COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 31, No. 2, 1982

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

On Minimal Imbedded Subgroups of Abelian p-Groups

by

William J. KEANE

(Received December 1, 1981)

1. Introduction

A subgroup H of an Abelian p-group G is *imbedded* in G if its p-adic topology coincides with the relative one inherited from G. Equivalently, H is l-imbedded in G (written $H<_l G$) if there is an increasing function l: $Z^+ \to Z^+$ with $H \cap p^{l(n)}G \subset p^n H$, for all $n \in Z^+$. Properties of these subgroups were investigated by Moore in [8], and in particular, l-quasi-complete groups were characterized according to the existence of a minimal imbedded subgroup. In this paper, we study these latter subgroups, and show that they differ significantly from minimal pure ones. We then use this to give a characterization of the imbedded-complete groups introduced in [7]. Finally, we extend a theorem of Benabdallah and Irwin on direct sums of cyclic groups to the imbedded case.

Throughout, "group" means reduced abelian p-group, and the notation is that of [5]. In particular, $\oplus_C$ denotes direct sum of cyclic groups, and Z^+ the positive integers.

2. Neat-imbedded hulls

Definition. If K is a subgroup of G, and $n \in Z^+$, $K_n = p^n K[p] = \{x \in p^n K \mid px = 0\}$.

We begin by determining when a subgroup is contained in no l-imbedded subgroups.

Definition. A subgroup N of a group G is l-dense in G if G/K is divisible whenever $N \subset K <_l G$.

PROPOSITION. *Let $K <_l G$, and suppose $G_n \subset K$. Then $p^{l(n+1)-1}G \subset K$.*

Proof. We have that $p^{l(n+1)-1}G[p] \subset G_n \subset K$, so suppose that $p^{l(n+1)-1}G[p^r] \subset K$, and let $p^{r+1}g = 0$, where $g = p^{l(n+1)-1}x$. Then $pg \in K$, so $pg = p^{l(n+1)}x = p^n y$, for $y \in K$. Thus $g - p^n y \in G_n \subset K$, so $g \in K$.

THEOREM 2.1. *A subgroup N of a group G is contained in no proper l-imbedded subgroup of G if and only if N is l-dense in G, and $N \supset G_n$, for some $n \in Z^+$.*

Proof. Suppose N is l-dense, $G_n \subset N$, and $N \subset K <_l G$. Then $K \supset p^{l(n+1)-1}G$, so

G/K is bounded and divisible, and $K=G$. Clearly, if no proper l-imbedded subgroup contains N, N is l-dense, and since a pure subgroup is l-imbedded, $N \supset G_n$ for some $n \in Z^+$, by Theorem 1.3 of [1].

Most interesting is the case when $l(1)=1$ (i.e., when l-imbedded subgroups are neat) since by Lemma 4.3 of [7], the neat hulls of an l-imbedded subgroup are l-imbedded. For the rest of this section, we will assume $l(1)=1$. In this case, we characterize the l-dense subgroups of a group.

THEOREM 2.2. *A subgroup N of G is l-dense in G if and only if*

$$N+p^{l(n+2)-1}G \supset G_n, \qquad \text{for} \quad n \in Z^+ .$$

Proof. Suppose N satisfies the condition, and let $N \subset K <_l G$. Choose $x+K \in (G/K)[p]$. Since K is neat in G, we may assume $x \in G_n$, for some n. Then $x+p^{l(n+2)-1}g \in N$, for some $g \in G$, so $x+K \in p^{l(n+2)-1}(G/K)$. Suppose $x+K \in p^{l(r)-1}(G/K)$, and let $x+K=p^{l(r)-1}y+K$. Then $p^{l(r)}y \in K$, so $p^{l(r)}y=p^r z$, for some $z \in K$. Then

$$p^{l(r)-1}y-p^{r-1}z \in G_{r-1} ,$$

so

$$p^{l(r)-1}y-p^{r-1}z=p^{l(r+1)-1}u+v,\ u \in G,\ v \in N .$$

Therefore

$$(p^{l(r)-1}y-x)-p^{r-1}z=p^{l(r+1)-1}u-x+v \in K ,$$

and $h_{G/K}(x+K) \geq l(r+1)-1$. By induction, $x+K \in p^{\omega}(G/K)$, and since $x+K$ was arbitrary, G/K is divisible.

Conversely, suppose $G_n \not\subset N+p^{l(n+2)-1}G$, and let $G_n=S \oplus T$, where $T=G_n \cap p^{l(n+2)-1}G$. Choose a subgroup R of G such that $R/(p^{l(n+2)-1}G)$ is $S \oplus p^{l(n+2)-1}G/p^{l(n+2)-1}G$-high. By Lemma 1.5 of [2], R is S-high in G. Since $S \subset p^n G$, R is p^{n+1}-pure in G. In fact, $R <_l G$, since if $x \in R \cap p^{l(n+t)}G$, for $t>1$, then because $p^{l(n+2)-1}G \subset R$, $x=p^{t-1}g$, for some $g \in R \cap p^{n+1}G$. Then $g=p^{n+1}g'$, $g' \in R$ and $x=p^{t-1}(p^{n+1}g')=p^{n+t}g'$. But $R/N \neq 0$ and is bounded, so is not divisible.

If we remove the restriction of a fixed function l, we depart drastically from the pure case (when l is the identity).

Definition. Let $N \subset K <_l G$. Then K is a neat-imbedded hull of N if whenever $N \subset K' \subset K$ and $K' <_{l'} G$, then $K'=K$.

THEOREM 2.3. *Let $N \subset K <_l G$. Then K is a neat-imbedded hull of N if and only if $N \supset K_n$, for some n, and $\bar{N} \supset K[p]$. ($\bar{N}$ is the closure of N in the p-adic topology.)*

Proof. K is an l-hull for every l if and only if $K[p] \subset N+p^r G$, for all r, so $\bar{N} \supset K[p]$.

COROLLARY. *Let N be a subsocle of G. If N is contained in a neat-imbedded hull K, then $N=K[p]$.*

Proof. If $x\in K[p]$, then there exists $y\in N$ such that $x+y=p^{l(n)}g$, for some $g\in G$. Then $p^{l(n)}g\in K[p]\cap p^{l(n)}G\subset K_n\subset N$, so $x\in N$, and N supports K.

That this last result is not true for minimal pure subgroups is seen in Theorem 2 of [6].

A group is imbedded-complete if and only if every subsocle supports an imbedded subgroup. It was shown in [7] that the class of these groups properly contains the pure-complete groups. We can characterize them in terms of minimal imbedded subgroups.

THEOREM 1.4. *A group is imbedded-complete if and only if every subsocle is contained in a neat-imbedded hull.*

Proof. The sufficiency follows from the corollary. If G is imbedded-complete, and $N\subset G[p]$, let $K<_l G$ where $K[p]=N$. Take a neat hull K' of K. Then $K'[p]=N$, and by Lemma 4.3 of [7], $K'<_l G$. Further, if $N\subset L\subset K'$, and L is neat in G, $L=K'$, by Lemmas 2 and 3 of [4]. Thus K is a neat-imbedded hull of N.

3. Direct sums of cyclic groups

We turn finally to the special case of direct sums of cyclic groups, and see that the assumption of imbeddedness can replace that of purity in many important situations.

THEOREM 3.1. *Let N be a subgroup of G such that $G/N=\bigoplus_C$ and suppose $N\subset K<_l G$. If $N\supset K_n$ for some $n\in Z^+$, then $G/K=\bigoplus_C$.*

Proof. Consider the canonical homomorphism from $(G_{l(n)}+N)/N$ to $(G_{l(n)}+K)/K$. Let $g\in G_{l(n)}$, and say $h_{(G/K)}(g+K)\geq l(m+2)$, for some $m\geq l(n)$. Then $g+K=p^{l(m+2)}x+K$, so $p^{l(m+2)+1}x\in K$. Now since $K<_l G$, $p^{l(m+2)+1}x=p^{m+2}y$, for $y\in K$, and hence $p^{l(m+2)}x-p^{m+1}y\in G_{m+1}$. Thus we can choose a representative of $p^{l(m+2)}x+K$ in $G_{m+1}\subset G_{l(n)}$, say z. Then $g-z\in K\cap G_{l(n)}\subset K_n\subset N$, so $g+N=z+N$, and therefore $h_{(G/N)}(g+N)\geq m+1$.

Now since $(G_{l(n)}+N)/N$ is the union of an ascending chain of subgroups of bounded height, so is $(G_{l(n)}+K)/K$. But $p^{l(l(n)+1)}(G/K)[p]\subset(G_{l(n)}+K)/K$, so $p^{l(l(n)+1)}(G/K)=\bigoplus_C$, and hence $G/K=\bigoplus_C$.

Theorems of Benabdallah and Irwin (cf. [3], [1]) can now be extended.

COROLLARY 1. *If N is a subgroup of G such that $G/N=\bigoplus_C$, and K is a minimal l-imbedded subgroup containing N, then $G/K=\bigoplus_C$.*

Proof. $K_n\subset N$ by Theorem 2.1.

COROLLARY 2. *Suppose $N\subset G[p]$ and $G/N=\bigoplus_C$. Then if N supports an* *imbedded subgroup K of G, $G/K=\bigoplus_C$. In fact, if K' is a neat hull of K, $G/K'=\bigoplus_C$.*

Proof. The first part follows since $K[p]=N$. If K' is a neat hull of K, and $K<_l G$, then $K'<_l G$, and $K'[p]=N$, so $G/K'=\bigoplus_C$.

References

[1] BENABDALLAH, K. and IRWIN, J.; On minimal pure subgroups, *Pub. Math. Debrecen*, **23** (1976), 111–114.

[2] ———; On quasi-essential subgroups of primary Abelian groups, *Can. J. Math.*, **12** (1970), 1176–1184.

[3] ———; Pure N-high subgroups, p-adic topology and direct sums of cyclic groups, *Can. J. Math.*, **26** (1974), 322–327.

[4] ——— and RAFIQ, M.; N-High subgroups of abelian p-groups, *Arch. Math. (Basel)*, **25** (1974), 29–34.

[5] FUCHS, L.; *Infinite Abelian Groups*, Vol. I, New York, Academic Press, 1970.

[6] HILL, P. and MEGIBBEN C.; Minimal pure subgroups in primary groups, *Bull. Soc. Math. France*, **92** (1964), 252–257.

[7] KEANE, W. J.; Imbedded N-high subgroups of Abelian groups," *Comment. Math. Univ. St. Pauli*, **29** (1980), 145–155.

[8] MOORE, J. D.; On quasi-complete Abelian p-groups," *Rocky Mtn. J. of Math.*, **5** (1975), 601–609.

Department of Mathematics
Boston College
Massachusetts, U.S.A.