

On Rings with Finite Rank Torsion Free Additive Group

by

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§I. All rings considered here are associative. The additive group of a ring R will be denoted by R^+ , the field of rational numbers by Q , and the ring of integers by Z .

Nilpotent rings with rank n torsion free additive group will be considered in §II. It will be shown that these rings are either commutative or satisfy the identity $x^n=0$. This fact in turn implies that when $n=2$ the ring must be commutative. In §III rings without zero divisors, and with finite rank torsion free additive group will be studied. These rings can be embedded in a division ring, or into a field in case the ring is commutative. The rank of the additive group of the division ring or field is the same as the rank of the additive group of the embedded ring. Again the ring is commutative in the rank 2 case.

§II.

PROPOSITION 2.1. *Let R be a nil ring with R^+ a rank n torsion free group. Then $R^{n+1}=0$.*

Proof. [2, Theorem 3.1.3].

THEOREM 2.2. *Let R be a nilpotent ring with R^+ a rank n torsion free group. Then either R is commutative, or $x^n=0$ for every $x \in R$.*

Proof. Suppose that R is not commutative. Then for $x \in R$ the elements x, x^2, \dots, x^n must be dependent over Z . Therefore there exist integers m_1, \dots, m_n , not all zero, such that $m_1x + \dots + m_nx^n = 0$. Let k be the least positive integer such that $m_k \neq 0$. Then $m_kx^k = -m_{k+1}x^{k+1} - \dots - m_nx^n$. Multiplying both sides of this equality by x^{n-k} and employing the above Proposition yield that $m_kx^n = 0$. Since R^+ is torsion free, $x^n = 0$.

COROLLARY 2.3. *Let R be a nilpotent ring with R^+ a rank 2 torsion free group. Then R is commutative.*

Proof. Suppose that R is not commutative. Let $x_1, x_2 \in R$ be independent over Z . By the above theorem $x_1^2 = x_2^2 = (x_1 + x_2)^2 = 0$. The last equality implies that $x_1x_2 + x_2x_1 = 0$, or that $x_1x_2 = -x_2x_1$. Therefore if $x_1x_2 = 0$, then $x_2x_1 = 0$, which

yields that $R^2=0$, contradicting the fact that R is not commutative. Hence $x_1x_2 \neq 0$. Since R^+ has rank 2, there exists a positive integer m , and integers m_1, m_2 such that $mx_1x_2 = m_1x_1 + m_2x_2$ and either $m_1 \neq 0$ or $m_2 \neq 0$. Assume that $m_1 \neq 0$. Then $mx_2x_1x_2 = m_1x_2x_1 = -m_1x_1x_2 \neq 0$. On the other hand $mx_2x_1x_2 = -mx_1x_2^2 = 0$, a contradiction. A symmetric argument yields a contradiction in case $m_2 \neq 0$.

§ III.

LEMMA 3.1. *Let G be a finite rank torsion free group. Then G is hopfian. G is cohopfian if and only if G is divisible.*

Proof. Let G have rank n , $x_1, \dots, x_n \in G$ be independent, and let $\varphi: G \rightarrow G$ be an epimorphism. It is readily seen that $\varphi(x_1), \dots, \varphi(x_n)$ are independent. Let $x \in \ker \varphi$. There exist a positive integer m , and integers m_1, \dots, m_n such that $mx = m_1x_1 + \dots + m_nx_n$. Therefore $0 = \varphi(mx) = m_1\varphi(x_1) + \dots + m_n\varphi(x_n)$. Since $\varphi(x_1), \dots, \varphi(x_n)$ are independent, $m_1 = m_2 = \dots = m_n = 0$, and $mx = 0$. Since G is torsion free $x = 0$ and so φ is a monomorphism.

Suppose that G is cohopfian. Let n be a positive integer. The map $\hat{n}: G \rightarrow G$ defined by $\hat{n}(x) = nx$ for $x \in G$ is a monomorphism. Since G is cohopfian, \hat{n} is an epimorphism, and so $\text{im } \hat{n} = nG = G$, i.e., G is divisible. Conversely, let G be divisible, and let $\varphi: G \rightarrow G$ be a monomorphism. Then $\varphi(G)$ is a rank n divisible subgroup of G . This clearly implies that $\varphi(G) = G$, and so G is cohopfian.

THEOREM 3.2. *Let R be a (commutative) ring without zero divisors, and with R^+ a finite rank torsion free group. Then $D = R \otimes_{\mathbb{Z}} Q$ is a division ring (field).*

Proof. R may be viewed as a subring of D by identifying $a \in R$ with $a \otimes 1 \in D$. It is readily seen that for any $x \in D$, there exists a positive integer m such that $mx \in R$. This fact implies that D has no zero divisors. Let n be the rank of R^+ . Then $D^+ \simeq \bigoplus_{n \text{ copies}} Q^+$, and so D^+ is cohopfian by Lemma 3.1. Let $a \in D$, $a \neq 0$. Left (right) multiplication by a is a monomorphism $D^+ \rightarrow D^+$, so by the cohopficity of D^+ , $aD = Da = D$. Let $e \in D$ be such that $ae = a$. For $x \in D$, $ax = aex$, and so $a(x - ex) = 0$. Since a is not a zero divisor $ex = x$, and e is a left unity in D . There exists $y \in D$ such that $x = ya$. Hence $xe = yae = ya = x$, i.e., e is a unity in R . There exists $b, c \in D$ such that $ab = ca = e$. Clearly $b = c = a^{-1}$, i.e. the nonzero elements of D form a multiplicative group, and so D is a division ring. It is readily seen that if R is commutative, then so is D , hence R commutative implies that D is a field.

COROLLARY 3.2. *Let R be a (commutative) ring with R^+ a finite rank torsion free group. R is a subring of a division ring (field) if and only if R has no zero divisors.*

COROLLARY 3.3 [1, Theorem 3]. *Let R be a ring with R^+ a rank 2 torsion free group. Then R is a subring of a quadratic extension of Q if and only if R has no zero divisors.*

Proof. Let R be a ring without zero divisors. In lieu of the above results it

suffices to show that R is commutative. Suppose that R is not commutative. Let $x, y \in R^+$ be independent over Z . There exist nonzero integers r_1, r_2, s_1, s_2 such that $r_1x + r_2x^2 = s_1y + s_2y^2 = 0$. Hence $r_1xy + r_2x^2y = 0$, and so $x(r_1y + r_2xy) = 0$. Since x is not a zero divisor, (A) $r_1y + r_2xy = 0$. Now the equations $0 = s_1xy + s_2xy^2 = (s_1x + s_2xy)y$ yields that (B) $s_1x + s_2xy = 0$. It follows from equalities (A), (B) that $r_2s_1x - r_1s_2y = 0$, contradicting the fact that x, y are independent.

The converse is obvious.

Question. Every ring R without zero divisors, and with rank 2 torsion free additive group satisfies the polynomial identity $XY - YX$. Does there exist a multilinear polynomial $P(X_1, \dots, X_n)$ of degree n with integer coefficients such that every ring R without zero divisors, and with rank n torsion free additive group satisfies the polynomial identity $P(X_1, \dots, X_n)$?

References

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