COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 33, No. 1, 1984

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

On Quotient Algebras in Generic Extensions

by

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(Received February 1, 1983)

§ 0. Introduction

Recently, Kakuda ([6]) has shown that precipitousness can be preserved under certain forcing extensions. Before then, he had also shown that saturation of ideals can be preserved under forcing extensions if several conditions are fulfilled (see [5]).

The notion of precipitousness is weaker than that of κ^+-saturatedness: that is, we can prove

LEMMA 1 (Solovay [8]). *Let I be a κ-complete non-trivial ideal over a regular uncountable cardinal κ. Then if the quotient algebra $P(\kappa)/I$ is κ^+-saturated, I is precipitous.*

It is also noticed that ω_0-distributivity of the quotient algebra $P(\kappa)/I$ implies precipitousness of I. Comparing with the above, we can propose the following question: *Can ω_0-distributivity be preserved under any generic extension*?

In this note, we answer the question and consider some other properties of the quotient algebra which are preserved under any generic extension.

We shall use standard set-theoretical notation and terminology.

We assume that the reader is familiar with the usual techniques in ideals, Boolean algebras and Boolean-valued models (see T. Jech [4]), so we shall just specify here some of the less standard notation and terminology.

A Boolean algebra is said to be *weakly λ-closed* if there is a dense subset which is $\leq\lambda$-closed. For each ideal I on a Boolean algebra B, the set of all elements having positive I-measure (resp. I-measure one) is denoted by "I^+" (resp. "I^*").

Throughout this note, κ will indicate an uncountable regular cardinal, I a κ-complete non-trivial ideal over κ and B a complete Boolean algebra. I is called *precipitous* if the associated Boolean ultrapower is well-founded.

Let B_I denote the quotient algebra $P(\kappa)/I$ and let $\boldsymbol{J}$ and $\boldsymbol{B_J}$ denote the B-valued sets such that $\|\boldsymbol{J}$ *is the ideal over* $\check{\kappa}$ *generated by* $\check{I}\| = 1_B$ and $\|\boldsymbol{B_J}$ *is the quotient algebra* $P(\check{\kappa})/\boldsymbol{J}\| = 1_B$. When we consider the relation between I and $\boldsymbol{J}$, the following lemma is very useful:

LEMMA 2 (Solovay [8]). (1) *If $sat(B)\leq add(I)$, then for every B-valued set X there is a set N in I such that* $\|X\in\boldsymbol{J}\| = \|X\subseteq\check{N}\|$. (2) $\|\boldsymbol{J}$ *is a $\check{\kappa}$-complete non-trivial ideal over* $\check{\kappa}\| = 1_B$. (3) *If I is normal,* $\|\boldsymbol{J}$ *is normal* $\| = 1_B$.

§ 1. The Algebras A_I, A_J and B_J

It is well-known that the set $\{X \in V^B\colon \|X \in \boldsymbol{B_J}\| = 1_B\}$ can be made into a Boolean algebra. Hereafter we shall indicate this algebra by "B_J" and its ordering by "$\leq_J$". Then it holds that $a \leq_J b$ iff $\|a \leq b\| = 1_B$ for any a and b in B_J.

The set ${}^\kappa B$ naturally corresponds to the power set of κ in V^B and if we set $\bar{I} = \{f \in {}^\kappa B\colon \{\alpha < \kappa\colon f(\alpha) > 0_B\} \in I\}$, the set $\bar{I}$ corresponds to $\boldsymbol{J}$ in V^B in a sense, that is:

LEMMA 3. (1) *${}^\kappa B$ is a Boolean algebra with the ordering induced by B.* (2) *$\bar{I}$ is an ideal on ${}^\kappa B$.* (3) *There is an order-preserving map h_J of the quotient algebra $A_J = {}^\kappa B/\bar{I}$ onto B_J.* (4) (Kakuda [5]) *If B is κ-saturated, h_J is an isomorphism of A_J onto B_J.*

Note that h_J is defined by $\|h_J([f]^*) = [X]_J\| = 1_B$ iff $\|\check{\alpha} \in X\| = f(\alpha)$ for all $\alpha < \kappa$, where $[f]^*$ and $[X]_J$ indicate the equivalence classes of f in V and of X in V^B respectively. Corresponding each $[X]_I$ in B_I to the equivalence class of the characteristic function C_X of X, we get a monomorphism h_I of B_I into A_J. Recall that $C_X(\alpha) = 0_B$ if $\alpha \in \kappa - X$ and $C_X(\alpha) = 1_B$ if $\alpha \in X$.

Moreover, we can easily prove that:

LEMMA 4. (1) *h_I is a complete monomorphism of B_I into A_J.* (2) *$h_J \circ h_I$ is a monomorphism of B_I into B_J.*

For (2), it should be noticed that for each subset X of κ, it is B-valid that $h_J \circ h_I([X]_I) = [\check{X}]_J$.

At the end of this section, we summarize the results about the relation between B_J and $\boldsymbol{B_J}$, which we shall need.

LEMMA 5 (Solovay-Tennenbaum [9]). (1) *Let λ be a cardinal number such that $sat(B) \leq \lambda$. Then B_J is λ-complete if and only if $\|\boldsymbol{B_J}$ is $\check{\lambda}$-complete $\| = 1_B$.* (2) *B_J is complete if and only if $\|\boldsymbol{B_J}$ is complete $\| = 1_B$.* (3) *Let λ be a regular cardinal number. Then B_J is λ-saturated if and only if B is λ-saturated and $\|\boldsymbol{B_J}$ is $\check{\lambda}$-saturated $\| = 1_B$.* (4) *Let λ and μ be cardinal numbers. Then B_J is (λ, μ)-distributive if and only if B is (λ, μ)-distributive and $\|\boldsymbol{B_J}$ is $(\check{\lambda}, \check{\mu})$-distributive $\| = 1_B$.*

It should be noticed that (4) in the above can be deduced from the following lemma, using the two-step generic extension.

LEMMA 6 (Scott-Solvay-Vopenka). *Let λ and μ be cardinal numbers. Then B is (λ, μ)-distributive if and only if it is B-valid that if f is a function of $\check{\lambda}$ into $\check{\mu}$ then f is in $\check{V}$.*

§ 2. Results

From now on, to simplify our arguments, we shall introduce some notations. For each $z \in A_J$, t_z indicates the representative of z, viz $[t_z]^* = z$, and for each $f\colon \kappa \to B$, T_f indicates the set $\{\alpha < \kappa\colon f(\alpha) > 0_B\}$, particularly T_{t_z} is abbreviated by S_z.

In this section, we shall begin by showing that there is another assumption under which saturation is preserved.

THEOREM 1. *Let γ and λ be any cardinal numbers such that $\lambda<\kappa$ and $\lambda=sat(B)\leq\gamma$. Assume that B_I is (λ,γ)-distributive. Then if B_I is γ-saturated it is B-valid that $\boldsymbol{B_J}$ is $\check{\gamma}$-saturated.*

Proof. We assume that B_I is γ-saturated and we have only to show that A_J is γ-saturated (by Lemma 3 and Lemma 5).

We may assume that γ is regular. Let us argue by contradiction and assume that $\{a_\xi\colon \xi<\gamma\}$ is a disjoint subset of A_J such that if $\alpha<\beta<\gamma$ then $a_\alpha\cdot a_\beta=0_{A_J}$.

We claim that there is a subset S of κ with positive I-measure such that for each $\xi<\gamma$ and for each subset D of S with positive I-measure, there exists a $\zeta<\gamma$ with $\xi\leq\zeta$ such that $D\cap S_{a_\zeta}$ has positive I-measure. The proof of this claim is not so difficult and we leave it to the reader.

For each $\zeta<\gamma$, we define a family $H_\zeta=\{\langle\xi^\zeta_\alpha, X^\zeta_\alpha\rangle\colon \alpha\in K_\zeta\}$ and an ordinal number $\delta(\zeta)$ as follows: (1) H_ζ is a maximal family such that i) for each $\alpha\in K_\zeta$, $\zeta<\xi^\zeta_\alpha<\gamma$ and X^ζ_α is a subset of $S\cap S_{a_{\xi^\zeta_\alpha}}$ with positive I-measure; ii) $\{X^\zeta_\alpha\}_{\alpha\in K_\zeta}$ is I-disjoint, and (2) $\delta(\zeta)=(\sup_{\alpha\in K_\zeta}\xi^\zeta_\alpha)+1$.

Since B_I is γ-saturated and γ is regular, it is evident that $|K_\zeta|<\gamma$ and $\delta(\zeta)<\gamma$ for each $\zeta<\gamma$.

Next, we define a sequence $\langle\pi(\alpha)\colon \alpha<\lambda\rangle$ in γ by $\pi(0)=0$, $\pi(\alpha+1)=\delta(\pi(\alpha))$ and if α is limit $\pi(\alpha)=\delta(\sup_{\zeta<\alpha}\pi(\zeta))$, this is well-defined because of $\lambda\leq\gamma$. Note that π is increasing and for each $\beta\in K_{\pi(\alpha)}$, $\pi(\alpha)<\xi^{\pi(\alpha)}_\beta<\pi(\alpha+1)$, where α is any ordinal number less than λ.

For each $\alpha<\lambda$, we set $F_\alpha=\{X\in I^+\colon (\exists\zeta\in K_{\pi(\alpha)})(\langle\xi^{\pi(\alpha)}_\zeta, X\rangle \textit{ is in } H_{\pi(\alpha)})\}$. By the claim, F_α is an I-partition of S, and so $|F_\alpha|<\gamma$ for each $\alpha<\lambda$.

Since B_I is κ-complete and (λ,γ)-distributive, there is a subset D of S with positive I-measure and a function v of λ into I^+ such that for each $\alpha<\lambda$, $v(\alpha)\in F_\alpha$ and $D-v(\alpha)\in I$.

Let $\tilde{\alpha}$ denote an ordinal number less than γ, for each $\alpha<\lambda$, such that $v(\alpha)$ is a subset of $S\cap S_{a_{\tilde{\alpha}}}$.

Noting the set $N(\beta,\alpha)=\{\xi<\kappa\colon t_{a_\beta}(\xi)\cdot t_{a_\alpha}(\xi)>0_B\}$ has I-measure zero, for any $\alpha<\beta<\gamma$, we can define a sequence $\langle D_\alpha\colon \alpha<\lambda\rangle$ by:

$$D_\alpha=(D\cap v(\alpha))-\bigcup_{\beta<\alpha}(N(\tilde{\alpha},\tilde{\beta})\cup(D-v(\beta)))\,.$$

Then obviously $D-D_\alpha\in I$ for every $\alpha<\lambda$, and so the set $D^*=\bigcap_{\alpha<\lambda}D_\alpha$ has positive I-measure.

Since D^* is non-empty, we can pick out an element ξ_0 in D^*.

Then the elements $t_{a_{\tilde{\alpha}}}(\xi_0)$ $(\alpha<\lambda)$ form a disjoint subset of B of cardinality λ. This is absurd. Hence A_J is γ-saturated.

On examining the above proof, we find out that if γ is a successor cardinal number μ^+ then the condition—(λ,γ)-distributivity of B_I—can be replaced by (λ,μ)-distributivity.

Moreover, to modify the proof, we can give a proof of Theorem 1 in [5] without using Boolean ultrapowers.

We can somewhat improve Lemma 6 in [5] and the proof we gave here is more simple.

THEOREM 2. *If* $sat(B)=\kappa$, *then* $\|\boldsymbol{B}_J$ *is* $\check{\kappa}$*-saturated* $\|<1_B$. *Moreover, if B is weakly homogeneous it is B-valid that* $\boldsymbol{B}_J$ *is not* $\check{\kappa}$*-saturated.*

Proof. Since $sat(B)=\kappa$ for each $\alpha<\kappa$ there is a disjoint subset $\{b^{\alpha}_{\xi}\colon \xi<\alpha\}$ of B such that if $\xi<\zeta<\alpha$, $b^{\alpha}_{\xi}\cdot b^{\alpha}_{\zeta}=0_B$. For each $\alpha<\kappa$, we define a function f_α of κ into B by $f_\alpha(\xi)=b^{\xi}_{\alpha}$ if $\xi>\alpha$ and otherwise $f_\alpha(\xi)=0_B$. Then it is clear that the family $\{[f_\alpha]^*\colon \alpha<\kappa\}$ guarantees that A_J is not κ-saturated and $\|\boldsymbol{B}_J$ *is* $\check{\kappa}$*-saturated* $\|<1_B$ (by Lemma 3 and Lemma 5). The rest of the thoerem is trivial.

Distributivity is also preserved under certain forcing extensions.

THEOREM 3. *Assume that B is* κ*-saturated and weakly* λ*-closed, where* λ *is a cardinal number less than* κ. *Then if* B_I *is* λ*-distributive it is B-valid that* $\boldsymbol{B}_J$ *is* $\check{\lambda}$*-distributive.*

Proof. Let P be a dense subset of B which is $\leq\lambda$-closed.

Suppose that B_I is λ-distributive and $\langle D_\alpha\colon \alpha<\lambda\rangle$ is a sequence of open dense subsets of A_J. Let a be any non-zero element of A_J.

Then we shall define by transfinite induction a sequence $\langle F_\alpha\colon \alpha\leq\lambda\rangle$ of length $\lambda+1$ satisfying the conditions (*):

(1) for each $f\in F_\alpha$, $0_{A_J}<[F]^*\leq a$, $(\forall\beta<\alpha)([f]^*\in D_\beta)$ and $(\forall\gamma\in T_f)(f(\gamma)\in P)$;

(2) $(\forall f\in F_\alpha)(\forall\beta<\alpha)(\exists g\in F_\beta)(\forall\gamma<\kappa)(f(\gamma)\leq g(\gamma))$;

(3) the set $\{T_f\colon f\in F_\alpha\}$ is an I-partition of S_a.

It is clear that for each γ in S_a, there is a b_γ in P such that $0_B<b_\gamma\leq t_a(\gamma)$. Define a function $f_0\colon \kappa\to B$ by $f_0(\gamma)=b_\gamma$ if $\gamma\in S_a$ and otherwise $f_0(\gamma)=0_B$, and set $F_0=\{f_0\}$. It is evident that F_0 satisfies the conditions (*).

Assume that for all $\beta<\alpha$, we have already defined F_β satisfying (*), where α is any ordinal number with $0<\alpha\leq\lambda$.

Now let us define F_α, using the above induction hypothesis.

1) The case: α is a successor ordinal number $\beta+1$.

Let F_α be a maximal family such that: (1) for each $f\in F_\alpha$, $[f]^*\in D_\beta$ and $(\forall\gamma\in T_f)(f(\gamma)\in P)$; (2) $(\forall f\in F_\alpha)(\exists g\in F_\beta)(\forall\gamma<\kappa)(f(\gamma)\leq g(\gamma))$; (3) the family $\{T_f\colon f\in F_\alpha\}$ is I-disjoint.

Then by the maximality of F_α, $\{T_f\colon f\in F_\alpha\}$ is an I-partition of S_a.

It is easy to see that F_α satisfies (*).

2) The case: α is a limit ordinal number.

Let $\Omega_\beta=\{z\in B_I-\{0_{B_I}\}\colon (\exists f\in F_\beta)(z\leq[T_f]_I)$ *or* $z\cdot[S_a]_I=0_{B_I}\}$.

Then Ω_β is clearly an open dense subset of B_I, for each $\beta<\alpha$.

Since $\alpha\leq\lambda$ and B_I is λ-distributive, $\Omega=\bigcap_{\beta<\alpha}\Omega_\beta$ is open dense.

Let Γ_α be a maximal disjoint subset of Ω such that for every z in Γ_α, $z\leq[S_a]_I$. For each $z=[S]_I\in\Gamma_\alpha$ and each $\beta<\alpha$, there exists an $f^z_\beta\in F_\beta$ such that $[S]_I\leq[T_{f^z_\beta}]_I$.

Let $S_z^* = S - \bigcup_{\beta<\alpha}(S - T_{f_\beta^z})$. Then, since $\alpha \le \lambda < \kappa$, $[S_z^*]_I = [S]_I = z$.

For each $\gamma \in S_z^*$, if $\xi < \eta < \alpha$ then we have that $0_B < f_\eta^z(\gamma) \le f_\xi^z(\gamma)$ and both $f_\xi^z(\gamma)$ and $f_\eta^z(\gamma)$ are in P by the induction hypothesis.

Thus $\langle f_\beta^z(\gamma): \beta < \alpha \rangle$ is a descending sequence in P for each $\gamma \in S_z^*$.

Thereby there exists a $b(\gamma)$ in P such that for all $\beta < \alpha$, $b(\gamma) \le f_\beta^z(\gamma)$.

Define a function $f_z: \kappa \to B$ by $f_z(\gamma) = b(\gamma)$ if $\gamma \in S_z^*$ and otherwise $f_z(\gamma) = 0_B$, and set $F_\alpha = \{f_z: z \in \Gamma_\alpha\}$. Then it is easily noticed that F_α satisfies the conditions (*).

Let f be any element of F_λ. Then it is obvious that $[f]^*$ is in $\bigcap_{\alpha<\lambda} D_\alpha$ and $[f]^* \le a$. Therefore $\bigcap_{\alpha<\lambda} D_\alpha$ is open dense in A_J. Thus A_J is λ-distributive, and by Lemma 3 and Lemma 5, $\| \boldsymbol{B_J}$ *is* $\check{\lambda}$*-distributive* $\| = 1_B$.

COROLLARY 1. *Let M be a transitive model of ZFC and let κ be a measurable cardinal number in M. Then there exists a generic extension $M[G]$ in which $\check{\kappa} = \omega_{n+2}$ and $\check{\kappa}$ carries a $\check{\kappa}$-complete non-trivial ideal I such that $P(\check{\kappa})/I$ is ω_n-distributive, and so I is precipitous, where n is any natural number.*

This corollary gives another proof of Theorem 4 in [1]. More generally, we can prove that:

COROLLARY 2. *Let M be a transitive model of ZFC and in M let κ be a measurable cardinal number and λ an infinite cardinal number less than κ. Then there exists a generic extension $M[G]$ in which $\check{\kappa} = \lambda^{++}$ and $\check{\kappa}$ carries a $\check{\kappa}$-complete non-trivial ideal I such that $P(\check{\kappa})/I$ is λ-distributive but not $\check{\kappa}^+$-saturated.*

This corollary gives an appraisal of Theorem 1 and the following is just one which is mentioned in §0.

COROLLARY 3. *Assume that B is κ-saturated and weakly ω-closed. Then if B_I is ω-distributive it is B-valid that $\boldsymbol{B_J}$ is ω-distributive.*

At the end of this section, we shall present a result about completeness.

THEOREM 4. *Let λ be a cardinal number not less than sat(B) and assume that B has a dense subset of cardinality less than κ. Then if B_I is λ-complete it is B-valid that $\boldsymbol{B_J}$ is $\check{\lambda}$-complete.*

Proof. Since sat$(B) < \kappa$, we have only to prove that A_J is λ-complete, by Lemma 5. Let $X = \{a_\xi: \xi < \nu\}$ be any subset of $A_J - \{0_{A_J}\}$ of cardinality less than λ. For brevity's, we shall write t_ξ instead of t_{a_ξ} and S_ξ instead of S_{a_ξ}.

Since B_I is λ-complete, there is a subset S of κ with positive I-measure such that $[S]_I = \sum_{\xi<\nu}[S_\xi]_I$. Let $\{b_\xi\}_{\xi<\gamma}$ be an enumeration of a dense subset of B, where γ is a cardinal number less than κ.

For each $(\xi, \eta) \in \gamma \times \nu$, we put $A(\xi, \eta) = \{\alpha \in S: b_\xi \le t_\eta(\alpha)\}$, and $D(\xi) = \{[X]_I \in B_I - \{0_{B_I}\}: (\exists \eta < \nu) X = A(\xi, \eta)\}$ for each $\xi < \gamma$.

Let $\delta = \{\xi < \gamma$: *$D(\xi)$ is non-empty*$\}$ and for each $\xi \in \delta$, let $T(\xi)$ be a subset of κ with positive I-measure such that $[T(\xi)]_I = \Sigma D(\xi)$.

Such a subset exists, because $|D(\xi)| \leq \nu < \lambda$.

Now we define a function f of κ into B by $f(\alpha)=\sum_{\xi\in\gamma(\alpha)} b_\xi$ if $\alpha \in S$ and $\gamma(\alpha)\neq\varnothing$ and otherwise $f(\alpha)=0_B$, where $\gamma(\alpha)=\{\xi\in\delta: \alpha\in T(\xi)\}$.

We claim that $[f]^*$ is the sup of X.

If for some $\xi_0<\nu$, $\alpha_{\xi_0}=[t_{\xi_0}]^* \not\leq^* [f]^*$, i.e., the set $F=\{\alpha<\kappa: d_\alpha=t_{\xi_0}(\alpha)-f(\alpha)>0_B\}$ has positive I-measure. Since F is a subset of S_{ξ_0}, the set $D=F\cap S$ has also positive I-measure. And as I is κ-complete and non-trivial, the set $E=\{\alpha\in D: b_{\eta_0}\leq d_\alpha\}$ has positive I-measure for some $\eta_0<\gamma$. Thus we have that $E\subseteq A(\eta_0, \xi_0)$ and $[A(\eta_0, \xi_0)]_I\in D(\eta_0)$, and so η_0 belongs to δ.

Hence, for each $\alpha\in E\cap T(\eta_0)$, $\eta_0\in\gamma(\alpha)$ and $b_{\eta_0}\leq\sum_{\xi\in\gamma(\alpha)}b_\xi=f(\alpha)$.

It follows that $0_B<b_{\eta_0}\leq(t_{\xi_0}(\alpha)-f(\alpha))\cdot f(\alpha)=0_B$. Since $E\cap T(\eta_0)$ has positive I-measure, this is absurd. Thus $[f]^*$ is an upper bound of X.

Let $[g]^*$ be any upper bound of X. If the set $D=\{\alpha<\kappa: f(\alpha)-g(\alpha)>0_B\}$ has positive I-measure, since D is a subset of S and $f(\alpha)>0_B$, there is a $\xi_\alpha\in\gamma(\alpha)$, for each $\alpha\in D$, such that $b_{\xi_\alpha}-g(\alpha)>0_B$.

By the fact $|\bigcup_{\alpha\in D}\gamma(\alpha)|\leq\gamma<\kappa$, there must be a $\xi_{\alpha_0}<\gamma$ such that $E=\{\alpha\in D: b_{\xi_{\alpha_0}}-g(\alpha)>0_B\}$ has positive I-measure.

Thus E is a subset of $T(\xi_{\alpha_0})$ with positive I-measure. Hence for some $\eta_0<\nu$, $A(\xi_{\alpha_0}, \eta_0)\cap E$ has positive I-measure and for all $\alpha\in A(\xi_{\alpha_0}, \eta_0)\cap E$, $0_B< b_{\xi_{\alpha_0}}-g(\alpha)\leq t_{\eta_0}(\alpha)-g(\alpha)$. This fact contradicts that $[g]^*$ is an upper bound of X. Therefore D has I-measure zero, and we get that $[f]^*\leq^*[g]^*$.

The following theorem is an easy exercise:

THEOREM 5. *Let λ be a cardinal number. Then*:

(1) *If* $\mathrm{sat}(B)\leq\kappa$, *$B$ is λ-distributive and B_I is not λ-distributive, then* $\|\boldsymbol{B_J}$ *is* $\check{\lambda}$-*distributive* $\|<1_B$. (2) *If* $\mathrm{sat}(B)\leq\kappa$, $\mathrm{sat}(B)\leq\lambda$ *and B_I is not λ-complete, then* $\|\boldsymbol{B_J}$ *is* $\check{\lambda}$-*complete* $\|<1_B$.

§ 3. Remarks

I. In this note, Lemma 3 (4), called *Basic Lemma* in [5], plays an important role. As far as we use the lemma, the condition —*κ-saturation of B*— can not be dropped in our theorems. For, it is easily noticed that:

LEMMA 7. *The following are equivalent*: (1) *B is κ-saturated.* (2) *For some* (*all*) *κ-complete non-trivial ideal I over κ, h_J is an isomorphism.* (3) *For some* (*all*) *κ-complete non-trivial ideal I over κ, it holds that for each B-valued set x there exists an X in I such that* $\|x\in\boldsymbol{J}\|=\|x\subseteq\check{X}\|$.

[*Proof*: (3)→(1) Let I be a κ-complete non-trivial ideal over κ satisfying the condition (3). If B is not κ-saturated, there exists a disjoint subset $\{b_\alpha\in B-\{0_B\}: \alpha<\kappa\}$ of B such that if $\alpha<\beta<\kappa$ then $b_\alpha\cdot b_\beta=0_B$. Let x be the mixture of $\{\{\alpha\}\check{}\}_{\alpha<\kappa}$ with respect to $\{b_\alpha\}_{\alpha<\kappa}$.

Then there exists an X in I such that $1_B=\|x\in\boldsymbol{J}\|=\|x\subseteq\check{X}\|$. For each $\alpha<\kappa$, $0_B<b_\alpha=\|\check{\alpha}\in x\|\leq\|\check{\alpha}\in\check{X}\|$ and α is in X. Thus we get that $X=\kappa$ is in I. This is absurd.]

II. On Theorem 3, we do not know whether the condition —*weakly* λ-*closedness*— can be weakened. For example: *Can we replace the condition by 'λ-distributivity'*?

Corollary 1 of the theorem says nothing about the standard Levy collapse which makes $\check{\kappa}=\omega_1$. About this, since we can prove that B_I *is* ω_0-*distributive iff* $2^{\omega_0}<\kappa$, there is no ω_1-complete non-trivial ideal I over ω_1 such that $P(\omega_1)/I$ is ω_0-distributive. More generally, the following holds:

LEMMA 8 (Pierce). *Let I be a κ-complete non-trivial ideal over κ, and let λ be a cardinal number less than κ. Then the following are equivalent*: (1) $P(\kappa)/I$ *is* $(\lambda, 2)$-*distributive.* (2) I *is* $(2^{\lambda})^{+}$-*complete.* (3) $2^{\lambda}<\kappa$.

COROLLARY. *The ideal of non-stationary set of ω_1 is not ω_0-distributive, but can be precipitous.*

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