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Birational Geometry of Birational Pairs II

by

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§ 0. Introduction

The present paper is a continuation of our previous paper [7], and the object is to characterize ruled and rational varieties in the sense of birational geometry of birational pairs. In Theorem 1.1, we characterize the general ruled varieties, and in Theorem 2.1, we have a stronger result. Using these results, we have two characterizations of rationality for algebraic 3-folds (Corollaries 1.5 and 2.4).

In this paper, we consider complete varieties over an algebraically closed field k of characteristic zero. For simplicity's sake, by an algebraic variety we mean a nonsingular complete algebraic variety over k.

Consider a linear system Λ on an algebraic variety X. We denote the rational map associated with Λ by $\Phi_\Lambda: X \to \mathbf{P}^{\dim \Lambda}$. If $\dim \Phi_{|mD|}(X) = \kappa(D, X)$ for a divisor D on X, then, by a result of Hironaka [2], there exist an algebraic variety $\tilde{X}$, a proper birational morphism $\mu: \tilde{X} \to X$ and a proper morphism $\psi_D: \tilde{X} \to \mathbf{P}^{\dim|mD|}$ such that $\psi_D = \Phi_{|mD|} \circ \mu$, where μ is a composition of successive blowings up with nonsingular centers. Taking the Stein factorization of ψ_D, we have an algebraic variety V_D and a proper surjective morphism with connected fibers $\varphi_D: \tilde{X} \to V_D$ and a finite morphism $f: V_D \to \mathbf{P}^{\dim|mD|}$. We call *$\varphi_D$ an algebraic fiber space associated with D.*

§ 1. A characterization of general ruled varieties

In this section, we shall characterize $\mathbf{P}^1$-bundles in the sense of birational geometry.

Our Theorem can be stated as follows.

THEOREM 1.1. *Let X be an n-dimensional algebraic variety. Then, the following two conditions are equivalent to each other.*

(i) *X is birationally equivalent to $B \times \mathbf{P}^1$ for some algebraic variety B.*

(ii) *There exist a positive integer m and a reduced effective divisor D on X such that*

$$\kappa(2/2m+1\, D \;\&\; X) = n-1\,, \qquad \kappa(1/m\; D \;\&\; X) = n$$

and

$$P_m(1/m\ D\ \&\ X)>0\ .$$

At first, we prove the following Lemma.

LEMMA 1.2. *Let X be an algebraic variety of dimension n and D an effective divisor on X. Suppose that $\kappa(D, X)=n-1$, the associated rational map $\Phi_{|D|}: X\to B$ is a morphism and* dim $B=n-1$. *Then for any general fiber f of $\Phi_{|D|}$, we have $D\cdot f=0$.*

Proof. By the fibering theorem [3, p. 302], we have $\kappa(D|_f, f)=0$. Since f is a curve, we obtain deg $(D|_f)=D\cdot f=0$. ■

Remark 1.3. By the definition of invariant $\kappa(\varepsilon D\ \&\ X)$, we may assume that X and D are projective nonsingular varieties. Let Z be a nonsingular subvariety of codimension $r\geqslant 2$ of X and consider a blowing up $\mu: \tilde{X}\to X$ with center Z. Then, for a rational number a with $0\leqslant a\leqslant 1$,

$$\mu^*(K(X)+aD)=K(X)+a\ (\mu^p D)+\alpha aE-R_\mu\ ,$$

where $E:=\mu^{-1}Z$, $R_\mu:=(r-1)E$, $\mu^p D$ denotes the proper transform of D and $\alpha=0$ or 1. We rewrite the above equality as

$$\mu^*(K(X)+aD)+R_\mu-\alpha aE=K(\tilde{X})+a\mu^p D\ .$$

Here, we have $R_\mu-\alpha aE\geqslant 0$, since $r\geqslant 2$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \ \mu\ \ } & X \\ & \searrow^{\varphi} \quad {}^{\psi}\swarrow & \\ & B & \end{array}$$

Here φ is the rational map associated with $K(\tilde{X})+a\mu^p D$, and ψ is the rational map associated with $K(X)+aD$. Therefore we may assume that the above rational map associated with $K(X)+aD$ is a morphism, and we can use the preceding Lemma 1.2 for this map.

Proof of Theorem 1.1.

(i) ⇒ (ii). Set $X=B\times \mathbf{P}^1$. Let A be a very ample divisor on B such that $(2m+1)K(B)+2A$ is also very ample, and that $|mK(B)+A|\neq\varnothing$, and let H be a general fiber of the second projection $\psi: X\to \mathbf{P}^1$. Let D be a general member of $|\varphi^*A+(2m+1)H|$, where $\varphi: X\to B$ is the first projection. By Bertini's theorem, D is nonsingular. Then we have $(2m+1)K(X)+2D\sim\varphi^*((2m+1)K(B)+2A)$, and hence

$$\kappa(2/2m+1\ D\ \&\ X)=n-1\ .$$

It is clear that $\kappa(1/m\ D\ \&\ X)=n$, and $|mK(X)+D|\neq\varnothing$.

(ii) ⇒ (i). Put $\Gamma=(2m+1)K(X)+2D$, and let $\varphi: X\to B$ be the algebraic fiber space associated with Γ, which is a surjective morphism with connected fiber. If f is a general fiber of φ, we have by Lemma 1.2, $f\cdot\Gamma=0$.

Considering the following short exact sequence,

$$0 \longrightarrow \Theta_f \longrightarrow \Theta_X|_f \longrightarrow N_{f|X} \longrightarrow 0,$$

we have $c_1(X)|_f = c_1(f) + c_1(N_{f|X})$, where $c_1(\cdot)$ is the first Chern classes, Θ_X is the tangent bundle of X and $N_{f|X}$ is the normal bundle of f in X. Since $c_1(N_{f|X})=0$, we have

$$f \cdot K(X) = 2g-2,$$

where g is the geometric genus of f. Since f is numerically effective, it holds that $(2m+1)(2g-2) = -2D \cdot f \leqslant 0$. Therefore we have $g=0$ or 1. If $g=1$, then $D \cdot f = 0$. In this case $(mK(X)+D) \cdot f = 0$; hence $\kappa(1/m\, D \,\&\, X) \leqslant n-1$, which contradicts the hypothesis. So, we obtain $g=0$ and $D \cdot f = 2m+1$. On the other hand, since $P_m(1/m\, D \,\&\, X) > 0$, there exists a member F of $|mK(X)+D|$ such that $f \cdot F = -2m + D \cdot f = 1$. Thus, F is a section of φ. This implies that X is birationally equivalent to $B \times \mathbf{P}^1$. ∎

Remark 1.4. X has a $\mathbf{P}^1$-fibration if and only if there exist a positive integer m with $m \geqslant 2$ and a reduced effective divisor D on X such that $\kappa(1/m\, D \,\&\, X) = n-1$ and $\kappa(2/m\, D \,\&\, X) = n$.

COROLLARY 1.5. *Let X be an algebraic variety of dimension 3. Then, the following two conditions are equivalent to each other.*

(i) *X is rational.*

(ii) *There exist positive integers m_1, m_2 and reduced effective divisors D_1 and D_2 on X which satisfy the following conditions*:

$$\kappa(2/2m_i+1\ D_i \,\&\, X) = 2, \qquad \kappa(1/m_i\, D_i \,\&\, X) = 3 \quad \textit{for} \quad i=1,2,$$

$$\kappa(D_1+D_2/m_1+m_2+1 \,\&\, X) = 3, \qquad P_{m_1}(1/m_1\, D_1 \,\&\, X) > 0, \quad \textit{and} \quad q(X)=0,$$

where $q(X) = \dim H^1(X, \mathcal{O}_X)$.

Proof.

(i) ⇒ (ii). Put $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, and let p_i be the natural i-th projection, i.e., $p_i(x_1, x_2, x_3) = x_i$. Let H_i be a general fiber of p_i, and take natural numbers a_1 and a_2 with $a_1, a_2 \geqslant 2(m_1+m_2+1)$. Consider D_1 and D_2 which are general nonsingular members of linear systems $|a_1(H_2+H_3)+(2m_1+1)H_1|$ and $|a_2(H_1+H_3)+(2m_2+1)H_2|$, respectively, and set $Z = D_1 \cap D_2$. It is easy to see that Z is nonsingular. To calculate $\kappa(D_1+D_2/m_1+m_2+1 \,\&\, X)$, we blow up X with center Z and we have $\mu: \tilde{X} \to X$,

$$\mu^p(D_1+D_2) = \mu^*(D_1+D_2) - 2E, \quad \text{and} \quad K(\tilde{X}) = \mu^* K(X) + E,$$

where $E = \mu^{-1}Z$. Then we have

$$(m_1+m_2+1)K(\tilde{X}) + \mu^p(D_1+D_2) = (m_1+m_2+1)K(X) + (m_1+m_2+1)E$$
$$+ D_1 + D_2 - 2E$$

$$
\begin{aligned}
&=(m_1+m_2+1)K(X)+D_1+D_2+(m_1+m_2-1)E\\
&\geqslant -2(m_1+m_2+1)(H_1+H_2+H_3)+a_1(H_2+H_3)\\
&\quad +a_2(H_1+H_3)+(2m_1+1)H_1+(2m_2+1)H_2\\
&\geqslant H_1+H_2+H_3.
\end{aligned}
$$

Therefore we have $\kappa(D_1+D_2/m_1+m_2+1\ \&\ X)=3$. We can obtain $\kappa(2/2m_i+1\ D_i\ \&\ X)=2$ and $\kappa(1/m_i\ D_i\ \&\ X)=3$ in the same way.

(ii) $\Rightarrow$ (i). Put $\Gamma_i=(2m_i+1)K(X)+2D_i$, and consider algebraic fiber spaces associated with Γ_i, $\varphi_i: X\to B_i$. Let f_i be a general fiber of φ_i. Since $\kappa(D_1+D_2/m_1+m_2+1\ \&\ X)=3$, f_1 and f_2 are not numerically equivalent to each other. Hence $\varphi_{1*}f_2$ is a curve on B_1. Therefore we conclude that B_1 is a ruled surface. Since $0=q(X)\geqslant q(B_1)$, we have $q(B_1)=0$ and B_1 is a rational surface. From Theorem 1.1, φ_1 is birationally equivalent to $B_1\times\mathbf{P}^1$; hence X is rational. ■

§2. A criterion for ruled varieties over conic bundles

First, we state our main Theorem of this section.

THEOREM 2.1. *Let X be an algebraic variety of dimension n. If there exist natural numbers m_1 and m_2 and irreducible divisors D_1 and D_2 on X such that, for $i=1$ and 2, $\kappa(2/2m_i+1\ D\ \&\ X)=n-i$, $\kappa(1/m_i\ D_i\ \&\ X)=n$, $P_{m_i}(1/m_i\ D_i\ \&\ X)>0$, and $\kappa(D_1+D_2/m_1+m_2+1\ \&\ X)=n-1$, then there exist two algebraic varieties B_1, B_2 of dimension $n-1$ and $n-2$, respectively, and the rational maps*

$$\pi_1: X\longrightarrow B_1\,,\qquad \pi_2: B_1\longrightarrow B_2$$

such that π_1 is birationally equivalent to a $\mathbf{P}^1$-bundle and π_2 is birationally equivalent to a $\mathbf{P}^1$-fibration (i.e., birationally equivalent to a conic bundle).

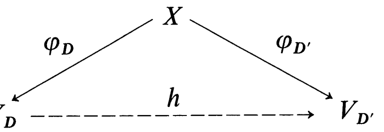
Maeda [5] has proved the following fundamental lemma (see also Kawamata [4, Lemma 14]).

LEMMA 2.2. *Let X, Y and Z be algebraic varieties over a field k. Let $f: X\to Y$ be a proper surjective morphism with geometrically connected fibers, i.e., the field extension induced by f, $\mathrm{Rat}(X)/\mathrm{Rat}(Y)$, is algebraically closed. Let $g: X\to Z$ be a morphism. Suppose that $g(f^{-1}(y_0))$ is a k-rational point $z_0\in Z$ for some k-rational point $y_0\in Y$. Then there exists a unique rational map $h: Y$---→ Z such that the following diagram is commutative.*

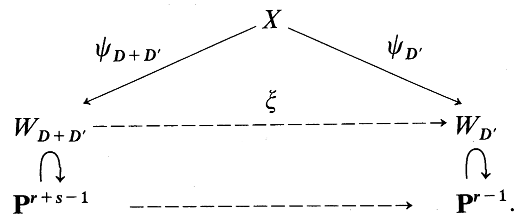
$$
\begin{array}{ccc}
X & \xrightarrow{\quad g \quad} & Z\\
{\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \\
Y & &
\end{array}
$$

Now we prove the following lemma.

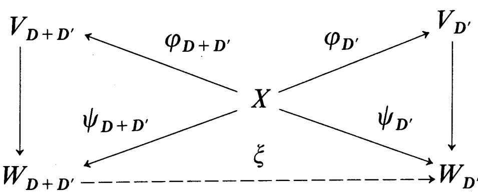
LEMMA 2.3. *Let X be an algebraic variety and D and D' be effective divisors on X. If $\kappa(D+D', X)=\kappa(D, X)$, then algebraic fiber spaces $\varphi_D: X\to V_D$ and $\varphi_{D+D'}: X\to V_{D+D'}$ associated with D and $D+D'$, respectively, are birationally equivalent to each other, and hence there exists a rational map $h: V_D\to V_{D'}$ such that $\varphi_{D'}=h\circ\varphi_D$.*



Proof. For any $m\geqslant 1$, $\mathcal{O}_X(mD')\subset\mathcal{O}_X(m(D+D'))$. Therefore, for any basis $f_1, \cdots, f_r$ of $H^0(X, \mathcal{O}_X(mD'))$, we can choose a basis $f_1, \cdots, f_r, g_1, \cdots, g_s$ of $H^0(X, \mathcal{O}_X(m(D+D')))$. Set $\psi_{D'}=(f_1:\cdots:f_r)$ and $\psi_{D+D'}=(f_1:\cdots:f_r:g_1:\cdots:g_s)$, and let $W_{D'}$ and $W_{D+D'}$ be the images of them, respectively. Then the natural projection $\mathbf{P}^{r+s-1} \dashrightarrow \mathbf{P}^{r-1}$ induces a rational map $\xi: W_{D+D'} \dashrightarrow W_{D'}$. We have the following diagram commutative:



Consider the Stein factorizations of $\psi_{D+D'}$ and $\psi_{D'}$ as follows.



Then, by Lemma 2.2, there exists a rational map $\zeta': V_{D+D'}\dashrightarrow V_{D'}$ such that $\varphi_{D'}=\zeta'\circ\varphi_{D+D'}$.

In the same way there exists a rational map $\zeta: V_{D+D'} \dashrightarrow V_D$ such that $\varphi_D=\zeta\circ\varphi_{D+D'}$. Since $\dim V_{D+D'}=\dim V_D=\kappa(D, X)$ and φ_D is a surjection with connected fibers, there exists a section of ζ, $v: V_D\to V_{D+D'}$ by Lemma 2.2 again. Therefore, ζ is a birational map. Put $h=\zeta'\circ v$, then this is the required map. ■

Proof of Theorem 2.1. For $i=1$ and 2, set $F_i=(2m_i+1)K(X)+2D_i$. By Theorem 1.1, F_1 induces a $\mathbf{P}^1$-bundle $\varphi_1: X\to B_1$. Let $\varphi_2: X\to B_2$ be an algebraic fiber space associated with F_2. Then, since $\kappa(F_1+F_2, X)=\kappa(F_1, X)=n-1$, we have a rational map $p: B_1 \dashrightarrow B_2$ by Lemma 2.3, such that the following diagram is commutative.

$$\begin{array}{ccc} & X & \\ {}^{\varphi_1}\swarrow & & \searrow^{\varphi_2} \\ B_1 & \overset{p}{\dashrightarrow} & B_2 \end{array}$$

Changing the models B_1, B_2 and X birationally, we may assume φ_1, φ_2 and p are all morphisms. By fibering theorem [3, p. 302], we have $\kappa(F_2|_F, F)=0$, where F is a general fiber of φ_2. Let f_1 be a general fiber of φ_1. By Lemma 1.2, we have $(F_1+F_2)\cdot f_1=F_1\cdot f_1=0$. Therefore we have $f_1\cdot F_2=0$. Let C be a general fiber of p. Then $\varphi:=\varphi_1|_F: F\to C$ is a ruled surface. Let f be a general fiber of φ and put $D=D_2|_F$. Since $F_2|_F=(2m_2+1)K(F)+2D$, we have $f\cdot(F_2|_F)=-2(2m_2+1)+2f\cdot D=0$ and $f\cdot D=2m_2+1$.

In general, D is not connected. We write $D=D^1+\cdots+D^s$. This represents a disjoint union of nonsingular irreducible divisors D^i. If C is a rational curve, then there is nothing to say. Therefore we may assume $q\neq 0$, where q denotes the geometric genus of C. Let $D_2\xrightarrow{g}B\to B_2$ be the Stein factorization of $D_2\subsetneq X\to B_2$. Then each D^i is a fiber of g. Hence $D^1, \cdots, D^s$ are numerically equivalent to each other on D_2. In particular, all D^i have the same geometric genera. If some D^i is contained in a fiber of φ, then C is rational by the Lüroth's theorem. Therefore no fiber of φ contains any D^i. Hence we may consider D to be the resolution of a singular curve, say $\bar{D}$, on $C\times\mathbf{P}^1$. Write $\bar{D}\approx(2m_2+1)H+nf$, then $D\approx(2m_2+1)H+nf-\sum n_iE_i$, where $\approx$ means numerically equivalence, H is a general fiber of the natural projection $C\times\mathbf{P}^1\to\mathbf{P}^1$, and $\sum n_iE_i$ is the exceptional divisors associated with resolution of $\bar{D}$. Applying elementary transformations to $C\times\mathbf{P}^1$, we may assume that each multiplicity n_i at singular point P_i of $\bar{D}$ satisfies the relation $n_i\leqslant 2m_2+1/2$, i.e., $n_i\leqslant m_2$ on a relatively minimal ruled surface $\bar{F}$. Using a locally free sheaf $\mathscr{E}$ of rank 2 over C with index $e=-\deg\mathscr{E}$, which is normalized in the sense of Hartshorn [2], we write $\bar{F}$ as $\mathbf{P}(\mathscr{E})$. Then there exists a section a with $a^2=-e$ such that $K(F)\approx -2a+(2q-2-e)f$, where $q=\dim H^1(F, \mathcal{O}_F)$. Write $\bar{D}\approx(2m_2+1)a+nf$, then

$$(2m_2+1)K(F)+2D\approx((2m_2+1)(2q-2-e)+2n)f+\sum(2m_2+1-2n_i)E_i\,.$$

Therefore we have, by $\kappa(2/2m_2+1\ D\ \&\ F)=0$,

$$2n=(2m_2+1)(e+2-2q)\,.$$

We continue the proof by examining the following two cases.

(I) $\bar{D}\cdot a\geqslant 0$,

(II) $\bar{D}\cdot a<0$.

Case (I). $\bar{D}\cdot a\geqslant 0$, i.e., $n\geqslant(2m_2+1)e$.

Then, $2n=(2m_2+1)(e+2-2q)\geqslant 2(2m_2+1)e$, and we have $0\geqslant 2-2q\geqslant e$. Since $e\geqslant -q$ by Nagata [6], we have $q=1$ or 2. If $q=1$, we have $0\geqslant e\geqslant -1$. If $e=0$, then $n=0$. Therefore $m_2K(\bar{F})+\bar{D}\approx a$, and we have $\kappa(1/m_2\ D\ \&\ F)\leqslant 1$. This contradicts $\kappa(1/m_2\ D_2\ \&\ X)=n$. If $e=-1$, then $2n=-(2m_2+1)$, this is absurd. If $q=2$, we have $e=-2$. So we obtain $n=-2(2m_2+1)$ and $\bar{D}\approx(2m_2+1)(a-2f)$.

Thus we obtain $m_2K(\bar{F})+\bar{D}\approx a-2f$, and by the assumption, there exists an irreducible member $Z\in|m_2K(F)+D|$ such that $Z^2=-2$. We have $\kappa(1/m_2\ D\ \&\ F)=0$, which also contradicts $\kappa(1/m_2\ D_2\ \&\ X)=n$.

Case (II). $\bar{D}\cdot a<0$. Hence $a\subset \mathrm{Supp}\,(\bar{D})$.

Put $D=D^1+\cdots+D^s$, where each D^i is a nonsingular irreducible connected component of D. Since $D\approx(2m_2+1)a+nf-\sum n_iE_i$ and each genus of D^i is q, we have by the adjunction formula,

$$(K(F)+D)\cdot D=s(2q-2)=-(4m_2{}^2-1)e+(2m_2+1)(2q-2-e+n)+n(2m_2-1)$$
$$-\sum n_i(n_i-1)\,.$$

Using the condition $2n=(2m_2+1)(e+2-2q)$, we have

$$2\sum n_i(n_i-1)=2(2-2q)(4m_2{}^2-1+s)\geqslant 0\,.$$

Therefore we may assumed $q=1$ and all $n_i=1$. Hence $D=\bar{D}$ and $\bar{D}\cdot a=a^2=-e=n-(2m_2+1)e<0$. We have $n=2m_2e$ and $2n=4m_2e=(2m_2+1)e$. Therefore $2m_2=1$. This is a contradiction. Hence C is a rational curve. ■

COROLLARY 2.4. *Let X be an algebraic variety of dimension* 3. *Then the following two conditions are equivalent to each other*:

(i) *X is rational.*

(ii) *There exist natural numbers m_1 and m_2 and irreducible divisors D_1 and D_2 on X such that, for $i=1$ and* 2,

$$\kappa(2/2m_i+1\ D_i\ \&\ X)=3-i\,,\quad \kappa(1/m_i\ D_i\ \&\ X)=3\,,\quad P_{m_i}(1/m_i\ D_i\ \&\ X)>0\,,$$

$$\kappa(D_1+D_2/m_1+m_2+1\ \&\ X)=2\,,\quad and\quad q(X)=0\,.$$

Proof of (i) $\Rightarrow$ (ii). Consider the following morphisms.

$$X=B_0=B_1\times\mathbf{P}^1\xrightarrow{p_1}B_1=B_2\times\mathbf{P}^1\xrightarrow{p_2}B_2=\mathbf{P}^1\,.$$

Let H_j be a general fiber of the natural projection $B_j\to\mathbf{P}^1$, $D_{2,2}$ be a general member of $|(2m_2+2)H_2|$, $D_{2,1}$ be a general member of $|p_2{}^*D_{2,2}+(2m_2+1)H_1|$, and D_2 be a general member of $|p_1{}^*D_{2,1}+(2m_2+1)H_0|$. Let $D_{1,1}$ be a general member of $|(2m_2+2)(H_2+H_1)|$ and D_1 be a general member of $|p_1{}^*D_{1,1}+(2m_1+1)H_0|$. Then, it is easy to check that D_1 and D_2 satisfy the conditions in (ii) of Corollary 2.4. ■

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