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Prime Submodules of Modules

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Introduction

A proper submodule N of a module M over a ring R is said to be prime (or P-prime) if $re \in N$ for $r \in R$ and $e \in M$ implies that either $e \in N$ or $r \in P = N:M$. In other words, N is a prime submodule if it is a primary submodule whose radical is identical with $P = N:M$. Clearly this is a generalization of the notion of prime ideals of a ring. In the theory of rings, prime ideals play important roles everywhere. But in the theory of modules, especially Noetherian modules, primary submodules are the major characters and a little attention has been paid to prime submodules.

The purpose of this paper is to introduce interesting and useful properties of prime submodules of modules and show various applications of the properties.

In § 1 we establish the most fundamental theorem of prime submodules. It lists seven conditions each of which is equivalent to that a proper submodule of a module is prime. The theorem results in several corollaries stating some basic properties of prime submodules.

In § 2 we first point out that the notion of maximal P-primary submodules and that of maximal P-prime submodules are equivalent. Applying this, we improve a known result ([7]) concerning maximal P-primary submodules and simplify its proof.

Section 3 is devoted to study extended submodules PM of an R-module M for prime ideals P of R. We prove that if M is a faithful Noetherian R-module, then $PM:M = P$ for every prime ideal P; consequently, there exists a P-prime submodule in M. We also prove that if M is a flat R-module and P is a prime ideal of R such that $PM \neq M$, then PM is P-prime; furthermore, QM is a P-primary submodule of M for every P-primary ideal Q of R.

The study of extended submodules is continued to § 4, where we deal with content R-modules M with the content $c(x) = \cap \{A;\ A$ is an ideal of R such that $x \in AM\}$ for each $x \in M$. We prove that if M is a content R-module, then $rc(x) \subseteq \sqrt{c(rx)}$ for every $r \in R$ and $x \in M \Leftrightarrow$ for each $P \in \mathrm{Spec}(R)$, either $PM = M$ or PM is P-prime. In this theorem we characterize, using the notion of prime submodules, a certain class of content R-modules which contains all flat content R-modules.

Section 5 is concerned with prime submodules of Noetherian modules. In [5], it was proved that a finitely generated module M is Noetherian if, and only if, every prime submodule of M is finitely generated. We give a proof of this theorem which is more concise than the one in [5] and introduce its application to prove that if M is a

Noetherian R-module, then $M[\![x]\!]$ is a Noetherian $R[\![x]\!]$-module.

Throughout this paper, all rings are assumed to be commutative rings with identity and all modules will be unitary.

§ 1. Basic properties of prime submodules

A proper submodule N of a module M over a ring R is said to be prime if $re \in N$ for $r \in R$ and $e \in M$ implies that $e \in N$ or $r \in N:M = \mathrm{Ann}_R(M/N)$.

Clearly, every prime ideal of a ring R is a prime submodule of the R-module R and every prime submodule of a module is primary. If N is a prime submodule of an R-module M whose residual $N:M$ by N is P, then P is a prime ideal of R and N is P-primary; under the circumstances, we shall call N a P-prime submodule of M. In the following Result 1-Result 5, we consider simple examples of prime submodules.

RESULT 1. *Every direct summand of a torsion free module is prime. In particular, every proper subspace of a vector space is prime.*

RESULT 2. *A proper submodule N of a torsion free R-module M is a pure submodule if, and only if, it is prime in M with $N:M=(0)$.*

Here by a pure sumbodule of M we mean a submodule N such that $rM \cap N = rN$ for every $r \in R$.

RESULT 3. *The torsion submodule $T(M)$ of a module M over an integral domain is a prime submodule if $T(M) \neq M$.*

RESULT 4. *Le B be an overring of a ring A. Then every prime ideal P of B is a prime submodule of the A-module B with $P:_A B = P \cap A$.*

RESULT 5. (0) *is a prime submodule of an R-module M, if and only if, $\mathrm{Ann}_R(M) = Z_R(M)$, the set of all zero divisors on M.*

PROPOSITION 1. (a) *Let N be a primary submodule of an R-module M. Then N is prime if, and only if, $N:M$ is a prime ideal of R.* (b) *If K is a P-primary submodule of M containing a P-prime submodule, then K is P-prime.*

According to [3], p. 169, Ex. 12, d), every primary ideal of an absolutely flat (von Neumann regular) ring is prime. Hence we have the following.

COROLLARY. *If M is a module over a von Neumann regular ring, then every primary submodule of M is prime.*

THEOREM 1. *Let N be a proper submodule of an R-module M with $N:M=P$. Then the following statements are equivalent:*

(a) *N is a prime submodule of M;*

(b) *M/N is a torsion-free R/P-module;*

(c) *$N:_M(r) = N$ for every $r \in R - P$;*

(d) *$N:_M J = N$ for every ideal $J \nsubseteq P$;*

(e) *$N:_R(e) = P$ for every $e \in M - N$;*

(f) $N:_R L=P$ *for every submodule* L *of* M *properly containing* N;
(g) $\mathrm{Ass}(M/N)=\{P\}$;
(h) $P=Z_R(M/N)$.

The proof of the theorem is straightforward, hence we omit it.

We remark that, for any submodule N of an R-module M and any prime ideal P of R, each of statements (c) and (d) in Theorem 1 is a necessary condition and (g) is a necessary and sufficient condition in order that N be P-primary (cf. [3], p. 140; [7], p. 99 and p. 100).

PROPOSITION 2. *If* N *is a submodule of an* R*-module* M *whose residual* $M:N$ *by* N *is a maximal ideal of* R*, then* N *is a prime submodule. In particular,* mM *is a prime submodule of an* R*-module* M *for every maximal ideal* m *of* R *such that* $mM\neq M$.

Proof. Since $N:M=P$ is a maximal ideal, M/N is a vector space over the field R/P, a torsion free R/P-module. Hence N is prime by Theorem 1.

Proposition 9 of [7], p. 200 states that if M is a Noetherian R-module and m is a maximal ideal of R, then a proper submodule N of M is an m-primary submodule if, and only if, $m^k M\subseteq N$ for some positive integer k. Without assuming that M is Noetherian, we have the following proposition which is an easy result of Proposition 2 and is similar to the above mentioned Proposition 9 of [7], p. 200.

PROPOSITION 3. *Let* N *be a proper submodule of an* R*-module* M *and let* m *be a maximal ideal of* R*. Then* N *is* m*-prime if, and only if,* $mM\subseteq N$*. Consequently, if* N *is an* m*-prime submodule of* M*, then so is every proper submodule of* M *containing* N.

PROPOSITION 4. *If* N *is a maximal submodule of an* R*-module* M*, then* N *is a prime submodule and* $N:M$ *is a maximal ideal of* R.

Proof. N is a maximal submodule if, and only if, M/N is a simple R-module. Hence M/N is a cyclic R-module $\bar{m}R$ and $\mathrm{Ann}_R\bar{m}=\mathrm{Ann}_R(M/N)=M:N$ is a maximal ideal of R due to [1], p. 29, Proposition 3. It follows that N is prime from Proposition 2.

Combining Proposition 4 and [1], p. 30, Proposition 4, we have

COROLLARY. *If* M *is a finitely generated module, then every proper submodule of* M *is contained in a prime submodule.*

We remark that if m is a maximal ideal of a ring R, then not every m-prime submodule of an R-module M is a maximal submodule. For example, (0) is a maximal ideal of any field F and all maximal or non-maximal subspaces of a vector space V over F are (0)-prime submodules in V.

PROPOSITION 5. *Let* $N_1, N_2, \cdots, N_k$ *be submodules of an* R*-module* M *and let* N *be a prime submodule of* M*. If* $N_1\cap N_2\cap\cdots\cap N_k\subseteq N$*, then there exists an* i *such that either* $N_i\subseteq N$ *or* $N_i:M\subseteq N:M$.

Proof. Assume the contrary, then there exist an $e\in N_1$ such that $e\notin N$ and a $p_i\in(N_i:M)$ such that $p_i\notin(N:M)$ for every $i\neq 1$. Consequently, $p_ie\in N_1\cap N_i$ for every $i\neq 1$ so that $p_2p_3\cdots p_ke\in N_1\cap N_2\cap\cdots\cap N_k\subseteq N$. However, $e\notin N$ and $p_2p_3\cdots p_k\notin(N:M)$, which contradicts that N is prime.

§2. Maximal P-prime submodules

Following [7], we shall call a submodule N of an R-module M a maximal P-primary submodule of M if N is a P-primary submodule which is not strictly contained by any other P-primary submodule of M. Maximal P-prime submodules of M can be defined in a similar way.

PROPOSITION 6. *N is a maximal P-primary submodule of a module M if, and only if, N is a maximal P-prime submodule of M.*

Proof. Assume that N is a maximal P-primary submodule of M. If $r\notin N:M$, then $N:(r)$ is a P-primary submodule by [7], p. 100, Proposition 21. Thus $N:(r)=N$ for every $r\notin N:M$ by the maximal property of N, whence N is a prime submodule by Theorem 1. Since $N:M$ is a prime ideal, $N:M=\sqrt{N:M}=P$, which means that N is a P-prime submodule. Now that N is a maximal p-prime submodule is easy to see. The converse can be verified by using Proposition 1.

In the following Proposition 7, we improve [7], p. 204, Proposition 12, in which modules are assumed to be Noetherian and no notion of prime submodules is involved. We also give a proof of the proposition, which is simpler than that of Proposition 12 of [7], p. 204, by applying some of the basic properties of prime submodules discussed previously.

PROPOSITION 7. *Le M be a finitely generated R-module and N a P-primary submodule. Then the following statements are equivalent:*

(a) *N is a maximal P-primary submodule of M;*

(a′) *N is a maximal P-prime submodule of M;*

(b) *for each submodule L of M satisfying $N\subset L\subseteq M$, we have $N:L=P$ and $L:M\supset P$ (strict inclusion).*

Proof. In view of Proposition 6, it suffices to prove only the equivalence of (a′) and (b). To prove (b)⇒(a′), let N be a submodule of M satisfying the two conditions (i) $N:L=P$ and (ii) $L:M\supset P$ for every submodule L of M such that $N\subset L\subseteq M$. Due to Theorem 1, condition (i) is equivalent to that N is P-prime. Condition (ii) implies that every submodule L containing N is not P-prime. Thus N is a maximal P-prime submodule of M, so (b) implies (a′). Conversely, we assume that N is a maximal P-prime submodule of M and let L be a submodule of M which contains N properly. Then clearly $L:M\supseteq P$ and we have $N:L=P$ from Theorem 1 again. Applying [7], p. 160, Theorem 10 and Proposition 6, we can see that N_P is a maximal PR_P-prime submodule of the R_P-module M_P. Now assume that $L:M=P$. Then since M is a finitely generated R-module, $L_P\neq M_P$ by virtue of [7], p. 158, Proposition 13. It follows that L_P is a proper submodule of M_P which contains the PR_P-prime submodule N_P, therefore L_P is a PR_P-prime submodule by Proposition 3. By the maximality of N_P we have $N_P = L_P$, whence $N = N_P \cap M = L_P \cap M \supseteq L$, a contradiction. Thus $L:M$ contains P properly, so (a′) implies (b). This completes the proof of Proposition 7.

§ 3. Extended submodules IM

In this section we consider extended submodules IM of an R-module M, where I are ideals of R, which are prime or primary submodules of M.

PROPOSITION 8. *Let M be a finitely generated R-module and let I be a radical ideal of R. Then $IM:M=I$ if, and only if, $\mathrm{Ann}_R(M)\subseteq I$.*

Proof. The necessity is obvious. Assume that $\mathrm{Ann}_R(M)\subseteq I$ and let r be an element of R which is contained in $IM:M$. If M is generated by n elements, then there exists a $y\in I$ such $r^n+y\in\mathrm{Ann}_R(M)\subseteq I$ by [6], p. 50, Theorem 75. Accordingly $r^n\in I$ and, therefore, $IM:M\subseteq\sqrt{I}=I$ as I is a radical ideal. Now we can see easily that $IM:M=I$.

COROLLARY 1 ([6], p. 7, Ex. 3). *Let P be a finitely generated prime ideal of a ring R such that $\mathrm{Ann}_R(P)\subseteq P$. Then $P^2:P=\mathrm{Ann}_R(P/P^2)=P$.*

COROLLARY 2. *Let P be a finitely generated prime ideal of a ring R such that $\mathrm{Ann}_R(P)\subseteq P$. Then the following statements are equivalent:*
(a) *P^2 is a P-primary ideal of R;*
(b) *P^2 is a P-primary submodule of the R-module P;*
(c) *P^2 is a P-prime submodule of the R-module P.*

Proof. The equivalence of (a) and (b) follows from [7], p. 199, Proposition 8 and that of (b) and (c) follows from Proposition 1 and the above Corollary 1 to Proposition 8.

Let P be a prime ideal of a ring R which is not maximal. Then it is known that P^2 is not necessarily a P-primary ideal of R. This together with the above Corollary 1 and Corollary 2 to Proposition 8 suggest that if N is a submodule of an R-module M such that $N:M=P$ for some prime ideal P of R, then N is not necessarily a P-prime submodule of M.

COROLLARY 3. *If M is a finitely generated R-module and m is a maximal ideal of R containing $\mathrm{Ann}_R(M)$, then $mM\neq M$ so that mM is a prime submodule of M. In particular, if M is a finitely generated faithful R-module, then mM is a prime submodule of M for every maximal ideal m of R.*

COROLLARY 4 ([7], p. 232, Ex. 8). *Let M be a finitely generated R-module having primary decomposition for submodules and let P be a prime ideal containing $\mathrm{Ann}_R(M)$. Then $pM:M=P$, consequently, there exists a submodule of M which is P-prime.*

Proof. We have seen that $PM:M=P$ in Proposition 8. Now let $\bigcap_{i=1}^{k} N_i$ be a primary decomposition of PM, where each N_i is a P_i-primary submodule. Then $P=PM:M=\bigcap_{i=1}^{k}(N_i:M)$, whence $P=N_j:M$ for some j because P is a prime ideal. It follows that $P=\sqrt{P}=\sqrt{N_j:M}=P_j$ and therefore N_j is a P-prime submodule of M by Proposition 1.

Both Corollary 1 and Corollary 4 to Proposition 8 are slight refinements of [6], p. 7, Ex. 3 and [7], p. 232, Ex. 8, respectively.

Applying Corollary 4 to Proposition 8, now, we can give some characterization of faithful Noetherian modules.

THEOREM 2. *Let M be a faithful Noetherian R-module. Then, for every prime ideal P of R, there exists a prime submodule N of M such that $N:M=P$.*

COROLLARY. *Let M be a Noetherian R-module and let P be a prime ideal of R containing another prime ideal Q. If H is a Q-prime submodule of M, then there exists a P-prime submodule K such that $H\subseteq K$.*

Proof. Put $M'=M/H$. The M' is a faithful Noetherian module over $R'=R/Q$. Since $P'=P/Q$ is a prime ideal of R', there exists a P'-prime submodule K' of M' by Theorem 2. Put $K=\eta^{-1}(K')$, where η is the natural homomorphism of M to M'. Then K is a P-prime submodule of M which contains H due to Proposition 1 and [7], p. 101, Proposition 25.

We remark that Theorem 2 and its corollary are, respectively, analogous to the lying-over theorem and the going-up theorem for prime ideals of integral extensions of rings.

THEOREM 3. *Let M be a flat R-module and let P be a prime ideal of R such that $PM\neq M$. If Q is a P-primary ideal of R, then QM is a P-primary submodule of M. Consequently, PM is a P-prime submodule of M, so that PM is the intersection of all P-prime submodules of M.*

Proof. If $re\in QM$ for $r\in R$ and $e\in M$, then there exist a finite index set I, $\{q_i\}_{i\in I}\subseteq Q$ and $\{e_i\}_{i\in I}\subseteq M$ such that $re=\sum_{i\in I}q_ie_i$. According to [2], p. 43, Corollary 1 to Proposition 13, there exist a finite set J, $\{x_j\}_{j\in J}\subseteq M$ and $\{a_{ji}\}_{j\in J,\, i\in I\cup\{0\}}\subseteq R$ such that $a_{j0}r=\sum_{i\in I}a_{ji}q_i\in Q$ and $e=\sum_{j\in J}x_ja_{j0}$. If $r\notin P$, then clearly $a_{j0}\in Q$ for every j, whence $e=\sum_{j\in J}x_ja_{j0}$ belongs to QM. On the other hand if $r\in P$, then $r^k\in Q$ for some positive integer k, hence $r^kM\subseteq QM$ and we can conclude that QM is a P-primary submodule of M by [7], p. 99, Lemma 8. As a consequence, PM is a P-primary submodule. Now it can be easily verified that $PM:M=P$, so PM is a P-prime submodule of M by Proposition 1.

COROLLARY. *Let M be a free R-module and let m be any maximal ideal of R. Then mM is the intersection of all m-prime submodules of M; it is also the intersection of all maximal submodules N with $N:M=m$.*

§ 4. Extended submodules of content modules

For any element x of an R-module M, the content $c(x)$ of x is defined by $c(x) = \bigcap\{A;\ A$ is an ideal of R such that $x \in AM\}$. M is called a content R-module if $x \in c(x)M$ for every $x \in M$. Content R-modules M can also be characterized by that, for every family $\{A_i\}_{i \in I}$ of ideals A_i of R, $(\bigcap A_i)M = \bigcap(A_iM)$. We remark that every projective module is a content module ([8]).

The following theorem 4 is motivated by Theorem 1.2 of [9].

THEOREM 4. *Let M be a content R-module. Then the next two statements are equivalent*:

(a) *$rc(x) \subseteq \sqrt{c(rx)}$ for every $r \in R$ and $x \in M$*;

(b) *For each $P \in \mathrm{Spec}(R)$, either $PM = M$ or PM is a P-prime submodule of M.*

Proof. (a)⇒(b): Let $P \in \mathrm{Spec}(R)$ such that $PM \neq M$. If $rx \in PM$ *for* $r \in R$ and $x \in M$, then clearly $c(rx) \subseteq P$ which implies that $rc(x) \subseteq \sqrt{c(rx)} \subseteq P$ by (a). Since P is a prime ideal, we have either $r \in P \subseteq PM : M$ or $c(x) \subseteq P$, i.e., $x \in PM$. Therefore PM is a prime submodule of M. To show that $PM : M = P$, we let s be an element of $PM : M$. Then $se \in PM$ for any $e \in M$ such that $e \notin PM$, whence $c(se) \subseteq P$ but $c(e) \nsubseteq P$. Since $sc(e) \subseteq \sqrt{c(se)} \subseteq P$ by (a), $s \in P$ so that we have $PM : M = P$. (b)⇒(a): For any $r \in R$ and $x \in M$, let P be a prime ideal of R containing $c(rx)$. Then $rx \in PM$. If $PM \neq M$, then PM is a P-prime submodule of M by (b), hence we have either $r \in P$ or $x \in PM$. It follows that $rc(x) \subseteq P$ in both cases. On the other hand, if $PM = M$, then evidently $c(x) \subseteq P$ so that $rc(x) \subseteq P$. We have seen that $rc(x) \subseteq P$ for every prime ideal P containing $c(rx)$, therefore, $rc(x) \subseteq \sqrt{c(rx)}$.

As we have seen in Theorem 3, condition (b) of Theorem 4 is a necessary condition for any module be flat. According to Corollary 1.6 of [8], p. 53, a content R-module M is flat if and only if $rc(x) = c(rx)$ for every $r \in R$ and $x \in M$. Hence the set of all content R-modules which satisfy the equivalent conditions listed in Theorem 4 contains all flat content R-modules. We can give other characterizations to content modules in this set if they are finitely generated.

COROLLARY. *If M is a finitely generated content R-module, then the following statements are equivalent*:

(a) *$rc(x) \subseteq \sqrt{c(rx)}$ for every $r \in R$ and $x \in M$*;

(b) *For each $P \in \mathrm{Spec}(R)$, either $PM = M$ or PM is a P-prime submodule of M*;

(c) *$M/(PM)$ is a projective R/P-module for every $P \in \mathrm{Spec}(R)$*;

(d) *$M/(PM)$ is a flat R/P-module for every $P \in \mathrm{Spec}(R)$.*

Proof. We have seen the equivalence of (a) and (b) in Theorem 4. Applying Theorem 1, we can also see that (b) is equivalent to the following statement: (b′) For each $P \in \mathrm{Spec}(R)$, $M/(PM)$ is a (zero or non-zero) torsion free module over the integral domain R/P. Now the equivalence of (b′), (c), and (d) follows directly from [4], p. 436, Theorem 2.14 due to the fact that $M/(PM)$ is a finitely generated content module over the integral domain R/P.

For any projective R-module M it is known that if J is the Jacobson radical of R, then JM is the intersection of the maximal submodules of M. Since every projective module is a content module, we consider a generalization of this fact of projective modules to content modules.

THEOREM 5. *Let J and I be, respectively, the Jacobson radical and the nilradical of a ring R, and let M be a non-zero content R-module. Then JM is the intersection of all maximal submodules of M. Furthermore, if M satisfies the condition that $rc(x) \subseteq \sqrt{c(rx)}$ for every $r \in R$ and $x \in M$, then IM is the intersection of all prime submodules of M.*

Proof. Let Ω be the set of all maximal ideals m of R. Since M is a content R-module, $JM = \bigcap_{m \in \Omega} mM$ and $JM \neq M$ by [8], p. 57, Lemma 3.2. Hence, there exists a maximal ideal m such that $mM \neq M$, whence $M^* = M/mM$ is a non-zero vector space over the field R/m. By virtue of Corollary to Theorem 3, the intersection of all maximal subspaces of M^* consists of only the zero vector, namely, mM is the intersection of all maximal submodules containing mM. Since each maximal submodule of M contains mM for some $m \in \Omega$, we can conclude that $JM = \bigcap_{m \in \Omega} mM$ is the intersection of the maximal submodules of M. That IM is the intersection of the prime submodules of m follows from Theorem 4 and the fact that $IM = \bigcap_{P \in \mathrm{Spec}(R)} PM$.

§5. Prime submodules of Noetherian modules

A theorem of I. S. Choen states that R is a Noetherian ring if every prime ideal of R is finitely generated. Here we consider a similar theorem for modules.

LEMMA. *Let N be a submodule of an R-module M. Then N is finitely generated (resp. countably generated (c.g.)) if any of the following two conditions is satisfied:* (a) *If there exists an element $r \in R$ such that both $N + rM$ and $N :_M (r)$ are finitely generated (resp. c.g.);* (b) *If there exists an element $e \in M$ such that both $N + eR$ and $N :_R eR$ are finitely generated (resp. c.g.).*

Proof. Clearly $N \cap rM$ and $N \cap eR$ are, respectively, R-homomorphic images of $N :_M (r)$ and $N :_R eR$. Now the lemma follows from the two isomorphic relations $(N + rM)/rM \cong N/(N \cap rM)$ and $(N + eR)/eR \cong N/(N \cap eR)$.

PROPOSITION 9. *Let N be a proper submodule of an R-module M. Suppose that N is not finitely generated (resp. not c.g.) and is maximal among all submodules that are not finitely generated (resp. not c.g.). Then N is a prime submodule.*

Proof. If $r \in R - (N : M)$, then $N \subseteq N :_M (r)$ and $N \subsetneqq N + rM$. Assume that $N \neq N :_M (r)$. Then both $N :_M (r)$ and $N + rM$ are finitely generated (resp. c.g.) by the maximal property of N, whence N is finitely generated (resp. c.g.) by Lemma, a contradiction. Therefore, $N = N :_M (r)$ for every $r \in R - (N : M)$ which means that N is a prime submodule.

In a finitely generated R-module M, any submodule that is not finitely generated (resp. not c.g.) can be enlarged to one that is maximal with this property. We are now ready for the next two theorems.

THEOREM 6. *A finitely generated R-module M is Noetherian if, and only if, every prime submodule of M is finitely generated.*

THEOREM 6′. *Let M be a finitely generated R-module. Then every submodule of M is countably generated if, and only if, every prime submodule is so.*

Theorem 6 was proved in [5], but our proof is more concise than that of [5].

THEOREM 7. *Let M be an R-module, N^* a prime submodule of the module $M[\![x]\!]$ over $R[\![x]\!]$, and N the image of N^* under the natural homomorphism η: $M[\![x]\!] \to M$ such that $\eta(f(x)) = f(0)$ for each $f(x) \in M[\![x]\!]$. Then N^* is finitely generated if, and only if, N is finitely generated. Furthermore, if M is generated by s elements and N is generated by t elements, then N^* is generated by either $t+s$ elements or t elements according as x belongs to $N^* : M[\![x]\!]$ or not.*

Proof. Suppose that $M = \sum_{i=1}^{s} m_i R$ and $N = \sum_{i=1}^{t} e_i R$. If $x \in N^* : M[\![x]\!]$, then $N^* = N + xM[\![x]\!] = \sum_{i=1}^{t} e_i R[\![x]\!] + \sum_{i=1}^{s} m_i x M[\![x]\!]$. If $x \notin N^* : M[\![x]\!]$, then $N^* = \sum_{i=1}^{t} f_i R[\![x]\!]$, where f_i is any element of N^* which leads off with e_i for each i (cf. Proof of Theorem 70, [6], p. 48).

COROLLARY ([7], p. 68, Ex. 10). *If M is a Noetherian module over R, then $M[\![x]\!]$ is a Noetherian module over $R[\![x]\!]$.*

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