T^*-Pure Archimedean Semigroups

by

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1. Introduction

A subsemigroup A of a semigroup S is called a bi-ideal of S if $ASA \subseteq A$. A bi-ideal A of a semigroup S is called two-sided pure if

$$A \cap xSy = xAy$$

holds for all $x, y \in S$. A semigroup S is called T^*-pure if every bi-ideal of it is two-sided pure, ([2]). A semigroup S is called weakly commutative if, for any $a, b \in S$, there exists a positive integer n such that

$$(ab)^n \in bSa \qquad ([5], \text{ p. 47}).$$

As is well-known, ([5], II, 5.6 Corollary), every weakly commutative semigroup is a semilattice of archimedean semigroups. Since every T^*-pure semigroup is weakly commutative ([4], p. 110), it is a semilattice of archimedean semigroups. In this paper, we shall give some properties of T^*-pure semigroups and characterize T^*-pure archimedean semigroups. And we shall give the minimum group congruence δ on a T^*-pure semigroup S, and prove that the set E of all idempotents of a T^*-pure semigroup S is a δ-congruence class if and only if E is a unitary subsemigroup of S.

2. Fundamental properties of T^*-pure semigroups

An element a of a semigroup S is called regular if there exists an element x in S such that

$$a = axa.$$

A semigroup S is called regular if every element of S is regular. An element a of a semigroup S is called completely regular if there exists an element x in S such that

$$a = axa \quad \text{and} \quad ax = xa.$$

LEMMA 2.1. *For any elements a and b of a T^*-pure semigroup S,*

$$aSb = a^2 Sb^2.$$

Proof. Let a and b be any elements of S. Then, since S is T^*-pure, the bi-ideal

aSb is two-sided pure. Thus we have

$$aSb = aSb \cap aSb = a(aSb)b = a^2Sb^2 \ .$$

LEMMA 2.2. *For any element a of a T^*-pure semigroup S, a^n $(n \geqq 3)$ is a completely regular element of S.*

Proof. Let a be any element of S. Then for any positive integer n $(n \geqq 3)$, by Lemma 2.1, we have

$$a^n = aa^{n-2}a \in aSa = (a^n)^2 S(a^n)^2 \ .$$

Then it follows from [5, IV. 1.2 Proposition, p. 104] that a is completely regular. This completes the proof.

We denote by $E(S)$ the set of all idempotents of a semigroup S, and $Z(S)$ the center of S. Then we have the following:

LEMMA 2.3. *For a T^*-pure semigroup S,*

$$E(S) \subseteq Z(S) \ .$$

Proof. Let e be any element of $E(S)$ and a any element of S. Since S is T^*-pure, the bi-ideal eSe is two-sided pure. Then we have

$$ae = aeeee \in a(eSe)e = aSe \cap eSe \subseteq eSe \ .$$

This implies that there exists an element x in S such that

$$ae = exe \ .$$

Similarly, there exists an element y in S such that

$$ea = eye \ .$$

Then we have

$$ae = exe = (ee)xe = e(exe) = e(ae)$$

$$= (ea)e = (eye)e = ey(ee) = eye$$

$$= ea \ .$$

Therefore, we have

$$e \in Z(S) \ ,$$

and so

$$E(S) \subseteq Z(S) \ .$$

This completes the proof.

As is stated in §1, any T^*-pure semigroup is weakly commutative. The following theorem shows that the converse of this and the converse of Lemma 2.3 hold for a

regular semigroup.

THEOREM 2.4. *For a regular semigroup* S *the following conditions are equivalent.*

(1) S *is* T^*-*pure.*
(2) S *is weakly commutative.*
(3) $E(S) \subseteq Z(S)$.

Proof. We shall prove that (2) implies (3), and (3) implies (1). Assume that (2) holds. Let e and a be any elements of $E(S)$ and S, respectively. Since S is regular, there exists an element x in S such that

$$ea = (ea)x(ea) \, .$$

Then, since S is weakly commutative, there exists a positive integer n such that

$$((xe)a)^n \in aS(xe) \, .$$

Then, since $x(ea) = (xe)a$ is idempotent, we have

$$ea = ea((xe)a) = ea((xe)a)^n$$

$$\in ea(aSxe) = (eaaSx)e \subseteq Se \, .$$

This implies that there exists an element y in S such that

$$ea = ye \, .$$

It can be seen in a similar way that

$$ae = ez$$

for some $z \in S$. Then we have

$$ea = ye = y(ee) = (ye)e = (ea)e$$

$$= e(ae) = e(ez) = (ee)z = ez$$

$$= ae \, .$$

Thus we have

$$e \in Z(S) \, ,$$

and so

$$E(S) \subseteq Z(S) \, .$$

Thus we obtain that (2) implies (3). Assume that (3) holds. Let A be any bi-ideal of S, and x and y any elements of S. Let $a = xsy$ ($a \in A$, $s \in S$) be any element of $A \cap xSy$. Then, since S is regular, there exist elements x', y' and a' in S such that

$$x = xx'x, \; y = yy'y \quad \text{and} \quad a = aa'a \, .$$

We note that

$$aa', \ a'a \in E(S) \,.$$

Since

$$E(S) \subseteq Z(S) \,,$$

we have

$$a = xsy = (xx'x)s(yy'y) = xx'(xsy)y'y$$

$$= xx'ay'y = xx'(aa'aa'a)y'y$$

$$= x(x'(aa'))a((a'a)y')y$$

$$= x((aa')x')a(y'(a'a))y$$

$$= x(a(a'x'ay'a')a)y$$

$$\in x(ASA)y \subseteq xAy \,,$$

and so we have

$$A \cap xSy \subseteq xAy \,.$$

Let xay $(a \in A)$ be any element of xAy. Then, since S is regular, there exists an element a' in S such that

$$a = aa'a \quad \text{and} \quad aa', \ a'a \in E(S) \,.$$

Then we have

$$xay = x(aa'aa'a)y = (x(aa'))a((a'a)y)$$

$$= ((aa')x)a(y(a'a)) = a(a'xaya')a$$

$$\in ASA \subseteq A \,,$$

and so we have

$$xAy \subseteq A \,.$$

Since the inclusion

$$xAy \subseteq xSy$$

always holds, we have

$$xAy \subseteq A \cap xSy \,.$$

Therefore we obtain that

$$xAy = A \cap xSy \,,$$

and that S is T^*-pure. Thus (3) implies (1). This completes the proof of the theorem.

THEOREM 2.5. *For a semigroup S the following conditions are equivalent.*
(1)   *S is a semilattice of groups.*

(2) *S is regular and weakly commutative.*
(3) *S is regular and $E(S) \subseteq Z(S)$.*

Proof. This follows from Theorem 2.4 and [3, Thoerem 9].

THEOREM 2.6. *The set $E(S)$ of all idempotents of a $T*$-pure semigroup S is a semilattice.*

Proof. Since S is a semigroup, it is non-empty. Let $a \in S$. Then it follows from Lemma 2.2 that a^3 is a completely regular element of S. Thus there exists an element x in S such that

$$a^3 = a^3 x a^3 .$$

Then, as is easily seen, $a^3 x$ is an idempotent element of S. Thus $E(S)$ is non-empty. On the other hand, it follows from Lemma 2.3 that $E(S)$ is commutative. Therefore $E(S)$ is a semilattice.

3. The minimum group congruence on $T*$-pure semigroups

A congruence relation β on a semigroup S is called a group congruence if S/β is a group. There is always at least one group congruence on a semigroup, namely the universal congruence. In this section we shall give the minimum group congruence δ on a $T*$-pure semigroup.

THEOREM 3.1. *Let S be a $T*$-pure semigroup, and let δ be a relation on S defined by the rule that*

$(a, b) \in \delta$ *if and only if $ea = eb$ for some idempotent e of S.*

Then δ is the minimum group congruence on S.

Proof. As is stated in the proof of Theorem 2.6, S has idempotents. Thus it is clear that δ is reflexive. It also clear that δ is symmetric. To show that δ is transitive, let

$$(a, b) \in \delta   \text{and}   (b, c) \in \delta   (a, b, c \in S) .$$

Then there exist idempotents e and f of S such that

$$ea = eb   \text{and}   fb = fc .$$

Then it follows from Theorem 2.6 that fe is idempotent. Thus we have

$$(fe)a = f(ea) = f(eb) = (fe)b$$
$$= (ef)b = e(fb) = e(fc)$$
$$= (ef)c = (fe)c .$$

This means that

$$(a, c) \in \delta .$$

Thus δ is transitive. To show that δ is compatible, let

$$(a, b) \in \delta \qquad (a, b \in S)$$

and let x be any element of S. Then there exists an idempotent e of S such that

$$ea = eb .$$

Then it follows from Lemma 2.3 that

$$e(xa) = (ex)a = (xe)a = x(ea)$$

$$= x(eb) = (xe)b = (ex)b$$

$$= e(xb) .$$

This means that

$$(xa, xb) \in \delta .$$

It is clear that

$$(ax, bx) \in \delta .$$

Thus we obtain that δ is a congruence on S. We denote by $x\delta$ the δ-congruence class mod δ to which x belongs. If e and f are any idempotents of S, then by Theorem 2.6 we have

$$(ef)e = e(fe) = e(ef) = (ee)f$$

$$= ef = e(ff) = (ef)f .$$

Since ef is idempotent, this implies that

$$(e, f) \in \delta \qquad \text{for all idempotents } e \text{ and } f \text{ of } S .$$

Let e be any idempotent of S, and a any element of S. Then we have

$$ea = (ee)a = e(ea) ,$$

and so we have

$$a\delta = (ea)\delta = (e\delta)(a\delta) .$$

Let a be any element of S. Then it follows from Lemma 2.2 that a^3 is a regular element of S. Thus there exists an element x in S such that

$$a^3 = a^3 x a^3 .$$

We note that xa^3 is idempotent. Then we have

$$((xa^2)\delta)(a\delta) = ((xa^2)a)\delta$$

$$= (xa^3)\delta = e\delta .$$

This means that s/δ is a group. Let γ be a congruence on S with the property that S/γ is a group. In order to prove that $\delta \subseteq \gamma$, let

$$(a, b) \in \delta \qquad (a, b \in S) .$$

Then there exists an idempotent e of S such that

$$ea = eb .$$

Then we have

$$(e\gamma)(a\gamma) = (ea)\gamma = (eb)\gamma = (e\gamma)(b\gamma) .$$

Since $e\gamma$ is an idempotent of the group S/γ, it is identity. Hence we have

$$a\gamma = b\gamma ,$$

that is,

$$(a, b) \in \gamma .$$

Therefore we obtain that $\delta \subseteq \gamma$, and that δ is the minimum group congruence on S. This completes the proof.

THEOREM 3.2.   *Let S be a T^*-pure semigroup with semilattice E of idempotents, and let δ be the minimum group congruence on S. Then the following conditions are equivalent.*
   (1) $(a, b) \in \delta$.
   (2) $ae = be$ *for some* $e \in E$.
   (3) $xa = xb$ *for some* $x \in S$.
   (4) $ax = bx$ *for some* $x \in S$.
   (5) $ea = fb$ *for some* $e, f \in E$.
   (6) $ae = bf$ *for some* $e, f \in E$.

Proof. It follows from Lemma 2.3 that (1) and (2) are equivalent, and that (5) and (6) are equivalent. It is clear that (1) implies (5). Conversely, assume that (5) holds, that is,

$$ea = fb$$

for some $e, f \in E$. Then, since E is commutative by Theorem 2.6, we have

$$(ef)a = (fe)a = f(ea) = f((ee)a)$$

$$= f(e(ea)) = (fe)(fb) = (ef)(fb)$$

$$= (e(ff))b = (ef)b .$$

Since $ef \in E$, this means that

$$(a, b) \in \delta .$$

Thus we obtain that (5) implies (1). It is clear that (1) implies (3). Conversely, assume

that (3) holds. Then

$$xa = xb$$

for some $x \in S$. Then it follows from Lemma 2.2 that x^3 is a regular element of S. Thus there exists an element y in S such that

$$x^3 = x^3 y x^3 .$$

We note that yx^3 is idempotent. Then we have

$$(yx^3)a = (yx^2)(xa) = (yx^2)(xb) = (yx^3)b ,$$

which means that

$$(a, b) \in \delta .$$

Therefore we obtain that (3) implies (1). It can be seen in a similar way that (2) and (4) are equivalent. This completes the proof of the theorem.

A non-empty subset A of a semigroup S is called right unitary [left unitary] if, for any $a \in A$ and any $s \in S$, $sa \in A$ [$as \in A$] implies $s \in A$, and unitary if it is both left and right unitary. Then we have the following:

THEOREM 3.3.   *Let S be a T^*-pure semigroup with semilattice E of idempotents. Then the following conditions are equivalent.*
    (1) *E is a δ-congruence class.*
    (2) *E is a unitary subsemigroup of S.*
    (3) *E is a right unitary subsemigroup of S.*
    (4) *E is a left unitary subsemigroup of S.*

Proof.   It follows from Lemma 2.3 that (2), (3) and (4) are equivalent. Assume that (1) holds. Let e be any element of E and s any elements of S such that $es \in E$. Then it follows from Lemma 2.3 that

$$(es)e = e(se) = e(es) = (ee)s = es .$$

Then it follows from Theorem 3.2 that

$$(e, s) \in \delta .$$

Then, by the assumption E is a δ-congruence class, we have

$$s \in E .$$

This means that E is left unitary. Since by Theorem 2.6 E is a subsemigroup of S, we obtain that (1) implies (4). Conversely, assume that (4) holds. Let e and f be any elements of E. Then, as is stated in the proof of Theorem 3.1,

$$(e, f) \in \delta .$$

Let a be any element of S and e any element of E such that

$$(a, e) \in \delta .$$

Then it follows from Theorem 3.1 that there exists an element g in E such that

$$ga = ge .$$

Then, since $ge \in E$, and since E is left unitary, we have

$$a \in E .$$

This means that E is a δ-congruence class. Therefore we obtain that (4) implies (1). This completes the proof.

4. T^*-Pure archimedean semigroups

A semigroup S is called archimedean if, for any $a, b \in S$, there exists a positive integer n such that

$$a^n \in SbS \qquad ([5] \text{ p. } 49) .$$

As is well-known ([1] p. 135), an archimedean commutative semigroup can contain at most one idempotent. For a T^*-pure archimedean semigroup we have the following:

THEOREM 4.1. *For a T^*-pure semigroup S the following conditions are equivalent.*
   (1) *S is archimedean.*
   (2) *Every bi-ideal of S is archimedean.*
   (3) *S has exactly one idempotent.*

Proof.   Assume that (1) holds. In order to prove that (3) holds, let e and f be any idempotents of S. Then, since S is archimedean, there exists a positive integer n such that

$$e^n \in SfS .$$

Since S is T^*-pure, the bi-ideal fSf is two-sided pure. Then we have

$$e = e^n \in SfS = S(fff)S \subseteq S(fSf)S$$

$$= SSS \cap fSf \subseteq fSf.$$

This implies that there exists an element x in S such that

$$e = fxf .$$

It can be seen in a similar way that there exists an element y in S such that

$$f = eye .$$

Then we have

$$e = fxf = (ff)xf = f(fxf) = fe$$
$$= (eye)e = ey(ee) = eye = f \ .$$

As is stated in the proof of Theorem 2.6, $E(S)$ is non-empty. Therefore we obtain that S has exactly one idempotent, and that (1) implies (3). Next assume that (3) holds. In order to prove that (2) holds, let A be any bi-ideal of S, and a and b any elements of A. Then, since S is T^*-pure, it follows from Lemma 2.2 that a^3 and b^3 are both regular. Thus there exist elements x and y in S such that

$$a^3 = a^3xa^3 \quad \text{and} \quad b^3 = b^3yb^3 \ .$$

We note that a^3x and b^3y are both idempotent. Thus by the assumption (3) we have

$$a^3x = b^3y \ .$$

Then by Lemma 2.1 we have

$$a^3 \in aSa = a^3Sa^3 = (a^3xa^3)Sa^3$$
$$= (a^3x)a^3Sa^3 = (b^3y)a^3Sa^3$$
$$= bb(b(ya^3Sa^2)a) \subseteq Ab(ASA)$$
$$\subseteq AbA \ .$$

This means that A is archimedean. Therefore we obtain that (3) implies (2). It is clear that (2) implies (1). This completes the proof of the theorem.

THEOREM 4.2. *Any cancellative archimedean semigroup without zero does not properly contain any two-sided pure bi-ideal.*

Proof. Let A be any pure bi-ideal of S. Let a and s be any elements of A and S, respectively. Then, since S is archimedean, there exist a positive integer n and elements x and y in S such that

$$a^n = xsy \ .$$

Since A is a two-sided pure bi-ideal of S, we have

$$a^n = xsy \in A \cap xSy = xAy \ .$$

This implies that there exists an element b in A such that

$$xsy = xby \ .$$

Since S is cancellative, and since x and y are both non-zero elements, we have

$$s = b \in A \ ,$$

and so we have

$$S \subseteq A \ .$$

Thus we obtain that

$$S = A \,,$$

which completes the proof.

THEOREM 4.3. *For a T^*-pure archimedean semigroup S the following conditions are equivalent.*

(1) *S is a group.*

(2) *S is a regular semigroup.*

Proof. It is clear that (1) implies (2). Assume that (2) holds. Let A be any bi-ideal of S. Let a and b be any elements of A and S, respectively. Then, since S is archimedean, there exists a positive integer n such that

$$b^n \in SaS \,.$$

On the other hand, since S is T^*-pure, the bi-ideal aSa is two-sided pure. Then by the regularity of S and by Lemma 2.1 we have

$$\begin{aligned}
b \in bSb &= b^n Sb^n \subseteq (SaS)S(SaS) \\
&= S\{a(SSS)a\}S \subseteq S(aSa)S \\
&= SSS \cap aSa \subseteq aSa \subseteq ASA \subseteq A \,,
\end{aligned}$$

and so we have

$$S \subseteq A \,.$$

Therefore we have

$$A = S \,.$$

Then it follows from [1, p. 84] that S is a group. Thus (2) implies (1). This completes the proof.

LEMMA 4.4. *Let S be a T^*-pure archimedean semigroup with idempotent e. Then $e \in A$ for any bi-ideal A of S.*

Proof. Let A be any bi-ideal of S. Then, since S is T^*-pure and archimedean, it follows from [4, Theorem 4.1] that A is absorbing. Thus for some positive integer n,

$$e = e^n \in A \,.$$

This completes the proof.

For each idempotent element f of a semigroup S, we put

$$G_f = \{a \in S: af = fa = a, \ xa = ay = f \text{ for some } x, y \in S\}$$

Then, as is well-known ([1], p. 23), G_f is the maximal subgroup of S. We note that by Theorem 4.1 a T^*-pure archimedean semigroup has exactly one idempotent.

THEOREM 4.5. *For a T^*-pure archimedean semigroup S with idempotent e,*

$$G_e = S^3 .$$

Proof.   Let a be any element of the group G_e. Then, since e is the identity of G_e, we have

$$a = eae \in S^3 ,$$

and so we have

$$G_e \subseteq S^3 .$$

Conversely, let $a = bcd$ $(b, c, d \in S)$ be any element of S^3. Then, by Lemma 2.2, b^3 and d^3 are both regular elements of S. Thus there exist elements x and y in S such that

$$b^3 = b^3 x b^3 \qquad \text{and} \qquad d^3 = d^3 y d^3 .$$

We note that $b^3 x$ and $y d^3$ are both idempotent. Then it follows from Theorem 4.1 that

$$b^3 x = y d^3 = e .$$

Thus by Lemma 2.1 we have

$$a = bcd \in bSd = b^3 S d^3 = (b^3 x b^3) S (d^3 y d^3)$$
$$= (b^3 x)(b^3 S d^3)(y d^3) \subseteq eSe .$$

This implies that there exists an element z in S such that

$$a = eze .$$

Then we have

$$ae = (eze)e = ez(ee) = eze = a .$$

Note that by Lemma 2.3 that

$$ae = ea.$$

On the other hand, by Lemma 2.2, a^3 is a regular element of S. Thus there exists an element u in S such that

$$a^3 = a^3 u a^3 .$$

Then it follows from Theorem 4.1 that

$$a^3 u = u a^3 = e .$$

This implies that

$$a(a^2 u) = (u a^2)a = e .$$

Thus we obtain that

$$a \in G_e ,$$

and so that

$$S^3 \subseteq G_e .$$

Therefore we obtain that

$$G_e = S^3 .$$

This completes the proof.

LEMMA 4.6.   *Let S be any semigroup with idempotent e such that $S^3 = G_e$. Then $G_e \subseteq A$ for any bi-ideal A of S.*

Proof.   Let A be any bi-ideal of S and let x be any element of G_e. Then, since for some $a \in A$,

$$a^3 \in S^3 = G_e ,$$

we have

$$x = exe = (a^3(a^3)^{-1})x((a^3)^{-1}a^3)$$
$$= a(a^2(a^3)^{-1}x(a^3)^{-1}a^2)a$$
$$\in ASA \subseteq A ,$$

and so we have

$$G_e \subseteq A .$$

This completes the proof.

Now we give a characterization of a T^*-pure semigroup which is archimedean.

THEOREM 4.7.   *For a semigroup S the following conditions are equivalent.*
(1)   S is a T^*-pure archimedean semigroup.
(2)   S has exactly one idempotent e and $S^3 = G_e$.

Proof.   Assume that (1) holds. Then it follows from Theorem 4.1 that S has exactly one idempotent. And it follows from Theorem 4.5 that $S^3 = G_e$. Conversely, assume that (2) holds. In order to prove that S is archimedean, let a and b be any elements of S. Then, since

$$a^3, b^3 \in S^3 = G_e ,$$

we have

$$a^3 = ea^3 = (b^3(b^3)^{-1})a^3$$
$$= b^2 b((b^3)^{-1}a^3) \in SbS .$$

This means that S is archimedean. In order to prove that S is T^*-pure, let A be any bi-ideal of S, and let x and y be any elements of S. Then, since

$$x^3, y^3 \in S^3 = G_e ,$$

and since by Lemma 4.6

$$G_e \subseteq A \,,$$

we have

$$
\begin{aligned}
A \cap xSy &\subseteq A \cap S^3 = A \cap G_e \\
&= G_e = eG_e e \\
&= (x^3(x^3)^{-1})G_e((y^3)^{-1}y^3) \\
&= x(x^2(x^3)^{-1})G_e((y^3)^{-1}y^2)y \\
&\subseteq x(S^3 G_e S^3)y = xG_e y \subseteq xAy \,.
\end{aligned}
$$

And conversely, it follows from Lemma 4.6 that

$$xAy \subseteq S^3 = G_e \subseteq A \,.$$

Since the inclusion

$$xAy \subseteq xSy$$

always holds, we have

$$xAy \subseteq A \cap xSy \,.$$

Thus we obtain that

$$A \cap xSy = xAy \,,$$

and that S is T^*-pure. Therefore we obtain that (2) implies (1). This completes the proof.

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