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Iterated Boolean Powers

by

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§ 0. Introduction

Under what conditions can we write an iterated Boolean power as a single Boolean power? This problem had been investigated by Mansfield ([4]), Banaschewski and Nelson ([1]). They essentially obtained the following

THEOREM A ([4]). *Let κ be a cardinal. If B satisfies the $(<\kappa, \infty)$-distributive law and A satisfies the $<\kappa$ chain condition with respect to B and $A[B]$ is complete, then the canonical embedding e: $(M^{(A)})^{(B)} \to M^{(A[B])}$ is an isomorphism for every structure M.*

THEOREM B ([1]). *Let κ be a cardinal and $P(\kappa)$ be the Boolean algebra of all subsets of κ. The canonical embedding e: $(M^{(P(\kappa))})^{(B)} \to M^{(P(\kappa)\otimes B)}$ is an isomorphism for every structure M if and only if B satisfies the (κ, ∞)-distributive law.*

In this paper, we use terminology of Boolean valued models of set theory. We assume that the reader is familiar with the notations of [6] for $V^{(B)}$. In the notations of $V^{(B)}$, the Boolean power of a structure M by a complete Boolean algebra (cBa) B is described by

$$M^{(B)} = \widehat{\check{M}} = \{f \in V^{(B)} \mid [\![f \in \check{M}]\!]^{(B)} = 1\}$$

and

$$M^{(B)} \models R(f_1, \cdots, f_n) \quad \text{iff} \quad [\![\check{M} \models R(f_1, \cdots, f_n)]\!]^{(B)} = 1$$

where $R(x_1, \cdots, x_n)$ is an atomic formula.

In this paper, we assume that $V^{(B)}$ is separated, i.e., $[\![f = g]\!]^{(B)} = 1$ implies $f = g$ for every $f, g \in V^{(B)}$.

Our main theorem is as follows:

THEOREM. *The following conditions are equivalent.*

(1) *The canonical embedding e: $(M^{(A)})^{(B)} \to M^{(A\otimes B)}$ is an isomorphism for every structure M.*

(2) $[\![$ *$\check{A}$ is complete and $\check{(M^{(A)})} = \check{M}^{(\check{A})}$* $]\!]^{(B)} = 1$ *for every structure M.*

(3) *B satisfies the $(<\mathrm{Sat}(A), \infty)$-distributive law and $[\![\check{\mathrm{Sat}(A)} = \mathrm{Sat}(\check{A})]\!]^{(B)} = 1$.*

(4) *B satisfies the $(<\mathrm{Sat}(A), \infty)$-distributive law and A satisfies the $<\mathrm{Sat}(A)$ chain condition with respect to B.*

(For precise definitions see blow.)

In [p. 33, 1], Banaschewski and Nelson remarked that there are topological spaces X and Y such that

(i) $\mathrm{Reg}(X)$ satisfies the $(|Y|, \infty)$-distributive law,

(ii) $(M^{(\mathrm{Reg}(Y))})^{(\mathrm{Reg}(X))} \not\cong M^{(\mathrm{Reg}(Y)\otimes\mathrm{Reg}(X))}$ for some structure M,

where $\mathrm{Reg}(X)$ (respectively $\mathrm{Reg}(Y)$) is the Boolean algebra of all regular open subsets of $X(Y)$. But they mistook at this point. We actually have the following

COROLLARY. *Let X and Y be topological spaces. If $\mathrm{Reg}(X)$ satisfies $(|Y|, \infty)$-distributive law, then the canonical embedding $e: (M^{(\mathrm{Reg}(Y))})^{(\mathrm{Reg}(X))} \to M^{(\mathrm{Reg}(Y)\otimes\mathrm{Reg}(X))}$ is an isomorphism for every structure M.*

§ 1. Preliminaries

We use letters α, β for ordinals and δ, κ, λ for infinite cardinals. The cardinality of a set X is denoted by $|X|$. We use letters A, B for infinite Boolean algebras. We denote the finite Boolean operations of B by $+_B$, $\cdot_B$, $\sim_B$ the least element by 0_B and the greatest element by 1_B. An element $(\sim_B a)+_B b$ is denoted by $a \Rightarrow_B b$. $\leqq_B$ is the canonical ordering of B. We shall omit the subscripts if there is no confusion. B is said to be κ-complete if the supremum $\bigvee S$ exits for every subset S of B such that $|S| < \kappa$. B is complete if it is κ-complete for every κ. We note that B is complete if and only if it is $\mathrm{Sat}(B)$-complete ([20.5, 5]). A partition of B is a maximal pairwise disjoint family of it. We denote the set of all pairwise disjoint families of B by $\mathrm{PDF}(B)$ and the set of all partitions of B by $\mathrm{PART}(B)$. B satisfies κ-chain condition if there is no partition P of B such that $|P| = \kappa$. $\mathrm{Sat}(B)$ is the least cardinal κ such that B satisfies the κ-chain conditions. A satisfies the $<\kappa$ chain condition with respect to B if for every function P from A to B such that $P(a)\cdot P(b) > 0$ implies that $a = b$ or $a \cdot b = 0$, there is a set $\{b_i \mid i \in I\}$ such that $\bigvee \{b_i \mid i \in I\} = 1$ and $|\{a \mid P(a)\cdot b_i > 0\}| < \kappa$ for every $i \in I$. A complete Boolean algebra B satisfies the (κ, λ)-distributive law if for every $\{\{b_{\alpha,\beta} \mid \beta < \lambda\} \in \mathrm{PART}(B) \mid \alpha < \kappa\}$

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha,\beta} = \bigvee_{f\in\lambda^\kappa} \bigwedge_{\alpha<\kappa} b_{\alpha,f(\alpha)}$$

B satisfies the (κ, ∞)-distributive law if it satisfies the (κ, λ)-distributive law for every λ. B satisfies the $(<\kappa, \infty)$-distributive law if it satisfies the (δ, ∞)-distributive law for every $\delta < \kappa$.

Let M be an L-structure for some (first order) language L and B be a cBa. The Boolean power $M^{(B)}$ of M by B is defined by

$$M^{(B)} = \{f \in B^M \mid \{f(m) \mid m \in M\} \in \mathrm{PART}(B)\}$$

and

$$M^{(B)} \models R(f_1, \cdots, f_k) \quad \text{if and only if}$$

$$\bigvee\{f_1(m_1)\cdot\cdots\cdot f_k(m_k)\,|\,M\models R(m_1,\cdots,m_k)\}=1$$

where R is an atomic L-formula.

In particular, we denote the Boolean power $A^{(B)}$ by $A[B]$. $A[B]$ is a Boolean algebra. We denote the normal completion of $A[B]$ by $A\otimes B$.

We refer the reader to [5] with respect to Boolean algebras and [1, 2] with respect to Boolean powers.

The following lemmas are very useful (see [4]).

LEMMA 1. *Suppose that* $\{a_i\,|\,i\in I\}\subset A$ *and* $\{b_i\,|\,i\in I\}\in \mathrm{PART}(B)$. *Then there is an* $f\in A[B]$ *such that* $\bigvee\{f(a)\,|\,a=a_i\}\geqq b_i$ *for every* $i\in I$. *Such an element* f *is unique, so that we denote it by* $\sum_{i\in I}b_ia_i$ or $\sum_i b_ia_i$. *In particular, if* $I=\{i_0,i_1\}$, *then we denote it by* $b_{i_0}a_{i_0}\oplus b_{i_1}a_{i_1}$.

LEMMA 2. (1) *For every* $f\in A[B]$, $f=\sum_{a\in A}f(a)a$.

(2) $\sum_i b_ia_i(a)=\bigvee\{b_i\,|\,a=a_i\}$.

(3) $\sum_i b_ia_i+\sum_j t_js_j=\sum_{i,j}(b_i\cdot t_j)(a_i+s_j)$,

$\sum_i b_ia_i\cdot\sum_j t_js_j=\sum_{i,j}(b_i\cdot t_j)(a_i\cdot s_j)$,

$\sim\sum_i b_ia_i=\sum_i b_i(\sim a_i)$.

(4) $\sum_i b_i(\bigvee_{j\in J} a_{ij})$ *is the supremum of* $\{\sum_i b_ia_{ij}\,|\,j\in J\}$.

Suppose that A, B be complete. The canonical embedding $e_{M,A,B}$ from $(M^{(A)})^{(B)}$ to $M^{(A\otimes B)}$ is defined by

$$e_{M,A,B}(F)(m)=\sum_{f\in M^{(A)}}F(f)f(m)$$

for every $F\in(M^{(A)})^{(B)}$ and $m\in M$. Usually we omit the subscripts.

PROPOSITION. (1) $e(F)\in M^{(A\otimes B)}$ *for every* $F\in(M^{(A)})^{(B)}$.

(2) e *is an embedding; i.e.,* $(M^{(A)})^{(B)}\models R(F_1,\cdots,F_k)$ *if and only if* $M^{(A\otimes B)}\models R(F_1),\cdots,e(F_k))$ *for every atomic L-formula R and* $F_1,\cdots,F_k\in(M^{(A)})^{(B)}$.

Proof. By virtue of Lemma 2, it is easy to show (1).

(2) Note that $1_{A\otimes B}=1_{A[B]}$ and $1_{A[B]}(1_A)=1_B$.

$$\begin{aligned}&\bigvee\{e(F_1)(m_1)\cdot\cdots\cdot e(F_k)(m_k)\,|\,M\models R(m_1,\cdots,m_k)\}\\ &=\bigvee\left\{\sum\nolimits_{f_1,\cdots,f_k}\left(\bigwedge_{i=1}^{k}F_i(f_i)\right)\left(\bigwedge_{i=1}^{k}f_i(m_i)\right)\middle|\,M\models R(m_1,\cdots,m_k)\right\}\\ &=\sum\nolimits_{f_1,\cdots,f_k}\left(\bigwedge_{i=1}^{k}F_i(f_i)\right)\left(\bigvee\left\{\bigwedge_{i=1}^{k}f_i(m_i)\middle|\,M\models R(m_1,\cdots,m_k)\right\}\right).\end{aligned}$$

Since $M^{(A)}\models R(f_1,\cdots,f_k)$ if and only if

$$\bigvee\left\{\bigwedge_{i=1}^{k}f_i(m_i)\middle|\,M\models R(m_1,\cdots,m_k)\right\}=1\,,$$

we have

$$(M^{(A)})^{(B)} \models R(F_1, \cdots, F_k)$$
$$\leftrightarrow \left(\sum_{f_1,\cdots,f_k} \left(\bigwedge_{i=1}^{k} F_i(f_i)\right)\left(\bigvee \left\{\bigwedge_{i=1}^{k} f_i(m_i) \,\middle|\, M \models R(m_1, \cdots, m_k)\right\}\right)\right)(1_A) = 1_B$$
$$\leftrightarrow \bigvee \{e(F_1)(m_1)\cdot \cdots \cdot e(F_k)(m_k) \mid M \models R(m_1, \cdots, m_k)\} = 1_{A\otimes B}$$
$$\leftrightarrow M^{(A\otimes B)} \models R(e(F_1), \cdots, e(F_k)) .$$

Hence e is an embedding.

§2. A proof of the theorem

In this section, we shall prove the theorem and its corollary.

LEMMA 3 ([7]). *If $A[B]$ is complete, then B satisfies the $(<\mathrm{Sat}(A), 2)$-distributive law.*

Proof. We give a sketch proof. Suppose that $\kappa < \mathrm{Sat}(A)$, $\{a_{\alpha,i} \mid \alpha < \kappa, i = 0, 1\} \in \mathrm{PART}(A)$ and $\{b_\alpha \mid \alpha < \kappa\} \subset B$. Put $f_\alpha = b_\alpha a_{\alpha,0} \oplus (\sim b_\alpha) a_{\alpha,1}$ for every $\alpha < \kappa$. Since $A[B]$ is complete, there exists $p = \bigvee f_\alpha$. $\{p(a) \mid a \in A\}$ is a common refinement of $\{\{b_\alpha, \sim b_\alpha\} \mid \alpha < \kappa\}$. Hence B satisfies the $(\kappa, 2)$-distributive law for every $\kappa < \mathrm{Sat}(A)$.

Proof of the theorem. (1)→(4). Let $\kappa < \mathrm{Sat}(A)$ and $\{\{b_{\alpha,\beta} \mid \beta < \lambda\} \mid \alpha < \kappa\} \subset \mathrm{PART}(B)$. We first show that $\{\{b_{\alpha,\beta} \mid \beta < \lambda\} \mid \alpha < \kappa\}$ has a common refinement. Then B satisfies the (κ, λ)-distributive law, so that it satisfies the $(<\mathrm{Sat}(A), \infty)$-distributive law. Since $A[B] \cong A \otimes B$, $A[B]$ is complete. Hence B satisfies the $(<\mathrm{Sat}(A), 2)$-distributive law. So for every

$$\beta < \lambda \bigvee_{g \in 2^\kappa} \bigwedge_{\alpha < \kappa} b_{\alpha,\beta}^{g(\alpha)} = 1$$

where $b_{\alpha,\beta}^0 = b_{\alpha,\beta}$ and $b_{\alpha,\beta}^1 = \sim b_{\alpha,\beta}$. Let $\{a_\alpha \mid \alpha < \kappa, a_\alpha > 0\} \in \mathrm{PDF}(A)$ and for every $\beta < \lambda$

$$F(\beta) = \sum_{g \in 2^\kappa} \left(\bigwedge_{\alpha < \kappa} b_{\alpha,\beta}^{g(\alpha)}\right)\left(\bigvee_{\alpha < \kappa} a_\alpha^{g(\alpha)}\right)$$

where $a_\alpha^0 = a_\alpha$ and $a_\alpha^1 = 0$. Then $F \in \lambda^{(A[B])}$. Hence, by the assumption, there is a $G \in (\lambda^{(A)})^{(B)}$ such that $e(G) = F$. Suppose that $f \in \lambda^{(A)}$, $G(f) > 0$ and $\alpha < \kappa$. Then $G(f) \cdot b_{\alpha,\beta} > 0$ for some $\beta < \lambda$.

$$e(G)(\beta) = \sum_{h \in \lambda^{(A)}} G(h) h(\beta) = \sum_{g \in 2} \kappa \left(\bigwedge_{\alpha < \kappa} b_{\alpha,\beta}^{g(\alpha)}\right)\left(\bigvee_{\alpha < \kappa} a_\alpha^{g(\alpha)}\right).$$

Since $\{a_\alpha \mid \alpha < \kappa, a_\alpha > 0\} \in \mathrm{PDF}(A)$, for every $g, g' \in 2^\kappa$, $g \neq g'$ implies $\bigvee_{\alpha < \kappa} a_\alpha^{g(\alpha)} \neq \bigvee_{\alpha < \kappa} a_\alpha^{g'(\alpha)}$. Hence $\{G(h) \mid h \in \lambda^{(A)}\}$ is a refinement of $\{\bigwedge_{\alpha < \kappa} b_{\alpha,\beta}^{g(\alpha)} \mid g \in 2^\kappa\}$. So $G(f) \leqq b_{\alpha,\beta}$. Hence $\{G(h) \mid h \in \lambda^{(A)}\}$ is a common refinement of $\{\{b_{\alpha,\beta} \mid \beta < \lambda\} \mid \alpha < \kappa\}$.

Next we show that A satisfies the $<\mathrm{Sat}(A)$ chain condition with respect to B. Let $P: A \to B$ such that $P(a)\cdot P(b)>0$ implies $a=b$ or $a\cdot b=0$. we define $F\in A^{(A[B])}$ by $F(a)=P(a)a\oplus(\sim P(a))0_A$ for every $a\in A$. By the assumption, there is a $G\in(A^{(A)})^{(B)}$ such that $e(G)=F$. Suppose that $f\in A^{(A)}$, $P(a)\cdot G(f)>0$ and $a>0$.

$$e(G)(a)(a)=\bigvee_{g(a)=a} G(g)=F(a)(a)=P(a)\,.$$

Since $P(a)\cdot G(f)>0$, $f(a)=a$.

$$\begin{aligned}|\{a\in A\mid P(a)\cdot G(f)>0\}| &\leqq |\{a\in A\mid f(a)=a\}|\\ &\leqq |\{f(a)\mid a\in A\}|\\ &<\mathrm{Sat}(A)\,.\end{aligned}$$

Hence $|\{a\in A\mid P(a)\cdot G(f)>0\}|<\mathrm{Sat}(A)$ for every $f\in A^{(A)}$. So A satisfies the $<\mathrm{Sat}(A)$ chain condition with respect to B.

(4)→(3). Suppose that $[\![\check{\mathrm{Sat}(A)}=\mathrm{Sat}(\check{A})]\!]^{(B)}<1$. Without loss of generality, we assume that $[\![\check{\mathrm{Sat}(A)}<\mathrm{Sat}(\check{A})]\!]^{(B)}=1$. Let X be an element of $V^{(B)}$ such that $[\![X\in\mathrm{PDF}(A)$ and $|X|=\check{\mathrm{Sat}(A)}]\!]^{(B)}=1$. We define $P: A\to B$ by $P(a)=[\![\check{a}\in X]\!]^{(B)}$.

$$P(a)\cdot P(b)=[\![\check{a}\in X \text{ and } \check{b}\in X]\!]^{(B)}\leqq[\![\check{a}=\check{b} \text{ or } \check{a}\cdot\check{b}=0]\!]^{(B)}\,.$$

Hence $P(a)\cdot P(b)>0$ implies $a=b$ or $a\cdot b=0$. By the assumption, there is a set $\{b_i\mid i\in I\}$ such that $\bigvee\{b_i\mid i\in I\}=1$ and $|\{a\mid P(a)\cdot b_i>0\}|<\mathrm{Sat}(A)$ for every $i\in I$. Fix $i\in I$. Put $Y=\{\{P(a)\cdot b_i,\ (\sim P(a))\cdot b_i\}\mid a\in A\}$. Since B satisfies the $(<\mathrm{Sat}(A),\infty)$-distributive law and $|Y|<\mathrm{Sat}(A)$, there is a common refinement $\{c_j\mid j\in J\}$ of Y. Fix $j\in J$ such that $c_j>0$. Put $X_j=\{a\mid c_j\leqq P(a)\}$. Note that $c_j\not\leqq P(a)$ if and only if $c_j\leqq\sim P(a)$.

Claim. $c_j\leqq[\![\check{X}_j=X]\!]^{(B)}$.

$$\begin{aligned}[\![\check{X}_j\subset X]\!]^{(B)}\cdot c_j &=\bigwedge_{a\in X_j}[\![\check{a}\in X]\!]^{(B)}\cdot c_j\\ &=\bigwedge_{c_j\leqq P(a)} P(a)\cdot c_j\\ &=c_j\,.\end{aligned}$$

$$\begin{aligned}[\![X\subset\check{X}_j]\!]^{(B)}\cdot c_j &=\bigwedge_{a\in A}(P(a)\Rightarrow[\![\check{a}\in\check{X}_j]\!]^{(B)})\cdot c_j\\ &=\bigwedge_{a\notin X_j}(\sim P(a))\cdot c_j\\ &=\bigwedge_{c_j\not\leqq P(a)}(\sim P(a))\cdot c_j\\ &=\bigwedge_{c_j\leqq\sim P(a)}(\sim P(a))\cdot c_j\\ &=c_j\,.\end{aligned}$$

Therefore claim is established.

Since $|X_j| \leqq |\{a \mid P(a)\cdot b_i > 0\}| < \mathrm{Sat}(A)$ and B satisfies the $(<\mathrm{Sat}(A), \infty)$-distributive law, $[\![|X_j| < \check{\mathrm{Sat}(A)}]\!]^{(B)} = 1$. Hence

$$\begin{aligned} c_j &\leqq [\![\check{X}_j = X]\!]^{(B)} \cdot [\![|\check{X}_j| < \check{\mathrm{Sat}(A)}]\!]^{(B)} \\ &\leqq [\![|X| < \check{\mathrm{Sat}(A)}]\!]^{(B)} . \end{aligned}$$

But this contradicts that $[\![|X| = \check{\mathrm{Sat}(A)}]\!]^{(B)} = 1$.

(3)$\to$(2). It is well known that if B satisfies the $(<\mathrm{Sat}(A), \infty)$-distributive law, then $[\![\check{A}$ is $\check{\mathrm{Sat}(A)}$-complete$]\!]^{(B)} = 1$. Hence

$$\begin{aligned} [\![\check{A} \text{ is complete}]\!]^{(B)} &= [\![\check{A} \text{ is } \mathrm{Sat}(\check{A})\text{-complete}]\!]^{(B)} \\ &= [\![A \text{ is } \check{\mathrm{Sat}(A)}\text{-complete}]\!]^{(B)} \\ &= 1 . \end{aligned}$$

It is clear that $[\![\check{M^{(A)}} \subset \check{M}^{(\check{A})}]\!]^{(B)} = 1$, so that we show that $[\![\check{M}^{(\check{A})} \subset M^{(A)}]\!]^{(B)} = 1$. Let

$$Fn(X, Y, \lambda) = \{p \mid |p| < \lambda,\ p \text{ is a function, } \mathrm{dom}(p) \subset X \text{ and } \mathrm{ran}(p) \subset Y\} .$$

It is well known that if B satisfies the $(<\lambda, \infty)$-distributive law, then $[\![\check{Fn(X, Y, \lambda)} = Fn(\check{X}, \check{Y}, \check{\lambda})]\!]^{(B)} = 1$ for every X and Y. Hence we have $[\![\check{Fn(A, M, \mathrm{Sat}(A))} = Fn(\check{A}, \check{M}, \mathrm{Sat}(\check{A}))]\!]^{(B)} = 1$ by the assumption.

Claim. If $[\![f \in Fn(\check{A}, \check{M}, \mathrm{Sat}(\check{A}))$ and $\mathrm{dom}(f) \in \mathrm{PART}(\check{A})]\!]^{(B)} = 1$, then $\bigvee \{[\![f = \check{g}]\!]^{(B)} \mid g \in Fn(A, M, \mathrm{Sat}(A))$ and $\mathrm{dom}(g) \in \mathrm{PART}(A)\} = 1$. Since $\bigvee \{[\![f = \check{g}]\!]^{(B)} \mid g \in Fn(A, M, \mathrm{Sat}(A))\} = 1$, it is enough to show that $\mathrm{dom}(g) \notin \mathrm{PART}(A)$ implies $[\![\mathrm{dom}(\check{g}) \notin \mathrm{PART}(\check{A})]\!]^{(B)} = 1$. Suppose that $\mathrm{dom}(g) \notin \mathrm{PART}(A)$. Then

(i) $\exists a \in A[a < 1$ and $\forall a' \in \mathrm{dom}(g)[a' \leqq a]]$,

or

(ii) $\exists a, a' \in \mathrm{dom}(g)[a \neq a'$ and $a \cdot a' > 0]$.

(i) implies that $[\![\exists a \in \check{A}[a < 1$ and $\forall a' \in \mathrm{dom}(\check{g})[a' \leqq a]]]\!]^{(B)} = 1$. (ii) implies that $[\![\exists a, a' \in \mathrm{dom}(\check{g})[a \neq a'$ and $a \cdot a' > 0]]\!]^{(B)} = 1$. Since $[\![\check{\mathrm{dom}(g)} = \mathrm{dom}(\check{g})]\!]^{(B)} = 1$, $[\![\mathrm{dom}(\check{g}) \notin \mathrm{PART}(\check{A})]\!]^{(B)} = 1$. Hence claim is established. Since we can canonically identify $M^{(A)}$ with $\{f \in Fn(A, M, \mathrm{Sat}(A)) \mid \mathrm{dom}(f) \in \mathrm{PART}(A)\}$, we obtain $[\![\check{M}^{(\check{A})} \subset \check{M^{(A)}}]\!]^{(B)} = 1$.

(2)$\to$(1). By virtue of [5.5, 0], $A[B]$ is complete if and only if $[\![\check{A}$ is complete$]\!]^{(B)} = 1$. Therefore it is enough to show that for every $G \in M^{(A[B])}$ there is an $F \in (M^{(A)})^{(B)}$ such that $e(G) = F$. Let $G \in M^{(A[B])}$. We define $\tilde{G} \in V^{(B)}$ by

$$\tilde{G} = \{\langle \check{m}, G(m) \rangle \mid m \in M\} \times \{1_B\} .$$

Then $[\![\tilde{G} \in \check{M}^{(\check{A})}]\!]^{(B)} = 1$. By the assumption $[\![\tilde{G} \in \check{M^{(A)}}]\!]^{(B)} = 1$, so that $G^* \in (M^{(A)})^{(B)}$ where G^* is defined by $G^*(f) = [\![\tilde{G} = \check{f}]\!]^{(B)}$ for every $f \in M^{(A)}$. Now we show that $e(G^*) = G$. $e(G^*)(m) = \sum_{f \in M}(A) G^*(f) f(m)$.

Note that

$$\begin{aligned}[\![\tilde{G}=\check{f}]\!]^{(B)} &= \bigwedge_{m\in M} [\![\tilde{G}(\check{m})=\check{f}(\check{m})]\!]^{(B)} \\ &= \bigwedge_{m\in M} [\![G(m)=f(m)]\!]^{(B)} \\ &= \bigwedge_{m\in M} G(m)(f(m)) .\end{aligned}$$

Hence

$$\begin{aligned}e(G^*)(m)(a) &= \bigvee \{G^*(f)\,|\,f(m)=a\} \\ &= \bigvee \{[\![\tilde{G}=\check{f}]\!]^{(B)}\,|\,f(m)=a\} \\ &= \bigvee \left\{\bigwedge_{m\in M} G(m)(f(m))\,|\,f(m)=a\right\} \\ &\leqq G(m)(a) .\end{aligned}$$

Since $\{e(G^*)(m)(a)\,|\,a\in A\}$, $\{G(m)(a)\,|\,a\in A\}\in \mathrm{PART}(B)$, $e(G^*)(m)(a)=G(m)(a)$ for every $a\in A$. Hence $e(G^*)=G$.

Proof of the corollary. Let $A=\mathrm{Reg}(Y)$ and $B=\mathrm{Reg}(X)$. Suppose that B satisfies the $(|Y|, \infty)$-distributive law. Note that $\mathrm{Sat}(A)\mathrm{Sat}(P(Y))=|Y|^+$. We show that $[\![\widecheck{\mathrm{Sat}(A)}=\mathrm{Sat}(\check{A})]\!]^{(B)}=1$. Then, by the theorem, $e\colon (M^{(A)})^{(B)}\to M^{(A\otimes B)}$ is an isomorphism.

If $\mathrm{Sat}(A)<|Y|^+$, then

$$[\![\widecheck{\mathrm{Sat}(P(Y))}=\widecheck{\mathrm{Sat}(A)}\leqq \mathrm{Sat}(\check{A})\leqq \mathrm{Sat}(\widecheck{P(Y)})]\!]^{(B)}=1 .$$

Since

$$[\![\widecheck{\mathrm{Sat}(P(Y))}=\mathrm{Sat}(\widecheck{P(Y)})]\!]^{(B)}=1, \qquad [\![\widecheck{\mathrm{Sat}(A)}=\mathrm{Sat}(\check{A})]\!]^{(B)}=1 .$$

So we assume that $\kappa=\mathrm{Sat}(A)<|Y|^+$. Since κ is regular ([Lemma 17.6, 3]),

$$\forall f\in A^{\kappa}[f(\kappa)\in \mathrm{PDF}(A)\to \exists\alpha<\kappa\forall\beta>\alpha[f(\beta)=0]] .$$

Since B satisfies the (κ, ∞)-distributive law,

$$[\![\forall f\in \check{A}^{\check{\kappa}}[f(\check{\kappa})\in \mathrm{PDF}(\check{A})\to \exists\alpha<\check{\kappa}\forall\beta>\alpha[f(\beta)=0]]]\!]^{(B)}=1 .$$

Hence $[\![\mathrm{Sat}(\check{A})\leqq\check{\kappa}=\widecheck{\mathrm{Sat}(A)}]\!]^{(B)}=1$. Since $[\![\widecheck{\mathrm{Sat}(A)}\leqq \mathrm{Sat}(\check{A})]\!]^{(B)}=1$, we have $[\![\widecheck{\mathrm{Sat}(A)}=\mathrm{Sat}(\check{A})]\!]^{(B)}=1$.

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