COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 31, No. 2, 1982

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

On a Family of Locally Finite Invariant Measures for a Cylinder Flow

by

Hitoshi NAKADA

(Received January 16, 1982)

Introduction

We consider the circle $\boldsymbol{T}=R/Z$ with addition mod. 1. For an irrational number θ, $0<\theta<1$, and a real number η, $0<\eta<1$, we define the transformation $T_{\theta,\eta}$ as follows:

$$T_{\theta,\eta}(x, y)=(x+\theta \text{ (mod. 1)}, y+\chi_{[0,\eta)}(x)-\eta) \qquad \text{for} \quad (x, y)\in T\times E_\eta$$

where E_η is the closed additive subgroup of R generated by η and $1-\eta$ and $\chi_{[0,\eta)}$ is the indicator function of $[0, \eta)$.

It is easy to see that $T_{\theta,\eta}$ preserves the Haar measure $\tilde{m}$ of $\boldsymbol{T}\times E_\eta$ and that each orbit of it is unbounded if $\eta\notin Z\theta$ (mod.1). So when $\eta\notin Z\theta$ (mod.1), there are following two problems arising from the theory of uniform distributions (see Veech [9]);

(i) Is $T_{\theta,\eta}$ ergodic with respect to $\tilde{m}$?

(ii) When $T_{\theta,\eta}$ is ergodic, does there exist a locally finite invariant measure for it which is singular to $\tilde{m}$?

These problems, mainly (i), have been considered by several authors (Conze [1], Stewart [7], [8], Veech [9], etc.).

In the case of the group extension of a uniquely ergodic transformation with a compact space, the relative unique ergodicity of the extended transformation follows from its ergodicity, in general. The problem (ii) is connected with this question for non-compact case.

In this note, we give the affirmative answer to the problem (ii). We show that there exists a locally finite invariant measure for $T_{\theta,\eta}$ which is singular with respect to $\tilde{m}$ whenever $\eta\notin Z\theta$ (mod.1) and the measure is a product measure (§2). Furthermore we show that there are uncountably many such measures which are singular with respect to each other for almost all (θ, η) (§3). Our arguments depend on the conjugacy problem of piecewise linear homeomorphisms on $\boldsymbol{T}$ considered by Herman [3]. So we discuss the properties of these transformations in §1.

The construction of the measure in §2 is related to the associated flow of $f_{\lambda,\beta,\alpha}$ defined in §1. The argument in §3 is modified from the proof of the non-existence of the absolutely continuous invariant measure for $f_{\lambda,\beta,\alpha}$ in Herman [3] and this method is closely related to the computation of T-set for $f_{\lambda,\beta,\alpha}$. Moreover, it is possible to calculate the ratio set of $f_{\lambda,\beta,\alpha}$, which is analogous to the calculation of the essential value of the cylinder flow. However, we do not treat this problem in this paper.

In the sequel, we denote by m the Haar measure of $\boldsymbol{T}$, by θ an irrational number of (0, 1) and denote by η a real number of (0, 1).

I would like to thank Professor Y. Ito for his helpful comments and Professor T. Kamae who brought this problem to my attention.

§1. Piecewise linear homeomorphisms of $\boldsymbol{T}$

In this section, we consider the essential properties of the piecewise linear homeomorphisms of $\boldsymbol{T}$ introduced by Herman [3], §6. For the properties of orientation preserving homeomorphisms of $\boldsymbol{T}$, we refer to Katznelson [4].

For $\lambda>1$ and $\beta>0$, we define the mapping $f_{\lambda,\beta}$ of [0, 1) onto itself as follows:

$$f_{\lambda,\beta}(x)=\begin{cases}\lambda\cdot x & \text{if} \quad x\in[0, a(\lambda, \beta)) \\ \lambda^{-\beta}(x-1)+1 & \text{if} \quad x\in[a(\lambda, \beta), 1)\end{cases}$$

where $a(\lambda, \beta)$ is determined by $\lambda\cdot a(\lambda, \beta)=\lambda^{-\beta}(a(\lambda, \beta)-1)+1$.

Next we define the orientation preserving homeomorphism $\hat{f}_{\lambda,\beta,b}$ of $\boldsymbol{T}$ by

$$\hat{f}_{\lambda,\beta,b}(x)=f_{\lambda,\beta}(x)+b \qquad (\mathrm{mod.}1) \qquad \text{for} \quad x\in[0, 1)$$

where $0<b<1$. We denote by $\rho(\hat{f}_{\lambda,\beta,b})$ the rotation number of $\hat{f}_{\lambda,\beta,b}$. Then the mapping

$$b \longmapsto \rho(\hat{f}_{\lambda,\beta,b})$$

is a continuous, non-decreasing function of [0, 1] onto [0, 1]. Indeed, the mappings $b\mapsto\hat{f}_{\lambda,\beta,b}$ and $\hat{f}_{\lambda,\beta,b}\mapsto\rho(\hat{f}_{\lambda,\beta,b})$ are continuous with respect to the uniform topology and the other properties are also evident.

Thus, for any α, $0<\alpha<1$, there exists a number b_0, $0<b_0<1$, such that $\rho(\hat{f}_{\lambda,\beta,b_0})=\alpha$. Let us define $\mathscr{R}_\alpha$, the rotation of $\boldsymbol{T}$, by

$$\mathscr{R}_\alpha(x)=x+\alpha \quad (\mathrm{mod.}\ 1) \qquad \text{for} \quad x\in T.$$

We have the following lemma from Denjoy's theorem:

LEMMA 1. *If $\rho(\hat{f}_{\lambda,\beta,b})=\alpha$ is irrational, then there exists an orientation preserving homeomorphism h of $\boldsymbol{T}$ such that*

$$h\circ\hat{f}_{\lambda,\beta,b}=\mathscr{R}_\alpha\circ h \qquad \text{and} \quad h(0)=0.$$

From this lemma, we have the following:

LEMMA 2. *If α, $0<\alpha<1$, is irrational, the number b_0 having the property $\rho(\hat{f}_{\lambda,\beta,b_0})=\alpha$ is determined uniquely.*

Proof. If $b_0<b_1$, then $d(\hat{f}_{\lambda,\beta,b_0}(x), \hat{f}_{\lambda,\beta,b_1}(x))=b_1-b_0$ where $d(\cdot, \cdot)$ is the usual metric of $\boldsymbol{T}$. On the other hand, it follows from Lemma 1 that $\{\hat{f}^n_{\lambda,\beta,b_0}(x), n\in Z\}$ is dense in $\boldsymbol{T}$ and that the order of the orbit coincides with that of $\{\mathscr{R}^n_\alpha(0), n\in Z\}$. From

these properties, it is easy to show that

$$\rho(\hat{f}_{\lambda,\beta,b_0}) \neq \rho(\hat{f}_{\lambda,\beta,b_1}) .$$

The same is true for the case of $b_1 < b_0$. q.e.d.

DEFINITION. For any irrational number α, $0<\alpha<1$, we define $f_{\lambda,\beta,\alpha}$ to be $\hat{f}_{\lambda,\beta,b_0}$ such that $\rho(\hat{f}_{\lambda,\beta,b_0}) = \alpha$.

Thus we rewrite Lemma 1 as follows:

LEMMA 3. *If α, $0<\alpha<1$, is irrational, then there exists an orientation preserving homeomorphism $h_{\lambda,\beta,\alpha}$ of $\boldsymbol{T}$ such that*

$$h_{\lambda,\beta,\alpha} \circ f_{\lambda,\beta,\alpha} = \mathcal{R}_\alpha \circ h_{\lambda,\beta,\alpha} \qquad \textit{and} \qquad h_{\lambda,\beta,\alpha}(0)=0 .$$

Moreover it is easy to show the following proposition (see Herman [3]).

PROPOSITION 1. *For any $\lambda>1$, $\beta>0$ and irrational α, $0<\alpha<1$, we have*

(i) $h_{\lambda,\beta,\alpha}(a(\lambda,\beta)) = \beta/(1+\beta)$,

(ii) $\lambda^{-2(1+\beta)} < Df^{q_n}_{\lambda,\beta,\alpha} < \lambda^{2(1+\beta)}$ *for all* $n>1$,

where q_n denotes the denominator of the n-th convergent of the continued fraction expansion of α,

(iii) *if* $\beta/(1+\beta) \notin Z\alpha$ (mod.1), *then $h_{\lambda,\beta,\alpha}$ is not absolutely continuous and so $f_{\lambda,\beta,\alpha}$ does not have any absolutely continuous invariant probability measure.*

Remark. In (ii), we adopt the right derivative when the right derivative differs from the left one.

First, we prove the ergodicity of $f_{\lambda,\beta,\alpha}$ with respect to the Haar measure m, which would simplify the subsequent discussions.

THEOREM 1. *For any irrational α, $0<\alpha<1$, $(\boldsymbol{T}, f_{\lambda,\beta,\alpha}, m)$ is ergodic.*

The proof of this theorem is essentially due to Katznelson [5], p. 160.

Proof. For simplicity, we write $\mathcal{R}$ and f instead of $\mathcal{R}_\alpha$ and $f_{\lambda,\beta,\alpha}$ respectively. And q_n denotes the denominator of the n-th convergent of α.

If n is even, then

$$\{\mathcal{R}^j[x, \mathcal{R}^{q_n}(x)),\ 0 \leqq j < q_{n+1}\} \cup \{\mathcal{R}^j[\mathcal{R}^{q_{n+1}}(x), x),\ 0 \leqq j < q_n\}$$

is a partition of $\boldsymbol{T}$ for any $x \in \boldsymbol{T}$. So

$$\{f^j[x, f^{q_n}(x)),\ 0 \leqq j < q_{n+1}\} \cup \{f^j[f^{q_{n+1}}(x), x),\ 0 \leqq j < q_n\}$$

is also a partition of $\boldsymbol{T}$ by Lemma 3. Moreover it follows from [5], Lemma 2-1 that

$$\lambda^{-2(1+\beta)} \leqq \frac{Df^j(y_2)}{Df^j(y_1)} \leqq \lambda^{2(1+\beta)}, \qquad 0 \leqq j < q_{n+1} \tag{1}$$

for any $y_1, y_2 \in [x, f^{q_n}(x))$ and the same inequalities hold for any $y_1, y_2 \in [f^{q_{n+1}}(x), x)$ and $0 \leqq j < q_n$.

Let U be a Borel subset of $\boldsymbol{T}$ with $f(U)=U$ and $m(U)>0$. From the density theorem and Lemma 3, we see that there exists $x\in U$ with the following property: for any $\varepsilon>0$ there exists an integer n such that

$$m([x, f^{q_n}(x))\cap U)\geqq(1-\varepsilon)\cdot m([x, f^{q_n}(x))$$

and

$$m([f^{q_{n+1}}(x), x)\cap U)\geqq(1-\varepsilon)\cdot m(f^{q_{n+1}}(x), x)) .$$

Hence we have

$$m(f^j[x, f^{q_n}(x))\cap U)\geqq(1-\varepsilon\lambda^{2(1+\beta)})\cdot m(f^j[x, f^{q_n}(x))) , \qquad 0\leqq j<q_{n+1} ,$$

and

$$m(f^j[f^{q_{n+1}}(x), x)\cap U)\geqq(1-\varepsilon\lambda^{2(1+\beta)})\cdot m(f^j[f^{q_{n+1}}(x), x)) , \qquad 0\leqq j<q_n ,$$

by the assumption made on U and (1). Thus we get

$$m(U)\geqq 1-\varepsilon\lambda^{2(1+\beta)}$$

and this implies $m(U)=1$. q.e.d.

Remark. The same result holds for mappings of class P by the same argument (see Herman [3], p. 74 and p. 86).

§2. A locally finite invariant measure for $T_{\theta,\eta}$

For a fixed η, we consider the real number $\beta>0$ such that $\eta=\beta/(1+\beta)$. Let F_β $(=F_{\lambda,\beta})$ be the closed subgroup of R generated by $\log\lambda$ and $\beta\cdot\log\lambda$ for some fixed $\lambda>1$. We define $S_{\theta,\beta}$ of $\boldsymbol{T}\times F_\beta$ by

$$S_{\theta,\beta}(x, y)=(f_{\lambda,\beta,\theta}(x), y+\log Df_{\lambda,\beta,\theta}(x)) \qquad \text{for} \quad (x, y)\in\boldsymbol{T}\times F_\beta .$$

We denote by ν the restriction to F_β of the measure $e^{-y}dy$ of R. It is easy to show that $m\times\nu$ is an invariant measure for $S_{\theta,\beta}$ equivalent to the Haar measure of $\boldsymbol{T}\times F_\beta$. And we have

$$\log Df_{\lambda,\beta,\theta}(x)=\log\lambda\cdot(1+\beta)\cdot\left[\chi_{[0,a(\lambda,\beta))}(x)-\frac{\beta}{1+\beta}\right] \tag{2}$$

from the definition of $f_{\lambda,\beta,\theta}$.

Now we define the homeomorphism $H_{\lambda,\beta,\theta}$ of $\boldsymbol{T}\times F_\beta$ to $T\times E_\eta$ by

$$H_{\lambda,\beta,\theta}(x, y)=\left(h_{\lambda,\beta,\theta}(x),\frac{y}{(1+\beta)\cdot\log\lambda}\right) \qquad \text{for} \quad (x, y)\in\boldsymbol{T}\times F_\beta .$$

From Lemma 3, (2) and Proposition 1–(i), we have

$$T_{\theta,\eta}\circ H_{\lambda,\beta,\theta}=H_{\lambda,\beta,\theta}\circ S_{\theta,\beta} . \tag{3}$$

Let us define measures $h_{\lambda,\beta,\theta}m$ and $H_{\lambda,\beta,\theta}(m\times\nu)$ by

$$\begin{cases}(h_{\lambda,\beta,\theta}m)(A)=m(h_{\lambda,\beta,\theta}^{-1}A) & \text{for} \quad A\subset \boldsymbol{T} \\ (H_{\lambda,\beta,\theta}(m\times v))(B)=(m\times v)(H_{\lambda,\beta,\theta}^{-1}B) & \text{for} \quad B\subset \boldsymbol{T}\times E_{\eta}\,,\end{cases}$$

then it follows from Theorem 1 and Proposition 1–(iii) that m and $h_{\lambda,\beta,\theta}m$ are singular to each other if $\eta\notin Z\theta$ (mod.1), so $\mu=H_{\lambda,\beta,\theta}(m\times v)$ and $\tilde{m}$ are also singular to each other in such a case.

THEOREM 2. *If $\eta\notin Z\theta$ (mod.1), then $\mu=H_{\lambda,\beta,\theta}(m\times v)$ is a locally finite conservative invariant measure for $T_{\theta,\eta}$ which is singular with respect to $\tilde{m}$.*

Proof. For simplicity, we write f and H instead of $f_{\lambda,\beta,\theta}$ and $H_{\lambda,\beta,\theta}$. It is easy to see that μ is an invariant measure for $T_{\theta,\eta}$ from (3), so we only need to prove the conservativeness. Let $K=\boldsymbol{T}\times[-M, M]$ for $M>(1+\beta)/2$. We show that the induced transformation $S_{\theta,\beta}|_K$ is well-defined with respect to $m\times v$. If this statement holds, then it is easy to see that $(T_{\theta,\eta}, \mu)$ is conservative.

We put

$$A_1=\{x:\log Df^n(x)>0 \quad \text{for all} \quad n>0\}$$

and

$$A_2=\{x:\log Df^n(x)<0 \quad \text{for all} \quad n>0\}.$$

Since (f, m) is ergodic, we have $m(A_1)=m(A_2)=0$. Thus there exists a positive integer $l=l(x)$ for m-a.a. x such that

$$\text{“}\log Df^l(x)>0 \quad \text{and} \quad \log Df^{l+1}(x)<0\text{”}$$

or

$$\text{“}\log Df^l(x)<0 \quad \text{and} \quad \log Df^{l+1}(x)>0\text{”}.$$

For such x and any y, $-M\leqq y\leqq M$, we get

$$S_{\theta,\beta}^l(x, y)\in K \quad \text{or} \quad S_{\theta,\beta}^{l+1}(x, y)\in K\,.$$

Thus the induced transformation $S_{\theta,\beta}|_K$ is well-defined $(m\times v)$-a.e. q.e.d.

§3. Singularity of $h_{\lambda,\beta,\theta}m$

In this section, we consider the problem of singularity between $h_{\lambda,\beta,\theta}m$ and $h_{\lambda',\beta,\theta}m$ for $\lambda\neq\lambda'$. If these are singular to each other, then the corresponding product measures defined in §2 are also singular to each other.

We put

$$\Phi=h_{\lambda',\beta,\theta}^{-1}\circ h_{\lambda,\beta,\theta}\,,$$

then we have

$$f_{\lambda',\beta,\theta}\circ\Phi=\Phi\circ f_{\lambda,\beta,\theta} \tag{4}$$

from Lemma 3. So $h_{\lambda,\beta,\theta}m$ and $h_{\lambda',\beta,\theta}m$ are absolutely continuous to each other if and only if Φ and Φ^{-1} are absolutely continuous to each other. Since both measures are ergodic, they would be equivalent to each other if one of them can be shown to be absolutely continuous with respect to the other.

THEOREM 3. *Let q_n be the denominator of the n-th convergent of θ. If $\{c_n: c_n = \beta/(1+\beta)\cdot q_n \pmod{1},\ n>0\}$ has a limit point $p\neq 0$, then $h_{\lambda,\beta,\theta}m$ and $h_{\lambda',\beta,\theta}m$ are singular to each other for all distinct λ and λ'.*

Proof. Suppose that Φ and Φ^{-1} are absolutely continuous. It follows from Proposition 1–(i) that

$$\begin{cases}\log Df_{\lambda',\beta,\theta}(x)=\log \lambda'\cdot[(1+\beta)\cdot\chi_{[0,a(\lambda',\beta))}(x)-\beta]\\ \log Df_{\lambda,\beta,\theta}(x)=\log \lambda\cdot[(1+\beta)\cdot\chi_{[0,a(\lambda,\beta))}(x)-\beta]\end{cases}\tag{5}$$

and that

$$\Phi(a(\lambda,\beta))=a(\lambda',\beta)\,.\tag{6}$$

Hence we have

$$\log D\Phi^{-1}(f_{\lambda',\beta,\theta}\Phi(x))+\log Df_{\lambda',\beta,\theta}(\Phi(x))+\log D\Phi(x)=\log Df_{\lambda,\beta,\theta}(x)\quad m\text{-a.e.}$$

by (4). Thus we get

$$(\log\lambda'-\log\lambda)\cdot[(1+\beta)\cdot\chi_{[0,a(\lambda,\beta))}(x)-\beta]=\log D\Phi(f_{\lambda,\beta,\theta}(x))-\log D\Phi(x)\ m\text{-a.e.}\,.\tag{7}$$

If we put

$$\psi(x)=\exp\left[-2\pi i\cdot\frac{\log D\Phi(x)}{(\log\lambda'-\log\lambda)(1+\beta)}\right],$$

then we get

$$\psi\circ f_{\lambda,\beta,\theta}(x)=\exp\left(2\pi i\cdot\frac{\beta}{1+\beta}\right)\cdot\psi(x)\qquad m\text{-a.e.}$$

by (7). So we have

$$\begin{aligned}\psi\circ f^{q_n}_{\lambda,\beta,\theta}(x)&=\exp\left(2\pi i\cdot\frac{\beta}{1+\beta}q_n\right)\cdot\psi(x)\\&=\exp(2\pi i\cdot c_n)\cdot\psi(x).\end{aligned}\tag{8}$$

Now it is possible to show that $\psi\circ f^{q_n}_{\lambda,\beta,\theta}$ converges to ψ strongly in $L^1(\boldsymbol{T}, m)$ (see Herman [3], §6). On the other hand, it follows from the assumption that there exists a subsequence $\{c_{n'}\}$ converging to p as n' tends to ∞. Thus (8) does not converge to ψ and this contradicts the above statement. q.e.d.

Remarks. 1) Since q_n/q_{n+1} are always irreducible, $\{c_n\}$ has a non-zero limit point for any rational β. Moreover it is well-known that $\{c_n\}$ is uniformly distributed for a.a. β (see Kuipers and Niederreiter [6]).

2) Recently Stewart [7] has shown that $\{c_n\}$ has a non-zero limit point for all β,

$\beta/(1+\beta) \notin Z\theta$ (mod.1), when θ is of constant type.

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Department of Mathematics
Keio University
3–14–1, Hiyoshi, Kohoku-ku
Yokohama 223
Japan