COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 34, No. 1, 1985

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

On the Meromorphy of Dirichlet Series Corresponding to Siegel Cusp Form of Degree 2 with respect to $\Gamma_0(N)$†

by

Isao MATSUDA MAKINO

(Received November 22, 1984)

§0. Introduction

0.1. In [1] and [2], Andrianov constructed the theory of Euler products of Dirichlet series corresponding to holomorphic automorphic forms for the Siegel modular group $Sp_2(\mathbf{Z})$ of degree 2. The method of his theory seems to be applicable even to the case of holomorphic automorphic forms for a certain class of subgroups of $Sp_2(\mathbf{Z})$. In the previous paper [7], we studied the case of congruence subgroups of $Sp_2(\mathbf{Z})$ which is also important from the arithmetic point of view. In this paper, we shall improve the formulation and theorems of [7] to be applicable to a wider class of automorphic forms and show some examples which make the meaning of our main theorems clearer.

0.2 As in [7], we investigate cusp forms of weight k with respect to a congruence subgroup

$$\Gamma_0(N)=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp_2(Z);\ C\equiv 0 \pmod N\right\} \qquad (0<N\in\mathbf{Z})$$

of $Sp_2(\mathbf{Z})$ and a Dirichlet character ψ of modulo N. We can define Hecke operators $T(m)_\psi$ $(m\in\mathbf{N})$ on the space $S_k(\Gamma_0(N),\psi)$ of the forms mentioned above. We take a non-zero automorphic form $F_0\in S_k(\Gamma_0(N)),\psi)$ which is a common eigen form of all Heck operators $T(m)_\psi$ with $(m,N)=1$:

$$F_0\,|\,T(m)_\psi=\lambda(m)F_0 \qquad \text{for} \quad (m,N)=1\,.$$

For the set $\{\lambda(m);\ (m,N)=1\}$ of eigen values, we define a subspace $S_k(\Gamma_0(N),\psi,\lambda)$ of the space $S_k(\Gamma_0(N),\psi)$ by putting

$$S_k(\Gamma_0(N),\psi,\lambda)=\{F\in S_k(\Gamma_0(N)),\psi);\ F\,|\,T(m)_\psi=\lambda(m)F \text{ for } (m,N)=1\}\,.$$

The space $S_k(\Gamma_0(N),\psi,\lambda)$ is invariant under the action of Hecke operators. Define a

† This paper is a large part of the doctorial dissertation of the author submitted to Tokyo University in 1981.

Dirichlet series by

$$D_\lambda^N(s, \psi) = \sum_{(m,N)=1} \lambda(m) m^{-s} \qquad (\operatorname{Re} s > k)$$

and let $L(s, \psi^2)$ be the Dirichlet L-function with character ψ^2. Set

$$Z_\lambda^N(s, \psi) = L(2(s-k+2), \psi^2) D_\lambda^N(s, \psi).$$

The function $Z_\lambda^N(s, \psi)$ has the following Euler products

$$Z_\lambda^N(s, \psi) = \prod_{p \nmid N} Q_p^\psi(p^{-s})^{-1}$$

in $\operatorname{Re} s > k$, where $Q_p^\psi(t)$ is a polynomial of t with degree 4. Each $F \in S_k(\Gamma_0(N), \psi)$ has a Fourier expansion of the form:

$$F(Z) = \sum_{T>0} a_F(T) \exp(2\pi i \sigma(TZ)),$$

where T runs through all positive definite half-integral symmetric matrices of size 2. The equivalence class of positive definite primitive half-integral symmetric matrices with given determinant $-D/4$ corresponds bijectively to the proper R-ideal class, where R is an order of $\mathbf{Q}(\sqrt{D})$ with discriminant D. Let $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_h$ be a complete set of representatives of proper R-ideal classes and let $T(\mathfrak{a}_1), T(\mathfrak{a}_2), \cdots, T(\mathfrak{a}_h)$ be the corresponding positive definite primitive half-integral symmetric matrices with determinant $-D/4$. We take a character χ of the group of proper R-ideal classes and consider the sum

$$a_F(1; \chi) = \sum_{i=1}^{h} a_F(T(\mathfrak{a}_i)) \chi(\mathfrak{a}_i).$$

Let $\{F_1, F_2, \cdots, F_l\}$ be a basis of the space $S_k(\Gamma_0(N), \psi, \lambda)$ and put

$$\mathbf{a}(1; \chi) = (a_{F_1}(1, \chi), \ a_{F_2}(1; \chi), \cdots, a_{F_l}(1; \chi)) \ (\in \mathbf{C}^l).$$

It is easy to show that there exist a certain order R of an imaginary quadratic field and a certain character χ of the group of proper R-ideal classes with the property $\mathbf{a}(1; \chi) \neq 0$. We assume that N is a prime number. We impose the following assumption on the space $S_k(\Gamma_0(N), \psi, \lambda)$:

(C1) for a suitable choice (see (4.2)′) of R and χ which satisfy the condition

$$\mathbf{a}(1; \chi) \neq 0, \ N \text{ does not divide the discriminant } D(R) \text{ of } R.$$

For each $F \in S_k(\Gamma_0(N), \psi, \lambda)$, we define a function $\hat{F}$ by

$$\hat{F}(Z) = \det(\sqrt{N} Z)^{-k} F(-N^{-1} Z^{-1}),$$

which in fact is an automorphic form of $S_k(\Gamma_0(N), \bar{\psi}, \bar{\psi}^2 \lambda)$. we define matrices U_ψ and $\hat{U}_{\bar{\psi}}$ by putting

$$(F_1 \mid T(N)_\psi, F_2 \mid T(N)_\psi, \cdots, F_l \mid T(N)_\psi) = (F_1, F_2, \cdots, F_l) U_\psi,$$

$$(\hat{F}_1 \mid T(N)_{\bar{\psi}}, \hat{F}_2 \mid T(N)_{\bar{\psi}}, \cdots, \hat{F}_l \mid T(N)_{\bar{\psi}}) = (\hat{F}_1, \hat{F}_2, \cdots, \hat{F}_l)\hat{U}_{\bar{\psi}} .$$

Moreover we put

$$\hat{\mathbf{a}}(1; \chi) = (a_{\hat{F}_1}(1; \chi) , \ a_{\hat{F}_2}(1; \chi), \cdots, a_{\hat{F}_l}(1; \chi)) .$$

We attach a suitable Γ-factor to the function $Z_{\lambda}^{N}(s, \psi)$ putting

$$\Phi^N(s, \lambda, \psi) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_{\lambda}^{N}(s, \psi) .$$

For a complex variable s, we put $\check{s} = 2k-2-s$.

Now our main theorems are formulated as follows. In the following theorems we assume that N is a prime number.

THEOREM 1. *The function $\Phi^N(s, \lambda, \psi)$ can be continued analytically to a meromorphic function in the whole complex plane and is holomorphic except for a possible simple pole at $s=k$.*

For the functional equation satisfied by the function $\Phi^N(s, \lambda, \psi)$, we divide two cases.

THEOREM 2. *Let ψ be a primitive Dirichlet character modulo N such that ψ^2 is non-trivial. Suppose that the space $S_k(\Gamma_0(N), \psi, \lambda)$ satisfies the assumption* (C1) *above. Then, $\Phi^N(s, \lambda, \psi)$ satisfies the functional equation:*

$$\begin{aligned} &N^{3s/2}\Phi^N(s, \lambda, \psi)\mathbf{a}(1; \chi)(E_l - N^{-s}U_{\psi})^{-1} \\ &\qquad = \psi(-1)\omega N^{3\check{s}/2}\Phi^N(\check{s}, \bar{\psi}^2\lambda, \bar{\psi})\hat{\mathbf{a}}(1; \bar{\chi})(E_l - N^{-\check{s}}\hat{U}_{\bar{\psi}})^{-1} , \end{aligned}$$

where

$$\omega = N^{-1}\left\{ \sum_{x \in R/NR} \psi(|x|^2) \exp\left(2\pi i \frac{x-\bar{x}}{N\sqrt{D}}\right)\right\} .$$

THEOREM 3. *Let ψ be the trivial Dirichlet character modulo N. We put $(\hat{F}_1, \hat{F}_2, \cdots, \hat{F}_l) = (F_1, F_2, \cdots, F_l)U_0$ with some $U_0 \in GL_l(\mathbf{C})$.*

Suppose that the space $S_k(\Gamma_0(N), \psi, \lambda)$ satisfies the condition (C1) above and moreover that

(C2) $\qquad$ N remains prime in $\mathbf{Q}(\sqrt{D(R)})/\mathbf{Q}$.

We put

$$\Phi^N(s, \lambda) = \Phi^N(s, \lambda, \psi)(E_l - N^{-s}U_{\psi})^{-1}(E_l - N^{-(s-k+2)}U_0)^{-1} .$$

Then we have

$$\mathbf{a}(1; \chi)\Phi^N(s, \lambda) = \mathbf{a}(1; \chi)(-1)^k\Phi^N(\check{s}, \lambda) .$$

In [3] Freitag proved the following fact: let Φ be the Siegel operator from $M_k(\Gamma_0(N), \psi)$ to the set of the elliptic modular forms of level N, character ψ. Suppose F is a common eigen function of all Hecke operators $T(m)_{\psi}$ with $(m, N)=1$ and

$F|\Phi\neq 0$. Then $Z_\lambda^N(s,\psi)$ is represented as some products of Dirichlet series corresponding to $F|\Phi$.

The functional equation satisfied by $\Phi^N(s,\psi,\lambda)$ should be independent of the choice of R and coefficient $\mathbf{a}(1;\chi)$. Up to now, the problem to obtain the functional equation of $\Phi^N(s,\lambda,\psi)$ in a better form seems to be rather difficult. However in some examples we can derive explicit functional equations from our theorems, where we employ some results of Ibukiyama [5].

Now we show an exmaple. Let m_1 and m_2 be elements in $\mathbf{Z}^2$ and let $\mathbf{m}=(m_1,m_2)\in\mathbf{Z}^4$. For $\mathbf{m}=(m_1,m_2)$, put

$$\begin{aligned}\theta_{\mathbf{m}}=\theta_{\mathbf{m}}(\tau)=\sum_{n\in\mathbf{Z}^2}&\exp(2\pi i((n+m_1/2)\tau(n+m_1/2)'\\&+(n+m_1/2)(m_2/2)')) \qquad (\tau\in\mathfrak{H}_2)\,.\end{aligned}$$

$$X=(\theta_{0000}{}^4+\theta_{0010}{}^4+\theta_{0001}{}^4+\theta_{0011}{}^4)/4\,,$$

$$Y=(\theta_{0000}\theta_{0010}\theta_{0001}\theta_{0011})^2\,,$$

$$Z=(E_4^*+3Y-4X^2)/12288\,,$$

where E_4^* is the normalized Eisenstein series for $Sp_2(Z)$ of weight 4, and

$$K=(\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1100}\theta_{1111})^2)/4049\,.$$

Then in the detail version of [5] (preprint), Ibukiyama showed the following fact: let ϕ_0 be the trivial Dirichlet character of modulo 2 and

$$F_1=Y^2Z+XYK-1024YZ^2=5120XYZ\,,$$

$$F_2=13XYK-2X^2YZ-4608YZ^2+5760K^2-9728XYZK-9F_1/4$$

Then F_1 and F_2 are cusp forms in $S_2(\Gamma_0(2),\phi_0)$ and

$$F_1\,|\,T(m)_{\phi_0}=\lambda(m)F_1\,,$$

$$F_2\,|\,T(m)_{\phi_0}=\lambda(m)F_2\,.$$

It can be easily seen that $\{F_1,F_2\}$ is a basis for $S_2(\Gamma_0(2),\phi_0,\lambda)$, and that

$$U_{\phi_0}=2^5\begin{pmatrix}-28&15\\-16&-28\end{pmatrix},\qquad U_0=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

Further he proved that the Fourier coefficients of F_1 and F_2 for $\left(\begin{smallmatrix}1&1/2\\1/2&1\end{smallmatrix}\right)$ are 6 and $-3/2$ respectively. Therefore we can choose that $R=\mathbf{Z}[(-1-\sqrt{-3})/2]$ and χ is the trivial character in the assumption (C1). Then 2, the level of the forms, remains prime in $\mathbf{Q}(\sqrt{D(R)})/\mathbf{Q}$. It is easy to see that $\mathbf{a}(1;\chi)=(6,-3/2)$. From Theorem 3, we have the functional equation

$$\Phi(s)=\Phi(2-s)\,,$$

for $\Phi(s)=(2\pi)^{-2s}\Gamma(s)^2 2^s Z_\lambda^2(s,\phi_0)(1+7\cdot 2^{8-s}+2^{20-2s})^{-1}$.

The author wishes to express his hearty thanks to Professor T. Ibukiyama who kindly showed him the manuscript [5] before published and allowed him to use examples in it. Further the author wishes to express his hearty thanks to Professor S. Ihara who encouraged him to accomplish the paper.

§ 1. Review on Siegel modular forms and Hecke operators

The Siegel upper half plane of degree $n \geqq 1$ is the complex manifold of the complex dimension $n(n+1)/2$ defined by the following set:

$$\mathfrak{H}_n = \{Z = X + iY \in M_n(\mathbf{C});\ X,\ Y \in M_n(\mathbf{R}),\ Z' = Z,\ Y > 0\},$$

where Z' is the transpose of a matrix Z and $Y>0$ means that Y is positive definite. Let E_n be the unit matrix of order n and J_n the alternating matrix $\left(\begin{smallmatrix} 0 & E_n \\ -E_n & 0 \end{smallmatrix}\right)$. Set

$$GSp_n(\mathbf{R}) = \left\{ \alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(\mathbf{R});\ \alpha' J_n \alpha = r(\alpha) J_n,\ r(\alpha) > 0 \right\}.$$

It is well-known that $GSp_n(\mathbf{R})$ acts transitively on $\mathfrak{H}_n$: if $\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GSp_n(\mathbf{R})$, then the map

$$Z \longmapsto {}_{\alpha}\langle Z \rangle = (AZ+B)(CZ+D)^{-1} \qquad (Z \in \mathfrak{H}_n)$$

is a holomorphic automorphism of $\mathfrak{H}_n$. For a function F on $\mathfrak{H}_n$, a positive integer k and $\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GSp_n(\mathbf{R})$, we define another function $F|[\alpha]_k$ on $\mathfrak{H}_n$ by the formula

$$(1.1) \qquad (F|[\alpha]_k)(Z) = r(\alpha)^{nk/2} |CZ+D|^{-k} F(\alpha\langle Z \rangle) \qquad (Z \in \mathfrak{H}_n),$$

where $|X|$ is the determinant of a matrix X.

For a positive integer N, we call the set

$$\Gamma_0^n(N) = \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_2(\mathbf{Z});\ C \equiv 0 \pmod N\}$$

a congruence subgroup of $Sp_n(\mathbf{Z})$. Let ψ be a Dirichlet character modulo N and k a positive integer. A modular form of degree n and weight k with respect to the pair $(\Gamma_0^n(N), \psi)$ is any holomorphic function on $\mathfrak{H}_n$ which satisfies the following condition: for every $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^n(N)$ and $Z \in \mathfrak{H}_n$, we have the identity

$$(F|[M]_k)(Z) = \bar{\psi}(|A|) F(Z).$$

In this definition, N is called the level of the form. All modular forms of degreen n and weight k with respect to $(\Gamma_0^n(N), \psi)$ form a vector space over $\mathbf{C}$. We denote the space by $M_k(\Gamma_0^n(N), \psi)$. For $F \in M_k(\Gamma_0^n(N), \psi)$, we can define another form $F|\Phi$ of degree $n-1$ by

$$(F|\Phi)(Z_0) = \lim_{\lambda \to \infty} F\begin{pmatrix} Z_0 & \\ & i\lambda \end{pmatrix} \qquad (Z_0 \in \mathfrak{H}_{n-1}).$$

The linear map Φ: $M_k(\Gamma_0^n(N), \psi) \in F \longmapsto F|\Phi \in M_k(\Gamma_0^{n-1}(N), \psi)$ is called the Siegel operator. An element of the set

$$S_k(\Gamma_0^n(N),\ \psi)=\{F\in M_k(\Gamma_0^n(N),\ \psi);\ (F|[M]_k)|\Phi=0 \text{ for all } M\in Sp_n(\mathbf{Z})\}$$

is called a cusp form of degreen n and weight k with respect to $(\Gamma_0^n(N),\ \psi)$.

It is easy to see that every modular form $F\in M_k(\Gamma_0^n(N)\ \psi)$ has the Fourier expansion

$$F(Z)=\sum_T a(T)\exp(2\pi i\sigma(TZ)),$$

where T runs through all elements of the set

$$P_n=\{T=(t_{ij})\in M_n(\Phi);\ T'=T\geqq 0,\ t_{ii},\ 2t_{ij}\in\mathbf{Z}\} \tag{1.2}$$

and $\sigma(X)$ is the trace of a matrix X. In the definition (1.2) of P_n, $T\geqq 0$ means that T is a positive semidefinite matrix. It is easy to see that

$$a(U'TU)=\psi(|U|)|U|^k a(T), \tag{1.3}$$

for all $U\in GL_n(\mathbf{Z})$ and all $T\in P_n$, and that

$$\begin{cases} a(T)=O(|T|^k)(|T|\neq 0), \\ F(Z)=O(|\operatorname{Im} Z|^{-k}), \end{cases} \tag{1.4}$$

where O depends only on F. Further if $F\in S_k(\Gamma_0^n(N),\ \psi)$, then

$$\begin{cases} a(T)=0 \text{ for all } T\in P_n \text{ with } |T|=0 \\ a(T)=O(|T|^{k/2}), \end{cases} \tag{1.5}$$

where O depends only on F.

In the following, we consider Hecke operators on $M_k(\Gamma_0^2(N),\ \psi)$. For further details on the facts and definitions cited below, see [2] and [7]. For any positive integer m, we put

$$\Delta_m(N)=\{\alpha=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in GSp_2(\mathbf{R})\cap M_4(\mathbf{Z});\ (|A|,\ N)=1,\ r(\alpha)=m,\ C\equiv 0 \pmod N\}.$$

Further we put $\Delta_0=\bigcup_{m=1}^{\infty}\Delta_m(N)$ and we write simply Γ_0 or $\Gamma_0(N)$ for $\Gamma_0^2(N)$. Since Δ_0 form a semi-group, we can define the Hecke ring $R(\Gamma_0,\ \Delta_0)$. Especially we put

$$T(m)=\sum_{\alpha\in\Gamma_0\backslash\Delta_m(N)/\Gamma_0}\Gamma_0\alpha\Gamma_0\in R(\Gamma_0,\ \Delta_0).$$

For each double coset $\Gamma_0\alpha\Gamma_0\in R(\Gamma_0,\ \Delta_0)$ with $\alpha\in\Delta_m(N)$, a Hecke operator $[\Gamma_0\alpha\Gamma_0]_{k,\psi}$ on $M_k(\Gamma_0,\ \psi)$ is given by the formula

$$F|[\Gamma_0\alpha\Gamma_0]_{k,\psi}=m^{k-3}\sum_{\gamma\in\Gamma_0\backslash\Gamma_0\alpha\Gamma_0}\psi(|A(\gamma)|)(F|[\gamma]_k)\in M_k(\Gamma_0,\ \psi),$$

where $\gamma=\begin{pmatrix} A(\gamma) & *\\ * & *\end{pmatrix}$ runs through a complete system of representatives $\Gamma_0\alpha\Gamma_0$ modulo Γ_0. Then, the Hecke operator

$$T(m)_{k,\psi}=\sum_{\alpha\in\Gamma_0\backslash\Delta_m(N)/\Gamma_0}[\Gamma_0\alpha\Gamma_0]_{k,\psi}$$

acts on $M_k(\Gamma_0, \psi)$ by

$$F\,|\,T(m)_{k,\psi} = \sum_{\alpha \in \Gamma_0 \backslash \Delta_m(N)/\Gamma_0} F\,|\,[\Gamma_0 \alpha \Gamma_0]_{k,\psi}\,. \tag{1.6}$$

We write simply $T(m)_\psi$ or $T(m)$ for $T(m)_{k,\psi}$.

Now let $\mathfrak{A}$ denote the set of all complex-valued functions φ on the set P_2 satisfying $\varphi(UTU') = \varphi(T)$ for all $U \in SL_2(\mathbf{Z})$ and $T \in P_2$. By (1.3), the Fourier coefficients of any modular form $F \in M_k(\Gamma_0, \psi)$ can be regarded as an element of $\mathfrak{A}$. Let $\Gamma^1 = SL_2(\mathbf{Z})$, g an element in $M_2(\mathbf{Z})$ with $|g| > 0$, $(|g|, N) = 1$, and let $\Gamma^1 g \Gamma^1 = \bigcup_j \Gamma^1 g_j$ be a decomposition into disjoint left cosets. For each function $\varphi \in \mathfrak{A}$, we set

$$(T_a(\Gamma^1 g \Gamma^1)\varphi)(T) = \sum_j \varphi(g_j T g_j') \qquad (T \in P_2)\,.$$

It is easy to see that T_a is independent of the choice of the system of representatives $\{g_j\}$ and that $T_a(\Gamma^1 g \Gamma^1)\varphi \in \mathfrak{A}$. Now define operators $\Delta^+(m)$, $\Delta^-(m)$ on $\mathfrak{A}$ for $m \in \mathbf{N}$ by

$$\begin{cases} (\Delta^+(m)\varphi)(T) = \varphi(mT)\,, & \\ (\Delta^-(m)\varphi)(T) = \begin{cases} \varphi(m^{-1}T) & (m^{-1}T \in P_2)\,, \\ 0 & (m^{-1}T \notin P_2)\,. \end{cases} \end{cases} \tag{1.7}$$

Further put, for $m \in \mathbf{N}$ with $(m, N) = 1$,

$$\Pi(m) = T_a\left(\Gamma^1 \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \Gamma^1\right) \Delta^1(m)\,. \tag{1.8}$$

Then $\Pi(m)$ is an operator on $\mathfrak{A}$.

Let p be a prime and $F \in M_k(\Gamma_0, \psi)$ a common eigen function of all the Hecke operators $T(p^\delta)$ $(\delta = 0, 1, 2, \cdots)$:

$$F\,|\,T(p^\delta) = \lambda(p^\delta)F\,.$$

Then, it is easy to show from [Shimura 8] that

$$\sum_{\delta=0}^{\infty} \lambda(p^\delta) t^\delta = \frac{P_p^\psi(t)}{Q_p^\psi(t)}\,, \tag{1.9}$$

where

$$\begin{aligned} &\begin{cases} P_p^\psi(t) = 1 - p^{2k-4}\psi^2(p)t\,, \\ Q_p^\psi(t) = 1 - \lambda(p)t + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4}\psi^2(p))t^2 \\ \qquad - p^{2k-3}\psi^2(p)\lambda t^3 + p^{4k-6}\psi^4(p)t^4 \end{cases} && (p \nmid N)\,, \\ &\begin{cases} P_p^\psi(t) = 1\,, \\ Q_p^\psi(t) = 1 - \lambda(p)t \end{cases} && (p \mid N)\,. \end{aligned} \tag{1.10}$$

Using operators $\Delta^+(m)$, $\Delta^-(m)$ and $\Pi(m)$ [7, Lemma 2.1], we can show the following proposition in the same manner as in [2, Proposition 2.2.1]

PROPOSITION 1.1. *Let the notation be as above. Suppose that*

$$F(Z)=\sum_{T\in P_2} a(T)\exp(2\pi i\sigma(TZ))\in M_k(\Gamma_0, \psi)$$

is a common eigen function of all the Hecke operators $T(p^\delta)$ for a prime p and $\delta\geqq 0$. Then, for any positive definite matrix

$$T=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\in P_2^+=\{T\in P_2;\ |T|>0\}$$

such that $(a, b, c, p)=1$, we have the following equalities:

$$\left\{\sum_{\delta=0}^{\infty} a(p^\delta T)t^\delta\right\}Q_p^\psi(t)=\begin{cases} a(T)-p^{k-2}\psi(p)(\Pi(p)a)(T)t \\ \quad +\{p^{2k-4}\psi^2(p)((\Pi(p)^2-\Pi(p^2)-1)a)(T) \\ \quad +p^{3k-5}\psi^3(p)((\Pi(p)\Delta^-(p)a)(T)\}t^2 & (p\nmid N), \\ a(T) & (p\,|\,M). \end{cases} \tag{1.11}$$

§ 2. Fourier coefficients and Euler products of a form

Let $K=\mathbf{Q}(\sqrt{d_0})$ be the imaginary quadratic extension of $\mathbf{Q}$ with discriminant d_0 (<0) and $R=R(K)$ the maximal order of K. Put $\omega=\sqrt{d_0}/2$ if $d_0\equiv 0 \pmod 4$ and $\omega=(1+\sqrt{d_0})/2$ if $d_0\not\equiv 0 \pmod 4$. Then $R=\mathbf{Z}+\mathbf{Z}\omega$. The order with conductor f in R has the form

$$R_f(K)=R_f=\mathbf{Z}+\mathbf{Z}f\omega\,.$$

Clearly $R=R_1$, and the discriminant of R_f is equal to $D=d_0f^2$. For any lattice $\mathfrak{a}$ in K, put

$$R_{\mathfrak{a}}=\{\alpha\in K;\ \alpha\mathfrak{a}\subset\mathfrak{a}\}\,.$$

Then $R_{\mathfrak{a}}$ is an order in K. We call $R_{\mathfrak{a}}$ the order of $\mathfrak{a}$ and $\mathfrak{a}$ a proper $R_{\mathfrak{a}}$-ideal. Two lattice $\mathfrak{a}_1$, $\mathfrak{a}_2$ in K are said to be similar if $\mathfrak{a}_1=\alpha\mathfrak{a}_2$ for some $\alpha\neq 0$ in K and then we denote $\mathfrak{a}_1\sim\mathfrak{a}_2$. For any two lattice $\mathfrak{a}_1$, $\mathfrak{a}_2$ in K,

$$\mathfrak{a}_1\cdot\mathfrak{a}_2=\{\alpha\beta\in K;\ \alpha\in\mathfrak{a}_1,\ \beta\in\mathfrak{a}_2\} \tag{2.1}$$

is also an lattice in K. If $R_{\mathfrak{a}_1}=R_{f_1}$, $R_{\mathfrak{a}_2}=R_{f_2}$ then

$$R_{\mathfrak{a}_1\mathfrak{a}_2}=R_f\,, \tag{2.2}$$

where f is the greatest common divisor of f_1 and f_2. Then norm $N(\mathfrak{a})$ of $\mathfrak{a}$ is defined by $N(\mathfrak{a})=[R_{\mathfrak{a}}:\mathfrak{a}]$. Then $N(\mathfrak{a}_1\mathfrak{a}_2)=N(\mathfrak{a}_1)N(\mathfrak{a}_2)$. Let $\mathfrak{a}$ be a lattice in K. Then $\bar{\mathfrak{a}}=\{\alpha\in K;\ \bar{\alpha}\in\mathfrak{a}\}$ ($\bar{\alpha}$ is the conjugate of $\alpha\in K$ over $\mathbf{Q}$) is also a lattice in K, for which we have $R_{\bar{\mathfrak{a}}}=R_{\mathfrak{a}}$ and

$$\mathfrak{a}\cdot\bar{\mathfrak{a}}=N(\mathfrak{a})R_{\mathfrak{a}}\,. \tag{2.3}$$

Fix an order R_0 in K. Then it follows from (2.2) and (2.3) that all proper R_0-ideals form a commutative group under the multiplication defined by (2.1). The quotient group of this group by the subgroup of ideals similar to R_0 is called the class group of the order R_0. The class group is denoted by $H(R_0)=H(D)$, where D is the discriminant of the order R_0. For any order R_0 in K, the number $h(D)$ of elements in the group $H(D)$ is finite. Suppose that $f'\,|f$. Then the map

$$R_f(\mathbf{Q}(\sqrt{d_0}))\supset \mathfrak{a}\longmapsto R_{f'}(\mathbf{Q}(\sqrt{d_0}))\mathfrak{a}$$

induces a surjective homomorphism from $H(d_0f^2)$ onto $H(d_0f'^2)$, which we denote by $\nu(f,f')$.

Every positive definite half-integral matrix

$$T=\begin{pmatrix} a & b/2\\ b/2 & c\end{pmatrix}\in P_2^+ \tag{2.4}$$

(see Proposition 1.1) can be regarded as the matrix of the positive definite integral binary quadratic form

$$Q(x,y)=ax^2+bxy+cy^2 \tag{2.5}$$

and vice versa. When we need to make this correspondence clear, we denote $T=T_Q$, $Q=Q_T$ for the matrix (2.4) and the binary quadratic form (2.5). For $T=\left(\begin{smallmatrix} a & b/2\\ b/2 & c\end{smallmatrix}\right)\in P_2^+$, let $e(T)=e(Q_T)=(a,b,c)$ be the greatest common divisor of a, b and c, and put $D(T)=D(Q_T)=b^2-4ac$.

Let d_0 be a negative integer which is the discriminant of a quadratic field $K=\mathbf{Q}(\sqrt{d_0})$. We denote by $\{\alpha,\beta\}$ the set $\mathbf{Z}\alpha+\mathbf{Z}\beta$ $(\alpha,\beta\in K)$. For any proper $R_f(\mathbf{Q}(\sqrt{d_0}))$-ideal $\mathfrak{a}$ and a $\mathbf{Z}$-basis $\{\alpha,\beta\}$ of $\mathfrak{a}$ such that $\operatorname{Im}(\alpha\bar\beta-\bar\alpha\beta)>0$, we can define a binary quadratic form $Q(\mathfrak{a})$ and a matrix $T(\mathfrak{a})\in P_2^+$ by

$$\begin{cases} Q(\mathfrak{a})(x,y)=N(\mathfrak{a})^{-1}(\alpha x+\beta y)(\bar\alpha x+\bar\beta y)\,,\\ T(\mathfrak{a})=T_{Q(\mathfrak{a})}\,. \end{cases} \tag{2.6}$$

Then

$$T(\mathfrak{a})=(N(\mathfrak{a}))^{-1}\begin{pmatrix} |\alpha|^2 & (\alpha\bar\beta+\bar\alpha\beta)/2\\ (\alpha\bar\beta+\bar\alpha\beta)/2 & |\beta|^2\end{pmatrix}. \tag{2.7}$$

Note that the class $\{T(\mathfrak{a})\}=\{UT(\mathfrak{a})U';\ U\in SL_2(\mathbf{Z})\}$ depends only on the ideal class $\{\mathfrak{a}\}$ of $\mathfrak{a}$. Further, we have $e(T(\mathfrak{a}))=1$ and $D(T(\mathfrak{a}))=d_0f^2$. Coversely, for a matrix $T=\left(\begin{smallmatrix} a & b/2\\ b/2 & c\end{smallmatrix}\right)\in P_2^+$ with $e(T)=1$ and $D(T)=d_0f^2$, put $\mathfrak{a}(T)=\{a,(b-\sqrt{D(T)})/2\}$. Then (T) is a proper $R_f(\mathbf{Q}(\sqrt{d_0}))$-ideal. Further, the ideal class $\{\mathfrak{a}(T)\}$ represented by $\mathfrak{a}$ depends only on the class $\{T\}$. It is well-known that this correspondence define a bijection between the set of all classes $\{T\}(T\in P_2^+,e(T)=1,\ D(T)=d_0f^2)$ and all proper $R_f(\mathbf{Q}(\sqrt{d_0})$-ideal classes.

Let $\mathfrak{M}$ be the set of all ideals in all imaginary quadratic extension of $\mathbf{Q}$ and let $\tilde{\mathfrak{A}}$ be the space of all complex valued function φ on the set $\mathbf{N}\times\mathfrak{M}$ with the property: $\tilde\varphi(m;\mathfrak{a}_1)=\tilde\varphi(m;\mathfrak{a}_2)$ if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are similar. Further, we set $\mathfrak{A}^*=\{\varphi\in\mathfrak{A};\ \varphi(T)=0$ if $|T|=0\}$. We say that $T\in P_2^+$ is primitive if $e(T)=1$. Then, for any $T\in P_2^+$, we have $T=e(T)T_0$ with some T_0 primitive. Hence we can associate $\varphi\in\mathfrak{A}^*$ with a function $\tilde\varphi\in\tilde{\mathfrak{A}}$ by putting $\varphi(T)=\varphi(e(T)T_0)=\tilde\varphi(e(T);\ \mathfrak{a}(T_0))$. Since the mapping $\varphi\longmapsto\tilde\varphi$ is an isomorphism from $\mathfrak{A}^*$ to $\tilde{\mathfrak{A}}$, we can regard any operator on $\mathfrak{A}^*$ as an operator on $\tilde{\mathfrak{A}}$ and conversely. In particular by definitions (1.7) for operators on $\mathfrak{A}^*$, we have

$$(\Delta^+(m)\tilde{\varphi})(n;\,\mathfrak{a})=\tilde{\varphi}(mn;\,\mathfrak{a})\,,$$

$$(\Delta^-(m)\tilde{\varphi})(n;\,\mathfrak{a})=\begin{cases}\tilde{\varphi}(nm^{-1};\,\mathfrak{a}) & (m\,|\,n)\,,\\ 0 & (m\nmid n)\,,\end{cases}$$

for $m,\,n\in\mathbf{N}$ and $\tilde{\varphi}\in\tilde{\mathfrak{A}}$. From [7, i) of Lemma 1–7] we can easily show that $\Pi(m)$ (see (1.8)) with $(m,\,N)=1$ has the same properties as $\Pi(m)$ in [2, Theorem 2.3.1, 2.3.2 and Lemma 2.3.2]. Namely we have the following facts.

Let $m\in\mathbf{N}$ and p a prime such that $(p,\,m)=(p,\,N)=1$. Suppose that $\mathfrak{a}$ be a proper R_f-ideal and let $\{\mathfrak{a}\}$ denote the class represented by $\mathfrak{a}$ in the proper R_f-ideal class group. Put $e_f=[R_f\colon R]$ and

$$A=\frac{e_{pf}}{e_f}\sum_{\{\mathfrak{a}_0\}}\tilde{\varphi}(m;\,\mathfrak{a}_0)\,,$$

where the sum extends over all $\{\mathfrak{a}_0\}$ such that $\{\mathfrak{a}_0\}\in H(d_0(pf)^2)$ and $\nu(pf,\,f)\{\mathfrak{a}_0\}=\{\mathfrak{a}\}$. Then the following formulae hold:

I) If $(p,f)=1$ and $p=\mathfrak{p}\bar{\mathfrak{p}}(\mathfrak{p}\neq\bar{\mathfrak{p}})$ in R_f, then

$$(\Pi(p^\beta)\tilde{\varphi})(m;\,\mathfrak{a})=\tilde{\varphi}(m;\,\mathfrak{p}^\beta\mathfrak{a})+\tilde{\varphi}(m;\,\bar{\mathfrak{p}}^\beta\mathfrak{a})\qquad(\beta\in\mathbf{N})\,,$$

$$(\Pi(p)\tilde{\varphi})(pm;\,\mathfrak{a})=\tilde{\varphi}(pm;\,\mathfrak{p}\mathfrak{a})+\tilde{\varphi}(pm;\,\bar{\mathfrak{p}}\mathfrak{a})+A\,.$$

II) If $(p,f)=1$ and $p=\mathfrak{p}^2$ in R_f, then

$$(\Pi(p^\beta)\tilde{\varphi})(m;\,\mathfrak{a})=\begin{cases}\tilde{\varphi}(m;\,\mathfrak{p}\mathfrak{a}) & (\beta=1)\,,\\ 0 & (\beta=2,\,3,\,\cdots)\,,\end{cases}$$

$$(\Pi(p)\tilde{\varphi})(pm;\,\mathfrak{a})=\tilde{\varphi}(pm;\,\mathfrak{p}\mathfrak{a})+A\,.$$

III) If $(p,f)=1$ and p remains prime in R_f, then

$$(\Pi(p^\beta)\tilde{\varphi})(m;\,\mathfrak{a})=0\qquad(\beta\in\mathbf{N})\,,$$

$$(\Pi(p)\tilde{\varphi})(pm;\,\mathfrak{a})=A\,.$$

IV) If $p\,|\,f$, then

$$(\Pi(p)\tilde{\varphi})(m;\,\mathfrak{a})=\tilde{\varphi}(pm;\,R_{f/p}\mathfrak{a})\,,$$

$$((\Pi(p)^2-\Pi(p^2)-1)\tilde{\varphi})(m;\,\mathfrak{a})=0\,,$$

$$(\Pi(p)\tilde{\varphi})(pm;\,\mathfrak{a})=\tilde{\varphi}(p^2m;\,R_{f/p}\mathfrak{a})+A\,.$$

Let $\mathfrak{a}_1.\ \mathfrak{a}_2,\,\cdots,\,\mathfrak{a}_h$ $(h=h(d_0f^2))$ be a complete system of representatives of $H(d_0f^2)$, χ be a character of $H(d_0f^2)$. Suppose that

$$F(Z)=\sum_{T\in P_2}a(T)\exp(2\pi i\sigma(TZ))\in M_k(\Gamma_0,\,\psi)$$

be a common eigen function of the Hecke operators $T(p^\beta)$ for a prime p and $\beta=0, 1, 2, \cdots$:

$$F|T(p^\beta)=\lambda(p^\beta)F.$$

Since $a(UTU')=a(T)$ for $U\in SL_2(\mathbf{Z})$ and the class $\{T(\mathfrak{a})\}=\{UT(\mathfrak{a})U';\ U\in SL_2(\mathbf{Z})\}$ depends only on the class $\{\mathfrak{a}\}$ represented by $\mathfrak{a}$,

$$a(m;\chi)=\sum_{i=1}^{h} a(mT(\mathfrak{a}_i))\chi(\mathfrak{a}_i) \tag{2.8}$$

is well-defined. By the equality (1.11) and properties I), II), III) and IV) of $\Pi(m)$, in some right half plane $\operatorname{Re} s>\sigma_0$, we can prove, by the similar way to [2, Theorem 2.4.1],

$$\left\{\sum_{\delta=0}^{\infty} a(mp^\delta;\chi)p^{-\delta s}\right\}Q_p^{\psi}(p^{-s}) \tag{2.9}$$

$$=\begin{cases} a(m;\chi)\prod_{\mathfrak{p}|p}\left(1-\dfrac{\chi(p)\psi(N(\mathfrak{p}))}{N(\mathfrak{p})^{s-k+2}}\right) & \text{for } p\nmid N,\ p\nmid f, \\ \displaystyle\sum_{i=1}^{h}\chi(\mathfrak{a}_i)\left\{\left(1-\frac{\psi(p)\Pi(p)}{p^{s-k+2}}\right)\left(1-\frac{\psi^2(p)\Delta^-(p)}{p^{s-k+3}}\right)a\right\}(mT(\mathfrak{a}_i)) & \\ & \text{for } p\nmid N,\ p|f, \\ a(m;\chi) & \text{for } p|N \end{cases}$$

for $(p, m)=1$, where $\mathfrak{p}$ runs through all the proper R_f-ideals such that $\mathfrak{p}|p$. For a integer n, if all prime factors of n divide N, then we denote $n|N^\infty$. Put

$$\Phi_F(s,\chi,\psi,n)=\sum_{i=1}^{h}\chi(\mathfrak{a}_i)\prod_{\substack{p|f\\ p\nmid N}}\left\{\left(1-\frac{\psi(p)\Pi(p)}{p^{s-k+2}}\right)\left(1-\frac{\psi^2(p)\Delta^-(p)}{p^{s-2k+3}}\right)a\right\}(nT(\mathfrak{a}_i)).$$

Then

$$\Phi_F(s,\chi,\psi,n)=\sum_{i=1}^{h}\chi(\mathfrak{a}_i)\sum_{\gamma|\delta|f}\frac{\psi(\delta)\psi^2(\gamma)\mu(\delta)\mu(\gamma)}{\delta^{s-k+2}\gamma^{s-2k+3}}\,a\left(\frac{\delta}{\gamma}\,nT(R_{f/\delta}\mathfrak{a}_i)\right) \tag{2.10}$$

(μ is the Möbius function). For $T\in P_2^+$, set

$$R_F(T,s)=\sum_{m=1}^{\infty} a(mT)m^{-s} \tag{2.11}$$

and

$$R_F(\chi,s)=\sum_{i=1}^{h} R_F(T(\mathfrak{a}_i),s)\chi(\mathfrak{a}_i). \tag{2.12}$$

Then we obtain

$$R_F(\chi, s) = \sum_{m=1}^{\infty} a(m; \chi) m^{-s}$$

(see (2.8)). Further we can prove that $R_F(T, s)$ and $R_F(\chi, s)$ converge in $\mathrm{Re}\, s > 2k+1$ (resp. $k+1$) if $F \in M_k(\Gamma_0, \psi)$ (resp. $S_k(\Gamma_0, \psi)$) by the equalities (1.4) and (1.5). For eigen values $\lambda(m)$ $(m=1, 2, \cdots)$ of a modular form F in $M_k(\Gamma_0, \psi)$, we can show $|\lambda(m)| = O(m^c)$, where O and c depend only on k. Define a Dirichlet series

$$D_F^N(s, \psi) = \sum_{(m,N)=1} \lambda(m) m^{-s}.$$

then, we can see that the Dirichlet series $D_F^N(s, \psi)$ converges in a right half plane $\mathrm{Re}\, s > c+1$. Let $L(s, \psi^2)$ be a Dirichlet series $\sum_{m=1}^{\infty} \psi^2(m) m^{-s}$. Then $L(s, \psi^2)$ has an Euler products:

$$L(s, \psi^2) = \prod_p (1 - p^{-s} \psi^2(p))^{-1}.$$

Hence we have, by (1.9) and (1.10),

$$L(2(s-k+2), \psi^2) D_F^N(s, \psi) = \prod_{p \nmid N} \frac{1}{Q_p^{\psi}(p^{-s})}.$$

Put

$$Z_F^N(s, \psi) = \prod_{p \nmid N} \frac{1}{Q_p^{\psi}(p^{-s})} \tag{2.13}$$

and

$$L_D(s, \chi\psi_f) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \psi_f(\mathfrak{a}) N(\mathfrak{a})^{-s} \qquad (D = d_0 f^2), \tag{2.14}$$

the L-series of R_f with the character $\chi\psi_f(\psi_f(\mathfrak{a}) = \psi(N(\mathfrak{a}))$, where $\mathfrak{a}$ runs through all the proper R_f-ideals whose norms are coprime to f. Then the identity

$$L(2(s-k+2), \psi^2) D_F^N(s, \psi) = Z_F^N(s, \psi) \tag{2.15}$$

holds and by the equality (2.9), we have immediately

PROPOSITION 2.1. *Suppose that* $F \in M_k(\Gamma_0, \psi)$ *satisfy* $F | T(m) = \lambda(m) F$ *for* $(m, N) = 1$. *Then*

$$L_D(s-k+2, \chi\psi_f) R_F(\chi, s) = \left\{ \sum_{n | N^\infty} \Phi_F(s, \chi, \psi, n) n^{-s} \right\} Z_F^N(s, \psi).$$

We also denote $Z_F^N(s, \psi)$ (resp. $D_F^N(s, \psi)$) by

$$Z_\lambda^N(s, \psi) \text{ (resp. } D_\lambda^N(s, \psi)) \tag{2.16}$$

for $F \in S_k(\Gamma_0(N), \psi)$ with the condition $F | T(m)_\psi = \lambda(m) F((m, N) = 1)$, because $Z_F^N(s, \psi)$(resp. $D_F^N(s, \psi)$) depend only on the set $\{\lambda(m); (m, N) = 1\}$.

Especially if F is an eigen function of all $T(m)$ $(m=1, 2, \cdots)$, then we get the equality:

$$\sum_{n\mid N^\infty} \Phi_F(s, \chi, \psi, n)n^{-s}=\Phi_F(s, \chi, \psi, 1)\prod_{p\mid N} \frac{1}{Q_p^{\psi}(p^{-s})}$$

Therefore if we put

$$(2.17)\qquad \begin{cases} Z_F(s, \psi)=Z_F^N(s, \psi)\prod\limits_{p\mid N} Q_p^{\psi}(p^{-s})^{-1}\,, \\ D_F(s, \psi)=\sum\limits_{m=1}^{\infty} \lambda(m)m^{-s} \qquad (F\mid T(m)=\lambda(m)F)\,, \end{cases}$$

then we can show

$$L(2(s-k+2), \psi^2)D_F(s, \psi)=Z_F(s, \psi)$$

and

$$L_D(s-k+2, \chi\psi_f)R_F(\chi, s)=\Phi_F(s, \chi, \psi, 1)Z_F(s, \psi)\,.$$

§3. Discontinuous groups and Eisenstein series of Picard type

The set $H=\{u=(z, v)\in \mathbf{C}\times\mathbf{R};\ v>0\}$ is said to be the upper half space of three dimension. It is well-known that $SL_2(\mathbf{C})$ acts on H transitively by the following way: if $g=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in SL_2(\mathbf{C})$, $u=(z, v)\in H$, then the map

$$(3.1)\qquad u \longmapsto g(u)=\left(\frac{(az+b)(\bar{c}\bar{z}+\bar{d})+a\bar{c}v^2}{\Delta_g(u)}, \frac{u}{\Delta_g(u)}\right)$$

is an automorphism of H, where $\Delta_g(u)=|cz+d|^2+|c|^2v^2$. An $SL_2(\mathbf{C})$-invariant metric (resp. volume element) of H is given by $(dx^2+dy^2+dv^2)^{1/2}/v$ (resp. $dxdydv/v^3$) $(z=x+iy)$.

Let $K=\mathbf{Q}(\sqrt{d_0})$ and $R_f=R_f(K)$ be as in §2, $\mathfrak{a}$ a proper R_f-ideal such that $(N(\mathfrak{a}), N)=1$, and let $\{\alpha_1, \alpha_2\}$ be a $\mathbf{Z}$-basis of $\mathfrak{a}$ satisfying $\mathrm{Im}(\alpha_1\bar{\alpha}_2-\bar{\alpha}_1\alpha_2)>0$. Further we put

$$J_{\mathfrak{a}}=\begin{pmatrix} \alpha_1 & \alpha_2 \\ \bar{\alpha}_1 & \bar{\alpha}_2 \end{pmatrix}\in GL_2(K)\,,\qquad \hat{J}_{\mathfrak{a}}=\begin{pmatrix} J'_{\mathfrak{a}} & 0 \\ 0 & J_{\mathfrak{a}}^{-1} \end{pmatrix}\in Sp_2(K)\,,$$

$$g_0=\begin{pmatrix} D^{-1/4} & 0 \\ 0 & D^{1/4} \end{pmatrix}\in SL_2(\mathbf{C}) \qquad (D=d_0f^2)$$

and

$$\hat{g}=\begin{pmatrix} a & 0 & b & 0 \\ 0 & \bar{a} & 0 & \bar{b} \\ c & 0 & d & 0 \\ 0 & \bar{c} & 0 & \bar{d} \end{pmatrix}\in Sp_2(\mathbf{C}) \qquad \text{for}\quad g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbf{C})\,.$$

Then we can define an injective homomorphism $M_{\mathfrak{a}}$ of $SL_2(\mathbf{C})$ into $Sp_2(\mathbf{R})$ by

$$SL_2(\mathbf{C})\ni g \longmapsto M_{\mathfrak{a}}(g)=\hat{J}_{\mathfrak{a}}\hat{g}_0\hat{g}\hat{g}_0^{-1}\hat{J}_{\mathfrak{a}}^{-1}\in Sp_2(\mathbf{R}) .$$

Set $g_0^*=\begin{pmatrix} D^{-1/4} & 0 \\ 0 & \bar{D}^{-1/4}\end{pmatrix}\in GL_2(\mathbf{C})$. Then we can also define a map $Z_{\mathfrak{a}}$ from H into $\mathfrak{H}_2$ by

$$H\ni u=(z, v) \longmapsto Z_{\mathfrak{a}}(u)=J'_{\mathfrak{a}}g_0^*\begin{pmatrix} z & iv \\ iv & \bar{z}\end{pmatrix}g_0^*J_{\mathfrak{a}}\in\mathfrak{H}_2 .$$

Further, we can prove

$$M_{\mathfrak{a}}(g)\langle \mathbf{Z}_{\mathfrak{a}}(u)\rangle=Z_{\mathfrak{a}}(g(u))$$

for any $g\in SL_2(\mathbf{C})$ and any $u\in H$. For a proper R_f-ideal $\mathfrak{a}$ such that $(N(\mathfrak{a}), N)=1$, put

$$\Gamma(\mathfrak{a})=\left\{g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(K);\ a,\ d\in R_f,\ c\in N\mathfrak{a}^2,\ b\in\mathfrak{a}^{-2}\right\}$$

$$\Gamma_{\infty}(\mathfrak{a})=\left\{g=\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1\end{pmatrix}\in\Gamma(\mathfrak{a})\right\}.$$

Then we can show

$$M_{\mathfrak{a}}(\Gamma(\mathfrak{a}))\subset\Gamma_0 .$$

We denote by $D_{\mathfrak{a}}$ (resp. $S_{\mathfrak{a}}$) a fundamental domain for $\Gamma(\mathfrak{a})$ (resp. $\Gamma_{\infty}(\mathfrak{a})$) on H with respect to the action (3.1). For a cusp form $F\in S_k(\Gamma_0, \psi)$ we put

$$F_{\mathfrak{a}}(u)=F(Z_{\mathfrak{a}}(u)) .$$

Though $Z_{\mathfrak{a}}$ depends on the choice of the basis $\{\alpha_1, \alpha_2\}$ of $\mathfrak{a}$, $F_{\mathfrak{a}}$ depends only on the ideal $\mathfrak{a}$. Further if we put $\psi_f(\mathfrak{a})=\psi(N(\mathfrak{a}))$ for a proper $R_f(\mathbf{Q}(\sqrt{d_0}))$-ideal $\mathfrak{a}$, then $F_{\mathfrak{a}}$ satisfies

$$F_{\mathfrak{a}}(g(u))=\psi_f((a_g))\Delta_g(u)^kF_{\mathfrak{a}}(u) \qquad ((a_g)=a_gR_f)$$

for any $g=\begin{pmatrix} a_g & * \\ * & *\end{pmatrix}\in\Gamma(\mathfrak{a})$. Moreover we have

$$\text{(3.2)}\qquad \begin{cases} F_{\mathfrak{a}}((z, v))=O(v^{-k}) , \\ F_{\mathfrak{a}}((z, v))=O(\exp(-cv)) \qquad (v\to\infty) \\ F_{\mathfrak{a}}((z, v))=O(\exp(-c'v^{-1})) \qquad (v\to 0,\ (z, v)\in D_{\mathfrak{a}}) , \end{cases}$$

where c, c' (>0) and O depend only on F and $\mathfrak{a}$. Especially we denote F_{R_f} by F_0 and D_{R_f} by D_0.

For a cusp form

$$F(Z)=\sum_{T\in P_2{}^+} a(T)\exp(2\pi i(\sigma(TZ))\in S_k(\Gamma_0, \psi) ,$$

we defined $R_F(T(\mathfrak{a}), s)$ (see (2.7) and (2.11)) and $F_{\mathfrak{a}}(u)$.We show a relation between $R_F(T(\mathfrak{a}), s)$ and $F_{\mathfrak{a}}(u)$ in the following

PROPOSITION 3.1. *Put $N(\mathfrak{a})=N_0$, $D=d_0f^2$. For any cusp form $F\in S_k(\Gamma_0, \psi)$, it holds that*

$$\int_{D_{\mathfrak{a}}} F_{\mathfrak{a}}(u)v^{s-1}du=(2\pi)^{-s}|D|^{-(s-1)/2}\Gamma(s)N_0^{-(s+2)}R_F(T(\mathfrak{a}), s)$$

$(du=dxdydv,\ u=(z, v)$ and $z=x+iy)$ *in* $\mathrm{Re}\, s>k+1$.

For the proof, see [7, Lemma 3.1].

Let $E_{\mathfrak{a}}(\psi_f, u, s)$ $(u\in H, s\in \mathbf{C})$ be an Eisenstein series of Picard type defined by

$$E_{\mathfrak{a}}(\psi_f, u, s)=\frac{1}{2}\, N_0^s\psi_f(\mathfrak{a}) \sum_{\substack{(c,d)=\mathfrak{a}\\ c\equiv 0(N)}} (\psi_f((d))\left(\frac{v}{|cz+d|^2+|cv|^2}\right)^s. \tag{3.3}$$

Then we get

PROPOSITION 3.2. *For any cusp form $F\in S_k(\Gamma_0, \psi)$, then equality*

$$(4\pi)^{-s}\Gamma(s)R_F(T(\mathfrak{a}), s)=\left|\frac{D}{4}\right|^{(k-3)/2}\int_{D_0} v^kF_0(u) \times\left|\frac{D}{4}\right|^{(s-k+2)/2} E_{\mathfrak{a}}(\psi_f, u, s-k+2)\frac{du}{v^3} \qquad (u=(z, v)) \tag{3.4}$$

holds in $\mathrm{Re}\, s>k+1$.

Proof. By a result in [6], we see that $E_{\mathfrak{a}}(\psi_f, u, s)$ converges absolutely and uniformly in $\mathrm{Re}\, s>2$. Furthermore $R(T(\mathfrak{a}), s)$ converges $\mathrm{Re}\, s>k+1$. For $u=(z, v)\in H$, we denote $v(u)=v$. Put

$$I=\int_{S_{\mathfrak{a}}} F_{\mathfrak{a}}(u)v(u)^{s-1}du\,.$$

Since $D_{\mathfrak{a}}$ is a fundamental domain for $\Gamma(\mathfrak{a})$ on H,

$$S'_{\mathfrak{a}}=\bigcup_{\tau\in\Gamma_\infty(\mathfrak{a})\backslash\Gamma(\mathfrak{a})} \tau(D_{\mathfrak{a}})$$

becomes a fandamental domain for $\Gamma(\mathfrak{a})$ in H. Note that $v^kF_{\mathfrak{a}}(u)$ is invariant under the transformation of $\Gamma(\mathfrak{a})$. Therefore we have

$$I=\int_{D_{\mathfrak{a}}} v(u)^kF_{\mathfrak{a}}(u)\sum_{\tau}\psi_f((d_\tau))(v(\tau(u)))^{s-k+2}\frac{du}{v^3},$$

where $\tau=\left(\begin{smallmatrix} a_\tau & b_\tau\\ c_\tau & d_\tau\end{smallmatrix}\right)$ runs through a complete system of representatives of $\Gamma_\infty(\mathfrak{a})\backslash\Gamma(\mathfrak{a})$. By a brief computation, we can see that there exists a matrix $g(\mathfrak{a})=\left(\begin{smallmatrix} \alpha_1^* & \alpha_2^*\\ \alpha_1 & \alpha_2\end{smallmatrix}\right)\in SL_2(K)$ such that $\alpha_1, \alpha_2\in\mathfrak{a}$, $\alpha_1\in N\mathfrak{a}$ and $\alpha_1^*, \alpha_2^*\in\mathfrak{a}^{-1}$. It is easy to see that $\hat{J}_{\mathfrak{a}}\hat{g}_0\widehat{g(\mathfrak{a})}\hat{g}_0^{-1}\hat{J}_{R_f}^{-1}$ is an element of Γ_0. If we put

$$\hat{J}_{\mathfrak{a}}\hat{g}_0\widehat{g(\mathfrak{a})}\hat{g}_0^{-1}\widehat{J_{R_f}}^{-1}=\begin{pmatrix} A_0 & B_0\\ C_0 & D_0\end{pmatrix},$$

then $|A_0|=N(\alpha_1^*\mathfrak{a})$ and

$$|C_0 Z_{R_f}(u)+D_0|=N_0^{-1}\Delta_{g(\mathfrak{a})}(u)\,.$$

Change the variable u to $g(\mathfrak{a})(u)$ in the integrand above and note that $g(\mathfrak{a})(D_0)$ is a fundamental domain for $\Gamma(\mathfrak{a})$ in H, and we have

$$I=\int_{D_0}(v(u)^k N_0^{-k}F_0(u))\bar{\psi}_f(\alpha_1{}^*\mathfrak{a})\sum_{\tau\in\Gamma_\infty(\mathfrak{a})\backslash\Gamma(\mathfrak{a})}\psi_f((d_\tau))(v(\tau g)(\mathfrak{a})(u)))^{s-k+2}\frac{du}{v^3}\,.$$

Further put $\tau g(\mathfrak{a})=\tau_0$. Then τ_0 runs through a complete system of representatives of $\Gamma_\infty(\mathfrak{a})\backslash\Gamma(\mathfrak{a})g(\mathfrak{a})$. Since $d_\tau=-c_{\tau_0}\alpha_2^*+d_{\tau_0}\alpha_1^*$, we get $\psi_f((d_\tau))=\psi_f((d_{\tau_0}\alpha_1^*))$ and

$$I=N_0^{-k}\int_{D_0}(v(u)^k F_0(u))\psi_f(\mathfrak{a})\sum_{\tau_0\in\Gamma_\infty(\mathfrak{a})\backslash\Gamma(\mathfrak{a})g(\mathfrak{a})}\psi_f((d_{\tau_0}))v(\tau_0(u))^{s-k+2}\frac{du}{v^3}\,.$$

Moreover we can show that

$$\Gamma(\mathfrak{a})g(\mathfrak{a})=\left\{\tau_0=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(K);\ a,\ b\in\mathfrak{a}^{-1},\ c\in N\mathfrak{a},\ d\in\mathfrak{a}\right\}$$

and that the inclusion $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma(\mathfrak{a})g(\mathfrak{a})$ is equivalent to the conditions $(c, d)=\mathfrak{a}$ and $c\equiv 0\ (\mathrm{mod}\ N)$. Thus we have

$$I=\frac{1}{2}N_0^{-k}\int_{D_0}(v(u)^k F_0(u))\psi_f(\mathfrak{a})\sum_{\substack{(c,d)=\mathfrak{a}\\ c\equiv 0(N)}}\psi_f((d))\left(\frac{v}{|cz+d|^2+|cv|^2}\right)^s\frac{du}{v^3}$$

and hence the equality (3.4) by Proposition 3.1.

It is easy to show the following two propositions.

PROPOSITION 3.3. *Let* $\mathfrak{a}$ *be a proper* R_f*-ideal such that* $(N(\mathfrak{a}), N)=(N(\mathfrak{a}), f)=1$. *Put* $\mathfrak{a}_N=\{\alpha\in\mathfrak{a};\ (N(\alpha), N)=1\}$ *and*

$$X_N^m(\mathfrak{a}')=\{(\gamma', \delta')\in K\times K;\ (N\gamma', \delta')=\mathfrak{a}',\ (N(\delta'), N)=1\}$$

for $m\in\mathbf{N}$ *and an ideal* $\mathfrak{a}'$ *in* R_m *such that* $R_{\mathfrak{a}'}\supset R_m$. *Then*

$$\mathfrak{a}\times\mathfrak{a}_N=\bigcup_{\substack{f'|f\\ (f',N)=1}}F'\bigcup_{\mathfrak{a}'}X_N^{f/f'}(\mathfrak{a}')\,,$$

where $\mathfrak{a}'$ *extends over all ideals in* $R_{f/f'}$ *such that* $R_{\mathfrak{a}'}=R_{f/f'}$ *and* $(N(\mathfrak{a}'), nf/f')=1$.

PROPOSITION 3.4. *In every class of the group* $H(d_0f^2)$, *there exists a proper* R_f*-ideal* $\mathfrak{a}$ *such that* $(N(\mathfrak{a}), f)=(N(\mathfrak{a}), N)=1$.

By this proposition, we can choose a complete system of representatives of the proper R_f-ideal classes $\mathfrak{a}_1, \mathfrak{a}_2, \cdots\mathfrak{a}_h$ $(h=h(d_0f^2))$ such that $(N(\mathfrak{a}_i), f)=(N(\mathfrak{a}_i), N)=1$ $(i=1, 2, \cdots, h)$. Let χ be a character of the group $H(d_0f^2)$ satisfying the following condition:

(3.5) for every $f' \in \mathbf{N}$ such that $f' | f, f' > 1$ and $(f', N) = 1$, the character χ is non-trivial on the kernel of the surjective homomorphism $\nu(f, f/f')$: $H(d_0 f^2) \longmapsto H(d_0(f/f')^2)$.

Put

$$E_{\mathfrak{a}}^*(\psi_f, u, s) = N(\mathfrak{a})^s \bar{\psi}_f(\mathfrak{a}) \sum_{c, d \in \mathfrak{a}} \psi_f((d)) \left(\frac{v}{|cNz+d|^2 + |cNv|^2} \right)^s \tag{3.6}$$

for $u = (z, v) \in H$, $s \in \mathbf{C}$ and a proper ideal $\mathfrak{a}$ such that $(N(\mathfrak{a}), Nf) = 1$. Let $L_{d_0 f^2}(s, \chi\psi_f)$ be the L-seies of R_f defined by (2.14) and $E_{\mathfrak{a}}(\psi_f, u, s)$ the Eisenstein series (3.3). Then we have

PROPOSITION 3.5. *Let χ satisfy the condition* (3.5). *Then, in the domain* $\{\mathrm{Re}\, s > 2, u \in H\}$, *it holds that*

$$\sum_{i=1}^{h} \chi(\mathfrak{a}_i) E_{\mathfrak{a}}^*(\psi_f, u, s) = 2 L_{d_0 f^2}(s, \chi\psi_f) \sum_{i=1}^{h} \chi(\mathfrak{a}_i) E_{\mathfrak{a}_i}(\psi_f, u, s). \tag{3.7}$$

Further the both sides of the equality (3.7) *converge absolutely in the domain* $\mathrm{Re}\, s > 2$.

Proof. Put $\mathfrak{a}$ be a proper R_f-ideal such that $N(\mathfrak{a}) = N_0$ is coprime to Nf. Then via Proposition 3.4, we get

$$\begin{aligned} E_{\mathfrak{a}}^*(\psi_f, u, s) &= \psi_f(\mathfrak{a}) N_0^s \sum_{(c, d) \in x\mathfrak{a}_N} \psi_f((d)) \left(\frac{v}{|cNz+d|^2 + |cNv|^2} \right)^s \\ &= \psi_f(\mathfrak{a}) N_0^s \sum_{f' | f} \psi^2(f') f'^{-2s} \Bigg\{ \sum_{\mathfrak{a}'} \psi_{f/f'}(\mathfrak{a}') N(\mathfrak{a}')^{-s} \\ &\quad \times \psi_{f/f'}(\mathfrak{a}') N(\mathfrak{a}')^s \sum_{(c, d) \in X f/f' N(\mathfrak{a}')} \psi_{f/f'}((d)) \left(\frac{v}{|cNz+d|^2 + |cNv|^2} \right)^s \Bigg\}, \end{aligned}$$

where $\mathfrak{a}'$ runs through all the proper $R_{f/f'}$-ideals in $R_{f/f'}\mathfrak{a}$ satisfying $(N(\mathfrak{a}'), f/f') = (N(\mathfrak{a}'), N) = 1$. Hence we obtain

$$E_{\mathfrak{a}}^*(\psi_f, u, s) = \psi_f(\mathfrak{a}) N_0^S \sum_{f' | f} \psi^2(f') f'^{-2s} 2 \left\{ \sum_{\mathfrak{a}'} \psi_{f/f'}(\mathfrak{a}') N(\mathfrak{a}')^{-s} E_{\mathfrak{a}'}(\psi_{f/f'}, u, s) \right\}.$$

Let $\{\mathfrak{a}_k'; 1 \leqq k \leqq h'\}$ $(h' = h(d_0(f/f')^2))$ be a complete system of representatives of the ideal classes in $H(d_0(f/f')^2)$ such that $(N(\mathfrak{a}_k'), N(f/f')) = 1$ for $k = 1, 2, \cdots, h'$ and put, for a proper R_m-ideal $\mathfrak{a}''$,

$$L_{d_0 m^2}(s, \mathfrak{a}'', \psi_m) = \sum_{\mathfrak{a}^*} \psi_m(\mathfrak{a}^*) N(\mathfrak{a}^*)^{-s} \qquad (N(\mathfrak{a}^*) = [R_m : \mathfrak{a}^*],\ \psi_m(\mathfrak{a}^*) = \psi(N(\mathfrak{a}^*))),$$

where the sum is extended over all the R_m-proper ideals $\mathfrak{a}^*$ satisfying the conditions $\mathfrak{a}^* \sim \mathfrak{a}''$ and $(N(\mathfrak{a}^*), m) = 1$. Then by the fact that the Eisenstein series $E_{\mathfrak{a}'}^*(\psi_{f/f'}, u, s)$ depends only on the class $\{\mathfrak{a}'\}$, we have

$$\sum_{\mathfrak{a}'}\psi_{f/f'}(\mathfrak{a}')N(\mathfrak{a}')^{-s}E_{\mathfrak{a}'}(\psi_{f/f'},u,s)=\sum_{k=1}^{h'}\sum_{\substack{\mathfrak{a}'\sim\mathfrak{a}_k'\\ \mathfrak{a}'\subset R_{f/f'}\mathfrak{a}}}\psi_{f/f'}(\mathfrak{a}')N(\mathfrak{a}')^{-s}E_{\mathfrak{a}_k'}(\psi_{f/f'},u,s)$$

$$=\psi_{f/f'}(\mathfrak{a}_0')N(\mathfrak{a}_0')^{-s}\sum_{k=1}^{h'}L_{\mathfrak{d}_0(f/f')^2}(s,\mathfrak{a}_k'\mathfrak{a}_0'^{-1},\psi_{f/f'})$$

$$\times E_{\mathfrak{a}_k'}(\psi_{f/f'},u,s)\qquad(\mathfrak{a}_0=R_{f/f'}\mathfrak{a}).$$

From the equalities $N(\mathfrak{a}_0')=N(\mathfrak{a})=N_0$ and $\psi_{f/f'}(\mathfrak{a}_0')=\psi_f(\mathfrak{a})$, it follows that

$$E_{\mathfrak{a}}^*(\psi_f,u,s)=2\sum_{f'|f}\psi^2(f')f'^{-2s}\sum_{k=1}^{h'}L_{\mathfrak{d}_0(f/f')^2}(s,\mathfrak{a}_k'(R_{f/f}\mathfrak{a})^{-1},\psi_{f/f'})E_{\mathfrak{a}_k'}(\psi_{f/f'},u,s).$$

Hence we have

$$\sum_{i=1}^{h}\chi(\mathfrak{a}_i)E_{\mathfrak{a}_i}^*(\psi_f,u,s)=2\sum_{f'|f}\psi(f')f'^{-2s}\sum_{j,k=1}^{h'}\sum_{\substack{1\leqq i\leqq h\\ R_{f/f'}\mathfrak{a}_i\sim\mathfrak{a}_j'}}\chi(\mathfrak{a}_i)$$

$$\times L_{\mathfrak{d}_0(f/f')^2}(s,\mathfrak{a}_k'\mathfrak{a}_j'^{-1},\psi_{f/f'})E_{\mathfrak{a}_k'}(\psi_{f/f'},u,s).$$

If $(f',N)>1$, then $\psi(f')=0$. If $(f',N)=1$ and $f'>1$, then, by the condition (3.5), for any j $(1\leqq j\leqq h')$ we have

$$\sum_{\substack{1\leqq i\leqq h\\ R_{f/f'}\mathfrak{a}_i\sim\mathfrak{a}_j'}}\chi(\mathfrak{a}_i)=0.$$

Namely, we have

$$\psi(f')\sum_{\substack{1\leqq i\leqq h\\ R_{f/f'}\mathfrak{a}_i\sim\mathfrak{a}_j'}}\chi(\mathfrak{a}_i)=0\qquad\text{for}\quad 1<f'|f.$$

Thus we have proved the equality (3.7).

It is not difficult to prove the convergence of left hand side of (3.7).

For a character ω of a group, put

$$\delta^*(\omega)=\begin{cases}1 & (\omega\text{ is the trivial character}),\\ 0 & (\text{otherwise}).\end{cases}$$

For $x\in\mathbf{Z}$, put

$$\delta(x)=\begin{cases}1 & (x=0),\\ 0 & (x\neq 0).\end{cases}$$

Further we denote by $r(N)$ the number of elements of the set $(\mathbf{Z}/N\mathbf{Z})^\times$.

In the same manner as in proving [7, Lemma 3.4], we can show

THEOREM 3.6. *Let K, R_f and ψ_f be as above. For a proper R_f ideal $\mathfrak{a}$ whose norm N_0 is coprime to Nf, put*

$$E_{\mathfrak{a}}^{*}(\psi_f, u, s)=\bar{\psi}_f(\mathfrak{a})N_0^s \sum_{c,d\in\mathfrak{a}} \psi_f((d)) \left(\frac{v}{|cNz+d|^2|cNv|^2}\right)^s$$

and

(3.8) $$\Psi_{\mathfrak{a}}(\psi_f, u, s)=\pi^{-s}\Gamma(s)\left|\frac{D}{4}\right|^{s/2} E_{\mathfrak{a}}^{*}(\psi_f, u, s).$$

Then the following assertions hold:

i) $\Psi_{\mathfrak{a}}(\omega_f, u, s)$ *can be continued to the whole s-plane as a holomorphic function except possibly for simple poles at* $s=2$ *and* $s=0$. *The possible residues are equal to* $r(N)\delta^*(\psi_f)$ *and* $-\delta(N-1)$ *respectively;*

ii) *if* $N \nmid d_0 f^2$ *and* ψ^2 *is non-trivial or if* $N=1$ *(and* ψ *must be the tivial character), then the equation*

$$N^{3s/2}\Psi_{\mathfrak{a}}(\psi_f, u, s)=(AN^{-1})N^{3(2-s)/2}\Psi_{\mathfrak{a}}(\bar{\psi}_f, \tau_N(u), 2-s)$$

holds, where

$$A=A(\psi_f)=\sum_{x\in R_f/NR_f} \psi_f(xR_f)\exp\left(2\pi i\frac{x-\bar{x}}{N\sqrt{D}}\right)$$

and

$$\tau_N=\begin{pmatrix} 0 & \sqrt{N}^{-1} \\ -\sqrt{N} & 0 \end{pmatrix}\in SL_2(\mathbf{C}).$$

From this Theorem we have immediately

COROLLARY. *Put*

(3.9) $$\Psi(\chi, \psi_f, u, s)=\sum_{i=1}^{h} \chi(\mathfrak{a}_i)\Psi_{\mathfrak{a}_i}(\psi_f, u, s) \qquad (h=h(D))$$

for a character χ *of* $H(D)$. *Then* $\Psi(\chi, \psi_f, u, s)$ *can be continued to the whole s-plane as a holomorphic function except possibly for simple poles at* $s=2$ *and* $s=0$, *and the possible residues are* $\delta^*(\psi_f)\delta^*(\chi)r(N)h(D)$ *and* $-\delta(N-1)\delta^*(\chi)h(D)$ *respectively. Further if* N *is a prime such that* $N \nmid d_0 f^2$ *and* ψ^2 *is a non-trivial Dirichlet character modulo* N, *or if* $N=1$ *(and* ψ *must be the trivial character), then we have the functional equation*

(3.10) $$N^{3s/2}\Psi(\chi, \psi_f u, s)=(AN^{-1})N^{3(s-2)/2}\Psi(\bar{\chi}, \bar{\psi}_f, \tau_N(u), 2-s).$$

Note that if either $N>1$, ψ_f is non-trivial or χ is non-trivial, then $\Psi(\chi, \psi_f, u, s)$ becomes an entire function of s. For the case that $N=1$ (and ψ must be the trivial character), Andrianov proved Eq. (3.10) in [2]. In this case , if we put

(3.11) $$\Psi(\chi, u, s)=\Psi(\chi, \psi_f, u, s),$$

then we have

(3.12) $$\Phi(\chi, u, s)=\Phi(\bar{\chi}, u, 2-s).$$

From the definitions of $R_F(\chi, s)$ ((2.12)), $E_{\mathfrak{a}}(\psi_f, u, s)$ ((3.3)), $E^*_{\mathfrak{a}}(\psi_f, u, s)$ ((3.7)), $\Psi_{\mathfrak{a}}(\psi_f, u, s)$ ((3.8)), $\Psi(\chi, \psi_f, u, s)$ ((3.9)) and from Propositions 3.2 and 3.5, we have

$$(3.13)\qquad (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L_D(s-k+2, \chi\psi_f)R_F(\chi, s)$$
$$=c^*\int_{D_0} v(u)^k F_0(u)\Psi(\chi, \psi_f, u, s-k+2)\frac{du}{v^3}\left(c^*=\frac{\pi^{2-k}}{2}\left|\frac{D}{4}\right|^{(k-3)/2}\right)$$

for any form $F\in S_k(\Gamma_0, \psi)$ and any character χ of $H(d_0f^2)$ satisfying the condition (3.5).

§4. Proof of the main theorems

In this section, we assume that N is a prime. Let

$$(4.1)\qquad F(Z)=\sum_{T\in P_2^+} a(T)\exp(2\pi i\sigma(TZ))\in S_k(\Gamma_0, \psi)$$

be a common eigen function for $T(m)_\psi((m, N)=1)$:

$$F|T(m)_\psi=\lambda(m)F \qquad ((m, N)=1) \qquad (\text{see } (1.6)).$$

If F is not identically zero, then we can find an integer $D=d_0f^2$ ($d_0\in\mathbf{Z}, f\in\mathbf{N}$) such that

(4.2) there esists a primitive matrix $T\in P_2^+$ satisfying $D(T)$ $(=-4|T|)=D$ and the series $R_F(T, s)\not\equiv 0$ (see (2.11));

(4.3) d_0 is the discriminant of the field $K=\mathbf{Q}(\sqrt{d_0})$, and for any integer $f'>1, f'|f$ and for any matrix $T_0\in P_2^+$ such that $D(T_0)=d_0(f/f')^2$, the series $R_F(T_0, s)\equiv 0$.

Let $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_h$ ($h=h(D)$) be a complete system of representatives for proper $R_f(K)$-ideal classes such that $(N(\mathfrak{a}_i), Nf)=1$ ($i=1, 2, \cdots, h$) (see Proposition 3.4), χ a character of $H(D)$. Then, by the condition (4.3), $\Phi_F(s, \chi, \psi, n)$ should be a constant:

$$(4.4)\qquad \Phi_F(s, \chi, \psi, n)=a(n;\chi) \qquad (\text{see } (2.8) \text{ and } (2.10)).$$

Since $R_F(T, s)=\sum_{m=1}^{\infty} a(mT)m^{-s}\not\equiv 0$ $(D(T)=d_0f^2)$, we can choose an integer $\nu\geqq 0$ such that

$$\sum_{(m,N)=1} a(mN^\nu T)m^{-s}\not\equiv 0.$$

From the properties of Hecke operators, for cusp forms $F|T(N^\nu)_\psi$, $F|T(N^{\nu+1})_\psi\in S_k(\Gamma_0, \psi)$, we can show equalities

$$(4.5)\qquad \begin{cases}(F|T(N^\nu)_\psi)|T(m)_\psi=\lambda(m)(F|T(N^\nu)_\psi) & ((m, N)=1),\\ (F|T(N^{\nu+1})_\psi)|T(m)_\psi=\lambda(m)(F|T(N^{\nu+1})_\psi) & ((m, N)=1).\end{cases}$$

Therefore, by using Proposition 2.1 and the equality (4.4), we get

$$\begin{cases} L_D(s-k+2,\ \chi\psi_f)R_{F|T(N^\nu)_\psi}(\chi,\ s)=Z_F^N(s,\ \psi)\sum_{\delta=0}^{\infty} a(N^{\nu+\delta};\ \chi)N^{-\delta s}, \\ L_D(s-k+2,\ \chi\psi_f)R_{F|T(N^{\nu+1})_\psi}(\chi,\ s)=Z_F^N(s,\ \psi)\sum_{\delta=0}^{\infty} a(N^{\nu+\delta+1};\ \chi)N^{-\delta s}. \end{cases}$$

Since

(4.7) $$R_{F|T(N^\nu)_\psi}(\chi,\ s)-N^{-s}R_{F|T(N^{\nu+1})\psi}(\chi,\ s)=\sum_{(m,N)=1} a\ (mN^\nu;\ \chi)m^{-s}$$

holds, we have

(4.8) $$L_D(s-k+2,\ \chi\psi_f)\sum_{(m,N)=1} a(mN^\nu;\ \chi)m^{-s}=Z_F^N(s,\ \psi)a(N^\nu;\ \chi).$$

By the condition (4.5), we can choose a character χ of $H(D)$ satisfying

(4.9) $$\sum_{(m,N)=1} a(mN^\nu;\ \chi)m^{-s}\left(=\sum_{i=1}^{h}\left(\sum_{(m,N)=1} a(mN^\nu T(\mathfrak{a}_i))m^{-s}\right)\chi(\mathfrak{a}_i)\right)\not\equiv 0$$

(see (2.8)), and then

(4.10) $$a(N^\nu;\ \chi)\neq 0.$$

Via property IV) of $\Pi(p)$ $(p|f)$ and [2, Lemma 3.8.1], we get the following

PROPOSITION 4.1. *Suppose that* $D=d_0f^2$ *satisfies* (4.2) *and* (4.3), *and* χ *is chosen so that* (4.9) *holds. Then the character* χ *satisfies the condition* (3.5).

Let χ satisfy (4.9) (so that (3.5)). Then we proved in § 3, for any cusp form $F\in S_k(\Gamma_0,\ \psi)$, the equality holds:

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L_D(s-k+2,\ \chi\psi_f)R_F(\chi,\ s)$$
$$=c^*\int_{D_0} v(u)^k F_0(u)\Psi(\chi,\ \psi_f,\ u,\ s-k+2)\frac{du}{v^3}\left(c^*=\left(\frac{n^{2-k}}{2}\right)\left|\frac{D}{4}\right|^{(k-3)/2}\right)$$

(see (3.13)). Take $F|T(N^\nu)_\psi$, $F|T(N^{\nu+1})_\psi$ for F. Then

$$c^*\int_{D_0} v(u)^k\{(F|T(N^\nu)_\psi)_0(u)-N^{-s}(F|T(N^{\nu+1})_\psi)_0(u)\}\Psi(\chi,\ \Psi_f,\ u,\ s-k+2)\frac{du}{v^3}$$
$$=(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L_D(s-k+2)(R_{F|T(N^\nu)_\psi}(\chi,\ s)-N^{-s}R_{F|T(N^{\nu+1})\psi}(\chi,\ s)).$$

Hence according to (4.7), we obtain

$$c^*\int_{D_0} v(u)^k\{(F|T(N^\nu)_\psi)_0(u)-N^{-s}(F|T(N^{\nu+1})_\psi)_0(u)\}\Psi(\chi,\ \psi_f,\ u,\ s-k+2)\frac{du}{v^3}$$
$$=(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L_D(s-k+2,\ \chi\psi_f)\sum_{(m,N)=1} a(mN^\nu;\ \chi)m^{-s}.$$

Therefore we have, by using the equality (4.8),

$$a(N^{\nu};\chi)(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F^N(s,\psi)$$

$$=c^*\int_{D_0} v(u)^k\{(F\mid T(N^{\nu})_{\psi})_0(u)-N^{-s}(F\mid T(N^{\nu+1})_{\psi})_0(u)\}\Psi(\chi,\psi_f,u,s-k+2)\frac{du}{v^3}.$$

Note that $a(N;\chi)\neq 0$ ((4.10)) and $F|T(N^{\nu})_{\psi}$, $F|T(N^{\nu+1})_{\psi}$ are elements in $S_k(\Gamma_0,\psi)$. From the equalities (3.2), the defnitions (2.16) of $Z_{\lambda}^N(s,\psi)$, (3.9) of $\Psi(\chi,\psi_f,u,s)$ and Theorem 3.6, we get

THEOREM 4.2. *The function* $(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)\ Z_{\lambda}^N(s,\psi)$ *can be continued meromorphically to the whole s-plane and has possibly simple poles at* $s=k$ *and* $s=k-2$.

If $F\in S_k(\Gamma_0,\psi)$ has the Fourier expansion of the form (4.1), and we put $W=\begin{pmatrix} & E_2\\ -NE_2 & \end{pmatrix}\in GSP_2(\mathbf{R})$, then $\hat{F}=F|[W]_k$ is an element in $S_k(\Gamma_0,\bar{\psi})$. We denote its Fourier expansion by

$$\hat{F}(Z)=\sum_{T\in P_2^+}\hat{a}(T)\exp(2\pi i\sigma(TZ)).$$

Further, for any positive integer m with the condition $(m,N)=1$, we have $\hat{F}|T(m)_{\psi}=\bar{\psi}^2(m)\lambda(m)\hat{F}$ if $\hat{F}|T(m)_{\psi}=\lambda(m)F$. We consider the eigen values $\{\lambda(m);(m,N)=1\}$ of a cusp form in the space $S_k(\Gamma_0,\psi)$. Put

$$S_k(\Gamma_0,\psi,\lambda)=\{F\in S_k(\Gamma_0,\psi);\ F|T(m)_{\psi}=\lambda(m)F\ (m,N)=1\}.$$

It is easy to see that $S_k(\Gamma_0,\psi,\lambda)$ is a subspace of $S_k(\Gamma_0,\psi)$. Let $\{F_1,F_2,\cdots,F_l\}$ ($l=\dim_{\mathbf{C}} S_k(\Gamma_0,\psi,\lambda)$) be a basis of $S_k(\Gamma_0,\psi,\lambda)$ and the Fourier expansion of F_j be

$$(F_j(Z))=\sum_{T\in P_2^+}a_j(T)\exp(2\pi i\sigma(TZ))\qquad(1\leqq j\leqq l).$$

Further, we put $\mathbf{F}=(F_1,F_2,\cdots,F_l)$ and $\mathbf{F}|T(m)_{\psi}=(F_1|T(m)_{\psi},\ F_2|T(m)_{\psi},\cdots,F_l|T(m)_{\psi})$ $(m\in\mathbf{N})$ and

$$\mathbf{a}(m;\chi)=(a_1(m;\chi),a_2(m;\chi),\cdots,a_l(m;\chi))$$

for a positive integer m (see (2.8)). Then by the definitions (2.13) of $Z_F^N(s,\psi)$, (2.16) of $Z_{\lambda}^N(s,\psi)$, we get

$$Z_{F_1}^N(s,\psi)=Z_{F_2}^N(s,\psi)=\cdots=Z_{F_l}^N(s,\psi)=Z_{\lambda}^N(s,\psi).\tag{4.11}$$

Since $T(m)_{\psi}T(N)_{\psi}=T(N)_{\psi}T(m)_{\psi}$ for $(m,N)=1$, the space $S_k(\Gamma_0,\psi,\lambda)$ is invariant by the action of the operator $T(N)_{\psi}$. Hence there exists a matrix $U_{\psi}\in M_l(\mathbf{C})$ such that

$$\mathbf{F}|T(N)_{\psi}=\mathbf{F}U_{\psi}.\tag{4.12}$$

Further, it is easy to show that $\{\hat{F}_1,\hat{F}_2,\cdots,\hat{F}_l\}$ is a basis for $S_K(\Gamma_0,\bar{\psi},\bar{\psi}^2\lambda)$. If $S_k(\Gamma_0,\psi,\lambda)\not\equiv\{0\}$, then we can choose an integer $D=d_0f^2$ (d_0 is the discriminant of the imaginary quadratic field $\mathbf{Q}(\sqrt{d_0})$ and f is a positive integer), a character χ of

$H(D)$ and an integer $v \geqq 0$ such that

(4.2)′ there exist a primitive matrix $T \in P_2^+$ satisfying

$D(T)=D$ and $R_{F_{j_0}}(T, s) \not\equiv 0$ for some j_0 $(1 \leqq j_0 \leqq l)$;

(4.3)′ $R_{F_j}(T_0, s)=R_{\hat{F}_j}(T_0, s) \equiv 0$ for all $j=1, 2, \cdots, l$ and all primitive matrix $T_0 \in P_2^+$ with $D(T_0)=d_0(f/f')^2$ $(1<f', f' | f)$

(if $R_{\hat{F}_j}(T_0, s) \not\equiv 0$ for some j, then we consider by changing roles between $S_k(\Gamma_0, \psi, \lambda)$ and $S_k(\Gamma_0, \bar{\psi}, \bar{\psi}^2\lambda))$, and

$$\mathbf{a}(N^v; \chi) \neq 0 \qquad (\text{see } (4.10)).$$

Compare the Fourier coefficients of $\mathbf{F} | T(N)_\psi^v$ and $\mathbf{F} U_\psi^v$. Then from the definition (4.12) of U_ψ and the equality [7, (2.5)], we can see

$$\mathbf{a}(N^v; \chi)=\mathbf{a}(1; \chi) U_\psi^v.$$

Therefore, we get

$$\mathbf{a}(1; \chi) \neq 0. \tag{4.10$'$}$$

It follows from the equality (2.10) and the condition (4.3)′ that equalities

$$\begin{cases} \Phi_{F_j}(s, \chi, \psi, N^\delta)=a_j(N^\delta; \chi), \\ \Phi_{\hat{F}_j}(s, \chi, \psi, N^\delta)=\hat{a}_j(N^\delta; \chi), \end{cases} \tag{4.13}$$

$(j=1, 2, \cdots, l$ and $\delta=0, 1, 2, \cdots)$ hold. Put

$$\mathbf{F}=(F_1, F_2, \cdots, F_l),$$

and

$$R_{\mathbf{F}}(\chi, s)=(R_{F_1}(\chi, s), R_{F_2}(\chi, s), \cdots, R_{F_l}(\chi, s)),$$

and define a matrix $\hat{U}_\psi \in M_l(\mathbf{C})$ by $\hat{\mathbf{F}} | T(N)_\psi=\mathbf{F}\hat{U}_\psi$. For the Fourier coefficients $\hat{a}_j(T)$ of $\hat{F}_j$ $(j=1, 2, \cdots, l)$, we set

$$\hat{a}_j(m; \chi)=\sum_{i=1}^{h} \hat{a}_j(T(\mathfrak{a}_i))\chi(\mathfrak{a}_i)$$

(see (2.8)). Further we put $\hat{\mathbf{a}}(m; \chi)=(\hat{a}_1(m; \chi), \hat{a}_2(m; \chi), \cdots, \hat{a}_l(m; \chi))$ for a positive integer m. Then we have $\mathbf{a}(mN; \chi)=\mathbf{a}(m; \chi)U_\psi$, $\hat{\mathbf{a}}(mN; \chi)=\hat{\mathbf{a}}(m; \chi)\hat{U}_\psi$ and hence

$$\begin{cases} R_{\mathbf{F} | T(N)_\psi}(\chi, s)=R_{\mathbf{F}}(\chi, s)U_\psi \\ R_{\mathbf{F} | T(N)_{\bar{\psi}}}(\chi, s)=R_{\mathbf{F}}(\chi, s)\hat{U}_{\bar{\psi}} \end{cases}$$

(see [7, (2.5)] and (2.12)).

From Proposition 2.1 and the first equality of (4.13), we can see

$$\begin{cases} L_D(s-k+2, \chi\psi_f)R_{F_j}(\chi, s)=\left\{\sum_{\delta=1}^{\infty} a_j(N^{\delta}; \chi)N^{-\delta s}\right\}Z_{\lambda}^N(s, \psi), \\ L_D(s-k+2, \chi\psi_f)R_{F_j|T(N)_{\psi}}(\chi, s)=\left\{\sum_{\delta=1}^{\infty} a_j(N^{\delta+1}; \chi)N^{-\delta s}\right\}Z_{\lambda}^N(s, \psi) \end{cases}$$

for $j=1, 2, \cdots, l$. Hence we get

$$L_D(s-k+2, \chi\psi_f)\{R_{\mathbf{F}}(\chi, s)-N^{-s}R_{\mathbf{F}|T(N)_{\psi}}(\chi, s)\}=\mathbf{a}(1; \chi)Z_{\lambda}^N(s, \psi)$$

Namely we have

$$L_D(s-k+2, \chi\psi_f)\{R_{\mathbf{F}}(\chi, s)(E_l-N^{-2}U_{\psi})\}=\mathbf{a}(1; \chi)Z_{\lambda}^N(s, \psi)\,. \tag{4.14}$$

Further, by Proposition 2.1 and second equality of (4.13), we can prove

$$L_D(s-k+2, \bar{\chi}\bar{\psi}_f)\{R_{\hat{\mathbf{F}}}(\bar{\chi}, \check{s})(E_l-N^{-s}U_{\psi})\}=\hat{\mathbf{a}}(1; \bar{\chi})Z_{\psi^2\lambda}^N(\check{s}, \bar{\psi}) \tag{4.15}$$

($\check{s}=2k-2-s$). From (3.13), we have

$$\begin{aligned}(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L_D(s-k+2, \chi\psi_f)R_{\mathbf{F}}(\chi, s)\\ =c^*\int_{D_0} v(u)^k\mathbf{F}_0(u)\Psi(\chi, \psi_f, u, s-k+2)\frac{du}{v^3},\end{aligned} \tag{4.16}$$

where $\mathbf{F}_0=((F_1)_0, (F_2)_0, \cdots, (F_l)_0)$, and

$$\begin{aligned}(2\pi)^{-2s}\Gamma(\check{s})\Gamma(\check{s}-k+2)L_D(\check{s}-k+2, \bar{\chi}\bar{\psi}_f)R_{\hat{\mathbf{F}}}(\bar{\chi}, \check{s})\\ =c^*\int_{D_0} v(u)^k(\hat{\mathbf{F}})_0(u)\Psi(\bar{\chi}, \bar{\psi}_f, u, \check{s}-k+2)\frac{du}{v^3}.\end{aligned} \tag{4.17}$$

First we consider the case that the conductor of ψ is N and ψ^2 is non-trivial. In this case, we have the following

THEOREM 4.3. *Let ψ be a primitive character modulo N with ψ^2 non-trivial and put*

$$\Phi^N(s, \lambda, \psi)=(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_{\lambda}^N(s, \psi)\,.$$

Suppose that

(C1) *N does not divide d_0f^2, the discriminant of $R_f(\mathbf{Q}(\sqrt{d_0}))$, where $D=d_0f^2$ satisfies the condition (4.2)′ and (4.3)′.*

Then we have the following equation:

$$\begin{aligned}\mathbf{a}(1; \chi)N^{3s/2}\Phi^N(s, \lambda, \psi)(E_l-N^{-s}U_{\psi})^{-1}\\ =\psi(-1)\omega\hat{\mathbf{a}}(1; \chi)N^{3\check{s}/2}\Phi^N(\check{s}, \bar{\psi}^2\lambda, \bar{\psi})(E_l-N^{-\check{s}}\hat{U}_{\bar{\psi}})^{-1},\end{aligned} \tag{4.18}$$

where $\check{s}=2k-2-s$ and

$$\omega=N^{-1}\left\{\sum_{x\in R_f/NR_f}\psi(|x|^2)\exp\left(2\pi i\,\frac{x-\bar{x}}{Nf\sqrt{d_0}}\right)\right\}.$$

(*Note that* $\mathbf{a}(1;\chi)\neq(0,\cdots,0)$ (*see* (4.10)$'$).)

Proof. If we note that

$$v(\tau_N(u))^k\mathbf{F}_0(\tau_N(u))=\psi(-1)v^k(\hat{\mathbf{F}})_0(u)\ \left(\tau_N=\begin{pmatrix} & \sqrt{N}^{-1}\\ -\sqrt{N} & \end{pmatrix}\right)$$

and that $\tau_N(D_0)$ is another fundamental domain for $\Gamma(R_f)$, then by Theorem 3.6, we get

$$\int_{D_0}v(u)^k\mathbf{F}_0(u)\Psi(\chi,\psi_f,u,s-k+2)\frac{du}{v^3}$$
$$=\psi(-1)\omega\int_{D_0}v(u)^k(\hat{\mathbf{F}})_0(u)\Psi(\bar{\chi},\bar{\psi}_f,u,\check{s}-k+2)\frac{du}{v^3}$$

for $\mathbf{F}=(F_1,F_2,\cdots,F_l)$. Hence from equalities (4.14), (4.15), (4.16) and (4.17), we have the equality (4.18).

Next we consider the case that $\psi=\phi_0$ is the trivial Dirichlet character modulo N. We can easily prove

PROPOSITION 4.4. *If N is a prime which remains prime in $\mathbf{Q}(\sqrt{d_0})/\mathbf{Q}$, then*

$$\Psi(\chi,(\phi_0)_f,u,s)=N^{-s}\{\Psi(\chi,\tau_N(u),s)-N^{-s}\Psi(\chi,u,s)\},$$

where $\Psi(\chi,u,s)$ is defined by (3.11).

If $F\in S_k(\Gamma_0,\phi_0,\lambda)$, then $\hat{F}\in S_k(\Gamma_0,\phi_0,\lambda)$. Hence we can find a matrix U_0 such that $\hat{\mathbf{F}}=\mathbf{F}U_0$ with $U_0^2=E_l$. Finally we have

THEOREM 4.5. *Let the notation be as above and put*

$$\Phi^N(s,\lambda)=(2\pi)^{-2s}N^s\Gamma(s)\Gamma(s-k+2)Z_\lambda^N(s,\phi_0)(E_l-N^{-s}U_{\phi_0})^{-1}(E_l-N^{-(s-k+2)}U_0)^{-1}$$

Suppose that

C2) *N remains prime in $\mathbf{Q}(\sqrt{D})/\mathbf{Q}$, where $D=d_0f^2$ satisfies the condition* (4.2)$'$ *and* (4.3)$'$ *for some f.*

Then the following equation holds:

$$\mathbf{a}(1;\chi)\Phi^N(s,\lambda)=(-1)^k\mathbf{a}(1;\chi)\Phi^N(\check{s},\lambda)\qquad(\check{s}=2k-2-s).\tag{4.19}$$

Proof. Put

$$I(s,\chi)=N^{s-k+2}\int_{D_0}v(u)^k\mathbf{F}_0(u)\Psi(\chi,(\phi_0)_f,u,s-k+2)\frac{du}{v^3}$$

Then from (4.14), (4.16) and the definition of $\Phi^N(s, \lambda)$, we have

$$cI(s, \chi)=\mathbf{a}(1; \chi)N^s\Phi^N(s, \lambda)(E_l-N^{-s}U_{\phi_0})^{-1} \tag{4.20}$$

where $c=(\pi/N)^{2-k}|D/4|^{(k-3)/2}/2$. Further according to Proposition 4.4, we get

$$I(s, \chi)=\int_{D_0} v(u)^k\mathbf{F}_0(u)\Psi(\chi, \tau_N(u), s-k+2)\frac{du}{v^3}$$
$$-N^{-(s-k+2)}\int_{D_0} v(u)^k\mathbf{F}_0(u)\Psi(\chi, u, s-k+2)\frac{du}{v^3}.$$

Since $v(\tau_N(u)^k\mathbf{F}_0(\tau_N(u))=v(u)^k(\hat{\mathbf{F}})_0(u)=v(u)^k\mathbf{F}_0U_0$, we can easily see

$$I(s, \chi)=\int_{D_0} v(u)^k\mathbf{F}_0(u)\Psi(\chi, \tau_N(u), s-k+2)\frac{du}{v^3}(E_l-N^{-(s-k+2)}U_0). \tag{4.21}$$

Further we get

$$\Psi(\chi, u, s-k+2)=\Psi(\check{\chi}, u, \check{s}-k+2) \tag{4.22}$$

by (3.12). Therefore we obtain

$$I(\check{s}, \check{\chi})=\int_{D_0} v^k\mathbf{F}_0(u)\Psi(\chi, \tau_N(u), s-k+2)\frac{du}{v^3}(E_l-N^{-(\check{s}-k+2)}U_0). \tag{4.23}$$

From the equalities (4.20), (4.21), (4.23) and the fact $\mathbf{a}(1; \check{\chi})=(-1)^k\mathbf{a}(1; \chi)$, finally we get (4.19).

§ 5. Examples

We shall exhibit some examples to make clear the meaning of our theorems.

Example 1. Put

$$\eta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{12},\ \theta(z)=\sum_{n=-\infty}^{\infty}q^{n^2},$$

$q=\exp(2\pi iz)$ and $f(z)=\theta(z)^{-1}\sqrt{\eta(2z)}$. Then

$$f(z)=a_1q+a_2q^2+a_3q^3+\cdots$$

is an elliptic cusp form of half integral weight in $\mathfrak{S}_{11}(\Gamma_0(4))$ (see [4]) and, for Hecke operators $T_{11}(m^2)$ on $\mathfrak{S}_{11}(\Gamma_0(4))$,

$$f\,|\,T_{11}(m^2)=\omega_m f \qquad (m=1, 2, 3, \cdots).$$

For $T=\begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix}\in P_2^+$, set

$$a(T)=\sum_{m|(a,b,c)}\phi_0(m)a_{(4ac-b^2)/m^2}.$$

Then, by Kojima [4, Theorem 3] and Ibukiyama [5],

$$\psi(f)(Z)=\sum_{T\in P_2^+} a(T)\exp(2\pi i\sigma)(TZ))$$

is the cusp form in $S_6(\Gamma_0(2),\phi_0)$. Note that Ibukiyama showed $\dim S_6(\Gamma_0(2),\phi_0)=1$ in [5]. So that we have

$$\psi(f)\,|\,T(m)_{\phi_0}=\lambda(m)\psi(f) \qquad (m=1,2,\cdots),$$
$$\widehat{\psi(f)}=\psi(f).$$

Further we can show $a\left(\begin{pmatrix}1 & 1/2\\ 1/2 & 1\end{pmatrix}\right)=a_3=-8\neq 0$ and 2 remains prime in $\mathbf{Q}(\sqrt{-3})$.

Put

$$\Phi(s)=(2\pi)^{-s}\Gamma(s)\Gamma(s-4)2^s(1-2^{-(s-4)})^{-1}(1-\lambda(2)2^{-s})^{-1}Z_\lambda^2(s,\phi_0).$$

Then, by Theorem 4.5, we get

(5.1) $$\Phi(s)=\Phi(10-s).$$

N.B. In [4], it was shown that

$$\begin{aligned}Z_F(s,\phi_0)&=(1-\lambda(2)2^{-s})^{-1}Z_\lambda^2(s,\phi_0)\\&=(1-2^{-(s-5)})(1-2^{-(s-4)})\zeta(s-5)\zeta(s-4)\\&\quad\times\prod_p(1-\omega_p p^{-s}+\phi_0(p)p^{2k-3-s})^{-1}\end{aligned}$$

(see (2.17)).

Equation (5.1) can also be proved by using this product.

Example 2. Put

$$S_1=\begin{pmatrix}2&1&0&0\\1&6&0&0\\0&0&2&1\\0&0&1&6\end{pmatrix},\quad S_2=\begin{pmatrix}4&0&2&1\\0&4&1&-2\\2&1&4&0\\1&-2&0&4\end{pmatrix},\quad S_3=\begin{pmatrix}2&0&1&0\\0&8&4&3\\1&4&8&7\\0&3&7&8\end{pmatrix}$$

and

$$\tilde{\vartheta}_j(Z)=\sum_{x\in M_{4,2}(\mathbf{Z})}\exp(2\pi i\sigma(x'S_jxZ)) \qquad (Z\in\mathfrak{H}_2,\ j=1,2,3).$$

Then, in his paper [11], Yoshida showed that

$$F(Z)=(3\tilde{\vartheta}_1(Z)+\tilde{\vartheta}_2(Z)-2\tilde{\vartheta}_3(Z))/24$$

is a cusp form in $S_2(\Gamma_0(11),\phi_0)$ and

(5.2) $$F\,|\,T(m)_{\phi_0}=\lambda(m)F$$

for $m\in\mathbf{N}$ with $(m,2\cdot 11)=1$. It is easy to see that if we put

$$U_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad S_2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$S_3=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \in GL_4(\mathbf{Z}),$$

then $11S_j^{-1}=U_j'S_jU_j$ $(j=1, 2, 3)$. Further, we can show

$$\begin{array}{ccc} \{x\in M_{4,2}(\mathbf{Z});\ x'S_jx=T\} & & \{y\in M_{4,2}(\mathbf{Z});\ y'S_jy=11T\} \\ \Downarrow & & \Downarrow \\ x & \longmapsto & y=U_jS_jx \end{array}$$

$(T\in P_2^+, j=1, 2, 3)$ is one-to-one and onto correspondence. By these facts, we get

$$\text{(5.3)} \qquad \begin{cases} \hat{F}=F\,, \\ F\,|\,T(11)_{\phi_0}=F\,. \end{cases}$$

Put

$$F(Z)=\sum_{T\in P_2^+} a(T)\exp(2\pi i\sigma(TZ))\,.$$

Then $a(E)=1\neq 0$. Hence $D=D(E)=-4$ and χ_0 (the trivial character of $H(-4)$) satisfy the conditions (4.2) and (4.3). Let

$$Q(s)=\sum_{j=0}^{4} 2^{-js}a_jT_j\,,$$

where $a_0=a_2=1$, $a_1=-1$, $a_3=-2$, $a_4=2^2$, $T_0=T_4=T(1)$, $T_1=T_3=T(2)$ and $T_2=T(2)^2-T(2^2)-T(1)$, and put

$$(F\,|\,Q(s))(Z)=\sum_{T\in P_2^+} b_s(T)\exp(2\pi i\sigma(TZ))\,.$$

Further set

$$Z_\lambda^*(s, \phi_0)=\prod_{p\neq 2} Q_p^{\phi_0}(p^{-s})^{-1} \qquad \text{(see (1.10))}\,.$$

Since $(F|\,T_j)\,|\,T(m)_{\phi_0}=\lambda(m)F\,|\,T_j$ for $(m, 2)=1$ and, $j=1, 2, 3, 4$, we get, by equations (2.9), (5.2), (5.3) and the definition of b_s,

$$L_{-4}(s,\ \phi_0\mathrm{Norm})\left\{\sum_{j=0}^{4} 2^{-js}a_jR_{F\mid T_j}(X_0,\ s)\right\}$$
$$=Z_\lambda^*(s,\ \phi_0)\left\{\sum_{\delta=0}^{\infty} b_s(2^\delta E)2^{-\delta s}\right\}\prod_{\substack{\mathfrak{p}\mid 2\\ \text{in } \mathbf{Q}(\sqrt{-4})}}\left(1-\frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$
$$=Z_\lambda^*(s,\ \phi_0)a(E)\,.$$

Hence we have, from (3.13),

$$(2\pi)^{-2s}\Gamma(s)^2Z_\lambda^*(s,\ \phi_0)=\sum_{j=0}^{4} 2^{-js}a_j\int_{D_0} v^2(F\mid T_j)_0(u)\Psi(\chi_0,\ \phi_0\mathrm{Norm},\ u,\ s)\,\frac{du}{v^3}\,.$$

Thus we have proved that $(2\pi)^{-2s}\Gamma(s)^2Z_\lambda^*(s,\ \phi_0)$ can be continued holomorphically to the whole s-plane except possibly for simple pole at $s=2$. The functional equation of $Z_\lambda^*(s,\ \phi_0)$ is shown in the following. Put

$$(F\mid Q(\check{s})Q(s))(Z)=\sum_{j,k=0}^{4} 2^{-js-ks}a_ja_k(F\mid T_jT_k)(Z)\qquad(\check{s}=2-s)$$
$$=\sum_{T\in P_2^+} c_s(T)\exp(2\pi i\sigma(TZ))\,.$$

Then we get, by using (2.9),

$$L_{-4}(s,\ \phi_0\mathrm{Norm})\sum_{j,k=0}^{4} 2^{-js-ks}a_ja_kR_{F\mid T_jT_k}(\chi_0,\ s)$$
$$=Z_\lambda^*(s,\ \phi_0)\left\{\sum_{\delta=0}^{\infty} c_s(2^\delta E)2^{-\delta s}\right\}\prod_{\mathfrak{p}\mid 2}(1-(N(\mathfrak{p}))^{-s})^{-s})^{-1}$$
$$=Z_\lambda^*(s,\ \phi_0)b_s(E)\,.$$

Since the equality (4.16) holds and 11 remains prime in $\mathbf{Q}(\sqrt{-4})$, we have

$$(2\pi)^{-2s}\Gamma(s)^2b_{\check{s}}(E)Z_\lambda^*(s,\ \phi_0)=\sum_{j,k=0}^{4} 2^{-js-ks}a_ja_k$$
$$\times\int_{D_0} v^2(F\mid T_jT_k)_0(u)\bar{\Psi}_1(\chi_0,\ \tau_{11}(u),\ s)\,\frac{du}{v^3}\,(1-11^{-s})\,.$$

Further, we can prove

$$b_s(E)=1-2^{-s}+2^{1-2s}-2^{1-3s}+2^{2-4s}$$
$$=(1+2^{1-s}+2^{1-2s})(1-2^{-s})(1-2^{1-s})\,.$$

Therefore, if we put

$$\Phi(s)=(2\pi)^{-s}\Gamma(s)^2 11^s(1-11^{-s})^{-1}(1+2^{1-s}+2^{1-2s})^{-1}$$
$$(1-2^{-s})^{-1}(1-2^{1-s})^{-1}Z_\lambda^*(s,\phi_0),$$

then we get the functional equation

$$\Phi(s)=\Phi(2-s) \tag{5.4}$$

by the same way as Theorem 4.5.

N.B. In [10], the p-factor ($p\neq 11$) of $Z_\lambda^*(s,\phi_0)$ is calculated explicitly. By using this result, it is again possible to get (5.4).

References

[1] ANDRIANOV, A. N.; Dirichlet series with Euler products in the theory of Siegel modular forms of genus 2, *Trudy Mat. Inst. Steklov*, **112** (1971), 73–94; *Proc. Steklov Inst.*, **112** (1971), 70–93.

[2] ANDRIANOV, A. N.; Euler products corresponding to Siegel modular forms of genus 2, *Uspekhi Mat. Nauk*, **29** (3) (1974), 43–110.

[3] FREITAG, E.; Die Invarianz gewisser von Thetareihen erzeugter Vektorräume unter Heckeoperatoren, *Math. Zeit.*, **156** (1977), 141–155.

[4] KOJIMA, H.; Siegel modular forms of degree 2, Symposium of Number Theory in Kinosaki, 1979, pp. 37–53. (in Japanese).

[5] IBUKIYAMA, T.; On symplectic Euler factors of genus 2, *Proc. Japan Acad. Ser. A*, **57** (1981), 271–275.

[6] KUBOTA, T.; Uber diskontinuerliche Gruppen Picardschen Typus und zugehörige Eisensteinsche Reihen, *Nagoya Math. J.*, **32** (1968), 259–271.

[7] MATSUDA, I.; Dirichlet series corresponding to Siegel modular forms of degree 2, level *N*, *Sci. Paper of Coll. Gen. Ed. Univ. Tokyo*, **28** (1978), 21–49.

[8] SHIMURA, G.; On modular correspondences for $Sp(n, \boldsymbol{Z})$ and their congruence relations, *Proc. Nat. Acad. Sci. U.S.A.*, **49** (1963), 824–828.

[9] SHIMURA, G.; *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten and Princeton Univ. Press, 1971.

[10] YOSHIDA, H.; Siegel modular forms and arithmetic of quadratic forms, *Inv. Math.*, **60** (1980), 193–248.

[11] ZHARKOVSKAYA, N. A.; The Siegel operator and Hecke operators, *Funktional Analiz. i Prilozhen*, **8** (2) (1974), 30–38; *Functional Anal. Appl.*, **8** (1974), 113–120.

Faculty of Engineering
Kogakuin University
Nishi Shinjuku
Tokyo 160, Japan