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Two Results on Straight Abelian p-Groups

by

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K. Benabdallah and K. Honda, in 1983, defined the concept of a straight basis of a p-group and pointed out the two important properties. If B is a straight basis of a p-group A, then any non-zero element of A can be expressed uniquely as a linear combination of elements of B with non-negative integer coefficients smaller than p. The other important property is that the relations between the elements of B give rise to a family of integers called an s-factor set which determines the group A up to isomorphism.

In [1], it is stated that the direct sum of cyclic p-groups is straight, and the direct sum of a bounded p-group and a divisible p-group is strongly straight. In the first part of this paper, we shall give a characterization of straight p-groups. Namely, we shall show that every straight p-group can be expressed in terms of a direct sum of cyclic p-groups. Next, we shall discuss the strongly straightness in a special case. Our main purpose is to prove that the direct sum of cyclic p-groups is strongly straight. This last result is also proved in [2], but the author got it independently by a direct method.

All groups considered in this paper will be additively written abelian p-groups. For all terminologies and notations without explanations, we refer to [1] and [3].

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We start with the following proposition.

PROPOSITION 1. *A p-group A is straight if and only if there exists a direct sum G of cyclic p-groups and a pure subgroup H of G such that:*

1°. $A \cong G/H$,

2°. *There exists a normal straight basis of G which extends a suitable normal straight basis of H.*

Proof. Let A be a straight p-group, and let B be a normal straight basis of A and write $B(n)=\{b_{n\lambda}\}_{\lambda\in\Lambda_n}$, for every $n\geq 0$. From Proposition 1.6 in p. 557 of [1] and Definition 3.1 in p. 559 of [1], every $pb_{n+1,\lambda}$ can be written uniquely as:

$$pb_{n+1,\lambda}=\sum_{\lambda'\in\Lambda_n} g_{n\lambda'}^{n+1\lambda}b_{n\lambda'}\,, \qquad (g_{n\lambda'}^{n+1\lambda}\in N_p)\,.$$

Next, for each $b_{n\lambda}$, we take a cyclic p-group $\langle x_{n\lambda}\rangle$ isomorphic to $\langle b_{n\lambda}\rangle$, and we let

$$G=\bigoplus_{n\geq 0}\Bigl(\bigoplus_{\lambda\in\Lambda_n}\langle x_{n\lambda}\rangle\Bigr).$$

Then, using Proposition 1.6 in p. 557 of [1], it is straightforward to check that the mapping θ, defined by

$$\theta\Bigl(\sum_{n\geq 0}\Bigl(\sum_{\lambda\in\Lambda_n}\alpha_{n\lambda}x_{n\lambda}\Bigr)\Bigr)=\sum_{n\geq 0}\Bigl(\sum_{\lambda\in\Lambda_n}\alpha_{n\lambda}b_{n\lambda}\Bigr)\qquad(\alpha_{n\lambda}\in Z),$$

is an epimorphism of G onto A. We can prove easily that Ker θ is pure in G, using Proposition 1.6 in p. 557 of [1] and the well-known characterization of pure subgroups. We put:

$$y_{n+1,\lambda}=px_{n+1,\lambda}-\sum_{\lambda'\in\Lambda_n}g_{n\lambda'}^{n+1\lambda}x_{n\lambda'},\qquad(n\geq 0,\ \lambda\in\Lambda_{n+1}),$$

and

$$H=\mathrm{Ker}\ \theta.$$

Then, it is obvious that the set of $y_{n\lambda}$'s $(n\geq 1,\ \lambda\in\Lambda_n)$ is a basis of H. On the other hand, from Proposition 1.4 in p. 557 of [1],

$$E=\bigcup_{n\geq 0}\Bigl(\bigcup_{k\geq n+1}\{p^{k-n-1}y_{k\lambda}\}_{\lambda\in\Lambda_k}\Bigr)$$

is a normal straight basis of H and, furthermore,

$$X=\Bigl(\bigcup_{n\geq 0}\{x_{n\lambda}\}_{\lambda\in\Lambda_n}\Bigr)\cup E$$

is a normal straight basis of G. Clearly, X extends E. Thus we have proved the "only if" part of the assertion.

Next, assume that

$$0\longrightarrow H\xrightarrow{\ i\ }G\xrightarrow{\ \theta\ }A\longrightarrow 0$$

is a pure-exact sequence of p-groups. If X is a subset of G^* such that any element of $p(X(n+1))$ $(n\geq 0)$ can be expressed as the sum of a linear combination of elements of $X(n)$ with coefficients in N_p and an element in Im i, and if $p^n(X(n))$ $(n\geq 0)$ is a basis of a complement of $(p^n(\mathrm{Im}\ i))[p]$ in $(p^nG)[p]$, then it follows by Proposition 1.4 in p. 556 of [1] and the purity of Im i in G that θX is a normal straight basis of A. Therefore A is straight. Our converse assertion is a special case of the above fact.

Now we give the following theorem which is the main purpose of this paper.

THEOREM 2. *Every direct sum of cyclic p-groups is strongly straight.*

Proof. Let

$$A=\bigoplus_{n\geq 0}\Bigl(\bigoplus_{i\in I_n}\langle a_{ni}\rangle\Bigr)$$

where $\langle a_{ni}\rangle \cong Z(p^{n+1})$ for every $n \geq 0$ and every $i \in I_n$. We put:

$$E=\bigcup_{n\geq 0}\left(\bigcup_{k\geq 0}\{p^k a_{n+k,i}\}_{i\in I_{n+k}}\right).$$

Using Proposition 1.4 in p. 557 of [1], it follows that E is a normal straight basis of A. Now, let $C=\{C^n\}_{n\geq 0}$ be an arbitrary sequence of bases of the socle of A and write

$$C^n=\{c_{n\lambda}\}_{\lambda\in\Lambda_n},$$

for every $n\geq 0$. Then, for every $n\geq 0$ and every $\lambda\in\Lambda_{n+1}$, $c_{n+1,\lambda}$ is expressible uniquely as:

$$c_{n+1,\lambda}=\sum_{\lambda'\in\Lambda_n} g_{n\lambda'}^{n+1\lambda}c_{n\lambda'}, \qquad (g_{n\lambda'}^{n+1\lambda}\in N_p,\ \lambda'\in\Lambda_n).$$

Using Theorem 3.5 in p. 560 of [1] and its proof, there exists a group A' and a normal straight basis B' of A' such that:

1°. For every $n\geq 0$, $(p^nA')[p]=(p^nA)[p]$,

2°. B' is a straight basis associated with C.

Put

$$B'(n)=\{b_{n\lambda}\}_{\lambda\in\Lambda_n}$$

for every $n\geq 0$. Then, we have:

$$pb'_{n+1,\lambda}=\sum_{\lambda'\in\Lambda_n} g_{n\lambda'}^{n+1\lambda}b'_{n\lambda'}$$

for every $n\geq 0$ and every $\lambda\in\Lambda_{n+1}$, and

$$p^n b'_{n\lambda'}=c_{n\lambda'}$$

for every $n\geq 0$ and every $\lambda'\in\Lambda_n$. On the other hand, from 1°, we can pick out elements a'_{ni} in A' such that

$$p^n a'_{ni}=p^n a_{ni}$$

for every $n\geq 0$ and every $i\in I_n$.

We let:

$$E'=\bigcup_{n\geq 0}\left(\bigcup_{k\geq 0}\{p^k a'_{n+k,i}\}_{i\in I_{n+k}}\right).$$

Then, it follows by Proposition 1.4 in p. 557 of [1] that E' is a normal straight basis of A'. Clearly, the s-factor set of A' relative to E' and that of A relative to E are identical. Therefore, by Theorem 2.2 in p. 558 of [1], there is an isomorphism $\theta\colon A'\to A$ such that:

$$\theta(a'_{ni})=a_{ni}$$

for every $n\geq 0$ and every $i\in I_n$. Then, it is easy to see that $\theta(c_{n\lambda})=c_{n\lambda}$ for every $n\geq 0$ and every $\lambda\in\Lambda_n$. Now, we put:

$$\theta(b'_{n\lambda})=b_{n\lambda}$$

for every $n\geq 0$ and every $\lambda\in\Lambda_n$. Let B be the set $b_{n\lambda}$'s ($n\geq 0$, $\lambda\in\Lambda_n$). Since we obtain

$$p^n b_{n\lambda}=c_{n\lambda}$$

for every $n\geq 0$ and $\lambda\in\Lambda_n$, $p^n(B(n))$ is a basis of $(p^nA)[p]$ for every $n\geq 0$. Hence, it follows by Proposition 1.4 in p. 557 of [1] that B is a straight basis of A associated with C. On the other hand, we have

$$pb_{n+1,\lambda}=\sum_{\lambda'\in\Lambda_n} g_{n\lambda'}^{n+1\lambda} b_{n\lambda'}$$

for every $n\geq 0$ and every $\lambda\in\Lambda_{n+1}$. This shows that B is a normal straight basis of A. Thus A is strongly straight.

References

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