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On Kennedy's Problems

by

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Kennedy defined the notion of a Krull ring for a commutative ring $R\ni 1$ which is not necessarily an integral domain, and developed the Krull ring theory in [5] and in [6]. At the ends of these articles he poses two open questions:

1. If R is a Krull ring with the trivial divisor class group, then is R a unique factorization ring?

2. If R is completely integrally closed and every nonempty set of divisorial ideals of R has a maximal elment, then is R a Krull ring?

The aim of this paper is to answer these problems.

1. On Problem 1

Let k be a field and $\{X_1, X_2, X_3, \cdots\}$ a system of indeterminates. We set $\bigcup_{n=1}^{\infty} k[[X_1, \cdots, X_n]]=k[[X_1, X_2, X_3, \cdots]]_1$ ([3] p. 6). Let $(X_iX_j;\ i\neq j)$ be the ideal of $k[[X_1, X_2, X_3, \cdots]]_1$ generated by all the elements X_iX_j for $i\neq j$. We make $A=k[[X_1, X_2, X_3, \cdots]]_1 / (X_iX_j;\ i\neq j)$.

LEMMA 1.

(1) *A is its own total quotient ring.*

(2) *A does not have idempotents except* 0 *and* 1.

(3) *A is not a principal ideal ring.*

The ideal of A generated by the set $\{X_n+(X_iX_j;\ j\neq j);\ n=1, 2, 3, \cdots\}$ is not principal.

Remark 2. Both Exercise 14 of the first edition of [4] p. 63 and Exercise 13 of [3] p. 110 state that, if a commutative ring with its own total quotient ring does not have nonzero nilpotent elements, it is a von Neumann regular ring. But Gilmer pointed out that the statement is false. The above ring A is due to Gilmer and author and gives a counter example for the statement.

PROPOSITION 3. *Let R be a Krull ring with the trivial divisor class group. Then R need not be a unique factorization ring.*

A Counter Example. Let D be a unique factorization domain which is not a field. We denote the divisor class group of a ring R by $C(R)$. We have $C(D)=0$. From our

A and D we make a direct sum: $R=D\oplus A$. By [6] Proposition 2.5, R is a Krull ring. By [5] Proposition 1.10, (iii), we have $C(R)=C(D)\oplus C(A)=0$. Next, let $r=d+a$ be an idempotent of $R(d\in D$ and $a\in A)$. It follows $d^2=d$ and $a^2=a$. We have $d=0$ or $d=1_D$; also we have by Lemma 1, (2), $a=0$ or $a=1_A$ (1_D (resp. 1_A) is the identity of D (resp. A)). We see that D and A are only the direct summands of R. If R is a unique factorization ring, by [2] R is of the form $D_1\oplus\cdots\oplus D_n\oplus A_1\oplus\cdots\oplus A_m$ for unique factorization domains D_i and special principal ideal rings A_j. Since A is not even a principal ideal ring (Lemma 1, (3)), we have a contradiction. Therefore R is not a unique factorization ring.

2. On Problem 2

LEMMA 4 ([1] §1, Th. 2). *Let G be a lattice ordered abelian group such that every nonempty set of positive elements has a minimal element. Then, if x is a minimal element among the strictly positive elements of G and $y+z\geq x$, $y\geq 0$, $z\geq 0$ for y, $z\in G$, it follows $y\geq x$ or $z\geq x$.*

We denote the set of regular fractional ideals of R by $F(R)$.

THEOREM 5. *If R is completely integrally closed distinct from its total quotient ring K such that every nonempty set of divisorial ideals of R has a maximal element, then R is a Krull ring.*

Proof. By [6] Proposition 1.1, the semigroup $D(R)$ of divisors of R is a lattice ordered abelian group. By the hypothesis every nonempty set of positive elements of $D(R)$ has a minimal element. Let $P(R)$ be the set of minimal elements among the strictly positive elements of $D(R)$. By Lemma 4, $D(R)$ is a free abelian group with a free basis $P(R)$ and a positive element of $D(R)$ may be written as $n_1P_1+\cdots+n_mP_m$ for $Z\ni n_i\geq 0$ and $P_i\in P(R)$. Let $P\in P(R)$ and $A\in F(R)$. We have $\operatorname{div} A=nP+n_1P_1+\cdots+n_mP_m$ for n, $n_i\in Z$, $P_i\in P(R)$ and $P\neq P_i$ $(1\leq i\leq m)$. Then we set $n=v_P(A)$. For $x\in K$, we set $v_P(x)=\sup\{v_P(xR+A); A\in F(R)\}$. We see that v_P is a Z-valued valuation on K. Thus, $v_P(xy)\geq v_P(x)+v_P(y)$ is immediate for x, $y\in K$. Conversely, let $n\in Z$ and $n\leq v_P(xy)$. We may take $A\in F(R)$ such that $v_P(xyR+A)\geq n$ and $A\subseteq R$. Choose a regular $d\in R$ such that dx, $dy\in R$. We have $\operatorname{div}(xR+dA)+\operatorname{div}(yR+dA)\geq\operatorname{div}(xyR+A)\geq n$. It follows $v_P(x)+v_P(y)\geq v_P(xy)$, hence $v_P(x)+v_P(y)=v_P(xy)$. The inequality $v_P(x+y)\geq\inf(v_P(x), v_P(y))$ is immediate. Let A, B be divisorial ideals such that $\operatorname{div} A=P$ and $\operatorname{div} B=2P$. Choose $a\in A\backslash B$. For each $A'\in F(R)$, we have $\operatorname{div} A\leq\operatorname{div}(aR+(A'\cap B))\ngeq\operatorname{div} B$. It follows $v_P(a)=1$. Therefore v_P is a Z-valued valuation on K. Let R_P be the valuation ring of v_P. Obviously $R\subseteq\bigcap_{P\in P(R)}R_P$. Conversely let $x\in\bigcap_{P\in P(R)}R_P$. We have $\operatorname{div}(xR+R)=n_1P_1+\cdots+n_mP_m$ for $n_i\in Z$ and $P_i\in P(R)$. Choose regular ideal $A_i\subseteq R$ such that $v_{P_i}(xR+A_i)\geq 0$. Since $n_1P_1+\cdots+n_mP_m\leq\operatorname{div}(xR+A_1\cdots A_m)$, we have $\operatorname{div}(xR+A_1\cdots A_m)\geq 0$. It follows $x\in R$; and hence $R=\bigcap_{P\in P(R)}R_P$. Therefore R is a Krull ring.

Remark 6. Let R be a Krull ring. Let $P \in P(R)$ and $P = \operatorname{div} Q$ for a divisorial ideal Q. Then, as for a domain, Q is the center of v_P on R. And the valuation ring of v_P is the large quotient ring $R_{[Q]} = \{x \in K; sx \in R \text{ for some } s \in R \backslash Q\}$.

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