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On Baskakov-type Operators

by

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The object of this note is to present an estimate on approximation of derivatives by derivatives of Baskakov-type operators. The convexity property is studied also.

1. Introduction

Recently Papanicolau [3] studied some Bernstein-type operators proposed by A. Lupas. Following [3], we define a sequence of linear positive operators $\{L_n f\}$,

$$(L_n f)(x)=\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\phi_n^k(t)f\left(x+\frac{k}{n}\right),\qquad x\in[0, R](R>0)\ \text{fixed} \tag{1.1}$$

and which map $C[0, R]$, the space of bounded continuous functions into itself. Here the sequence of functions $\{\phi_n(t)\}$ $(n=1, 2, \cdots)$ possess the following properties on $[0, R]$:

(a) ϕ_n is analytic on the interval $[0, R]$ including the end points.

(b) $\phi_n(0)=1$

(c) $(-1)^k\phi_n^{(k)}(t)\geq 0$ if $k=0, 1, \cdots$ and $t\in[0, R]$

(d) There exists a positive integer $m(n)$ not depending on k, such that

$$\phi_n^{(k)}(t)=-n\phi_{m(n)}^{(k-1)}(t)[1+\alpha_{k,n}(t)]\ (k=1, 2, \cdots)$$

where $\alpha_{k,n}(t)$ converges to zero uniformly in k when $n\to\infty$.

(e) $\lim_{n\to\infty}\dfrac{n}{m(n)}=1$.

The object of this note is to give some results on operators (1.1).

These are the special cases of operators (1.1).

[A] $\phi_n(t)=(1-t)^n$. Then we get the Bernstein-type polynomials [3],

$$(L_n f)(x)=\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}f\left(x+\frac{k}{n}\right). \tag{1.2}$$

[B] $\phi_n(t)=e^{-nt}$. Then we get the operators of Szâsz and Mirakian-type,

$$(1.3) \qquad (L_nf)(x)=e^{-nt}\sum_{k=0}^{\infty}\frac{(nt)^k}{k!}f\left(x+\frac{k}{n}\right).$$

[C] $\phi_n(t)=(1+t)^{-n}$. Then we get some Bernstein-type operators as shown in [3],

$$(1.4) \qquad (L_nf)(x)=(1+t)^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{t}{1+t}\right)^k f\left(x+\frac{k}{n}\right)$$

2. Basic result

Let p be a positive integer, then

$$(2.1) \quad \sum_{k=0}^{\infty}\left(t-\frac{k}{n}-\frac{r}{2n}\right)^{2p}\frac{(-1)^{r+k}}{n^r}\frac{t^k}{k!}\phi_n^{(k+r)}(t)=\frac{(-1)^r}{n^r}\sum_{1=0}^{2p-1}(-1)^1\binom{2p}{1}\frac{\left(t-\dfrac{r}{2n}\right)^1}{n^{2p-1}}$$

$$\times\sum_{s=1}^{2p-1}\lambda_s(-t)^{2p-1-s+1}\phi_n^{(2p-1-s+r+1)}(0)+\cdots+\frac{(-1)^r\phi_n^{(r)}(0)}{n^r}\left(t-\frac{r}{2n}\right)^{2p}$$

$$=\mu_{n,p,r}(t) \qquad \text{(say)}.$$

where the constants λ_s are determined by the relation

$$(2p-1-s+1)^{2p-1}=\lambda_s(2p-1-s+1)(2p-1-s)\cdots 1+\cdots+\lambda_{2p-1}\cdot(2p-1-s+1).$$

Proof. Using

$$k^{2p}=\sum_{s=1}^{2p}\lambda_s k(k-1)\cdots(k-2p+s)$$

where

$$(2p-s+1)^{2p}=\lambda_s(2p-s+1)(2p-s)\cdots 1+\cdots$$
$$+\lambda_{2p-1}(2p-s+1)(2p-s)+\lambda_{2p}(2p-s+1).$$

The evaluation of (2.1) follows easily by the binomial expansion.

3.

We prove the following theorems.

THEOREM 1. *Let $f\in C^r[0,\infty)$ with its modulus of continuity $\omega_r(f^{(r)};\delta)$ $(\delta>0)$ and let p be a positive integer. Then for $n\in N$,*

$$(3.1) \qquad |(L_n^r f)(x) - f^{(r)}(x+t)| \leq |1 - a_{n,r}| \cdot |f^{(r)}(x+t)| + \omega_r(f^{(r)};\delta)\left[\left(1+\frac{r}{2n\delta}\right)a_{n,r} + \frac{\mu_{n,p,r}(t)}{\delta\left(\delta-\frac{r}{2n}\right)^{2p-1}}\right]$$

where $\mu_{n,p,r}(t)$ *is given by* (2.1) *and*

$$a_{n,r} = \frac{(-1)^r \phi_n^{(r)}(0)}{n^r}$$

Proof. We note that for each $n \geq 1$ and $x \in [0, \infty)$ fixed, $(L_n f)(x)$ is an infinitely differentiable function of t.

After differentiating (1.1) r times w.r.t.t, we get

$$(L_n^{(r)} f)(x) = \sum_{k=0}^{\infty} \frac{(-1)^r(-t)^k}{k!}\, \phi_n^{(k+r)}(t) \Delta_{n^{-1}}^r f\left(x+\frac{k}{n}\right)$$

where $\Delta_{n^{-1}}^r f(x+(k/n))$ represents the difference of order r of function f with step $1/n$ starting from value $x+(k/n)$. Using the mean value theorem

$$\Delta_{n^{-1}}^r f\left(x+\frac{k}{n}\right) = \frac{1}{n^r} f^{(r)}\left(x+\frac{k+r\theta_k}{n}\right) \qquad 0 < \theta_k < 1$$

we get

$$(L_n^r f)(x) = \frac{(-1)^r}{n^r} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}\, \phi_n^{(k+r)}(t) f^{(r)}\left(x+\frac{k+r\theta_k}{n}\right).$$

Clearly

$$\begin{aligned} |(L_n^r f)(x) - f^{(r)}(x+t)| &\leq |1 - a_{n,r}| \quad |f^{(r)}(x+t)| \\ &\quad + \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-t)^k}{k!}\, \phi_n^{(k+r)}(t) \left|\left[f^{(r)}(x+t) - f^{(r)}\left(x+\frac{k+r\theta_k}{n}\right)\right]\right| \\ &= S_1 + S_2 \qquad \text{(say)}. \end{aligned}$$

Using (2.1) and the inequality

$$\left|f^{(r)}(x+t) - f^{(r)}\left(x+\frac{k+r\theta_k}{n}\right)\right| \leq [1+] \frac{\left|t-\frac{k}{n}-\frac{r}{2n}\right| + \frac{r}{2n}}{\delta} [\quad] \cdot \omega_r(f^{(r)};\delta)$$

we get

$$S_2 \le \omega_r(f^{(r)};\delta)\Bigg[\left(1+\frac{r}{2n\delta}\right)a_{n,r}$$

$$+\frac{1}{\delta}\sum_{|t-(k/n)-(r/2n)|\ge\delta-(r/2n)}\left|t-\frac{k}{n}-\frac{r}{2n}\right|\frac{(-1)^r(-t)^k}{n^r\cdot k!}\cdot\phi_n^{(k+r)}(t)\Bigg]$$

$$\le\omega_r(f^{(r)};\delta)\Bigg[\left(1+\frac{r}{2n\delta}\right)a_{n,r}$$

$$+\frac{1}{\delta\left(\delta-\frac{r}{2n}\right)^{2p-1}}\sum_{k=0}^{\infty}\left(t-\frac{k}{n}-\frac{r}{2n}\right)^{2p}\frac{(-1)^r(-t)^k}{n^r\cdot k!}\phi_n^{(k+r)}(t)\Bigg]$$

$$=\omega_r(f^{(r)};\delta)\left[\left(1+\frac{r}{2n\delta}\right)a_{n,r}+\frac{\mu_{n,p,r}(t)}{\delta\left(\delta-\frac{r}{2n}\right)^{2p-1}}\right]$$

This completes the proof.

Remarks.

1. In the case $p=1$, the result (3.1) is analogous to the result of Martini [1].
2. Let $\phi_n(t)=(1-t)^n$ and $p=2$ in (3.1). Then after a little calculation, we get

$$\mu_{n,2,r}(t)=\frac{(-1)^r}{n^r}\Bigg[\frac{1}{n^4}(t^4\phi_n^{(r+4)}(0)-6t^3\phi_n^{(r+3)}(0)+7t^2\phi_n^{(7+2)}(0)-t\phi_n^{(r+1)}(0))$$

$$-\frac{4\left(t-\frac{r}{2n}\right)}{n^3}(-t^3\phi_n^{(r+3)}(0)+3t^2\phi_n^{(r+2)}(0)-t\phi_n^{(r+1)}(0))$$

$$+\frac{6\left(t-\frac{r}{2n}\right)^2}{n^2}(t^2\phi_n^{(r+2)}(0)-t\phi_n^{(r+1)}(0))$$

$$+\frac{4t\left(t-\frac{r}{2n}\right)^3}{n}\phi_n^{(r+1)}(0)+\left(t-\frac{r}{2n}\right)^4\phi_n^{(r)}(0)\Bigg]$$

$$=\frac{d_{n,r}}{n^4}\Bigg[3(n-r)^2T^2+(n-r)T(1-6T)$$

$$+\frac{r(n-r)(4+3r)}{2}T\cdot(1-2t)^2+\frac{r^4}{16}(1-2t)^4\Bigg]$$

where $d_{n,r}=n(n-1)\cdots(n-r+1)/n^r$ and $T=t(1-t)$. Since $d_{n,r}\le 1$, then for $0\le t\le 1$,

$$\mu_{n,2,r}(t) \leq \frac{1}{16n^4}\left[3(n-r)^2 + \frac{r(n-r)(3r+4)}{2} + r^4\right]$$

$$= \mu^*_{n,r} \qquad \text{(say)}.$$

We get for polynomials (1.2),

$$|(L_n^{(r)}f)(x) - f^{(r)}(x+t)| \leq \frac{r(r-1)}{2n}\,|f^{(r)}(x+t)| + \omega_r(f^{(r)};\,\delta)\left[\left(1+\frac{r}{2n\delta}\right) + \frac{\mu^*_{n,r}}{\delta\left(\delta - \frac{r}{2n}\right)^3}\right] \tag{3.2}$$

We note that estimate (3.2) in case $r=0$ reduces to

$$|(L_nf)(x) - f(x+t)| \leq \frac{19}{16}\,\omega\left(f;\,\frac{1}{\sqrt{n}}\right)$$

The above result may be compared to the estimate [2],

$$|(L_nf)(x) - f(x)| \leq \frac{5}{4}\omega\left(f;\,\frac{1}{\sqrt{n}}\right).$$

THEOREM 2. *If $f(t)$ is convex of order s, then (L_nf) (0) is also a convex function of t of order s.*

Proof. We get from (1.1) that

$$\left(\frac{d}{dt}\right)^s (L_nf)(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+s} t^k \phi_n^{(k+s)}(t)}{k!}\cdot b_n \cdot \Delta^s_{n-1} f\left(x+\frac{k}{n}\right)$$

where b_n is a constant function of n and

$$\Delta_{n-1} f\left(x+\frac{k}{n}\right) = n\left[f\left(x+\frac{k+1}{n}\right) - f\left(x+\frac{k}{n}\right)\right]$$

and Δ^s_{n-1} is the s-th iterate of difference operator. By the use of mean value theorem, we know that divided differences of any order are equal to a constant multiple times derivatives of corresponding order evaluated at intermediate points.

Hence

$$\left(\frac{d}{dt}\right)^s (L_nf)(0) \geq 0 \qquad (s=1, 2, \cdots)$$

whenever $f(t)$ is convex of order s. This completes the proof.

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