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An Example of Infinite Measure Preserving Ergodic Geodesic Flow on a Surface with Constant Negative Curvature

by

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§0. Introduction

E. Hopf has proposed in [1] (see the footnotes in p. 272 and p. 276) a problem of constructing an infinite measure preserving ergodic geodesic flow on a surface with constant negative curvature. As is known from [1], [2], this problem is equivalent to the one of constructing a Fuchsian group which is of the first class and whose non-Euclidean (NE) area of the fundamental domain is infinite. In the footnote in p. 276 in [1] E. Hopf has indicated also that an example of a Fuchsian group of certain type can be expected to satisfy these conditions. The purpose of this paper is to give a concrete example of a Fuchsian group satisfying these conditions indicated by E. Hopf, and thus we have succeeded in giving an example of infinite measure preserving ergodic geodesic flow on a surface with constant negative curvature.

In §1, we show that the geodesic flow on a surface with constant negative curvature is ergodic if and only if the Riemann surface associated with the Fuchsian group is of null boundary in R. Nevanlinna's sense. In §2, we show that the example of a Fuchsian group given by E. Hopf ([1]) is of the first class and the NE area of the fundamental domain for it is infinite. Therefore, the geodesic flow associated with the Fuchsian group of the example is ergodic and infinite measure preserving. In §3, we show that the entropy of any time changed flow obtained from the geodesic flow of our example is infinite.

§1. Ergodic theory and Fuchsian groups

Let G be the group of all linear fractional transformations S taking the unit disk $U=(|z|<1)$ onto itself:

$$S\colon w=e^{i\alpha}\frac{z-a}{1-\bar{a}z} \qquad (|a|<1),$$

then

$$\frac{|dw|}{1-|w|^2}=\frac{|dz|}{1-|z|^2}$$

for any $S\in G$. We introduce an NE metric ds and an NE surface element dV in U by

$$ds=\frac{2|dz|}{1-|z|^2}, \qquad dV=\frac{4dxdy}{(1-|z|^2)^2} \qquad (z=x+iy),$$

then ds and dV are invariant by any $S\in G$. Let Γ be a countable subgroup of G. z, $z'\in U$ are called equivalent by Γ if $z'=S(z)$ for some $S\in\Gamma$.

DEFINITION 1. A countable subgroup Γ of G is called a Fuchsian group if it is properly discontinuous, namely, equivalents of each $z\in U$ have no cluster points in U.

For simplicity, in this paper we will be concerned with a Fuchsian group Γ for which 0, the center of U, is not a fixed point of any $S\in\Gamma$, different from the identity. A domain D_0 in U is called a fundamental domain of Γ, if any two points of D_0 are not equivalent and any point $z\in U$ has its equivalent in D_0. Now we shall construct a fundamental domain D_0 of Γ. Let $\Gamma=\{S_n: n=0, 1, \cdots\}$, $S_0=I$ (identity), $z_n=S_n(0)$ be equivalents of $z=0$ and Δ_0 be the set of z, such that $s(z, 0)<s(z, z_n)$ $(n=1, 2, \cdots)$. We call Δ_0 a normal domain of Γ. The boundary B of Δ_0 consists of arcs on circles orthogonal to the unit circle $|z|=1$ and a certain closed set on $|z|=1$, which may be empty. Δ_0 has the following properties:

(i) any two points of Δ_0 are not equivalent,

(ii) any point $a\in U$ has its equivalent in $\Delta_0\cup B$,

(iii) any point $a\in B$ has its unique equivalent $a'\in B$ $(a'\neq a)$ such that $|a|=|a'|$.

By (iii), $B\cap U$ consists of equivalent pairs B_k, B_k' $(k=1, 2, \cdots)$ of arcs on circles orthogonal to $|z|=1$. We see easily that a fundamental domain D_0 is obtained from Δ_0, by adding B_k $(k=1, 2, \cdots)$ to Δ_0. We shall call the fundamental domain D_0 thus constructed the canonical fundamental domain. Now we shall define a geodesic flow associated with a Fuchsian group Γ. Let $\Omega=D_0\times \boldsymbol{T}^1$, where $\boldsymbol{T}^1$ is the 1-dimensional torus. $(z, \varphi)\in\Omega$ is called a line element which makes an angle φ at z with the positive real axis. We define a metric element $d\sigma$ and a volume element dm in Ω by

$$d\sigma^2=\frac{4|dz|^2}{(1-|z|^2)^2}+d\varphi^2, \qquad dm=\frac{4dxdyd\varphi}{(1-|z|^2)^2},$$

then $m(\Omega)=2\pi V(D_0)$. The measure m is invariant by any $S\in G$. For $(z, \varphi)\in\Omega$ we see easily that there exists a unique circular arc $C(z, \varphi)$ orthogonal to $|z|=1$, which is tangent to the line element (z, φ) at z. $C(z, \varphi)$ is called a geodesic determined by (z, φ). Let $\eta_1=e^{i\theta_1}$, $\eta_2=e^{i\theta_2}$ be its two end points on $|z|=1$, such that η_1 corresponds to $-\infty$ point on $C(z, \varphi)$ and η_2 $+\infty$ point. Let z_0 be the mid-point of the arc $\widehat{\eta_1\eta_2}=C(z, \varphi)$ and s be the NE length of the arc $\widehat{z_0 z}$ of $C(z, \varphi)$, where $s>0$ if z lies on $\widehat{z_0\eta_2}$ and $\leqq 0$, otherwise. Then we have a one to one correspondence between (z, φ) and (η_1, η_2, s), so that we write $(z, \varphi)=(\eta_1, \eta_2, s)$. Now we consider a flow $(T^t$;

$-\infty<t<\infty$) in Ω defined by

$$T^t\colon (z, \varphi)=(\eta_1, \eta_2, s)\to(z', \varphi')=(\eta_1, \eta_2, s+t),$$

then T^t is a measure preserving transformation of Ω onto itself. (T^t) is called a geodesic flow associated with Γ. $(z, \varphi)\in\Omega$ is said to disappear positively (negatively), if for $t\to\infty$ $(t\to-\infty)$ $\sigma(T^t(z, \varphi), (z_0, \varphi_0))\to\infty$ holds, where (z_0, φ_0) is an arbitrarily fixed point in Ω.

DEFINITION 2. A flow (T^t) is called ergodic if whenever $f(\omega)$ and $g(\omega)>0$ are m-integrable in Ω

$$\lim_{t\to\infty}\frac{\int_0^t f(T^s\omega)ds}{\int_0^t g(T^s\omega)ds}=\frac{\int f dm}{\int g dm}$$

holds for almost all $\omega\in\Omega$ in the sense of the measure m. (T^t) is called dissipative if almost every $(z, \varphi)\in\Omega$ disappears positively as well as negatively.

Now we shall define an open Riemann surface of null boundary or of positive boundary. Let Γ be a Fuchsian group and D_0 be its fundamental domain. If we identify the equivalent points on the boundary of D_0, then D_0 can be considered as a Riemann surface $F=U/\Gamma$, which we call the Riemann surface associated with Γ ([3]). Let F be an open Riemann surface. We exhaust F by a sequence of compact Riemann surfaces F_n $(n=0, 1, \cdots)$, $\bigcup_{n=0}^{\infty} F_n=F$, where $\bar{F}_n\subset F_{n+1}$ and the boundary Γ_n of F_n consists of a finite number of analytic Jordan curves. Let $\omega_n(z)$ be the harmonic measure of Γ_n with respect to F_n-F_0; namely, $\omega_n(z)$ is harmonic in F_n-F_0, $\omega_n=1$ on Γ_n, $\omega_n=0$ on Γ_0. Then by the maximum principle $\omega_{n+1}(z)<\omega_n(z)$ in F_n-F_0, so that by Harnack's theorem, $\lim_{n\to\infty}\omega_n(z)=\omega(z)$ uniformly in the wider sense in F_n-F_0. $\omega(z)$ is called the harmonic measure of the ideal boundary of F with respect to $F-F_0$. There occur two cases: either (i) $\omega(z)\equiv 0$ or (ii) $\omega(z)\not\equiv 0$. In case (ii), $0<\omega(z)<1$ in $F-F_0$, $\omega=0$ on Γ_0.

DEFINITION 3 (R. Nevanlinnan [4]). F is said to be null boundary, or of positive boundary, according as (i) $\omega(z)\equiv 0$ or (ii) $\omega(z)\not\equiv 0$ take place.

THEOREM 1. *Let Γ be a Fuchsian group, $F=U/\Gamma$ the Riemann surface associated with Γ and (T^t) the geodesic flow associated with Γ. Then (T^t) is ergodic if and only if $F=U/\Gamma$ is of null boundary. And (T^t) is dissipative if and only if $F=U/\Gamma$ is of positive boundary.*

Now we shall start to prove Theorem 1. Assume that $F=U/\Gamma$ is of positive boundary. Then by the Theorem A we deduce that there exists a Green's function on $F=U/\Gamma$, namely, $F=U/\Gamma\in P_G$ which is defined as follows (M. Tsuji [5]). We exhaust F by a sequence of compact Riemann surfaces F_n $(n=0, 1, \cdots)$ $\bigcup_{n=0}^{\infty} F_n=F$, where $\bar{F}_n\subset F_{n+1}$ and the boundary Γ_n of F_n consists of a finite number of analytic Jordan curves. Let $a\in F_0$ and $g_n(z, a)$ be the Green's function of F_n, with a as its pole, then by the maximum principle $g_n(z, a) < g_{n+1}(z, a)$ on F_n, so that by Harnack's theorem, (i) $\lim_{n\to\infty} g_n(z, a) \equiv \infty$ or (ii) $\lim_{n\to\infty} g_n(z, a) = g(z, a)$ uniformly in the wider sense on F. In case (i), we say there exists no Green's function on F and in case (ii), we call $g(z, a)$ the Green's function of F. $g(z, a)$ is harmonic on F, except at a, while

$$\omega(z) = g(z, a) - \log \frac{1}{z-a}$$

is harmonic at a. We denote $F \in P_G$, or $F \in O_G$, according as there does or does not exist a Green's function on F.

THEOREM A (K. I. Virtanen [6]). *Let F be an open Riemann surface. Then $F \in O_G$ if and only if F is of null boundary. Hence $F \in P_G$ if and only if F is of positive boundary.*

It follows, by Theorem A, from the assumption that $F = U/\Gamma$ is of positive boundary that we can deduce that $F = U/\Gamma \in P_G$. Then by the next Theorem B we deduce that the Fuchsian group Γ is of convergence type, which is defined as follows (M. Tsuji [5]). Let $\Gamma = \{S_n: n = 0, 1, \cdots\}$, D_0 be its canonical fundamental domain, $a_n = S_n(a)$ $(a \in D_0)$ and $z_n = S_n(0)$. We introduce another metric $[a, b]$ for two points a, b in U:

$$[a, b] = \left| \frac{a-b}{1-\bar{a}b} \right| .$$

Then $[a, b] = [b, a]$ and $[S(a), S(b)] = [a, b]$ for any $S \in G$. Since $|a| = [a, 0] = [a_n, z_n]$, we have

$$1 - |a|^2 = \frac{(1-|a_n|^2)(1-|z_n|^2)}{|1-\bar{a}_n z_n|^2} \leqq \frac{4(1-|a_n|)(1-|z_n|)}{(1-|z_n|)^2},$$

$$1 - |a|^2 \leqq \frac{4(1-|a_n|)(1-|z_n|)}{(1-|a_n|)^2},$$

so that

$$\frac{(1-|a|^2)(1-|z_n|)}{4} \leqq 1 - |a_n| \leqq \frac{4(1-|z_n|}{1-|a|^2} .$$

From this we have that either

$$\text{(i)} \quad \sum_{n=0}^{\infty} (1-|a_n|) < \infty \quad \text{or} \quad \text{(ii)} \quad \sum_{n=0}^{\infty} (1-|a_n|) = \infty$$

independently of $a \in D_0$. A Fuchsian group Γ is said to be of convergence, or of divergence type, according as the case (i) or (ii) occur.

THEOREM B (P. J. Myrberg [7], M. Tsuji [5]). *Let $F = U/\Gamma$ be the Riemann surface associated with a Fuchsian group Γ. Then $F = U/\Gamma \in P_G$ if and only if Γ is of* *convergence type. Hence $F=U/\Gamma\in O_G$ if and only if Γ is of divergence type.*

Thus by Theorem B, from the assumption that $F=U/\Gamma\in P_G$ we deduce that Γ is of convergence type. Then by the next Theorem C we deduce that $\lim_{r\to 1}|((re^{i\theta}))|=1$ for almost all $e^{i\theta}$ on $\boldsymbol{T}^1$, where $((z))$ is in D_0 and equivalent to $z\in U$ and $|\cdot|$ is the absolute value of a complex number.

THEOREM C (Z. Yûjôbô [8], M. Tsuji [5]). (i) *If Γ is of divergence type, then $E(\theta)=\{((re^{i\theta}));\ 0\leqq r<1\}\cap D_0$ is everywhere dense in D_0 for almost all $e^{i\theta}$ on $\boldsymbol{T}^1$.*
(ii) *If Γ is of convergence type, then $\lim_{r\to 1}|((re^{i\theta}))|=1$ for almost all $e^{i\theta}$ on $\boldsymbol{T}^1$.*

Thus by Theorem C, from the assumption that Γ is of convergence type we deduce that $\lim_{r\to 1}|((re^{i\varphi}))|=1$ for almost all $e^{i\varphi}$ on $\boldsymbol{T}^1$. Now for $\varphi\in[0, 2\pi)$ we have

$$\lim_{r\to 1}|((re^{i\varphi}))|=1 \Leftrightarrow \lim_{t\to\infty} s(\mathrm{Proj}\,(T^t(0, \varphi)), 0)=\infty\,,$$

where $\quad \mathrm{Proj}\,(z, \theta)=z \quad$ for $\quad (z, \theta)\in\Omega\,,$

$$\Leftrightarrow \lim_{t\to\infty}\sigma(T^t(0, \varphi), (0, 0))=\infty\,,$$

$$\Leftrightarrow (0, \varphi) \text{ disappears positively}\,.$$

In this way, from the assumption that $\lim_{r\to 1}|((re^{i\varphi}))|=1$ for almost all $e^{i\varphi}$ on $\boldsymbol{T}^1$ we deduce that $(0, \varphi)$ disappears positively for almost all $e^{i\varphi}$ on $\boldsymbol{T}^1$. Then, by the next Theorem D we deduce that Γ is of the second class, which is defined as follows by E. Hopf ([1], [2]). A Fuchsian group Γ is of the first class, if the positively disappearing orbits issuing from a fixed point z of D_0 form a set of directions at z of the angular measure zero, i.e., $|\Delta_z|=0$, where $\Delta_z=\{\varphi;\ (z, \varphi)$ disappears positively$\}$ and the symbol $|\cdot|$ denotes the angular measure. If this is true for one fixed point $z\in D_0$, it is true for any other fixed point $z\in D_0$. Γ is said to be of the second class if it is not of the first class.

THEOREM D (E. Hopf [1], [2]). *For a Fuchsian group of the first class the geodesic flow (T^t) associated with it is ergodic. For a Fuchsian group of the second class (T^t) is dissipative. In this case we obtain a somewhat sharper result: if a Fuchsian group is of the second class, then for any point $z\in D_0$, (z, φ) disappears positively and negatively for almost all directions φ.*

Thus by Theorem D, from the assumption that $(0, \varphi)$ disappears positively for almost all $e^{i\varphi}$ on $\boldsymbol{T}^1$ we deduce that Γ is of the second class, which is equivalent to say that (T^t) is dissipative. Therefore from the assumption that $F=U/\Gamma$ is of positive boundary we have deduced that (T^t) is dissipative.

Secondly, from the assumption that (T^t) is dissipative we shall deduce that $F=U/\Gamma$ is of positive boundary. This can be obtained by analogous discussions from Theorems A, B, C and D. Thus the proof of Theorem 1 is completed.

COROLLARY. *Let Γ be a Fuchsian group, (T^t) the geodesic flow and $F=U/\Gamma$ the Riemann surface associated with Γ. Then the following are equivalent;*

(i) *$F=U/\Gamma$ is of null boundary,*

(ii) *Γ is of divergence type,*

(iii) *(T^t) is ergodic,*

(iv) *Γ is of the first class.*

§ 2. The construction of an example

We shall define an isometric circle of a linear fractional transformation. For a linear transformation

$$w=S(z)=\frac{az+b}{cz+d}, \qquad ad-bc=1 \tag{1}$$

we have

$$\frac{dw}{dz}=S'(z)=\frac{1}{(cz+d)^2},$$

whence, lengths are multiplied by $|S'(z)|$ or $|cz+d|^{-2}$ and areas are multiplied by $|S'(z)|^2$, or $|cz+d|^{-4}$. Then circle I: $|cz+d|=1$, $c\neq 0$, which is the complete locus of points in a neighbourhood of which lengths and areas are unaltered in magnitude by the transformation (1), is called the isometric circle of the transformation (1). Its center is $-d/c$, its radius $1/|c|$. The inverse transformation

$$w=\frac{-dz+b}{cz-a}$$

has the isometric circle I': $|cz-a|=1$. Its center is a/c, its radius $1/|c|$. The transformation S carries I into I'. It is a well known fact that the isometric circles of the transformations of a Fuchsian group are orthogonal to the principal circle $|z|=1$ (L. R. Ford [9], Theorem 1 in p. 67). We now explain a method of forming a Fuchsian group by combination.

THEOREM E (L. R. Ford [9]). *Given an infinite number of circles I_1, I_1'; I_2, I_2'; $\cdots$; I_j, I_j'; $\cdots$, which are of equal radius in pairs, and which are exterior to one another or are externally tangent we set up the transformation S_j so that S_j and S_j^{-1} have the isometric circles I_j and I_j' respectively. Then a group Γ generated by $S_1, S_2, \cdots, S_j, \cdots$ is a Fuchsian group and has for a fundamental domain D_0*

$$(*) \qquad \{z;\ s(0, z)<s(z_n, z)\ n=1, 2, \cdots\}=\bigcap_{j=1}^{\infty} (\mathrm{Ext}(I_j)\cap \mathrm{Ext}(I_j')),$$

where $z_n=S_n(0)$ are all equivalents to $z=0$, $(n=1, 2, \cdots)$, $\mathrm{Ext}(I_j)=$exterior region of I_j. Here we remark that 0, the center of $|z|=1$, is not a fixed point of any $S\in\Gamma$, different from the identity.

This Theorem follows from (b) in p. 58 and Theorem 9 in p. 71 of L. R. Ford [9].

Example (E. Hopf's conjecture. See the footnote in p. 276 of [1]). Let a, K be constants $0<a<1$, $0<K$ such that

$$\sum_{j=1}^{\infty} \theta_j = \pi ,$$

where $\theta_j = a^j/K$. Let

$$A = \exp\{i(3\pi/2)\} , \qquad A_j = \exp\left\{ i\left(\frac{3\pi}{2} + \sum_{k=1}^{j} \theta_k\right)\right\}, \qquad (j=1, 2, \cdots)$$

and

$$A'_j = \exp\left\{ i\left(\frac{3\pi}{2} - \sum_{k=1}^{j} \theta_k\right)\right\}, \qquad (j=1, 2, \cdots)$$

be infinite sequences of points on $|z|=1$. Let $\widehat{A_{j-1}A_j}$ be the circle orthogonal to $|z|=1$ through A_{j-1} and A_j, and $I_1=\widehat{AA_1}$, $I_2=\widehat{A_1A_2}$, $\cdots$, $I_j=\widehat{A_{j-1}A_j}$, $\cdots$. Similarly let $I'_1=\widehat{AA'_1}$, $I'_2=\widehat{A'_1A'_2}$, $\cdots$, $I'_j=\widehat{A'_{j-1}A'_j}$, $\cdots$. An infinite number of these circles I_1, I'_1; I_2, I'_2; $\cdots$; I_j, I'_j; $\cdots$ are of equal radius in pairs, and are exterior to one another or are externally tangent. (See Fig. 1.) Let S_j be the transformation so that S_j and S_j^{-1} have the isometric circles I_j and I'_j respectively. By Theorem E, we have that Γ, a group generated by $\{S_1, S_2, \cdots, S_j, \cdots\}$, is a Fuchsian group and for its fundamental domain D_0 (*) holds, and 0 is not a fixed point of any $S \in \Gamma$, different from the identity. E. Hopf has conjectured that Γ is of the first class, that is, the geodesic flow (T^t) associated with Γ is ergodic.

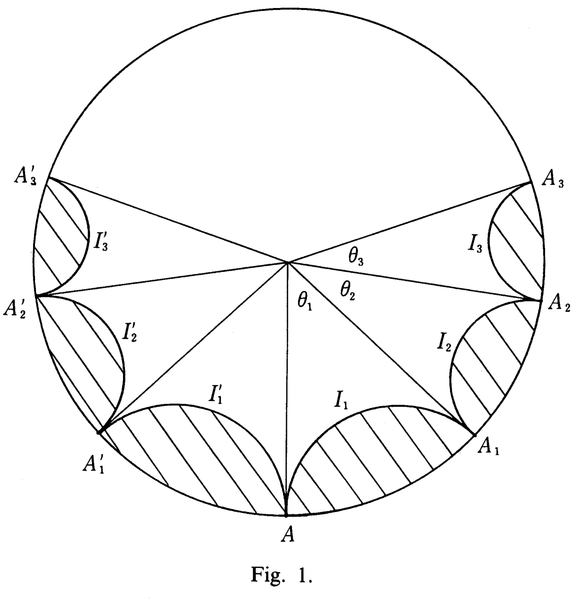

Fig. 1.

THEOREM 2. *The example of infinite measure preserving geodesic flow on a surface with constant negative curvature given above is ergodic.*

For the proof of Theorem 2, first we shall show that the total mass $m(\Omega)=2\pi V(D_0)$ of the invariant measure m is infinite. Let $\varDelta$ be a triangle in U bounded by three circles orthogonal to $|z|=1$ and A, B, C, be the inner angles at its vertices. Let $V(\varDelta)$ be the NE area of $\varDelta$, then $V(\varDelta)=\pi-(A+B+C)$. The fundamental domain D_0 is a union of a countable number of triangles $\varDelta_j$, the inner angles of which are 0, 0 and θ_j. So that we have $V(D_0)=\infty$ and $m(\Omega)=2\pi V(D_0)=\infty$. For the proof of ergodicity of (T^t), which is equivalent to the assertion that $F=U/\Gamma$ is of null boundary by Theorem 1, we begin with a sufficient condition that a Riemann surface associated with a Fuchsian group may be of null boundary. A Fuchsian group which has an infinite number of generators is called a Fuchsoid group.

THEOREM F (P. Laasonen [10]). *Let $\Gamma=\{S_0, S_1, S_2, \cdots, S_n, \cdots\}$ be a Fuchsoid group. Suppose that* 0, *the center of $|z|=1$, is not a fixed point of any $S\in\Gamma$, different from the identity. Let $z_n=S_n(0)$ $(n=1, 2, \cdots)$ be equivalents to $z=0$ and D_0 be the canonical fundamental domain, that is, $D_0=\{z;\ s(0, z)<s(z_n, z),\ n=1, 2, \cdots\}$. Let $l(r)=|(z; |z|=r)\cap D_0|$, where $|\cdot|$ denotes the ordinary Euclidean metric. Suppose that*

$$\int^1 \frac{1}{l(r)}dr=\infty ,$$

then the Riemann surface $F=U/\Gamma$ associated with Γ is of null boundary.

For the Fuchsian group of our example we have the following lemma.

LEMMA 1.

$$l(r)<C(1-r) \qquad in \quad 0<r<1 ,$$

where $C=4(1+\pi)/1-a$ is a constant independent of r.

The proof of this lemma will be given later.

Now we shall prove that (T^t) associated with the Fuchsian group of our example is ergodic. By Lemma 1,

$$\int^1 \frac{1}{l(r)}dr=\infty ,$$

and hence by Theorem F, $F=U/\Gamma$ is of null boundary. Consequently, Theorem 1 yields that (T^t) is ergodic. The proof of Theorem 2 is now complete.

Remark. In [10], P. Laasonen has given an example of a Fuchsoid group such that the Riemann surface associated with it is of null boundary and the NE area of its fundamental domain is infinite and

$$\int^1 \frac{1}{l(r)}dr<\infty .$$

For the proof of Lemma 1 we shall divide the proof into several steps.

Step 1. The circle $S(r)=\{z; |z|=r\}$ $(0<r<1)$ cuts the positive real axis at C_1 and the radius OB, $B=(\cos\theta, \sin\theta)$ $(0<\theta<\pi)$, of the unit circle $S=\{|z|=1\}$ at C_2. The circle orthogonal to S through $A(1, 0)$ and B is denoted by $C(A, B)$ and $C(A, B)$ cuts the circle $S(r)$ at D_1 and D_2. (See Fig. 2.) It is easily established that a necessary and sufficient condition for the circle $S(r)$ to intersect $C(A, B)$ is that

$$\cos(\theta/2) \leqq 2r/(1+r^2)\,. \tag{2}$$

In the following, we assume that $C(A, B)$ intersects $S(r)$. Let $l(r, \theta)$ be the difference of the Euclidean lengths of circular arcs $\widehat{C_1C_2}$ and $\widehat{D_1D_2}$ on $S(r)$, that is,

$$l(r, \theta)=\widehat{C_1C_2}-\widehat{D_1D_2}=|\{re^{i\psi}; 0\leqq\psi\leqq\theta\}\cap \mathrm{Ext}\,\{C(A, B)\}|\,.$$

In Step 1 we shall show that $l(r, \theta)<4\pi(1-r)^2/\theta$. Draw the tangents at C_1 and D_1 to $S(r)$ and extend them until they meet each other at E. $C(A, B)$ cuts a straight line C_1E at F. (See Fig. 2.) Then we have easily that

$$\widehat{C_1D_1}<\overline{C_1E}+\overline{D_1E}=2\overline{C_1E}<2\overline{C_1F}\,. \tag{3}$$

On the other hand, the equation for $C(A, B)$ is

$$(x-1)^2+(y-\tan\varphi)^2=\tan^2\varphi\,,$$

where $\varphi=\theta/2$ and $l(r, \theta)=2\widehat{C_1D_1}$. So that from (3) we have

$$l(r, \theta)<4\{\tan\varphi-\sqrt{\tan^2\varphi-(r-1)^2}\}\,. \tag{4}$$

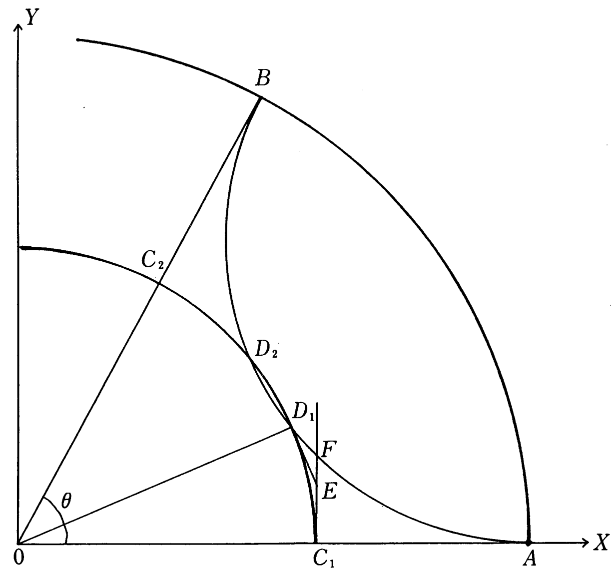

Fig. 2.

For an arbitrary fixed constant $k>0$, we have an inequality

$$k-\sqrt{k^2-x^2}\leqq x^2/k \quad \text{in} \quad 0\leqq x\leqq k.$$

Setting $k=\tan\varphi$, $x=1-r$, we have

$$\tan\varphi-\sqrt{\tan^2\varphi-(r-1)^2}\leqq(1-r)^2/\tan\varphi \tag{5}$$

for $\tan\varphi\geqq 1-r>0$, i.e., $1-\tan\varphi\leqq r<1$. Since the inequality $2\varphi/\pi<\sin\varphi$ $(0<\varphi<\pi/2)$ yields

$$\cot\varphi<\pi/2\varphi \quad \text{in} \quad 0<\varphi<\pi/2, \tag{6}$$

we have from (4), (5) and (6),

$$l(r, \theta)<4\pi(1-r)^2/\theta. \tag{7}$$

Here we remark that (7) is valid if (2) holds.

Step 2. Now we recall θ_i and I_i in our example and set

$$l_1(r)=\sum_{I_i\cap S(r)\neq\phi} l(r, \theta_i)+\sum_{I_i'\cap S(r)\neq\phi} l(r, \theta_i).$$

Since I_i and I_i' are of equal radius,

$$l_1(r)=2\sum_{I_i\cap S(r)\neq\phi} l(r, \theta_i).$$

In Step 2 we shall show that

$$l_1(r)<\frac{4\pi}{1-a}(1-r).$$

By (2) the condition $I_i\cap S(r)\neq\phi$ is equivalent to

$$\theta_i\geqq 2\,\mathrm{Cos}^{-1}(2r/1+r^2). \tag{8}$$

Setting

$$N=\mathrm{Max}\,\{i;\ \theta_i=a^i/K\geqq 2\,\mathrm{Cos}^{-1}(2r/1+r^2)\}, \tag{9}$$

we have

$$a^N\geqq 2K\,\mathrm{Cos}^{-1}(2r/1+r^2), \tag{10}$$

$$a^{N+1}<2K\,\mathrm{Cos}^{-1}(2r/1+r^2). \tag{11}$$

By (7), (8), (9) and $\theta_i=a^i/K$, we have

$$l_1(r)<8\pi(1-r)^2\sum_{i=1}^{N}\theta_i^{-1}$$

$$=8\pi K(1-r)^2\sum_{i=1}^{N}(a^{-1})^i$$

$$<8\pi K(1-r)^2\left\{(a^{-1})\frac{1-(a^{-1})^N}{1-(a^{-1})}\right\},$$

and from $a^{-1}>1$,

$$<\frac{8\pi K}{1-a}(1-r)^2(a^{-1})^N .$$

So that, by (10)

$$l_1(r)<\frac{4\pi(1-r)^2}{1-a}\times\frac{1}{\mathrm{Cos}^{-1}(2r/1+r^2)}. \tag{12}$$

The derivative of $f(r)=1-r-\mathrm{Cos}^{-1}(2r/1+r^2)$ is positive, so that $f(r)$ is increasing in $0<r<1$. Since $f(1)=0$, we have $f(r)<0$ in $0<r<1$, that is,

$$\frac{1}{\mathrm{Cos}^{-1}(2r/1+r^2)}<\frac{1}{1-r} \quad \text{in} \quad 0<r<1 . \tag{13}$$

By (12) and (13), we have

$$l_1(r)<\frac{4\pi}{1-a}(1-r). \tag{14}$$

Step 3. Set

$$l_2(r)=\sum_{I_i\cap S(r)=\phi} l(r,\theta_i)+\sum_{I_i'\cap S(r)=\phi} l(r,\theta_i),$$

then, since I_i and I_i' are of equal radius,

$$=2\sum_{I_i\cap S(r)=\phi} l(r,\theta_i),$$

and from (8) and (9) we have

$$=2\sum_{i=N+1}^{\infty} r\theta_i .$$

In Step 3 we shall show that $l_2(r)<4(1-r)/1-a$. From $\theta_i=a^i/K$ we have

$$l_2(r)=\frac{2ra^{N+1}}{K}\left(\sum_{i=0}^{\infty} a^i\right)=\frac{2ra^{N+1}}{(1-a)K},$$

so that by (11),

$$<\frac{4r}{1-a}\mathrm{Cos}^{-1}(2r/1+r^2). \tag{15}$$

The derivative of $g(r)=1-r-r\,\mathrm{Cos}^{-1}(2r/1+r^2)$ is negative in $0<r<1$, so that $g(r)$ is decreasing in $0<r<1$. Since $g(1)=0$, we have $g(r)>0$, that is,

$$r\,\mathrm{Cos}^{-1}(2r/1+r^2)<1-r \quad \text{in} \quad 0<r<1 . \tag{16}$$

By (15) and (16),

$$l_2(r) < 4(1-r)/1-a\,. \tag{17}$$

Therefore, from (14) and (17), we have

$$l(r) < \frac{4(1+\pi)}{1-a}(1-r)\,.$$

The proof of Lemma 1 is completed.

§3. The entropy of time changed flows obtained from an infinite measure preserving ergodic geodesic flow

Let (T^t) be an ergodic flow on a σ-finite measure space (Ω, m).

DEFINITION 4 (G. Maruyama [11], H. Totoki [12]). Let Ω be divided into two disjoint (T^t)-invariant measurable sets $\Omega-N$ and N such that $m(N)=0$. Let $\varphi(t, \omega)$ be a real measurable function defined on $(-\infty, \infty)\times(\Omega-N)$ with the following properties: For every fixed $\omega\in\Omega-N$

(A.1) $\varphi(t, \omega)$ is finite for all t,
(A.2) $\varphi(t, \omega)$ is continuous and nondecreasing in t,
(A.3) $\varphi(s+t, \omega)=\varphi(s, \omega)+\varphi(t, T^s\omega)$ for all t and s,
(A.4) $\varphi(0, \omega)=0$, $\lim_{t\to\infty}\varphi(t, \omega)=\infty$, $\lim_{t\to-\infty}\varphi(t, \omega)=-\infty$.

Then φ is called an additive functional of (T^t) with the carrier $\Omega-N$. An additive functional is said to be integrable, if it is integrable in ω for every fixed t. The regular set $\hat{\Omega}$ of φ is the totality of points $\omega\in\Omega-N$ for which the property

(R) $\qquad \varphi(t, \omega)>0 \quad$ for all $\quad t>0$

holds. Define

$$\hat{T}^t\omega = T^{\tau(t,\omega)}\omega\,, \qquad \tau(t, \omega)=\sup\{s;\ \varphi(s, \omega)\leqq t\}$$

for all $-\infty<t<\infty$ and all $\omega\in\hat{\Omega}$. Define

$$\hat{m}(B)=\int_\Omega\left(\int_0^1 \chi_B(T^u\omega)d\varphi(u, \omega)\right)dm$$

for $B\subset\hat{\Omega}$. Then $(\hat{T}^t)$ is a flow on the σ-finite measure space $(\hat{\Omega}, \hat{m})$ and is called the time changed flow obtained from (T^t) by φ. If φ is integrable, then $(\hat{\Omega}, \hat{m})$ is a finite measure space. Let $a(\omega)$ be non-negative, integrable and $\int_\Omega adm>0$. Define

$$\varphi(t, \omega)=\int_0^t a(T^u\omega)du$$

for all t. Then φ is an integrable additive functional of (T^t), $\hat{\Omega}=\{\omega;\ a(\omega)>0\}$ and $d\hat{m}=a(\omega)dm$.

THEOREM 3. *Let (T^t) be our example of infinite measure preserving ergodic* *geodesic flow on a surface with constant negative curvature. Let φ be any integrable additive functional of (T^t) and $(\hat{T}^t)$ be the time changed flow on $(\hat{\Omega}, \hat{m})$ obtained from (T^t) by φ. Then the measure theoretical entropy $h_{\hat{m}}(\hat{T}^t)$ is infinite, that is, $h_{\hat{m}}(\hat{T}^t)=\infty$ for any $t \neq 0$.*

For the proof of Theorem 3, first we shall have Lemma 2 and Lemma 3 concerning horocycles, which are defined as follows. A horocycle in U is an ordinary Euclidean circle which is internally tangent to the unit circle S. If $(z, \varphi) \in U \times \boldsymbol{T}^1$, then there is a unique horocycle such that (z, φ) is an inward normal to the horocycle, which is denoted by $S^+(z, \varphi)$. Similarly there is a unique horocycle such that (z, φ) is an outward normal to the horocycle, which is denoted by $S^-(z, \varphi)$. To each (z, φ) we may associate a unique line element (z, φ') so that (z, φ'), (z, φ) have the same orientation as the positive real and positive imaginary axes respectively. The horocycle flow (H^t) in $U \times \boldsymbol{T}^1$ is described as follows: (z, φ) moves with the unit velocity along the horocycle $S^+(z, \varphi)$ in the direction of (z, φ'). Then, for a Fuchsian group Γ in UH^t is a measure preserving transformation of $\Omega = D_0 \times \boldsymbol{T}^1$ onto itself. (H^t) is called the horocycle flow on Ω.

LEMMA 2.

$$H^{ve^{-t}}T^t = T^t H^v \qquad \textit{for any} \quad v, t \in (-\infty, \infty),$$

where (T^t) is the geodesic flow.

The proof is omitted. (See (10.2) in p. 285 in [1].)

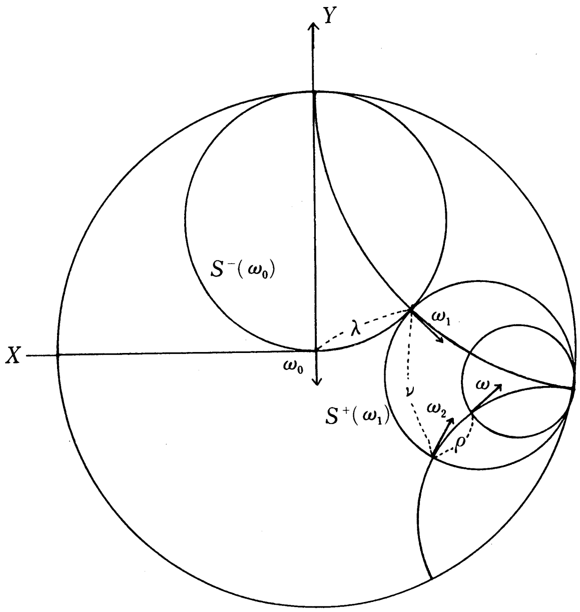

Fig. 3

We shall introduce a new coordinate system in $U \times \boldsymbol{T}^1$. First the line element $\omega_0=(0+0i, 3\pi/2)$ is translated by the NE distance λ along $S^-(\omega_0)$ counterclockwise into a line element ω_1. Secondly ω_1 is translated by the NE distance ν along $S^+(\omega_1)$ counterclockwise into a line element ω_2. Lastly ω_2 is translated by the NE distance ρ along the geodesic curve determined by ω_2 into a line element ω, the coordinates of which are given by (z, φ). Then (λ, ν, ρ) become new coordinates for $\omega=(z, \varphi)$ in $U \times \boldsymbol{T}^1$. (See Fig. 3.)

LEMMA 3.

$$dm=\frac{4dxdyd\varphi}{(1-|z|^2)^2}=2d\lambda d\nu d\rho\,.$$

The proof of Lemma 3 is given later.

For the special representation of the geodesic flow (T^t), now we shall define the base transformation which plays an important role in the proof of Theorem 3. Let Γ be the Fuchsian group of our example which is constructed in § 2, D_0 be its canonical fundamental domain, Ω be $D_0 \times \boldsymbol{T}^1$ and (T^t) be the infinite measure preserving ergodic geodesic flow associated with Γ. Let a, b and c be small positive constants and I be the interval $[0, a]$, J be $[0, b]$ and K be $[0, c)$, then the set $A=I\times J\times K=\{(\lambda, \nu, \rho); \lambda\in I, \nu\in J, \rho\in K\}$ may be regarded as a subset in Ω. Let $I\times J\times\{0\}=\{(\lambda, \nu, 0); \lambda\in I, \nu\in J\}$, a subset in A, be denoted by M. By the ergodicity of the flow (T^t) we can define

DEFINITION 5. A base transformation on M of the geodesic flow (T^t) is a transformation S: $M\to M$ defined by

$$S(\omega)=T^{\theta(\omega)}\omega\,,\ \theta(\omega)=\text{Minimum}\,\{t>0;\ T^t\omega\in M\} \qquad \text{for} \quad \omega\in M\,.$$

Let $d\mu$ be the normalized measure $d\lambda d\nu/ab$ on M. Since (T^t) preserveds dm and $dm=2d\lambda d\nu d\rho$ on A by Lemma 3, S preserves μ. Since (T^t) is ergodic, S is ergodic on (M, μ). Since $dm=2d\lambda d\nu d\rho$ on A by Lemma 3, (T^t) is isomorphic to the special flow constructed under the function θ on the dynamical system (M, μ, S).

PROPOSITION. *The base transformation S of the geodesic flow (T^t) on M is invertible, μ-preserving and ergodic. The measure theoretical entropy $h_\mu(S)$ of the invertible dynamical system (S, μ) is infinite, that is, $h_\mu(S)=\infty$. (T^t) is isomorphic to the special flow (M, μ, S, θ).*

Remark. Since $m(\Omega)=\infty$, we have $\int_M\theta d\mu=\infty$ for the ceiling function θ.

The idea of the proof is due to Ya. Sinai [13]. In order to prove $h_\mu(S)=\infty$, we shall construct an increasing partition ζ with respect to S such that the conditional entropy $H_\mu(S\zeta\,|\,\zeta)$ is infinite. Let η be a partition of $A=I\times J\times K$ into the horocycle arcs $\{(\lambda)\times J\times(\rho); \lambda\in I, \rho\in K\}$ and $\hat{\eta}$ be a partition of Ω whose elements are elements of η and a single set $\Omega-A$. Let T be a transformation in the flow (T^t) at $t=c$, that is, $T=T^c$, where c is the length of the interval K. Let us denote

$$(\hat{\eta})^0_{-\infty}=\text{the subdividing partition into connected components of } \bigvee_{n=-\infty}^{0} T^n(\hat{\eta})\,.$$

Then from the analogous discussions as in the proofs of Theorem 5.1 and Theorem 5.2 of [13], it follows that almost every element of $(\hat{\eta})^0_{-\infty}$ in A is a subarc of an element of η and that $(\hat{\eta})^0_{-\infty}$ is (T^t)-increasing, that is, $T^t((\hat{\eta})^0_{-\infty})>(\hat{\eta})^0_{-\infty}$ for any $t>0$. Set

$$\zeta = \text{the restriction of } (\hat{\eta})^0_{-\infty} \text{ to the set } M\,.$$

Then ζ satisfies following conditions;

(i) $S\zeta>\zeta$,

(ii) almost every element of ζ is a subarc of an element of the partition η_M which is the restriction of η to M,

(iii) $|C_{S\zeta}(\omega)|=e^{-\theta(S^{-1}\omega)}|C_\zeta(S^{-1}\omega)|$ a.a. $\omega\in M$,

and by iterations, for $n>0$

$$\left|C_{S^n\zeta}(\omega)\right| = \exp\left\{-\sum_{k=1}^{n}\theta(S^{-k}\omega)\right\}\left|C_\zeta(S^{-n}\omega)\right| \qquad \text{a.a.} \quad \omega\in M\,, \tag{15}$$

where $C_{S^n\zeta}(\omega)$, $C_\zeta(S^{-n}\omega)$ are elements of the partitions $S^n\zeta$ and ζ of M containing ω and $S^{-n}\omega\in M$ respectively and $|C|$ is the NE length of a subarc C.

In fact, (i) follows from the fact that $(\hat{\eta})^0_{-\infty}$ is (T^t)-increasing. (ii) follows from the fact that almost every element of $(\hat{\eta})^0_{-\infty}$ is a subarc of an element of η. (iii) follows from Lemma 2.

Now let us show that the conditional entropy $H_\mu(S\zeta\,|\,\zeta)=\infty$. $H_\mu(S\zeta\,|\,\zeta)$, the conditional entropy of $S\zeta$ with respect to ζ, is defined by

$$H_\mu(S\zeta\,|\,\zeta) = -\int_M \log_2\mu(C_{S\zeta}(\omega)\,|\,C_\zeta(\omega))d\mu\,,$$

where $\mu(\cdot\,|\,C_\zeta(\omega))$ is the conditional probability measure of μ with respect to the partition ζ. By $d\mu=d\lambda dv/ab$ and (ii), we have

$$\mu(C_{S^n\zeta}(\omega)\,|\,C_\zeta(\omega)) = |C_{S^n\zeta}(\omega)|/|C_\zeta(\omega)|\,. \tag{16}$$

Since S is ergodic,

$$H_\mu(S\zeta\,|\,\zeta) = -\lim_{n\to\infty}\frac{1}{n}\log_2\mu(C_{S^n\zeta}(\omega)\,|\,C_\zeta(\omega)) \tag{17}$$

for a.a. $\omega\in M$. Let p and q be positive constants such that

$$\mu(E)>0, \qquad \text{where} \quad E=\{\omega\in M;\, p<|C_\zeta(\omega)|<q\}.$$

From the recurrence property of the transformation S to the set E with $\mu(E)>0$, we have

$$\mu(E\cap\overline{\lim_{n\to\infty}}\, S^nE)>0\,. \tag{18}$$

For a fixed point $\omega\in E\cap\overline{\lim}_{n\to\infty} S^nE$ there exists an increasing sequence of positive integers $\{m_1, m_2, \cdots, m_i, \cdots\}$ such that

$$\omega \in S^{m_i}E \qquad (i=1, 2, \cdots). \tag{19}$$

From (15), (16) and (19) we have

$$\begin{aligned}\mu(C_{S^{m_i}\zeta}(\omega)|\, C_\zeta(\omega)) &= |\, C_{S^{m_i}\zeta}(\omega)\,| / |\, C_\zeta(\omega)\,| \\ &\leqq \frac{1}{p}|\, C_{S^{m_i}\zeta}(\omega)\,| = \frac{1}{p}\exp\left\{-\sum_{k=1}^{m_i}\theta(S^{-k}\omega)\right\} \times |\, C_\zeta(S^{-m_i}\omega)\,| \\ &\leqq \frac{q}{p}\exp\left\{-\sum_{k=1}^{m_i}\theta(S^{-k}\omega)\right\},\end{aligned}$$

so that

$$-\frac{1}{m_i}\log_2\mu(C_{S^{m_i}\zeta}(\omega)\,|\, C_\zeta(\omega)) \geqq \frac{\log_2 e}{m_i}\sum_{k=1}^{m_i}\theta(S^{-k}\omega) - \frac{1}{m_i}\log_2(q/p).$$

Since S is ergodic and $\int_M \theta d\mu = \infty$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\theta(S^{-k}\omega) = \infty \qquad \text{a.a.} \quad \omega \in M,$$

hence

$$\lim_{i\to\infty}\frac{1}{m_i}\sum_{k=1}^{m_i}\theta(S^{-k}\omega) = \infty.$$

Therefore from (17) and (18) we have $H_\mu(S\zeta\,|\,\zeta) = \infty$. Since $S\zeta > \zeta$ from (i), we have $h_\mu(S) \geqq H_\mu(S\zeta\,|\,\zeta)$ and hence $h_\mu(S) = \infty$. This completes the proof of Proposition.

Proof of Theorem 3. By Proposition, (T^t) is isomorphic to the special flow (S, M, μ, θ). By Theorem 9.1 of [12] the time changed flow $(\hat{T}^t)$ by φ obtained from the special flow (θ, S) is isomorphic to the special flow $\{\varphi(\theta(x), x, 0), S_D\}$, where $D = \{x \in M;\ \varphi(\theta(x), x, 0) > 0\}$. Since the base transformation S on (M, μ) is ergodic from Proposition, the induced transformation S_D on (D, μ) is ergodic. Thus by Lemma 10.2 of [12] and the formula of L. M. Abramov [14]

$$h_{\hat{m}}(\hat{T}^t) = \frac{|\,t\,| h_\mu(S_D)\mu(D)}{\int_D \varphi(\theta(x), x, 0)d\mu} = \frac{|\,t\,| h_\mu(S)\mu(M)}{\int_D \varphi(\theta(x), x, 0)d\mu}.$$

By Proposition $h_\mu(S) = \infty$, so that $h_{\hat{m}}(\hat{T}^t) = \infty$ for $t \neq 0$. This completes the proof of Theorem 3.

Proof of Lemma 3. We map U on to the upper half plane $(\mathrm{Im}(z) > 0)$ by $f(z) = i(i+z/i-z)$ and let the corresponding new and old coordinates in $(\mathrm{Im}(z) > 0) \times \boldsymbol{T}^1$ be denoted by the same notations (λ, v, ρ) and $(x+iy, \varphi)$, then we have $ds = |\,dz\,|/y$, $dm = dxdyd\varphi/y^2$. First $\omega_0 = (0+i, 3\pi/2) \in (\mathrm{Im}(z) > 0) \times \boldsymbol{T}^1$ is translated by the NE distance λ along $S^-(\omega_0)$, the straight line $y=1$ parallel to the real axis, counterclockwise into a

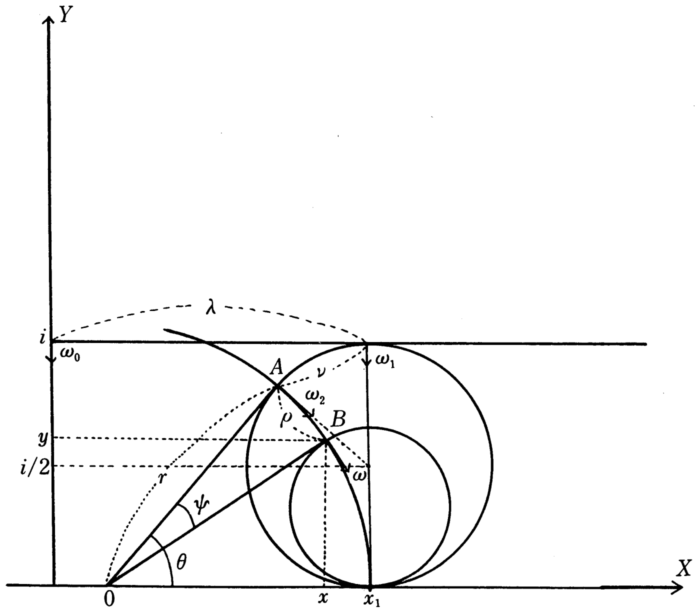

Fig. 4

line element ω_1 whose carrier point is denote by $(x_1, +i)$. Secondly ω_1 is translated by the NE distance ν along $S^+(\omega_1)$, the Euclidean circle with the center $(x_1, +(i/2))$, and radius 1/2, counterclockwise into a line element ω_2, the carrier point of which is denoted by A. Lastly ω_2 is translated by the NE distance ρ along the geodesic curve determined by ω_2 into a line element ω, the coordinates of which in $(\mathrm{Im}(z)>0)\times \boldsymbol{T}^1$ are given by $(x+iy, \varphi)$. (See Fig. 4.) The geodesic curve determined by ω_2 is a Euclidean circle with its center on the real axis, the radius of which is denoted by r. For the proof of Lemma 3, we shall show that a functional determinant $D(x, y, \varphi)/D(\lambda, \nu, \rho)=2y^2$ and therefore we have $dm=dxdyd\varphi/y^2=2d\lambda d\nu d\rho$. Let a carrier point of ω be B, the angle between OA and the real axis be θ and $\angle AOB=\psi$. Then the coordinates $(z, \varphi)=(x, y, \varphi)$ of ω are expressed as follows;

$$x=(x_1-r)+r\cos(\theta-\psi)\,, \tag{1}$$

$$y=r\sin(\theta-\psi)\,, \tag{2}$$

$$\varphi=\theta-\psi-\frac{\pi}{2}\,. \tag{3}$$

Since ω_0 is translated by the NE distance λ along $S^-(\omega_0)$, we have

$$\lambda=\int_{\widehat{\omega_0\omega_1}} 1ds=\int_0^{x_1} 1dx=x_1\,. \tag{4}$$

Since $S^+(\omega_1)$ is a Euclidean circle with its center $(x_1, +(i/2))$ and radius 1/2, the parametric representation of the circular arc $\widehat{\omega_1\omega_2}$ on $S^+(\omega_1)$ is given by

$$x=x_1-\frac{1}{2}\sin u\,, \qquad y=\frac{1}{2}+\frac{1}{2}\cos u \qquad (0\leqq u\leqq\theta)\,.$$

So that

$$dx=-\frac{1}{2}\cos u du\,, \qquad dy=-\frac{1}{2}\sin u du\,, \qquad ds=|dz|/y=du/(1+\cos u)\,,$$

hence we have

$$v=\int_{\widehat{\omega_1\omega_2}} 1ds=\int_0^\theta \frac{du}{1+\cos u}=\tan(\theta/2)\,.$$

Since $r=\overline{OA}$ is the radius of the circle and θ is the angle between OA and the real axis, we have

$$r=(2\tan(\theta/2))^{-1}\,.$$

Hence by the above discussions, we have

$$r=(2v)^{-1}\,. \tag{5}$$

Similarly the parametric representation of the circular arc $\widehat{\omega_2\omega}$ on $C(\omega_2)$ is given by

$$x=(x_1-r)+r\cos u\,, \qquad y=r\sin u \qquad (\theta-\psi\leqq u\leqq\theta)\,.$$

So that

$$dx=-r\sin u du\,, \qquad dy=r\cos u du\,, \qquad ds=|dz|/y=(\sin u)^{-1}du\,,$$

hence we have

$$\rho=\int_{\widehat{\omega_2\omega}} 1ds=\int_{\theta-\psi}^\theta \frac{du}{\sin u}=\frac{1}{2}\left[\log\left(\frac{1-\cos u}{1+\cos u}\right)\right]_{\theta-\psi}^\theta,$$

that is,

$$\cos(\theta-\psi)=\frac{(1+\cos\theta)-e^{-2\rho}(1-\cos\theta)}{(1+\cos\theta)+e^{-2\rho}(1-\cos\theta)}\,.$$

Since $v=\tan(\theta/2)$,

$$1+\cos\theta=2/(1+v^2)\,, \qquad 1-\cos\theta=2v^2/(1+v^2)\,.$$

From above discussions, we have

$$\cos(\theta-\psi)=\frac{1-v^2e^{-2\rho}}{1+v^2e^{-2\rho}}. \tag{6}$$

From (6) we have

$$\sin(\theta-\psi)=\frac{2ve^{-\rho}}{1+v^2e^{-2\rho}}. \tag{7}$$

Now we shall calculate the functional determinant $D(x, y, \varphi)/D(\lambda, v, \rho)$. By (1), (4), (5) and (6) we have

$$\frac{\partial x}{\partial \lambda}=1. \tag{8}$$

By (2), (5) and (7) we have

$$\frac{\partial y}{\partial \lambda}=0, \tag{9}$$

$$\frac{\partial y}{\partial v}=\frac{-2ve^{-3\rho}}{(1+v^2e^{-2\rho})^2}, \tag{10}$$

$$\frac{\partial y}{\partial \rho}=-\frac{e^{-\rho}(1-v^2e^{-2\rho})}{(1+v^2e^{-2\rho})^2}. \tag{11}$$

By (3) and (6) we have

$$\frac{\partial \varphi}{\partial \lambda}=0, \tag{12}$$

$$\frac{\partial \varphi}{\partial v}=\frac{2e^{-\rho}}{1+v^2e^{-2\rho}}, \tag{13}$$

$$\frac{\partial \varphi}{\partial \rho}=\frac{-2ve^{-\rho}}{1+v^2e^{-2\rho}}. \tag{14}$$

From (8) ~ (14), we have

$$\frac{D(x, y, \varphi)}{D(\lambda, v, \rho)}=\frac{2e^{-2\rho}}{(1+v^2e^{-2\rho})^2},$$

and from (2), (5) and (7)

$$=2y^2,$$

so that $dxdyd\varphi/y^2=2d\lambda dvd\rho$. This completes the proof of Lemma 3.

Addendum: After this paper had been completed, Professor T. Kuroda informed the author that the equivalence of the first class and the divergence type for the Fuchsian group had been already shown by P. J. Nicholls [15].

Correction: Coordinates (ϕ, s, ρ) in $U\times \boldsymbol{T}^1$ in my papers [16] and [17] were given erroneously. Replace them by the coordinates (λ, v, ρ) appearing in § 3 of this paper.

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