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The Category of Boolean-Valued Models and Its Applications†

by

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Introduction

This paper is divided into two parts. In part one, the basic definitions and several theorems from [2] are referred to and some errors are corrected. In part two, the notion of elementary equivalence between two Boolean-valued models is introduced, and Frayne's Theorem in the model theory is generalized, as in the case of the Boolean-valued models, somewhat categorically.

PART I

§1. The category BVM(T)

Throughout this paper, T will denote a given theory. Our theory T consists of three parts. The first part is the language of T Lang(T). It is the quadruple (X, LS, F, P), where X is the countable set of all individual variables, LS is the set $\{\neg, \rightarrow\} \cup \{(\forall x) \mid x \in X\}$, F is the set of function letters and P is the set of predicate letters. Both F and P are graded sets (i.e. the sets with arity functions). An O-placed function letter is often called a constant.

The set of terms Tm, the set of well-formed formulas wff and the set of closed well-formed formulas cwff are constructed as usual in Lang(T). Note, especially, that Tm is the free F-(type) algebra generated by the set X, and is also an object of the category Alg_F of F-algebras. The second part is the logic of T Logic (T). It is the triple (LA, EA, RI), where LA is the set of logical axioms, EA is the set of equality axioms and RI is the set of inference rules which are necessary and sufficient for the development of classical logic. The third and last part of T is the set of mathematical axioms Ax(T).

$$T = \mathrm{Lang}(T) + \mathrm{Logic}(T) + \mathrm{Ax}(T)$$

The Lindenbaum algebra of T will be denoted by $L(T)$. Since our logic is classical,

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$L(T)$ is an object of the category Bool of Boolean algebras. If the theory T is inconsistent, then $L(T)$ is terminal in Bool.

By the above notation, our category BVM(T) of Boolean-valued models for T will be defined as a full subcategory of the comma category

$$(\mathrm{Alg}_F(\mathrm{Tm}, -)\downarrow \mathrm{Bool}(L(T), -))$$

where $\mathrm{Alg}_F(\mathrm{Tm}, -)$ and $\mathrm{Bool}(L(T), -)$ mean the hom functors. So the object μ of BVM(T) is the mapping

$$\mu\colon \mathrm{Alg}_F(\mathrm{Tm}, M) \longrightarrow \mathrm{Bool}(L(T), A)$$

which satisfies certain conditions. The above μ is called the A-valued model of T with the domain (algebra) M. If we want to emphasize the domain M and the value-algebra A of μ, we would denote it by

$$\mu_{(M,A)}\,.$$

DEFINITION 1.1. An object

$$\mu\colon \mathrm{Alg}_F(\mathrm{Tm}, M) \longrightarrow \mathrm{Bool}(L(T), A)$$

of $(\mathrm{Alg}_F(\mathrm{Tm}, -)\downarrow \mathrm{Bool}(L(T), -))$ will be called a Boolean-valued model or, more precisely, an A-valued model with the domain M if and only if

i) $(\forall\varphi\in \mathrm{wff})(\forall x_1, \cdots, x_n\in X)(\forall t_1, \cdots, t_n\in \mathrm{Tm})(\forall\sigma, \tau\colon \mathrm{Tm}\to M)$

$$[\mathrm{Fv}(\varphi)\subseteq\{x_1, \cdots, x_n\}\wedge\sigma(x_1)=\tau(t_1)\wedge\cdots\wedge\sigma(x_n)=\tau(t_n)$$
$$\to\mu(\sigma)[\varphi]=\mu(\tau)[\varphi(\tbinom{x_1, \cdots, x_n}{t_1, \cdots, t_n})]]$$

ii) $(\forall\sigma\colon \mathrm{Tm}\to M)(\forall x\in X)(\forall\varphi\in \mathrm{wff})[\mu(\sigma)[(\forall x)\varphi]=\bigwedge_{a\in M}\mu(\sigma(\tbinom{x}{a}))[\varphi]]$.

Where $\mathrm{Fv}(\varphi)$ is the set of the free variables of φ, $\varphi(\tbinom{x}{t})$ is the formula which results from first replacing each bound occurrence of the variables z_i in φ which appear in t by some y_i which does not occur in φ, and then replacing all free occurrences of x by t. $[\varphi]$ is the equivalence class $\{\psi\in \mathrm{wff}\mid\vdash_T\varphi\leftrightarrow\psi\}$ and also is an element of $L(T)$. The symbol $\bigwedge_{a\in A}$ means the infirmum in A. The arrow $\sigma(\tbinom{x}{a})\colon \mathrm{Tm}\to M$ in Alg_F will be defined uniquely by the condition

$$(\forall y\in X)\left[y\neq x\to\sigma\begin{pmatrix}x\\a\end{pmatrix}(y)=\sigma(y)\right]\wedge\sigma\begin{pmatrix}x\\a\end{pmatrix}(x)=a\,.$$

The full subcategory of $(\mathrm{Alg}_F(\mathrm{Tm}, -)\downarrow \mathrm{Bool}(L(T), -))$ determined by the Boolean-valued models of T will be denoted by BVM(T).

THEOREM 1.2. *Let $\mu_{(M,A)}\in$ BVM(T). Then there is a unique function ρ on P (the set of predicate letters of T)*

$$\rho\colon P\longrightarrow\bigcup_{n\in\omega}\mathrm{Set}(|M|^n, |A|)$$

$$p \longmapsto \rho(p): |M|^n \to |A|$$

(where the arity of p *is* n*, the symbols* $|M|$, $|A|$ *will denote the underlying sets of* M, A*) such that, for any* $\sigma: \mathrm{Tm} \to M$,

i) $\mu(\sigma)[p(t_1, \cdots, t_n)] = \rho(p)(\sigma(t_1), \cdots, \sigma(t_n))$.

By the very definition of μ*, we also have*

ii) $\mu(\sigma)[\neg\varphi] = -\mu(\sigma)[\varphi]$

iii) $\mu(\sigma)[\varphi \wedge \psi] = \mu(\sigma)[\varphi] \wedge \mu(\sigma)[\psi]$

iv) $\mu(\sigma)[(\forall x)\varphi] = \bigwedge_{a \in M} \mu(\sigma(^x_a))[\varphi]$.

(i.e. nothing but the usual recursive definition of Boolean-valued structures.) Moreover,

v) *If* $\vdash_T \varphi$*, then* $\mu(\sigma)[\varphi] = 1$.

(i.e. $\mu_{(M,A)}$ *is a Boolean-valued model of* T *in the usual sense.)*

Proof. We only have to show i). Let

$$\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}: \mathrm{Tm} \to M$$

be one of the arrows in Alg_F with the property

$$\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}(x) = \begin{cases} a_i & \text{if } x = x_i \\ \text{arbitrary in } M & \text{otherwise} \end{cases}$$

for each $x \in X$, and let

$$\rho(p)(a_1, \cdots, a_n) = \mu\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}[p(x_1, \cdots, x_n)]$$

for all $a_1, \cdots, a_n \in M$. Then, we have

$$\begin{aligned} \mu(\sigma)[p(t_1, \cdots, t_n)] &= \mu\begin{pmatrix} x_1 & , \cdots, & x_n \\ \sigma(t_1) & , \cdots, & \sigma(t_n) \end{pmatrix}[p(x_1, \cdots, x_n)] \\ &= \rho(p)(\sigma(t_1), \cdots, \sigma(t_n)), \end{aligned}$$

as desired. □

Remarks. Our category BVM(T) contains the Natural Model. The Natural Model will be defined as follows:

$$N: \mathrm{Alg}_F(\mathrm{Tm}, \mathrm{Tm}) \longrightarrow \mathrm{Bool}(L(T), L(T)).$$

For each $\sigma: \mathrm{Tm} \to \mathrm{Tm}$ and formula φ whose free variables are among $x_1, \cdots, x_n$

$$N(\sigma)[\varphi] \overset{\Delta}{=} \left[\varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ \sigma(x_1) & , \cdots, & \sigma(x_n) \end{pmatrix}\right].$$

We can easily verify that N is an $L(T)$-valued model with the domain Tm. If the theory T is inconsistent, then $L(T)$ is the one-point algebra $\{1\}$. It is still a Boolean-valued model of T.

THEOREM 1.3. *If $\mu_{(M,A)}$ is a Boolean-valued model which satisfies the Maximum Principle, then, for each $h: A \to B$ in* Bool, *the mapping* $\mathrm{Bool}(L(T), h) \circ \mu$ *is a B-valued model of T with the domain M.* $\square$

THEOREM 1.4. *Let $v_{(N,B)} \in \mathrm{BVM}(T)$, and let $g: M \xrightarrow[\text{onto}]{} N$ in Alg_F. Then, the mapping* $v \circ \mathrm{Alg}_F(\mathrm{Tm}, g)$ *is a B-valued model of T with the domain M.* $\square$

Next, we will show that the category $\mathrm{BVM}(T)$ has all products. Let "I" be some arbitrary fixed index set. For each $i \in I$, let $\mu_{i(M_i, A_i)}$ be a Boolean-valued model of T. Since the product $\prod_{i \in I} M_i \in \mathrm{Alg}_F$, and for each $i \in I$, the projection $pr_i: \prod_{i \in I} M_i \to M_i$ is an onto arrow in Alg_F, we have

$$\mu_{i(M_i, A_i)} \circ \mathrm{Alg}_F(\mathrm{Tm}, pr_i) \in \mathrm{BVM}(T) \qquad (i \in I).$$

We also have the product $\prod_{i \in I} A_i \in \mathrm{Bool}$ and its projection $p_i: \prod_{i \in I} A_i \to A_i$ in Bool for each $i \in I$. Since the functor $\mathrm{Bool}(L(T), -)$ preserves all limits, we have

$$\prod_{i \in I} \mathrm{Bool}(L(T), A_i) \cong \mathrm{Bool}\left(L(T), \prod_{i \in I} A_i\right).$$

Hence, there exists a unique mapping π such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Alg}_F(\mathrm{Tm}, \prod_{i \in I} M_i) \arrow[rr, dashed, "\pi"] \arrow[d, "\mathrm{Alg}_F(\mathrm{Tm}, pr_i)"'] & & \mathrm{Bool}(L(T), \prod_{i \in I} A_i) \arrow[d, "\mathrm{Bool}(L(T), p_i)"] \\
\mathrm{Alg}_F(\mathrm{Tm}, M_i) \arrow[rr, "\mu_i"'] & & \mathrm{Bool}(L(T), A_i).
\end{tikzcd}$$

Here, we will show that the mapping π is $\prod_{i \in I} A_i$-valued model of T with the domain $\prod_{i \in I} M_i$. Moreover, π becomes the product of the factors $\mu_{i(M_i, A_i)}$ in $\mathrm{BVM}(T)$.

Now, let's suppose φ is a formula,

$$x_1, \cdots, x_n \in X,$$

$$t_1, \cdots, t_n \in \mathrm{Tm},$$

$$\sigma, \tau: \mathrm{Tm} \to \prod_{i \in I} M_i, \ Fv(\varphi) \subseteq \{x_1, \cdots, x_n\}$$

(i.e. all of the free variables of φ are among $x_1, \cdots, x_n$) and $\sigma(x_1) = \tau(t_1) \wedge \cdots \wedge \sigma(x_n) = \tau(t_n)$. Then, we have $pr_i \circ \sigma(x_1) = pr_i \circ \tau(t_1) \wedge \cdots \wedge pr_i \circ \sigma(x_n) = pr_i \circ \tau(t_n)$ and since μ_i is a Boolean-valued model of T,

$$\begin{aligned} p_i \circ \pi(\sigma)[\varphi] &= \mu_i(pr_i \circ \sigma)[\varphi] \\ &= \mu_i(pr_i \circ \tau)\left[\varphi\begin{pmatrix} x_1, & \cdots, & x_n \\ t_1, & \cdots, & t_n \end{pmatrix}\right] \\ &= p_i \circ \pi(\tau)\left[\varphi\begin{pmatrix} x_1, & \cdots, & x_n \\ t_1, & \cdots, & t_n \end{pmatrix}\right] \qquad (i \in I). \end{aligned}$$

Therefore,

$$\pi(\sigma)[\varphi] = \pi(\tau)\left[\varphi\begin{pmatrix} x_1, & \cdots, & x_n \\ t_1, & \cdots, & t_n \end{pmatrix}\right].$$

Furthermore, let $\sigma\colon \mathrm{Tm} \to \prod_{i \in I} M_i$, and let φ be a formula, then,

$$\begin{aligned} p_i \circ \pi(\sigma)[(\forall x)\varphi] &= \mu_i(pr_i \circ \sigma)[(\forall x)\varphi] \\ &= \bigwedge_{a_i \in M_i} \mu_i\left((pr_i \circ \sigma)\begin{pmatrix} x \\ a_i \end{pmatrix}\right)[\varphi] \\ &= \bigwedge_{a \in \prod_{i \in I} M_i} \mu_i\left(pr_i \circ \sigma\begin{pmatrix} x \\ a \end{pmatrix}\right)[\varphi] \\ &= \bigwedge_{a \in \prod_{i \in I} M_i} p_i \circ \pi\left(\sigma\begin{pmatrix} x \\ a \end{pmatrix}\right)[\varphi] \\ &= p_i\left(\bigwedge_{a \in \prod_{i \in I} M_i} \pi\left(\sigma\begin{pmatrix} x \\ a \end{pmatrix}\right)[\varphi]\right) \qquad (i \in I). \end{aligned}$$

Therefore,

$$\pi(\sigma)[(\forall x)\varphi] = \bigwedge_{a \in \prod_{i \in I} M_i} \pi\left(\sigma\begin{pmatrix} x \\ a \end{pmatrix}\right)[\varphi].$$

To establish the universal property, let $(g_i, h_i)\colon \nu \to \mu_i$ $(i \in I)$ be a family of arrows in $\mathrm{BVM}(T)$. By the universal properties for the products in Alg_F and Bool, there exists g and h such that

$$pr_i \circ g = g_i, \; p_i \circ h = h_i.$$

Moreover, for each $\sigma\colon \mathrm{Tm} \to N$ and $i \in I$,

$$\begin{aligned} p_i \circ h \circ \nu(\sigma) &= h_i \circ \nu(\sigma) \\ &= \mu_i(g_i \circ \sigma) \\ &= \mu_i(pr_i \circ g \circ \sigma) \\ &= p_i \circ \pi(g \circ \sigma). \end{aligned}$$

Hence, $h \circ \nu(\sigma)=\pi(g \circ \sigma)$ for each σ: Tm$\to N$, that is (g, h) in BVM(T). Such (g, h) is unique, so we have the universal property for π.

We can summarize the result of this discussion in the following theorem:

THEOREM 1.5. *The category* BVM(T) *has all products.* □

Let $\mu_{(M,A)}$ be any object of BVM(T). Since T includes the axioms of equality, we have

$$\mu(\sigma)[x=x]=1$$

$$\mu(\sigma)[x=y] \wedge \mu(\sigma)[\varphi(x)] \leq \mu(\sigma)[\varphi(y)] .$$

Note that $\mu(\sigma)[x=y]$ may be different from 0 and from 1. Also, we may have $\mu(\sigma)[x=y]=1$ but $\sigma(x) \neq \sigma(y)$. To exclude this last possibility we introduce the separated models.

Let $\mu_{(M,A)} \in$ BVM(T). Then,

$$\mu: \text{separated}$$
$$\overset{\Delta}{\leftrightarrow}(\forall \sigma: \mathrm{Tm} \to M)(\forall t, s \in \mathrm{Tm})[\mu(\sigma)[t=s]=1 \to \sigma(t)=\sigma(s)] .$$

Every Boolean-valued model μ is equivalent to a separated model Sep(μ) obtained from μ by considering the equivalence relation

$$R \overset{\Delta}{=} \{(a, b) \in M^2 \mid (\forall x, y \in X)[\mu(^{x,\,y}_{a,\,b})[x=y]=1]\} .$$

This model Sep($\mu_{(M,A)}$) is an A-valued model of T with the domain M/R.

$$\mathrm{Sep}(\mu): \mathrm{Alg}_F(\mathrm{Tm}, M/R) \longrightarrow \mathrm{Bool}(L(T), A)$$

$$\mathrm{Sep}(\mu)(q \circ \sigma)[\varphi]=\mu(\sigma)[\varphi]$$

where $q: M \to M/R$ is the natural surjection.

Using this terminology, we have the following result: Let $\mu_{i(M_i,2)}$, $i \in I$ be a family of separated 2-valued models of T. Then, the product of μ_i is a separated 2^I-valued model of T. Since each 2-valued model satisfies the Maximum Principle, this product also satisfies the Maximum Principle. So, for any given homomorphism $h: 2^I \to 2$ in Bool, the mapping

$$\mathrm{Bool}(L(T), h) \circ \prod_{i \in I} \mu_i : \mathrm{Alg}_F\Big(\mathrm{Tm}, \prod_{i \in I} M_i\Big) \leftrightarrow \mathrm{Bool}(L(T), 2)$$

becomes a 2-valued model of T. Furthermore, the separated model Sep(Bool($L(T), h) \circ \prod_{i \in I} \mu_i$) obtained from Bool($L(T), h) \circ \prod_{i \in I} \mu_i$ is isomorphic to the ultraproduct of μ_i defined by the ultrafilter D on I determined by h.

§ 2. The adjunctions (ξ, U_1', ε): $\mathrm{Alg}_F' \to \mathrm{BVM}(T)'$

DEFINITION 2.1.

$$\mathrm{Alg}_F' \overset{\Delta}{=} \{g \text{ in } \mathrm{Alg}_F \mid g : \text{onto}\}$$

$$\mathrm{BVM}(T)' \overset{\Delta}{=} \{(g, h) \text{ in } \mathrm{BVM}(T) \mid g : \text{onto}\}$$

The forgetful functors are introduced as follows:

$$\begin{aligned} U_1 &: \mathrm{BVM}(T) \longrightarrow \mathrm{Alg}_F & (\mu_{(M,A)} &\longmapsto M) \\ U_2 &: \mathrm{BVM}(T) \longrightarrow \mathrm{Bool} & (\mu_{(M,A)} &\longmapsto A) \\ U_1' &: \mathrm{BVM}(T)' \longrightarrow \mathrm{Alg}_F' & (\mu_{(M,A)} &\longmapsto M) \\ U_2' &: \mathrm{BVM}(T)' \longrightarrow \mathrm{Bool} & (\mu_{(M,A)} &\longmapsto A). \end{aligned}$$

In this section, we will construct the functor

$$\xi : \mathrm{Alg}_F' \longrightarrow \mathrm{BVM}(T)'$$

that is left adjoint for the forgetful functor U_1'.

First, we expand the theory T to a new theory T_C. The language of T_C, $\mathrm{Lang}(T_C)$, is that of T plus C as the set of new constants. The set of terms in $\mathrm{Lang}(T_C)$ and the set of well-formed formulas will be denoted Tm_C and wff_C, respectively. The logic of T_C, $\mathrm{Logic}(T_C)$, is not changed except in the influence of adding new constants. The set of mathematical axioms, $\mathrm{Ax}(T_C)$, is the same as that of T.

From a given $\mu_{(M,A)} \in \mathrm{BVM}(T)$ and a mapping $\gamma : C \to |M|$ ($|M|$ means the underlying set of M), we can construct a new Boolean-valued model (μ, γ) of T_C, as follows:

$$\mathrm{Alg}_{F \cup C}(\mathrm{Tm}_C, (M, \gamma)) \xrightarrow{(\mu, \gamma)} \mathrm{Bool}(L(T_C), A)$$

$$(\mu, \gamma)(\sigma)[\varphi] \overset{\Delta}{=} \mu\left(\sigma\begin{pmatrix} y_1, & \cdots, & y_m \\ \gamma(c_1), & \cdots, & \gamma(c_m) \end{pmatrix}\right)\left[\varphi\begin{pmatrix} c_1, & \cdots, & c_m \\ y_1, & \cdots, & y_m \end{pmatrix}\right]$$

(New constants of φ are among $c_1, \cdots, c_m$.)

where (M, γ) is $F \cup C$-algebra with a new interpretation γ of C. The symbol $\varphi\binom{c_1, \cdots, c_m}{y_1, \cdots, y_m}$ means the substitution of c_i that occurs in φ for the variable y_i which is not contained in φ. Since $\mathrm{Alg}_F(\mathrm{Tm}, M) \cong \mathrm{Alg}_{F \cup C}(\mathrm{Tm}_C, (M, \gamma))(\sigma \longmapsto \sigma^+)$, and we have just simplified the notation. It should be written as $(\mu, \gamma)(\sigma^+)[\varphi]$.

LEMMA 2.2. $(\mu, \gamma) \in \mathrm{BVM}(T_C)$.

Next, we further expand the theory $T_{|M|}$, for each $M \in \mathrm{Alg}_F$, to a new theory $T(M)$ as follows:

$$\mathrm{Lang}(T(M)) \overset{\Delta}{=} \mathrm{Lang}(T_{|M|})$$

$$\mathrm{Logic}(T(M)) \stackrel{\Delta}{=} \mathrm{Logic}(T_{|M|})$$

$$\mathrm{Ax}(T(M)) \stackrel{\Delta}{=} \mathrm{Ax}(T_{|M|}) \cup \{f_M(a_1, \cdots, a_n) = f(a_1, \cdots, a_n)\}_{\substack{f \in F \\ \text{arity of } f = n \\ a_1, \cdots, a_n \in M}}$$

LEMMA 2.3. *If* $\varphi \in \mathrm{wff}$, $\mathrm{Fr}(\varphi) \subseteqq \{x_1, \cdots, x_n\}$ *and all of the free variables occurring in the terms* $t_1, \cdots, t_n$ *are among* $y_1, \cdots, y_m$, *then, for each* $\tau : \mathrm{Tm} \to M$ *in* Alg_F, *we have*

$$T(M) \vdash \varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ \tau(t_1) & , \cdots, & \tau(t_n) \end{pmatrix} \leftrightarrow \varphi\begin{pmatrix} x_1 , \cdots, x_n \\ t_1 , \cdots, t_n \end{pmatrix}\begin{pmatrix} y_1 & , \cdots, & y_m \\ \tau(y_1) & , \cdots, & \tau(y_m) \end{pmatrix}.$$

DEFINITION 2.4. A theory T' in the language $\mathrm{Lang}(T_{|M|})$ said to be $|M|$-complete iff for every formula φ with at most one free variable x, we have $(\forall a \in |M|)[T' \vdash \varphi(\begin{smallmatrix} x \\ a \end{smallmatrix})]$ implies $T' \vdash (\forall x)\varphi$.

THEOREM 2.5. *Let* T' *be an* $|M|$*-complete theory in the language* $\mathrm{Lang}(T_{|M|})$ *and let* $\mathrm{Ax}(T(M)) \subseteqq \mathrm{Ax}(T')$. *Then,*

$$\mathrm{Alg}_F(\mathrm{Tm}, M) \xrightarrow{\mu} \mathrm{Bool}(L(T), L_0(T'))$$

$$\sigma \longmapsto \mu(\sigma) : L(T) \longrightarrow L_0(T')$$

$$\mu(\sigma)[\varphi] \stackrel{\Delta}{=} \left[\varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ \sigma(x_1) & , \cdots, & \sigma(x_n) \end{pmatrix}\right] \qquad (\mathrm{Fv}(\varphi) \subseteqq \{x_1, \cdots, x_n\})$$

is a Boolean-valued model of T. Further

$$\mathrm{Alg}_{F \cup |M|}(\mathrm{Tm}_{|M|}, (M, \mathrm{id}_{|M|})) \xrightarrow{(\mu, \mathrm{id}_{|M|})} \mathrm{Bool}(L(T_{|M|}), L_0(T'))$$

is a Boolean-valued model of $T_{|M|}$ *and*

$$(\forall \varphi \in \mathrm{Ax}(T'))(\forall \sigma : \mathrm{Tm} \to M)[(\mu, \mathrm{id}_{|M|})(\sigma)[\varphi] = 1]$$

(*where* $L_0(T')$ *is the Lindenbaum algebra of all sentences of* T').

Proof. By Lemma 2.3 and $\mathrm{Ax}(T(M)) \subseteqq \mathrm{Ax}(T')$, we have

$$T' \vdash \varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ \tau(t_1) & , \cdots, & \tau(t_n) \end{pmatrix} \leftrightarrow \varphi\begin{pmatrix} x_1 , \cdots, x_n \\ t_1 , \cdots, t_n \end{pmatrix}\begin{pmatrix} y_1 & , \cdots, & y_m \\ \tau(y_1) & , \cdots, & \tau(y_m) \end{pmatrix}.$$

If $\sigma(x_1) = \tau(t_1) \wedge \cdots \wedge \sigma(x_n) = \tau(t_n)$, $\mathrm{Fv}(\varphi) \subseteqq \{x_1, \cdots, x_n\}$ and all of variables occurring in $t_1, \cdots, t_n$ are among $y_1, \cdots, y_m$, then

$$\begin{aligned} \mu(\tau)\left[\begin{pmatrix} x_1 , \cdots, x_n \\ t_1 , \cdots, t_n \end{pmatrix}\right] &= \left[\varphi\begin{pmatrix} x_1 , \cdots, x_n \\ t_1 , \cdots, t_n \end{pmatrix}\begin{pmatrix} y_1 & , \cdots, & y_m \\ \tau(y_1), & \cdots, & \tau(y_m) \end{pmatrix}\right] \\ &= \left[\varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ \tau(t_1) & , \cdots, & \tau(t_n) \end{pmatrix}\right] \\ &= \left[\varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ \sigma(x_1) & , \cdots, & \sigma(x_n) \end{pmatrix}\right] \\ &= \mu(\sigma)[\varphi]. \end{aligned}$$

Next, suppose $\mathrm{Fv}(\varphi)=\{x, y_1, \cdots, y_m\}$. Then, we have

$$\begin{aligned}\mu(\sigma)[(\forall x)\varphi] &= \left[(\forall x)\varphi\begin{pmatrix} y_1 & , \cdots, & y_m \\ \sigma(y_1), & \cdots, & \sigma(y_m)\end{pmatrix}\right] \\ &= \bigwedge_{a\in|M|}\left[\varphi\begin{pmatrix} x, & y_1 & , \cdots, & y_m \\ a, & \sigma(y_1), & \cdots, & \sigma(y_m)\end{pmatrix}\right] \\ &= \bigwedge_{a\in|M|}\mu\left(\sigma\begin{pmatrix} x \\ a\end{pmatrix}\right)[\varphi]\,.\end{aligned}$$

Therefore, $\mu\in\mathrm{BVM}(T)$. By using Lemma 2.2 we have

$$(\mu, \mathrm{id}_{|M|})\in\mathrm{BVM}(T_{|M|})\,.$$

Furthermore, if $\varphi\in\mathrm{Ax}(T')$, then $[\varphi]=1$ in $L_0(T')$. So the proof is complete. $\square$

The $|M|$-rule is the following rule of proof: From $\{\varphi(\begin{smallmatrix}x\\a\end{smallmatrix})\mid a\in|M|\}$ infers $(\forall x)\varphi$. $|M|$-logic is formed by adding the $|M|$-rule to the usual first-order logic.

Here, again, we expand the theory $T(M)$ to a new theory $T(M)$ as follows:

$$\mathrm{Lang}(\widetilde{T(M)})\overset{\Delta}{=}\mathrm{Lang}(T(M))$$

$$\mathrm{Logic}(\widetilde{T(M)})\overset{\Delta}{=}\mathrm{Logic}(T(M))$$

$$\mathrm{Ax}(\widetilde{T(M)})\overset{\Delta}{=}\{\varphi\mid T(M)\vdash\varphi \text{ in } |M|\text{-logic}\}\,.$$

LEMMA 2.6. *$T(M)$ is $|M|$-complete.*

COROLLARY 2.7. *For each $M\in\mathrm{Alg}_F$, $T(M)$ is consistent iff*

$$(\exists\mu\in\mathrm{BVM}(T))[U_1(\mu)=M\wedge U_2(\mu)\neq\{1\}]\,.$$

Remarks. If $\widetilde{T(M)}$ is inconsistent, then the above μ in Corollary 2.7 is $\{1\}$-valued model with the domain M. $\{1\}$-valued models will be called trivial models. Corollary 2.7 is a generalized form of ω-completeness theorem.

Now, we construct the functor ξ that is left adjoint for U_1'.

First of all, we define the functor L by

$$\begin{array}{ccc} L\colon \mathrm{Alg}_F' & \longrightarrow & \mathrm{Bool} \\ M & & L_M \\ g\big\downarrow & \longmapsto & \big\downarrow L_g \\ N & & L_N \end{array}$$

as follows:

$$L_M\overset{\Delta}{=}L_0(\widetilde{T(M)})$$

$$L_g\left[\varphi\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n\end{pmatrix}\right]\overset{\Delta}{=}\left[\varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ g(a_1), & \cdots, & g(a_n)\end{pmatrix}\right]$$

where φ is a formula of Lang(T) whose free variables are among $x_1, \cdots, x_n$.

By the definition of L_M, its element may be expressed by the form $[\varphi\binom{x_1, \cdots, x_n}{a_1, \cdots, a_n}]$ where the symbol [ ] means the equivalence class defined by the relation $\widetilde{T(M)} \vdash * \leftrightarrow *$.

As a final step, we construct the functor

$$\xi : \mathrm{Alg}'_F \longrightarrow \mathrm{BVM}(T)'$$

This functor ξ assigns to each $M \in \mathrm{Alg}'_F$ the mapping

$$\xi(M) : \mathrm{Alg}_F(\mathrm{Tm}, M) \longrightarrow \mathrm{Bool}(L(T), L_M)$$

which sends each $\sigma : \mathrm{Tm} \to M$ to the Boolean homomorphism

$$\xi(M)(\sigma) : L(T) \longrightarrow L_M$$

$$\xi(M)(\sigma)[\varphi] \stackrel{\Delta}{=} \left[\varphi\begin{pmatrix} x_1 \ , & \cdots, & x_n \\ \sigma(x_1) \ , & \cdots, & \sigma(x_n) \end{pmatrix}\right].$$

$\xi(M)$ is well defined and it is easy to see that $\xi(M)$ becomes a Boolean-valued model of T (see, Theorem 2.5). Further, $\xi(g)$ is defined as follows: $\xi(g) \stackrel{\Delta}{=} (g, L_g)$. It is also easy to see that (g, L_g) is in $\mathrm{BVM}(T)'$.

In order to show that ξ is left adjoint for the forgetful functor U'_1, we define the arrow

$$\xi \circ U'_1(v) \xrightarrow{(1_N, \varepsilon_v)} v \text{ in } \mathrm{BVM}(T)'$$

for each $v_{(N,B)}$ as follows:

$$\varepsilon_v : L_N \longrightarrow B$$

$$\varepsilon_v\left[\varphi\begin{pmatrix} x_1 \ , & \cdots, & x_n \\ b_1 \ , & \cdots, & b_n \end{pmatrix}\right] \stackrel{\Delta}{=} v\begin{pmatrix} x_1 \ , & \cdots, & x_n \\ b_1 \ , & \cdots, & b_n \end{pmatrix}[\varphi].$$

We can prove that for any $v \in \mathrm{BVM}(T)'$, the arrow $(1_N, \varepsilon_v)$ in $\mathrm{BVM}(T)'$ is universal from ξ to v, (i.e. $(1_N, \varepsilon_v)$ is the counit.)

$$\begin{tikzcd} & \xi(M) \arrow[dl, "{(g, L_g)}"'] \arrow[dr, "{(g, h)}"] & \\ \xi \circ U'_1(v) \arrow[rr, "{(1_N, \varepsilon_v)}"'] & & v \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "g"] \\ U'_1(v) \end{tikzcd}$$

To prove universality we have to show that for every pair $(M, (g, h) : \xi(M) \to v)$ there is a unique $g' : M \to U'_1(v)$ with $(g, h) = (1_N, \varepsilon_v) \circ \xi(g')$. Since $\xi(g') = (g', L_{g'})$, we have $g = g'$. It suffices to show that $h = \varepsilon_v \circ L_g$.

$$\begin{tikzcd} & L_M \arrow[dl, "L_g"'] \arrow[dr, "h"] & \\ L_{U_1'(v)} \arrow[rr, "\varepsilon_v"'] & & B \end{tikzcd}$$

But,

$$\begin{aligned}\varepsilon_v \circ L_g\left(\left[\varphi\begin{pmatrix}x_1, & \cdots, & x_n\\ a_1, & \cdots, & a_n\end{pmatrix}\right]\right) &= \varepsilon_v\left(\left[\varphi\begin{pmatrix}x_1 &, \cdots, & x_n\\ g(a_1) &, \cdots, & g(a_n)\end{pmatrix}\right]\right)\\ &= v\begin{pmatrix}x_1 &, \cdots, & x_n\\ g(a_1) &, \cdots, & g(a_n)\end{pmatrix}[\varphi]\\ &= h \circ \xi(M)\begin{pmatrix}x_1, & \cdots, & x_n\\ a_1, & \cdots, & a_n\end{pmatrix}[\varphi]\\ &= h\left[\varphi\begin{pmatrix}x_1, & \cdots, & x_n\\ a_1, & \cdots, & a_n\end{pmatrix}\right]\end{aligned}$$

so, we have $h = \varepsilon_v \circ L_g$.

The following theorem is the final result of the aforementioned discussion:

THEOREM 2.8. *The forgetful functor* U_1': BVM$(T)' \to \mathrm{Alg}_F'$ *has a left adjoint* ξ. $\square$

Theorem 2.8 will be refined as follows:

Let μ: $\mathrm{Alg}_F(\mathrm{Tm}, M) \to \mathrm{Bool}(L(T), A)$ be a Boolean-valued model of T. If we substitute any Boolean algebra B under the condition $A \subseteq B$ for A in μ above, then the A-valued model μ becomes a B-valued model at the same time. Clearly, the element of $B - A$ cannot be used under the given interpretation. Here, we call these elements "dummy values". The purpose of the next discussion is to construct a model with no dummy values or with as few as possible, and to pursue its behavior.

One needs at least the set of values

$$\bigcup_{\sigma : \mathrm{Tm} \to M} \mu(\sigma)''L(T) \qquad (\subseteq A)$$

to assign the Boolean-values to all formulas at any sequence σ: $\mathrm{Tm} \to M$. So, the subalgebra of A generated by the set $\bigcup_{\sigma : \mathrm{Tm} \to M} \mu(\sigma)''L(T)$ must fulfill the requirement.

LEMMA 2.9. *The set* $A_0 = \bigcup_{\sigma : \mathrm{Tm} \to M} \mu(\sigma)''L(T)$ *is a Boolean algebra.*

Proof. We will show that A_0 is closed under the Boolean operators. First of all, clearly, $0, 1 \in A_0$ and for any $\mu(\sigma)[\varphi] \in A_0$, we have its Boolean complement $-\mu(\sigma)[\varphi] \in \mu(\sigma)''L(T) \subseteq A_0$. Next, let $\mu(\sigma)[\varphi]$, $\mu(\tau)[\psi] \in A_0$. Then

$$\begin{aligned}\mu(\sigma)[\varphi]\wedge\mu(\tau)[\psi]&=\mu\begin{pmatrix}x_1 & ,\cdots, & x_n\\ \sigma(x_1), & \cdots, & \sigma(x_n)\end{pmatrix}[\varphi]\wedge\mu\begin{pmatrix}y_1 & ,\cdots, & y_m\\ \tau(y_1), & \cdots, & \tau(y_m)\end{pmatrix}[\psi]\\
&=\mu\begin{pmatrix}x_1 & ,\cdots, & x_n, & z_1 & ,\cdots, & z_m\\ \sigma(x_1), & \cdots, & \sigma(x_n), & \tau(y_1), & \cdots, & \tau(y_m)\end{pmatrix}[\varphi]\\
&\quad\wedge\mu\begin{pmatrix}x_1 & ,\cdots, & x_n, & z_1 & ,\cdots, & z_m\\ \sigma(x_1), & \cdots, & \sigma(x_n), & \tau(y_1), & \cdots, & \tau(y_m)\end{pmatrix}\left[\psi\begin{pmatrix}y_1, & \cdots, & y_m\\ z_1, & \cdots, & z_m\end{pmatrix}\right]\\
&=\mu\begin{pmatrix}x_1 & ,\cdots, & x_n, & z_1 & ,\cdots, & z_m\\ \sigma(x_1), & \cdots, & \sigma(x_n), & \tau(y_1), & \cdots, & \tau(y_m)\end{pmatrix}\left[(\varphi\wedge\psi)\begin{pmatrix}y_1, & \cdots, & y_m\\ z_1, & \cdots, & z_m\end{pmatrix}\right]\\
&\in A_0\end{aligned}$$

where the free variables of φ are among $x_1, \cdots, x_n$, these ψ are among $y_1, \cdots, y_m$ and

$$z_j=\begin{cases}x_i & \text{there exists } i \text{ for which } \sigma(x_i)=\tau(y_j)\\ y_j & \text{otherwise.}\end{cases}\quad\square$$

According to Lemma 2.9 the generating step falls into disuse. Therefore, the next definition comes into effect.

DEFINITION 2.10. A Boolean-valued model

$$\mu\colon \mathrm{Alg}_F(\mathrm{Tm}, M) \longrightarrow \mathrm{Bool}(L(T), A)$$

is said to be strict if and only if $A=\bigcup_{\sigma:\mathrm{Tm}\to M}\mu(\sigma)''L(T)$.

The full subcategory of $\mathrm{BVM}(T)$ determined by the strict Boolean-valued models will be denoted by $\mathrm{SBVM}(T)$. Clearly, these well-known two valued models are strict.

LEMMA 2.11. *For any Boolean-valued model $\mu_{(M,A)}$, there is exactly one strict Boolean-valued model*

$$s(\mu)\colon \mathrm{Alg}_F(\mathrm{Tm}, M) \longrightarrow \mathrm{Bool}(L(T), s(A)),$$

where

$$s(A)\overset{\Delta}{=}\bigcup_{\sigma:\mathrm{Tm}\to M}\mu(\sigma)''L(T),$$

such that

$$s(\mu)(\sigma)[\varphi]=\mu(\sigma)[\varphi].\quad\square$$

This $s(\mu)$ will be called the strict model obtained from μ.

In Theorem 2.8, we introduced the adjunction

$$\mathrm{BVM}(T)' \underset{\xi}{\overset{U_1'}{\rightleftarrows}} \mathrm{Alg}_F'.$$

In this notation, we have

LEMMA 2.12. *For any F-type algebra M, $\xi(M)$ is strict.*

Proof. Clearly,

$$\bigcup_{\sigma:\mathrm{Tm}\to M} \xi(M)(\sigma)''L(T) \subseteq L_M .$$

The converse inclusion also holds true. Since any element of L_M can be written in the form

$$\left[\varphi\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}\right],$$

there is a σ: Tm$\to M$ and a formula φ such that

$$\left[\varphi\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}\right] = \xi(M)(\sigma)[\varphi] .$$

This means

$$\left[\varphi\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}\right] \in \bigcup_{\sigma:\mathrm{Tm}\to M} \xi(M)(\sigma)''L(T) . \qquad \square$$

By this Lemma, the full subcategory SBVM(T) contains all the objects $\xi(M)$ for $M \in \mathrm{Alg}_F$, and it leads to another adjunction

$$\mathrm{SBVM}(T)' \underset{\xi_s}{\overset{V_1'}{\rightleftarrows}} \mathrm{Alg}_F' .$$

where SBVM$(T)' = \{(g, h)$ in SBVM$(T) \mid g$: onto$\}$ and the functor ξ_S is just ξ with its codomain restricted from BVM$(T)'$ to SBVM$(T)'$. V_1' is U_1' with a domain restricted to SBVM$(T)'$.

Considering the information presented thus far in the above lemmas, we have the following theorem:

THEOREM 2.13. *The forgetful functor*

$$V_1' : \mathrm{SBVM}(T)' \longrightarrow \mathrm{Alg}_F'$$

has a left adjoint ξ_S.

PART II

§ 3. Generalization of Frayne's theorem

In this last section, the basic theorem in the model theory will be generalized to the Boolean-valued case somewhat categorically. We shall introduce two new functors Bar and Core, and extend our former definition of elementary equivalence to the Boolean-valued case.

$$\mathrm{Bar}: \mathrm{BVM}(T) \longrightarrow (L_0(T) \downarrow \mathrm{Bool})$$

$$\begin{tikzcd} \mu_{(M,A)} \arrow[d, "{(g,h)}"'] & & L_0(T) \arrow[r, "\mathrm{Bar}(\mu)"] \arrow[dr, "\mathrm{Bar}(\nu)"'] & A \arrow[d, "{\mathrm{Bar}(g,h)}"] \\ \nu_{(N,B)} & \longmapsto & & B \end{tikzcd}$$

$$\mathrm{Bar}(\mu)[\varphi] \overset{\Delta}{=} \mu(\sigma)[\varphi] \qquad (\text{for some } \sigma: \mathrm{Tm} \to M)$$

$$\mathrm{Bar}(g, h) \overset{\Delta}{=} h$$

where $L_0(T)$ is the Lindenbaum algebra of all sentences (closed well formed formulas) of T, and since φ is closed, the definition is sound.

$$\mathrm{Core}: \mathrm{BVM}(T) \longrightarrow (L_0(T) \downarrow \mathrm{Bool})$$

$$\begin{tikzcd} \mu_{(M,A)} \arrow[d, "{(g,h)}"'] & & L_0(T) \arrow[r, "\mathrm{Core}(\mu)"] \arrow[dr, "\mathrm{Core}(\nu)"'] & \mathrm{Im}\,\mathrm{Bar}(\mu) \arrow[d, "{\mathrm{Core}(g,h)}"] \\ \nu_{(N,B)} & \longmapsto & & \mathrm{Im}\,\mathrm{Bar}(\nu) \end{tikzcd}$$

$$\mathrm{Core}(\mu)[\varphi] \overset{\Delta}{=} \mathrm{Bar}(\mu)[\varphi]$$

$\mathrm{Core}(g, h) \overset{\Delta}{=}$ just h with a domain restricted to $\mathrm{Im}\,\mathrm{Bar}(\mu)$

and with a codomain restricted to $\mathrm{Im}\,\mathrm{Bar}(\nu)$.

where $\mathrm{Im}\,\mathrm{Bar}(\mu)$ is the image of $\mathrm{Bar}(\mu)$.

By the above notation,

DEFINITION 3.1.

$$\mu \equiv \nu \overset{\Delta}{\longleftrightarrow} \mathrm{Core}(\mu) \cong \mathrm{Core}(\nu)$$

(isomorphic in $(L_0(T) \downarrow \mathrm{Bool})$).

The symbol "$\mu \equiv \nu$" is read "μ is elementarily equivalent to ν". If both μ and ν are 2-valued models, the symbol $\mu \equiv \nu$ means the usual elementary equivalence. Furthermore, let $(g, h): \mu \to \nu$ in $\mathrm{BVM}(T)$, and let $h \upharpoonright \mathrm{Im}\,\mathrm{Bar}(\mu)$ be injective, then $\mu \equiv \nu$.

DEFINITION 3.2. $\mu_{(M,A)}$: strict in the strong sense

$$\overset{\Delta}{\longleftrightarrow} \mathrm{Im}\,\mathrm{Bar}(\mu) = A .$$

If μ is strict in the strong sense, μ is clearly strict

$$(A = \mathrm{Im}\,\mathrm{Bar}(\mu) \subseteq \bigcup_{\sigma:\mathrm{Tm}\to M} \mu(\sigma)''L(T) \subseteq A) .$$

We come now to the main theorem of this section.

THEOREM 3.3. *Let* $\mu_{(M,A)} \equiv \nu_{(N,B)}$, *$\mu$ satisfy the Maximum Principle, ν be strict in the strong sense and let the condition*

$$(\exists g: I \to \mathrm{Alg}_F(N, M))(\forall \psi \in I)$$

$$[i \circ \mathrm{Bar}(\nu, 1_{|N|})[\psi] \leq \mathrm{Bar}(\mu, g(\psi))[\psi]] \text{---}(*)$$

be satisfied by these factors. Where

$$I \overset{\Delta}{=} \{\psi \in \mathrm{cwff}_{|N|} \mid \mathrm{Bar}(\nu, 1_{|N|})[\psi] \neq 0\}$$

(*the symbol* $\mathrm{cwff}_{|N|}$ *means the set of closed well formed formulas of* $T_{|N|}$) *and i is the isomorphism* $\mathrm{Core}(\nu) \to \mathrm{Core}(\mu)$. *Then,*

$$(\exists \hat{g}: N \to M^I \textit{ in } \mathrm{Alg}_F)(\exists P \in \mathrm{Bool})(\exists k: A^I \to P \textit{ in } \mathrm{Bool})$$
$$[\mathrm{Bool}(L(T), k) \circ \mu^I \in \mathrm{BVM}(T)$$
$$\wedge (\hat{g}, k \circ \Delta_A \circ i): \nu_{(N,B)} \to \mathrm{Bool}(L(T), k) \circ \mu^I \textit{ in } \mathrm{BVM}(T)$$
$$\wedge\, k \circ \Delta_A \circ i \textit{ is injective in } \mathrm{Bool}]$$

(*where Δ_A is the diagonal* $A \to A^I$).

$$\begin{tikzcd}
\mathrm{Alg}_F(\mathrm{Tm}, N) \arrow[rr, "\nu"] \arrow[d, dashed, "\mathrm{Alg}_F(\mathrm{Tm}, \hat{g})"'] & & \mathrm{Bool}(L(T), B) \arrow[d, dashed, "\mathrm{Bool}(L(T), k \circ \Delta_A \circ i)"] \\
\mathrm{Alg}_F(\mathrm{Tm}, M^I) \arrow[dr, "\mu^I"'] & & \mathrm{Bool}(L(T), P) \\
& \mathrm{Bool}(L(T), A^I) \arrow[ur, dashed, "\mathrm{Bool}(L(T), k)"'] &
\end{tikzcd}$$

Proof. We define the function $\hat{g}: |N| \to |M|^I$ in the following way:

$$\hat{g}(b)(\psi) = g(\psi)(b) \qquad (b \in |N|,\ \psi \in I)\,.$$

Then, in fact, $\hat{g}$ is an arrow in Alg_F. Next, let $k: A^I \to P$ be a coequalizer of the pair $(\Delta_A \circ i \circ \mathrm{Bar}(\nu, 1_{|N|}), \mathrm{Bar}(\mu^I, \hat{g}))$ of homomorphism in Bool.

$$\begin{tikzcd}
B \arrow[r, "i"] & A \arrow[d] & \\
L_0(T_{|N|}) \arrow[u, "\mathrm{Bar}(\nu, 1_{|N|})"] \arrow[r, "\mathrm{Bar}(\mu^I, \hat{g})"'] & A^I \arrow[r, "k"] & P
\end{tikzcd}$$

Using $\hat{g}$, P and k, we will show that

i) $\mathrm{Bool}(L(T), k) \circ \mu^I \in \mathrm{BVM}(T)$

ii) $(\hat{g}, k \circ \Delta_A \circ i)$ in $\mathrm{BVM}(T)$

iii) $k \circ \Delta_A \circ i$ is injective in Bool.

Proof of i). Since μ satisfies the Maximum Principle, μ^I also satisfies the Maximum Principle. Hence, we have i).

Proof of ii). Let $\tau: \mathrm{Tm} \to N$ and $[\varphi] \in L(T)$ where all free variables of φ are among $x_1, \cdots, x_n$. Then

$$\begin{aligned} k \circ \mu^I(\hat{g} \circ \tau)[\varphi] &= k \circ \mathrm{Bar}(\mu^I, \hat{g})\left[\varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ \tau(x_1) & , \cdots, & \tau(x_n) \end{pmatrix}\right] \\ &= k \circ \Delta_A \circ i \circ \mathrm{Bar}(v, 1_{|N|})\left[\varphi\begin{pmatrix} x_1 & , \cdots, & x_n \\ \tau(x_1) & , \cdots, & \tau(x_n) \end{pmatrix}\right] \\ &= k \circ \Delta_A \circ i \circ v(\tau)[\varphi] . \end{aligned}$$

Hence,

$$k \circ \mu^I(\hat{g} \circ \tau) = k \circ \Delta_A \circ i \circ v(\tau) .$$

That is

$$\mathrm{Bool}(L(T), k) \circ \mu^I(\hat{g} \circ \tau) = k \circ \Delta_A \circ i \circ v(\tau)$$

and this proves

$$(\hat{g}, k \circ \Delta_A \circ i) \quad \text{in} \quad \mathrm{BVM}(T) .$$

Proof of iii). Let

$$D_0 \overset{\Delta}{=} \{J(v) \mid v \in L_0(T_{|N|})\} \subseteqq A^I$$

where $J(v) = \Delta_A \circ i \circ \mathrm{Bar}(v, 1_{|N|})(v) \Leftrightarrow \mathrm{Bar}(\mu^I, \hat{g})(v)$ and let D be the filter generated by D_0. Then, by the definition of coequalizer, $P = A^I/D$ and k = the natural surjection $A^I \to A^I/D$. Furthermore,

$$\begin{aligned} & k \circ \Delta_A \circ i(w) = 1 \\ & \leftrightarrow \Delta_A \circ i(w) \in D \\ & \leftrightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|}) \\ & \quad [J([\psi_0]) \wedge \cdots \wedge J([\psi_{n-1}]) \leq \Delta_A \circ i(w)] \\ & \leftrightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|})(\forall \eta \in I) \\ & \quad [p_\eta(J([\psi_0])) \wedge \cdots \wedge p_\eta(J([\psi_{n-1}])) \leq i(w)] \\ & \leftrightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|})(\forall \eta \in I) \\ & \quad \left[\bigvee_{e \in 2^n} \bigwedge_{j \in n} e_\eta^\#(j) \leq i(w)\right] \end{aligned}$$

(where

$$e^{\#}_{\eta}(j) \overset{\Delta}{=} \begin{cases} i \circ \mathrm{Bar}(v, 1_{|N|})[\psi_j] \wedge \mathrm{Bar}(\mu, g(\eta))[\psi_j] \text{ if } e(j)=0 \\ -i \circ \mathrm{Bar}(v, 1_{|N|})[\psi_j] \wedge -\mathrm{Bar}(\mu, g(\eta))[\psi_j] \text{ if } e(j)=1) \end{cases}$$

$$\leftrightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|})(\forall \eta \in I)(\forall e \in 2^n)$$

$$\left[\bigwedge_{j \in n} e^{\#}_{\eta}(j) \leq i(w)\right]$$

$$\leftrightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|})(\forall \eta \in I)(\forall e \in 2^n)$$

$$\left[i \circ \mathrm{Bar}(v, 1_{|N|})\left[\bigwedge_{j \in n} {}^{e(j)}\psi_j\right] \wedge \mathrm{Bar}(\mu, g(\eta))\left[\bigwedge_{j \in n} {}^{e(j)}\psi_j\right] \leq i(w)\right]$$

(where ${}^0\psi_j \overset{\Delta}{=} \psi_j$ and ${}^1\psi_j = \neg \psi_j$), set $\eta = \bigwedge_{j \in n} {}^{e(j)}\psi_j$, then

$$\rightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|})(\forall e \in 2^n)$$

$$\left[i \circ \mathrm{Bar}(v, 1_{|N|})\left[\bigwedge_{j \in n} {}^{e(j)}\psi_j\right] \wedge \mathrm{Bar}\left(\mu, g\left(\bigwedge_{j \in n} {}^{e(j)}\psi_j\right)\right)\left[\bigwedge_{j \in n} {}^{e(j)}\psi_j\right] \leq i(w)\right]$$

$$\leftrightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|})(\forall e \in 2^n)$$

$$\left[i \circ \mathrm{Bar}(v, 1_{|N|})\left[\bigwedge_{j \in n} {}^{e(j)}\psi_j\right] \leq i(w)\right]$$

(since $i \circ \mathrm{Bar}(v, 1_{|N|})[\psi] \leq \mathrm{Bar}(\mu, g(\psi))[\psi]$)

$$\leftrightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|})$$

$$\left[i \circ \mathrm{Bar}(v, 1_{|N|})\left[\bigvee_{e \in 2^n} \bigwedge_{j \in n} {}^{e(j)}\psi_j\right] \leq i(w)\right]$$

$$\leftrightarrow (\exists \psi_0, \cdots, \psi_{n-1} \in \mathrm{cwff}_{|N|})$$

$$\left[\mathrm{Bar}(v, 1_{|N|})\left[\bigvee_{e \in 2^n} \bigwedge_{j \in n} {}^{e(j)}\psi_j\right] \leq w\right].$$

But

$$\bigvee_{e \in 2^n} \bigwedge_{j \in n} {}^{e(j)}\psi_j$$

is a tautology, hence,

$$\rightarrow w = 1 .$$

Therefore, $k \circ \Delta_A \circ i$ is injective and the proof is complete. □

Since $k \circ \Delta_A \restriction \mathrm{Im}\,\mathrm{Bar}(\mu)$ is injective, we have

$$\mu \equiv \mathrm{Bool}(L(T), k) \circ \mu^I$$

and P is not terminal in Bool.

So, applying any Boolean homomorphism $P \to 2$ to this model $\mathrm{Bool}(L(T), k) \circ \mu^I$ gives a usual 2-valued model. This line of argument will complete the proof of 2-valued Frayne's Theorem.

COROLLARY 3.4 (Generalized Frayne's theorem). *Suppose the theory T has no function letters. Then the condition* $(*)$ *can be omitted, and Theorem* 3.3 *will be rewritten as follows*:

If $\mu \equiv \nu$, μ satisfies the Maximum Principle, and if ν is strict in the strong sense, then

$$(\exists I)(\exists \hat{g} : N \to M^I)(\exists P \in \mathrm{Bool})(\exists k : A^I \to P \textit{ in } \mathrm{Bool})$$
$$[\mathrm{Bool}(L(T), k) \circ \mu^I \in \mathrm{BVM}(T)$$
$$\wedge (\hat{g}, k \circ \Delta_A \circ i) \textit{ in } \mathrm{BVM}(T)$$
$$\wedge k \circ \Delta_A \circ i \textit{ is injective in } \mathrm{Bool}] .$$

Proof. We will show that the condition $(*)$ is satisfied automatically. For each $\psi \in I$ with constants $b_1, \cdots, b_n$,

$$\begin{aligned} i \circ \mathrm{Bar}(\nu, 1_{|N|})[\psi] &= i \circ \nu\begin{pmatrix} x_1, & \cdots, & x_n \\ b_1, & \cdots, & b_n \end{pmatrix}\left[\psi\begin{pmatrix} b_1, & \cdots, & b_n \\ x_1, & \cdots, & x_n \end{pmatrix}\right] \\ &\leq i \circ \nu\begin{pmatrix} x_1, & \cdots, & x_n \\ b_1, & \cdots, & b_n \end{pmatrix}\left[(\exists x_1, \cdots, x_n)\psi\begin{pmatrix} b_1, & \cdots, & b_n \\ x_1, & \cdots, & x_n \end{pmatrix}\right] \\ &= i \circ \mathrm{Core}(\nu)\left[(\exists x_1, \cdots, x_n)\psi\begin{pmatrix} b_1, & \cdots, & b_n \\ x_1, & \cdots, & x_n \end{pmatrix}\right] \\ &= \mathrm{Core}(\mu)\left[(\exists x_1, \cdots, x_n)\psi\begin{pmatrix} b_1, & \cdots, & b_n \\ x_1, & \cdots, & x_n \end{pmatrix}\right] \\ &= \mu(\sigma)\left[(\exists x_1, \cdots, x_n)\psi\begin{pmatrix} b_1, & \cdots, & b_n \\ x_1, & \cdots, & x_n \end{pmatrix}\right] \qquad \text{(for some } \sigma\text{)} . \end{aligned}$$

Since μ satisfies the Maximum Principle, there exists $a_1, \cdots, a_n \in M$ such that

$$= \mu\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}\left[\psi\begin{pmatrix} b_1, & \cdots, & b_n \\ x_1, & \cdots, & x_n \end{pmatrix}\right].$$

Let, for each ψ, $g(\psi) : N \to M$ be

$$g(\psi)(b) = \begin{cases} a_i & \text{if } b = b_i \ (1 \leqq i \leqq n) \\ \text{any element of } M & \text{otherwise.} \end{cases}$$

Then, since the set of function letters F of T is empty, there exists

$$g : I \to \mathrm{Set}(N, M) \cong \mathrm{Alg}_\phi(N, M) ,$$

such that for all $\psi \in I$

$$\mathrm{Bar}(\mu,\, g(\psi))[\psi] = \mu\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}\left[\psi\begin{pmatrix} b_1, & \cdots, & b_n \\ x_1, & \cdots, & x_n \end{pmatrix}\right].$$

But

$$i \circ \mathrm{Bar}(\nu,\, 1_{|N|})[\psi] \leq \mu\begin{pmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{pmatrix}\left[\psi\begin{pmatrix} b_1, & \cdots, & b_n \\ x_1, & \cdots, & x_n \end{pmatrix}\right].$$

and, it follows that we have

$$i \circ \mathrm{Bar}(\nu,\, 1_{|N|})[\psi] \leq \mathrm{Bar}(\mu,\, g(\psi))[\psi]\,.$$

Carefully examined, this Corollary is just a generalized version of Frayne's theorem which asserts that "$M \equiv N$ if and only if N is elementarily embedded in some ultrapower M^I/D of M."

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