1, p. 352].

We begin with the positive answer to the problem raised in [3, p. 366].

THEOREM 1. *If f is meromorphic in D with finite image area,*

$$\iint_D f^{\#}(z)^2 dx\,dy<\infty \qquad (z=x+iy)\,, \tag{1.1}$$

then $f\in UBC_0$.

For $w\in D$ we set

$$R(w)=D\,, \qquad \text{if} \quad w=0\,;$$
$$=\{z\in D;\ |w|<|z|<1 \quad \text{and} \quad |\arg(z/w)|<\pi(1-|w|)\}\,, \qquad \text{if} \quad w\neq 0\,.$$

Then, the length of the arc

$$\partial D\cap\partial R(w)=\{e^{it};\ |t-\arg w|<\pi(1-|w|)\}$$

is $2\pi(1-|w|)$ if $w\neq 0$; this is also true for $w=0$. For a measure $\mu\geqq 0$ in D we set

$$Q(\mu, w)=\mu(R(w))/\{2\pi(1-|w|)\}\,, \qquad w\in D\,.$$

Then, μ is called a Carleson measure if $\sup_{w\in D} Q(\mu, w)<\infty$; the definition is equivalent to that in [2, p. 238].

For f meromorphic in D we consider the measure μ_f defined in the differential form:

$$d\mu_f(z)=f^{\#}(z)^2(1-|z|)dx\,dy\,.$$

THEOREM 2. *Let f be meromorphic in D and suppose that $f\in UBC$. Then μ_f is a Carleson measure.*

A partial answer to the problem whether or not the converse of Theorem 3 is true, is supplied by

THEOREM 3. *Let f be meromorphic in D and suppose that*

$$\limsup_{|w|\to 1} Q(\mu_f, w)<1/2\,. \tag{1.2}$$

Then $f\in UBC$.

The condition (1.2) implies that μ_f is a Carleson measure; see Lemma 4.1.

2. Proof of Theorem 1

We have already proved in [3, Theorem 6.1, p. 362] that (1.1) implies

$$\lim_{|z|\to 1}(1-|z|^2)f^{\#}(z)=0\,.$$

Accordingly, for each $r>0$, there exists $\delta=\delta(r)$, $0<\delta<1$, such that

$$\sup_{\delta<|z|<1}(1-|z|^2)f^{\#}(z)<r\,. \tag{2.1}$$

Let A be the area integral in (1.1), let

$$\rho=\rho(r)=1-\exp(-2/r)\,, \quad \text{and} \quad \phi(r)=(\rho+\delta)/(1+\delta\rho)\,.$$

Let M be the supremum of the function

$$\psi(t)=-t^2(\log t)/(1-t^2)-(1/2)\log(1+t)\,,$$

bounded in $0<t<1$, and finally let

$$g(z, w) = \log|(1-\bar{w}z)/(z-w)|$$

be the Green function of D with its pole at $w \in D$. Apparently,

$$M \geqq \lim_{t\to 1} \psi(t) = (1/2)\log(e/2) > 0 .$$

Then we shall show that

$$\iint_D f^{\#}(z)^2 g(z, w)dx\,dy \leqq -A\log\rho(r) + \pi r + \pi M r^2 \tag{2.2}$$

for each w in the annulus $\{\phi(r) < |w| < 1\}$. Since the right-hand side of (2.2) tends to zero as r tends to zero, we observe that

$$\lim_{|w|\to 1} \iint_D f^{\#}(z)^2 g(z, w)dx\,dy = 0 ,$$

whence $f \in UBC_0$ by the criterion [3, Theorem 2.2, (II), p. 352].

For the proof we set

$$\Delta(w_0, \tau) = \{z \in D;\ |z-w_0|/|1-\overline{w_0}z| < \tau\}$$

for each $w_0 \in D$ and $0 < \tau < 1$. First, it is easily proved that

$$I_1 \equiv \iint_{D\setminus\Delta(w,\rho)} f^{\#}(z)^2 g(z, w)dx\,dy \leqq A\log(1/\rho) .$$

Next, since $\Delta(w, \rho) \subset \{\delta < |z| < 1\}$ by $\phi(r) < |w|$, it follows from (2.1) that

$$f^{\#}(z)^2 \leqq r^2(1-|z|^2)^{-2}, \qquad z \in \Delta(w, \rho) ,$$

whence

$$\begin{aligned} I_2 &\equiv \iint_{\Delta(w,\rho)} f^{\#}(z)^2 g(z, w)dx\,dy \leqq r^2 \iint_{\Delta(w,\rho)} (1-|z|^2)^{-2} g(z, w)dx\,dy \\ &= r^2 \iint_{|\zeta|<\rho} (1-|\zeta|^2)^{-2}\log|1/\zeta|\,d\xi\,d\eta \end{aligned}$$

by the change of variable

$$z = (\zeta + w)/(1+\bar{w}\zeta) , \qquad |\zeta| < \rho , \qquad \zeta = \xi + i\eta .$$

Therefore,

$$\begin{aligned} I_2 &\leqq \pi r^2 \int_0^\rho 2t(1-t^2)^{-2}\log(1/t)dt \\ &= \pi r^2\{-(1/2)\log(1-\rho) + \psi(\rho)\} \leqq \pi r^2(1/r + M) . \end{aligned}$$

Since the left-hand side of (2.2) is $I_1 + I_2$, we obtain the estimate (2.2).

3. Proof of Theorem 2

Our aim is to show that

$$\sup_{w \in D} Q(\mu_f, w) \leqq (\pi+2)^2 \|f\|_T . \tag{3.1}$$

First of all, for each $w \in D$, and for each $z \in R(w)$, the inequality

$$|1-\bar{w}z| < (\pi+2)(1-|w|) \tag{3.2}$$

holds. Actually, we may suppose for the proof that $w=a$, $0<a<1$, with the aid of rotation. Let z_1 and z_2 be the vertices of the annular trapezoid $R(a)$ in the upper half-plane. Then,

$$\begin{aligned} |z-1/a| &< \max(|z_1-1/a|, |z_2-1/a|) \\ &< (1/a-a)+(1/a)\pi(1-a) ; \end{aligned}$$

the second term in the right-most is the length of the arc on the circle $\{|\zeta|=1/a\}$. On multiplying both sides by a we have (3.2) for $w=a$.

Since $-\log t > (1-t^2)/2$ for $0<t<1$, it follows from (3.2) that for each $z \in R(w)$,

$$g(z, w) \geqq \frac{(1-|z|^2)(1-|w|^2)}{2|1-\bar{w}z|^2} \geqq \frac{1-|z|}{2(\pi+2)^2(1-|w|)} .$$

Therefore,

$$\begin{aligned} \mu_f(R(w)) &\leqq 2(\pi+2)^2(1-|w|) \iint_D f^{\#}(z)^2 g(z, w) dx\, dy \\ &\leqq 2\pi(\pi+2)^2(1-|w|)\|f\|_T \end{aligned}$$

by [3, (2.6), p. 353]. We thus have

$$Q(\mu_f, w) \leqq (\pi+2)^2 \|f\|_T , \qquad w \in D ,$$

which yields (3.1).

4. Lemmas

We prepare two lemmas for the proof of Theorem 3.

LEMMA 4.1. *Suppose that*

$$\limsup_{|w| \to 1} Q(\mu_f, w) < \infty$$

for f meromorphic in D. Then μ_f is a Carleson measure.

Remark. The only one property of μ_f required in the proof is that $\mu_f(E) < \infty$ for each closed disk E with center 0 and contained in D.

Proof of Lemma 4.1. There exist δ, $0<\delta<1$, and $K>0$ such that

$$\mu_f(R(w)) \leqq 2\pi K(1-|w|) \tag{4.1}$$

for each w in the annulus $\{\delta \leqq |w| < 1\}$.

We fix a natural number $N_0 > 1/(1-\delta)$. Then, for each w, $|w|<\delta$, we may find w_j with $|w_j|=\delta$, such that

$$R(w) \cap \{\delta < |z| < 1\} \subset \bigcup_{j=1}^{N_0} R(w_j)\,.$$

Since (4.1) is true for each w_j, it follows that

$$\mu_f(R(w) \cap \{\delta < |z| < 1\}) \leqq 2\pi K N_0(1-\delta)\,, \qquad |w|<\delta\,. \tag{4.2}$$

Furthermore,

$$\mu_f(R(w) \cap \{|z| \leqq \delta\}) \leqq \mu_f(\{|z| \leqq \delta\}) \equiv A_\delta\,, \qquad |w|<\delta\,. \tag{4.3}$$

Therefore, for w, $|w|<\delta$, (4.2) and (4.3) yield

$$\begin{aligned} Q(\mu_f, w) &\leqq \mu_f(R(w))/\{2\pi(1-\delta)\} \\ &\leqq KN_0 + A_\delta/\{2\pi(1-\delta)\} \equiv A'_\delta\,. \end{aligned} \tag{4.4}$$

Combining this with (4.1) for $\delta \leqq |w| < 1$, we have

$$Q(\mu_f, w) \leqq A'_\delta + K$$

for each $w \in D$. Therefore μ_f is a Carleson measure.

We fix a constant B, $1/2 < B < 1$, once and for all, so that

$$(1-t)/(1+t) < \sin\{\pi(1-t)\} \qquad \text{for} \quad B<t<1\,. \tag{4.5}$$

LEMMA 4.2. *Let f be meromorphic in D, let $B<\delta<1$, and $0<r<1$. Then for each ζ,*

$$(\delta+r)/(1+\delta r) < |\zeta| < 1\,,$$

we have

$$\iint_{\Delta(\zeta,r)} f^{\#}(z)^2 dx\,dy \leqq 2\pi\{(1+r)/(1-r)\}^2 \sup_{\delta<|w|<1} Q(\mu_f, w)\,. \tag{4.6}$$

Proof. We may suppose that the supremum in the right-hand side of (4.6), denoted by C, is finite. Set

$$w = \{(|\zeta| - r)/(1 - r|\zeta|)\} e^{i \arg \zeta}\,.$$

Since $B<\delta<|w|<1$, (4.5) yields that

$$(1-|w|)/(1+|w|) < \sin\{\pi(1-|w|)\}\,.$$

Consequently, the Euclidean disk with the diameter between w and $w/|w|$ is contained in $R(w)$, so that

$$\Delta(\zeta, r) \subsetneq R(w) .$$

On the other hand, for each $z \in \Delta(\zeta, r)$,

$$1-|z|>1-\frac{|\zeta|+r}{1+|\zeta|r}=\frac{(1-|\zeta|)(1-r)}{1+|\zeta|r} .$$

Therefore

$$\begin{aligned}\frac{2\pi C(1+r)(1-|\zeta|)}{1-r|\zeta|}&=2\pi C(1-|w|)\\ &\geqq \iint_{R(w)} f^{\#}(z)^2(1-|z|)dx\,dy\\ &\geqq \frac{(1-|\zeta|)(1-r)}{1+|\zeta|r}\iint_{\Delta(\zeta,r)} f^{\#}(z)^2 dx\,dy ,\end{aligned}$$

whence follows (4.6).

COROLLARY. *Let f be meromorphic in D, let $B<\delta<1$, and suppose that*

$$\sup_{\delta<|w|<1} Q(\mu_f, w)<1/2 . \tag{4.7}$$

Then

$$\sup_{z\in D}(1-|z|^2)f^{\#}(z)<\infty .$$

For the proof, choose r, $0<r<1$, such that

$$2\pi C(1+r)^2/(1-r)^2<\pi ,$$

where C is the supremum in (4.7). Then it follows from (4.6) that

$$\sup \iint_{\Delta(\zeta,r)} f^{\#}(z)^2 dx\,dy<\pi , \qquad \text{for} \quad (\delta+r)/(1+\delta r)<|\zeta|<1 .$$

By [3, Remark, p. 355], the conclusion of the Corollary holds.

5. Proof of Theorem 3

By Lemma 4.1, μ_f is a Carleson measure, or, equivalently, $f^{\#}(z)^2(1-|z|^2)dx\,dy$ is a Carleson measure in the differential form. It then follows from [2, Lemma 3.3, p. 239] that

$$C\equiv\sup_{w\in D}\iint_D \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^2}f^{\#}(z)^2 dx\,dy<\infty .$$

Since there exists a constant Γ, $0<\Gamma<1$, such that

$$-\log t \leqq (1-t^2)/(2\Gamma^2) \qquad \text{for} \quad \Gamma \leqq t < 1\,,$$

it follows that, for each $w \in D$,

$$(5.1) \qquad \iint_{D \setminus \Delta(w,\Gamma)} f^{\#}(z)^2 g(z,w)\,dx\,dy \leqq \{1/(2\Gamma^2)\} \iint_D f^{\#}(z)^2 \left(1-\left|\frac{z-w}{1-\bar{w}z}\right|^2\right) dx\,dy$$
$$\leqq C/(2\Gamma^2)\,.$$

On the other hand, by the Corollary of Lemma 4.2, there exists $Q>0$ such that

$$f^{\#}(z)^2 \leqq Q(1-|z|^2)^{-2}\,, \qquad z \in D\,.$$

Therefore, for each $w \in D$,

$$(5.2) \qquad \iint_{\Delta(w,\Gamma)} f^{\#}(z)^2 g(z,w)\,dx\,dy \leqq Q \iint_{\Delta(w,\Gamma)} (1-|z|^2)^{-2} g(z,w)\,dx\,dy$$
$$= Q \iint_{|\zeta|<\Gamma} (1-|\zeta|^2)^{-2} \log|1/\zeta|\,d\xi\,d\eta$$
$$\equiv \Phi(Q,\Gamma)\,.$$

Therefore,

$$\pi\|f\|_T = \sup_{w\in D} \iint_D f^{\#}(z)^2 g(z,w)\,dx\,dy \leqq C/(2\Gamma^2) + \Phi(Q,\Gamma) < \infty\,,$$

which shows that $f \in UBC$.

Appendix

The "if" parts of (I) and (II) in [3, Lemma 3.2, p. 354] can be ameliorated. For $0<r<1$, and $w \in D$, let $\alpha(r, w, f)$ be the spherical area of the image $f(\Delta(w, r))$ (namely, the projection of the Riemannian image) of $\Delta(w, r)$ by f meromorphic in D; $f(\Delta(w, r))$ is a subset of the Riemann sphere of diameter one.

(a) *Suppose that there exist* r, $0<r<1$, *and* δ, $0<\delta<1$, *such that*

$$\sup_{\delta<|w|<1} \alpha(r, w, f) < \pi\,.$$

Then

$$\sup_{z\in D} (1-|z|^2) f^{\#}(z) < \infty\,.$$

(b) *Suppose that there exists* r, $0<r<1$, *such that*

$$\lim_{|w|\to 1} \alpha(r, w, f) = 0\,.$$

Then

$$\lim_{|z|\to 1}(1-|z|^2)f^{\#}(z)=0\,.$$

The proofs are the same as those of (I) and (II) on replacing [3, Lemma 3.1, p. 354] by the obvious version in terms of α; see [1, Lemma II, (6), p. 216].

A function f meromorphic in $\mathbf{C}=\{|z|<\infty\}$ is called Yosida if

$$\sup_{z\in\mathbf{C}} f^{\#}(z)<\infty\,,$$

or equivalently, f is of class (A) in K. Yosida's sense [4]. Similar propositions for f meromorphic in $\mathbf{C}$ to be Yosida hold. In this case, $\alpha(r, w, f)$ should be replaced by the spherical area of the image by f of the Euclidean disk of center $w\in\mathbf{C}$ and radius $r>0$.

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