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On the Values of the Dedekind Zeta Function of an Imaginary Quadratic Field at $s=1/3$

by

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In [1] and [2], the author obtained interesting formulae for the values of the Dedekind zeta function of an imaginary quadratic field at half integers. The aim of this note is to give similar results.

Let K be an imaginary quadratic field with the discriminant d, $\chi=(\frac{d}{\cdot})$ the Kronecker symbol and $\zeta_K(s)$ the Dedekind zeta function of K. For any positive integer n, we define

$$g(n)=\sum\prod_{j=0}^{2}\chi(n_j)n_j^{-j/3}m_j^{-j/3},$$

where the summation extends over all positive integers n_j, m_j $(0\leq j\leq 2)$ with

$$\prod_{j=0}^{2} n_jm_j=n.$$

Then, from Theorem 15 of [2], we have the following:

PROPOSITION. *Define, for $x>0$,*

$$E(x)=\sum_{n=1}^{\infty} g(n)\exp(-\sqrt[3]{n}\,\pi x/\sqrt{|d|}).$$

Then for any positive number a,

$$E(6/a)-aE(6a)=\sum_{k=0}^{2} A(k)(a^{2k+1}-1)/a^k, \tag{1}$$

where

$$A(k)=3\cdot k!(\sqrt{|d|}/6\pi)^{k+1}L(1,\chi)\prod_{r=k-2}^{k}{}'\,\zeta_K(1+r/3)$$

and the prime on the product indicates that $r\neq 0$. Here $L(s,\chi)$ denotes the L-function associated with χ.

The proof is easy and will be omitted.

By setting $a=2$, 3 and 6 in (1), we get

$$A(0)+(7/2)A(1)+(31/2^2)A(2)=E(3)-2E(12)\,, \tag{2}$$

$$2A(0)+(26/3)A(1)+(242/3^2)A(2)=E(2)-3E(18) \tag{3}$$

and

$$5A(0)+(215/6)A(1)+(7775/6^2)A(2)=E(1)-6E(36)\,. \tag{4}$$

From (2) and (4), we have

$$22A(0)+(176/3)A(1)-(242/6^2)A(2)=27E(3)-54E(12)-E(1)+6E(36)\,.$$

From this and (3), we find that

$$\begin{aligned}&90A(0)+(730/3)A(1)\\&\quad=-4E(1)+E(2)+108E(3)-216E(12)-3E(18)+24E(36)\,.\end{aligned} \tag{5}$$

On the other hand, from (3) and (4), we get

$$11A(0)+(67/2)A(1)-(31/6^2)A(2)=8E(2)-24E(18)-E(1)+6E(36)\,.$$

From this and (2), we can easily deduce that

$$\begin{aligned}&100A(0)+305A(1)\\&\quad=-9E(1)+72E(2)+E(3)-2E(12)-216E(18)+54E(36)\,.\end{aligned} \tag{6}$$

Therefore, from (5) and (6), we obtain

$$\begin{aligned}1870A(0)=582E(1)-10329E(2)+19618E(3)-39236E(12)\\+30987E(18)-3492E(36)\,.\end{aligned}$$

Then, from the definition of $A(0)$ and the functional equation of $\zeta_K(s)$, we have the following:

COROLLARY 1.

$$\begin{aligned}\zeta_K(1/3)^2=C_1(582E(1)-10329E(2)+19618E(3)\\-39236E(12)+30987E(18)-3492E(36))\,,\end{aligned}$$

where

$$C_1=(2\pi)^{5/3}/1870\sqrt{3}\,|d|^{1/3}L(1,\chi)\Gamma(1/3)^2\,.$$

COROLLARY 2.

$$\begin{aligned}\zeta_K(2/3)^2=C_2(582E(1)-10329E(2)+19618E(3)\\-39236E(12)+30987E(18)-3492E(36))\,,\end{aligned}$$

where

$$C_2=(2\pi)^{7/3}/1870\sqrt{3}\,|d|^{2/3}L(1,\chi)\Gamma(2/3)^2\,.$$

Here $\Gamma(s)$ denotes the gamma function.

Put

$$\omega=2\int_1^\infty \frac{dx}{\sqrt{4x^3-4}},$$

which is a period of the elliptic curve $y^2=4x^3-4$. Then it is easily verified that

$$\omega=\Gamma(1/3)^3/2^{8/3}\pi .$$

With the aid of this result and the formula for the class number, the above corollaries can be rewritten as follows:

COROLLARY 3. *Let h be the class number of K and w the number of roots of unity in K. Then*

$$\zeta_K(1/3)^2=C_3\omega^{-2/3}(582E(1)-10329E(2)+19618E(3) -39236E(12)+30987E(18)-3492E(36)),$$

where

$$C_3=w|d|^{1/6}/3740\cdot 2^{1/9}3^{1/2}h .$$

COROLLARY 4. *Let h and w be as above. Then*

$$\zeta_K(2/3)^2=C_4\omega^{2/3}(582E(1)-10329E(2)+19618E(3) -39236E(12)+30987E(18)-3492E(36)),$$

where

$$C_4=2^{1/9}3^{1/2}w/935|d|^{1/6}h .$$

References

[1] TOYOIZUMI, M.; Formulae for the values of zeta and L-functions at half integers, *Tokyo J. Math.*, **4** (1981), 193–201.

[2] TOYOIZUMI, M.; Ramanujan's formulae for certain Dirichlet series, *Comment. Math. Univ. St. Pauli*, **30** (1981), 149–173.

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