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Precipitous Ideals Over κ and the Estimate of the Size of 2^{κ}

by

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§0. Introduction

To determine the behavior of the continuum function $2^{\aleph_\alpha}$ is one of the main problems in set theory. If κ is a singular cardinal with an uncountable cofinality, then by [8], 2^{κ} is determined by the continuum function below κ. If κ is a regular cardinal, then by [1], 2^{κ} is independent of the continuum function below κ. Nevertheless, if κ carries a κ^+-saturated ideal and if the generalized continuum hypothesis (GCH) holds below κ, then by [6], it holds at κ; more generally, if κ carries a precipitous ideal and if GCH holds below κ, then 2^{κ} is less than or equal to the saturation number of the ideal. In [6], Jech and Prikry applied the method of generic ultrapowers which is a combination of Cohen's method of forcing and the method of ultraproducts used in model theory and in the theory of large cardinals.

In this paper, it is showed how 2^{κ} is determined by the continuum function below κ and the saturation number of a precipitous ideal over κ. In estimating 2^{κ}, the main tool is the functionals which are canonical representatives for elements of the generic ultrapower. In §2, we introduce the concepts of functionals and degrees of functionals, and show some fundamental properties. Degrees of functionals correspond to norms of ordinal functions introduced by Galvin and Hajnal in [2]. In §3, we investigate degrees of various functionals. In §4, we prove the main theorem, which asserts that 2^{κ} is less than or equal to the degrees of a certain functional depending upon the ideal over κ and the continuum function below κ, and apply it to some particular cases.

Our set theoretical notation is standard. In this paper, we define the precipitous ideal in terms of functionals. For properties of ideals which are equivalent to "precipitous," the reader may refer to [3].

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§1. Preliminaries

Let κ be a regular uncountable cardinal number. A collection I of subsets of κ is an *ideal* over κ if

(i) $\phi \in I$ and $\kappa \notin I$
(ii) if $X \in I$ and $Y \subseteq X$ then $Y \in I$
(iii) if $X \in I$ and $Y \in I$ then $X \cup Y \in I$.

In this paper we deal only with *nontrivial κ-complete* ideals over κ:
(iv) $\{\alpha\} \in I$ for all $\alpha < \kappa$
(v) if $\gamma < \kappa$ and $X_\alpha \in I$ for all $\alpha < \gamma$, then $\bigcup_{\alpha<\gamma} X_\alpha \in I$.

Throughout the paper, *ideal* means a nontrivial κ-complete ideal over κ.

Let I be a given ideal, and X a subset of κ. If $X \in I$, $X \notin I$ or $\kappa - X \in I$ then we say that X has *measure* 0, *positive measure* or *measure* 1 respectively.

An ideal is *prime* if every subset of κ has measure 0 or 1. An ideal is *normal* if for any set S of positive meausre, if f is a function on S and $f(\alpha) < \alpha$ for all $\alpha \in S - \{0\}$, then f is a constant on some subset of S of positive measure. An example of normal ideal is the ideal of *thin* sets; a subset X of κ is *thin* if $\kappa - X$ contains some closed unbounded subset of κ.

A family W of subsets of κ is *almost disjoint* if $X \cap Y$ has measure 0 for any distinct $X, Y \in W$. Let λ be a cardinal number. An ideal is *λ-saturated* if there is no almsot disjoint family of size λ of sets of positive measure. A *saturation number* of an ideal I, denoted by $\mathrm{sat}(I)$, is the least cardinal number λ such that I is λ-saturated. If $\mathrm{sat}(I)$ is infinite, $\mathrm{sat}(I)$ is a regular uncountable cardinal.

Let S be a set of positive measure. An *I-partition* of S is a maximal almost disjoint family of subsets of S of positive measure. Let W_0 and W_1 be I-partitions of S. W_0 is a refinement of W_1 if every element of W_0 is a subset of some element of W_1. The family $\{X_0 \cap X_1 \notin I: X_0 \in W_0, X_1 \in W_1\}$ of subsets of S is a common refinement of W_0 and W_1.

For each ordinal γ, let c_γ denote the constant function on κ with value γ.

Let α^+ denote the least cardinal number such that $\alpha < \alpha^+$. We define the *β-th cardinal successor* of α, denoted by $\alpha^{+\beta}$, by transfinite induction:

$$\alpha^{+0} = \alpha\,,$$
$$\alpha^{+(\beta+1)} = (\alpha^{+\beta})^+\,,$$
$$\alpha^{+\beta} = \sup_{\gamma<\beta} \alpha^{+\gamma} \qquad \text{if} \quad \beta \quad \text{is a non-zero limit ordinal}\,.$$

If $\alpha = \aleph_\xi$ then $\alpha^{+\beta} = \aleph_{\xi+\beta}$.

Throughout the paper, $+$ and $\cdot$ denote the ordinal sum and the ordinal product respectively. Let f and g be ordinal functions. Let $f+g$, $f \cdot g$ and f^{+g} denote the functions on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ defined by

$$(f+g)(\alpha) = f(\alpha) + g(\alpha)\,,$$
$$(f \cdot g)(\alpha) = f(\alpha) \cdot g(\alpha), \qquad \text{and}$$
$$f^{+g}(\alpha) = f(\alpha)^{+g(\alpha)} \qquad \text{respectively}\,.$$

Let $f \circ g$ denote the function on $\{\alpha \in \mathrm{dom}(g) : g(\alpha) \in \mathrm{dom}(f)\}$ defined by

$$f \circ g(\alpha) = f(g(\alpha)) .$$

§ 2. Fundamental properties of functionals

Let I be an ideal over κ. A *functional* is a collection F of ordinal functions such that $\{\mathrm{dom}(f) : f \in F\}$ is an I-partition of κ and $\mathrm{dom}(f) \neq \mathrm{dom}(g)$ for any distinct $f, g \in F$.

Let S be a set of positive measure. We now define four binary relations $\simeq_S$, $\#_S$, $<_S$ and $\leq_S$ between functionals as follows: $F \simeq_S$ (resp. $\#_S$, $<_S$ and $\leq_S$)G if and only if

$$\{\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(\alpha) \neq (\text{resp. } =, \geq \text{ and } >) g(\alpha)\} \cap S \in I$$

for each $f \in F$ and for each $g \in G$. If $S = \kappa$, we drop the subscript S.

PROPOSITION. *The relation $\simeq_S$ is an equivalence relation, and moreover a congruence with respect to $\#_S$, $<_S$ and $\leq_S$.*

Proof. First we shall prove that $\simeq_S$ is transitive. Suppose, on the contrary, that $F \simeq_S G$, $G \simeq_S H$ and $F \not\simeq_S H$; i.e., for some $f \in F$ and for some $h \in H$ $\{\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(h) : f(\alpha) \neq h(\alpha)\} \cap S$, denoted by $[f \neq h]_S$, has positive measure. Since $\{\mathrm{dom}(g) : g \in G\}$ is an I-partition of κ, for some $g \in G$ $\mathrm{dom}(g) \cap [f \neq h]_S$ has positive measure. However

$$\mathrm{dom}(g) \cap [f \neq h]_S \subseteq [f \neq g]_S \cup [g \neq h]_S \in I , \qquad \text{a contradiction} .$$

It is proved similarly that $\simeq_S$ is a congruence w.r.t. $\#_S$, $<_S$ and $\leq_S$. □

In this paper we shall consider only operations over functionals which are compatible with the equivalence relation $\simeq$. In other words, operations may be regarded as operations defined on equivalence classes. If f is an ordinal function on κ, we identify the function f with the functional $\{f\}$. Between ordinal functions the relation $<$ is not other than the well-founded relation introduced by Galvin and Hajnal in [2], where they considered the ideal of thin sets (see Preliminaries).

An ideal I is *precipitous* if $<_S$ is well-founded for any set S of positive measure. A κ^+-saturated ideal over κ is precipitous, see [6]. More precisely, if an ideal is κ^+-saturated then every functional is equivalent to some ordinal function on κ. But a precipitous ideal is not always κ^+-saturated: Jech and Mitchell constructed a model in which $\aleph_1$ carries a precipitous and not $\aleph_2$-saturated ideal, in [5]. If κ carries a κ-saturated ideal, then by [10], κ is weakly Mahlo and hence weakly inaccessible. Nevertheless the existence of a κ^+-saturated ideal over κ does not necessarily entail that κ is a large cardinal: Kunen constructed a model in which $\aleph_1$ carries an $\aleph_2$-saturated ideal, in [7]. The existence of a precipitous ideal is equiconsistent with the existence of a measurable cardinal, see [4].

For the remainder of this paper, we always assume that I is a precipitous ideal over κ. For a functional F and for a set S of positive measure, we define the *S-degree* of F, denoted by $\deg_S(F)$, as follows:

$$\deg_S(F) = \sup\{\deg_S(G)+1 \colon G <_S F\}\,.$$

It is obvious that if $F \simeq_S G$ then $\deg_S(F) = \deg_S(G)$. If f is an ordinal function on κ, then $\deg_S(f) = \deg_S(\{f\})$. If $S = \kappa$, we drop the subscript S.

LEMMA 1. *Let F be a functional, and S a set of positive measure. Then for any I-partition W of S,*

$$\deg_S(F) = \min_{X \in W} \deg_X(F)\,.$$

Proof. By $<_S$-induction.

If X is a subset of S of positive measure, then $G <_S F$ implies $G <_X F$, and hence $\deg_S(F) \le \deg_X(F)$.

Suppose, on the contrary, that $\gamma = \deg_S(F) < \min_{X \in W} \deg_X(F)$. For $X \in W$, we take a functional $F_X <_X F$ such that $\deg_X(F_X) = \gamma$. Let $F_X \restriction X$ denote the collection $\{f \restriction X \colon f \in F_X,\ \mathrm{dom}(f) \cap X \notin I\}$ of functions. We put

$$F^* = \bigcup_{X \in W} F_X \restriction X \cup \{c_0 \restriction (\kappa - S)\}\,.$$

For each $X \in W$, $\{\mathrm{dom}(f) \cap X \notin I \colon f \in F_X\}$ is an I-partition of X, and hence

$$\bigcup_{X \in W} \{\mathrm{dom}(f) \cap X \notin I \colon f \in F_X\}$$

is an I-partition of S. Thus F^* is a functional, and moreover $F^* <_S F$. From induction hypothesis,

$$\gamma > \deg_S(F^*) = \min_{X \in W} \deg_X(F_X) = \gamma\,, \qquad \text{a contradiction}\,. \qquad \square$$

Let S be a set of positive measure, and γ an ordinal number. We say that a functional F has S-degree γ *uniformly* if $\deg_X(F) = \gamma$ for any subset X of S of positive measure.

LEMMA 2. *Let F be a functional. Then there is an I-partition W of κ such that for each $X \in W$, F has the uniform X-degree.*

Proof. Let P denote the collection of all sets of positive measure. Let us consider the partially ordered set $(P, \subset)$. If $D = \{X \in P \colon F \text{ has the uniform } X\text{-degree}\}$ is dense, then a maximal almost disjoint family of D is a required I-partition of κ.

Suppose, on the contrary, that D is not dense. Then there is a set $S \in P$ such that $D^* = \{X \in P \colon X \subseteq S,\ \deg_X(F) > \deg_S(F)\}$ is dense below S. Here we take a maximal almost disjoint family W of D^*, then W is an I-partition of S. By Lemma 1,

$$\deg_S(F) = \min_{X \in W} \deg_X(F) > \deg_S(F)\,, \qquad \text{a contradiction}\,. \qquad \square$$

THEOREM 1. *For each ordinal γ, there is a functional with uniform degree γ.*

Proof. Let P denote the collection of all sets of positive measure. First we shall prove that $D = \{X \in P \colon \text{there is a functional with uniform } X\text{-degree } \gamma\}$ is dense in

$(P, \subset)$. Let $S \in P$. Since $\deg_S(c_\gamma) \geq \gamma$, there is a functional $F \leq_S c_\gamma$ such that $\deg_S(F) = \gamma$. By Lemma 1 and Lemma 2, there is a set $X \in P$ such that $X \subseteq S$ and F has X-degree γ uniformly.

Thus we get an I-partition W of κ such that for each $X \in W$, there is a functional F_X with uniform X-degree γ. Then $\bigcup_{X \in W} F_X \upharpoonright X$ (see the proof of Lemma 1) is a functional and moreover, by the application of Lemma 1, has degree γ uniformly. □

The functional with uniform degree γ is unique;

LEMMA 3. *Let F and G be functionals with uniform degrees β and γ respectively. Then*

(a) *If $\beta = \gamma$, then $F \simeq G$.*
(b) *If $\beta < \gamma$, then $F < G$.*

Proof. a) Suppose, on the contrary, that $F \not\simeq G$; i.e., $\{\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g): f(\alpha) \neq g(\alpha)\}$ has positive measure for some $f \in F$ and for some $g \in G$. Without loss of generality, we can assume that $S = \{\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g): f(\alpha) < g(\alpha)\}$ has positive measure. Then $F <_S G$ and hence $\beta = \deg_S(F) < \deg_S(G) = \gamma$, a contradiction.
b) Similar. □

Remark that the assumption that I is precipitous can be weakened at all lemmas in this section.

§ 3. Degrees of various functionals

What form does the functional with a uniform degree have? It follows from κ-completeness that for any $\gamma < \kappa$ the functional with uniform degree γ is c_γ. If I is normal (see preliminaries), then the functional with uniform degree κ is the *diagonal function* d on κ defined by $d(\alpha) = \alpha$.

For two functionals F and G, we put

$$F + G = \{f + g: f \in F, g \in G, \mathrm{dom}(f+g) \notin I\},$$

$$F \cdot G = \{f \cdot g: f \in F, g \in G, \mathrm{dom}(f \cdot g) \notin I\}, \quad \text{and}$$

$$F^{+G} = \{f^{+g}: f \in F, g \in G, \mathrm{dom}(f^{+g}) \notin I\}, \qquad \text{see preliminaries}.$$

It is obvious that $F+G$, $F \cdot G$ and F^{+G} are functionals. Notice that the above operations are compatible with $\simeq$.

For brevity we denote $\{C_\gamma\}$ by γ, and F^{+1} by F^+.

When we write that $F = \{f_X: X \in W\}$ is a functional, it is understood that W is an I-partition of κ and $\mathrm{dom}(f_X) = X$ for each $X \in W$. For two functionals F and G, take a common refinement W of $\{\mathrm{dom}(f): f \in F\}$ and $\{\mathrm{dom}(g): g \in G\}$ (cf. preliminaries), then $F \simeq \{f \upharpoonright X: X \in W, f \in F, X \subseteq \mathrm{dom}(f)\}$ and $G \simeq \{g \upharpoonright X: X \in W, g \in G, X \subseteq \mathrm{dom}(g)\}$. Thus, finitely many functionals have a common I-partition. In general, if $\{X_\xi: \xi < \nu\}$ is an I-partition of κ and $\{Y_\xi: \xi < \nu\}$ is a family of sets with measure 0, then $\{X_\xi - Y_\xi:$

$\xi<\nu\}$ is an I-partition of κ. Thus when $F<_S G$, we can take functionals $F^*=\{f_X^*: X\in W\}\simeq F$ and $G^*=\{g_X^*: X\in W\}\simeq G$ such that for each $X\in W$, $f_X^*(\alpha)<g_X^*(\alpha)$ for all $\alpha\in X\cap S$.

THEOREM 2. *Let F and G be the functionals with uniform degrees β and γ, respectively. Then*

(a) *$F+G$ has degree $\beta+\gamma$ uniformly.*

(b) *$F\cdot G$ has degree $\beta\cdot\gamma$ uniformly.*

Proof. By induction on γ.

a) For $\xi<\gamma$, let E^ξ denote the functional with uniform degree ξ. Let S be a set of positive measure. From induction hypothesis,

$$\beta+\xi=\deg_S(F+E^\xi)<\deg_S(F+G) \qquad \text{for all} \quad \xi<\gamma\,,$$

and hence $\beta+\gamma\leq\deg_S(F+G)$. Suppose, on the contrary, that $\beta+\gamma<\deg_S(F+G)$. We take a functional $H<_S F+G$ such that $\deg_S(H)=\beta+\gamma$. Since F has S-degree β uniformly, $F\leq_S H$. Thus we can take a functional G^* such that $F+G^*\simeq_S H$. For some subset X of S of positive measure, $G^*\simeq_X E^\xi$ where $\xi=\deg_S(G^*)<\gamma$. From induction hypothesis,

$$\beta+\xi=\deg_X(F+E^\xi)=\deg_X(H)\geq\deg_S(H)=\beta+\gamma\,, \qquad \text{a contradiction}\,.$$

b) *Case I.* $\gamma=\delta+1$. Let E be the functional with uniform degree δ. By (a) of this theorem, $G\simeq E+1$ and hence $F\cdot G\simeq(F\cdot E)+F$. From induction hypothesis, $F\cdot G$ has degree $\beta\cdot\delta+\beta=\beta\cdot\gamma$ uniformly.

Case II. γ is a limit ordinal. First we shall prove that every value of function of G is a limit ordinal; i.e., $\{\alpha\in\mathrm{dom}(g)$: $g(\alpha)$ is a successor ordinal$\}$ has measure 0 for each $g\in G$.

Suppose, on the contrary, that $S=\{\alpha\in\mathrm{dom}(g)$: $g(\alpha)$ is a successor ordinal$\}$ has positive measure for some $g\in G$. We take a function h on S such that $h+1=g$. By the application of Lemma 1 and Lemma 2,

$$\gamma=\deg_S(G)=\deg_S(h+1)=\deg_S(h)+1\,, \qquad \text{a contradiction}\,.$$

For $\xi<\gamma$, let E^ξ denote the functional with uniform degree ξ. From induction hypothesis,

$$\deg_S(F\cdot G)\geq\sup_{\xi<\gamma}\deg_S(F\cdot E^\xi)=\sup_{\xi<\gamma}\beta\cdot\xi=\beta\cdot\gamma$$

for any set S of positive measure.

Assume that S has positive measure and $H<_S F\cdot G$. Then we can take a functional $G^*<_S G$ such that $H<_S F\cdot G^*$. For some subset X of S of positive measure, $G^*\simeq_X E^\xi$ where $\xi=\deg_S(G^*)<\gamma$. From induction hypothesis,

$$\deg_S(H)<\deg_S(F\cdot G^*)\leq\deg_X(F\cdot E^\xi)=\beta\cdot\xi\leq\beta\cdot\gamma\,.$$

Consequently, $\deg_S(F\cdot G)\leq\beta\cdot\gamma$. $\square$

LEMMA 4. *Let F be a functional, and S a set of positive measure. Assume that either* sat(I) *or* $\deg_S(F)$ *is infinite. If $\{F^\xi\colon \xi<\nu\}$ is a family of functionals, $F^\xi<_S F$ for all $\xi<\nu$ and $F^\xi \#_S F^\eta$ for any distinct ξ, $\eta<\nu$, then*

$$\nu<\max(\mathrm{sat}(I), (\deg_S(F))^+)\,.$$

Proof. We put $\mu=\deg_S(F)$. For $\xi<\nu$, $\gamma_\xi=\deg_S(F^\xi)<\deg_S(F)=\mu$. Then F^ξ has the uniform S_ξ-degree γ_ξ for some subset S_ξ of S of positive measure. For $\gamma<\mu$, $\{S_\xi\colon \gamma_\xi=\gamma\}$ is almost disjoint and hence $|\{\xi<\nu\colon \gamma_\xi=\gamma\}|<\mathrm{sat}(I)$. From the regularity of $\max(\mathrm{sat}(I), \mu^+)$,

$$\nu=\bigcup_{\gamma<\mu}\{\xi<\nu\colon \gamma_\xi=\gamma\}<\max(\mathrm{sat}(I), \mu^+)\,. \qquad \square$$

THEOREM 3. *Let F and G be the functionals with uniform degrees β and γ respectively. If* sat$(I)\le\beta$, *then*

$$F^{+G}\le \text{ the functional with uniform degree } \beta^{+\gamma}\,.$$

Proof. We shall prove by induction on γ that $\deg_S(F^{+G})\le\beta^{+\gamma}$ for any set S of positive measure.

Case I. $\gamma=0$. Then $F^{+G}\simeq F^{+0}\simeq F$.

Case II. γ is a non-zero limit ordinal. Similar to the proof of Theorem 2 (b) case II.

Case III. $\gamma=\delta+1$. If $\beta^{+\delta}$ is finite then obvious. Thus we assume that $\beta^{+\delta}$ is infinite. Let E denote the functional with uniform degree δ. Then $G\simeq E+1$.

Assume that S has positive measure and $H<_S F^{+G}\simeq_S(F^{+E})^+$. Here we can assume that E, F and H have the same I-partition W of κ; i.e., $E=\{e_X\colon X\in W\}$, $F=\{f_X\colon X\in W\}$, $H=\{h_X\colon X\in W\}$ and moreover for any $X\in W$ and for any $\alpha\in X\cap S$,

$$h_X(\alpha)<(f_X{}^{+e_X}(\alpha))^+$$

hence there is a one-to-one mapping $\pi_{X,\alpha}$ from $h_X(\alpha)$ to $f_X{}^{+e_X}(\alpha)$. We put $\nu=\deg_S(H)$. For $\xi<\nu$, let $E^\xi=\{e^\xi_Y\colon Y\in W^\xi\}$ denote the functional with uniform degree ξ. Since $E^\xi<_S H$, we can assume that W^ξ is a refinement of W, see preliminaries, and for each $Y\in W^\xi$, $e^\xi_Y(\alpha)\in h_X(\alpha)$ for all $\alpha\in Y\cap S$ where X is the unique element of W such that $Y\subseteq X$. For $Y\in W^\xi$, let $\pi(e^\xi_Y)$ denote the function on Y defined by

$$\pi(e^\xi_Y)(\alpha)=\begin{cases}\pi_{X,\alpha}(e^\xi_Y(\alpha))\,, & \text{if } \alpha\in S\,,\\ e^\xi_Y(\alpha)\,, & \text{otherwise}\,,\end{cases}$$

where X is the unique element of W such that $Y\subseteq X$. We put

$$F^\xi=\{\pi(e^\xi_Y)\colon Y\in W^\xi\}\,.$$

Then $F^\xi<_S F^{+E}$ for all $\xi<\nu$ and moreover, since $\pi_{X,\alpha}$ is one-to-one, $F^\xi\#_S F^\eta$ for any distinct ξ, $\eta<\nu$. From induction hypothesis, $\deg_S(F^{+E})\le\beta^{+\delta}$. By Lemma 4,

$$\deg_S(H) = \nu < \max(\mathrm{sat}(I), (\beta^{+\delta})^+) = \beta^{+\gamma}.$$

Consequently, $\deg_S(F^{+G}) \leq \beta^{+\gamma}$. $\square$

The degree of F^{+G} is not necessarily $\beta^{+\gamma}$. Let κ be a measurable cardinal, and I a normal prime ideal over κ. If d^{++} has degree κ^{++}, then $\kappa^{++} = \deg(d^{++}) < \deg(\kappa) < (\kappa^\kappa)^+ = (2^\kappa)^+$ and hence $\kappa^+ < 2^\kappa$. This contradicts the consistency of the GCH with the existence of a measurable cardinal, see [9].

By Lemma 2, every functional consists of some functionals with uniform degrees. If eigher F or G has the uniform S-degree, by the application of Lemma 1,

$$\deg_S(F+G) = \deg_S(F) + \deg_S(G),$$

$$\deg_S(F \cdot G) = \deg_S(F) \cdot \deg_S(G), \quad \text{and}$$

$$\deg_S(F^{+G}) \leq \deg_S(F)^{+\deg_S(G)} \qquad (\text{if } \deg_S(F) \geq \mathrm{sat}(I)).$$

THEOREM 4. *Let F be the functional with uniform degree β. If $\beta < \mathrm{sat}(I)$, then*

$$F^+ \leq \text{ the functional with uniform degree } \mathrm{sat}(I).$$

Proof. If β is finite, then the theorem is obvious. Thus we assume that β is infinite. We put $\lambda = \mathrm{sat}(I)$. Let E denote the functional with uniform degree λ. Suppose, on the contrary, that $F^+ \nleq E$; i.e., $E <_S F^+$ for some set S of positive measure. By the similar argument in the proof of Theorem 3, we get a family $\{F^\xi: \xi < \lambda\}$ such that $F^\xi <_S F$ for all $\xi < \lambda$ and $F^\xi \#_S F^\eta$ for any distinct $\xi, \eta < \nu$. By Lemma 4, $\lambda < \max(\mathrm{sat}(I), \beta^+) = \mathrm{sat}(I)$, a contradiction. $\square$

§4. Degrees of functionals and the size of 2^κ

For an ordinal function f and a functional G, we put

$$f \circ G = \{f \circ g: g \in G\}, \qquad \text{see preliminaries}.$$

Although $f \circ G$ is not necessarily a functional, if $\{\alpha \in \mathrm{dom}(g): g(\alpha) \notin \mathrm{dom}(f)\}$ has measure 0 for all $g \in G$ then $f \circ G$ is a functional. Moreover, if $f \circ G$ is a functional and $G \simeq H$, then $f \circ H$ is a functional and $f \circ G \simeq f \circ H$.

Recall that we assume that an ideal I is precipitous.

THEOREM 5. *Let E be the functional with uniform degree κ, and f the function on κ such that $2^\alpha = (\alpha^+)^{+f(\alpha)}$ for all $\alpha < \kappa$. Then $f \circ E$ is a functional and*

$$2^\kappa \leq \lambda^{+\deg(f \circ E)}$$

where $\lambda = \max(\mathrm{sat}(I), \kappa^+)$.

Proof. $\deg_S(d) \geq \kappa$ for any set S of positive measure, and hence $E \leq d < \kappa$. Without loss of generality, we can assume that $E = \{e_X: X \in W\}$ and for each $X \in W$, $e_X(\alpha) < \kappa$ for all $\alpha \in X$. For each $\alpha < \kappa$, we take a one-to-one mapping π_α from 2^α onto

$(\alpha^+)^{+f(\alpha)}$. Let $\{X_\xi\colon \xi<2^\kappa\}$ be an enumeration of the power set of κ. For $\xi<2^\kappa$, we put $E^\xi=\{e_X^\xi\colon X\in W\}$, where e_X^ξ is the function on X defined by

$$e_X^\xi(\alpha)=\pi_{e_X(\alpha)}(X_\xi\cap e_X(\alpha))<(e_X(\alpha)^+)^{+f(e_X(\alpha))}.$$

Then $E^\xi<(E^+)^{+f\circ E}$ for all $\xi<2^\kappa$ and for any distinct $\xi,\eta<2^\kappa$, since $\alpha<E$, where $\alpha=\min(X_\xi-X_\eta)\cup(X_\eta-X_\xi)$, $E^\xi\,\#\,E^\eta$. We put $\beta=\deg(f\circ E)$. By Theorem 3 and Theorem 4, $E^+\leq$the functional with uniform degree λ and hence $\deg((E^+)^{+f\circ E})\leq\lambda^{+\beta}$. By Lemma 4,

$$2^\kappa<\max(\operatorname{sat}(I),(\lambda^{+\beta})^+)=(\lambda^{+\beta})^+\quad\text{and hence}\quad 2^\kappa\leq\lambda^{+\beta}.\qquad\square$$

We put $\lambda=\max(\operatorname{sat}(I),\kappa^+)$ in the following corollaries.

COROLLARY 1. *Assume that I is normal. Let f be as above. Then*

$$2^\kappa\leq\lambda^{+\deg(f)}.$$

Proof. The functional with uniform degree κ is d. $\square$

It seems to be open whether the existence of a precipitous ideal implies the existence of a normal one, although the existence of a normal precipitous ideal is equiconsistent with the existence of a precipitous ideal, cf. [4]. When the above function f has one of some special forms, we can omit the assumption "normal";

COROLLARY 2. *Let $\gamma<\kappa$.*
(a) *If $2^\alpha\leq(\alpha^+)^{+\gamma}$ for all $\alpha<\kappa$, then $2^\kappa\leq\lambda^{+\gamma}$.*
(b) *If $2^\alpha\leq\alpha^{+(\alpha+\gamma)}$ for all $\alpha<\kappa$, then $2^\kappa\leq\lambda^{+(\kappa+\gamma)}$.*
(c) *If $2^\alpha\leq\alpha^{+\alpha\cdot\gamma}$ for all $\alpha<\kappa$, then $2^\kappa\leq\lambda^{+\kappa\cdot\gamma}$.*
(d) *If $2^\alpha\leq\alpha^{+\alpha\cdot\alpha}$ for all $\alpha<\kappa$, then $2^\kappa\leq\lambda^{+\kappa\cdot\kappa}$.*
(e) *If $2^\alpha\leq\alpha^{+(\alpha^+)}$ for all $\alpha<\kappa$, then $2^\kappa\leq\lambda^{+\lambda}$.*
(f) *If $2^\alpha\leq\alpha^{+(\alpha^{+\alpha})}$ for all $\alpha<\kappa$, then $2^\kappa\leq\lambda^{+(\lambda^{+\kappa})}$.*

Proof. Let E and f be as before.
a) $f(\alpha)\leq\gamma$ for all $\alpha<\kappa$, and hence $f\circ E\leq\gamma$.
b) $f\circ E\leq(d+\gamma)\circ E\simeq E+\gamma$ and $\deg(E+\gamma)=\kappa+\gamma$.
c) $f\circ E\leq(d\cdot\gamma)\circ E\simeq E\cdot\gamma$ and $\deg(E\cdot\gamma)=\kappa\cdot\gamma$.
d) $f\circ E\leq(d\cdot d)\circ E\simeq E\cdot E$ and $\deg(E\cdot E)=\kappa\cdot\kappa$.
e) $f\circ E\leq d^+\circ E\simeq E^+$ and $\deg(E^+)\leq\lambda$.
f) $f\circ E\leq d^{+d}\circ E\simeq E^{+E}$ and $\deg(E^{+E})\leq\deg(F^{+E})\leq\lambda^{+\kappa}$,

where F is the functional with uniform degree λ. $\square$

COROLLARY 3. *Let $\kappa=\nu^+$.*
(a) *If $2^\nu=\kappa^{+\gamma}$ where $\gamma<\kappa$, then $2^\kappa\leq\lambda^{+\gamma}$.*
(b) *If $2^\nu=\kappa^{+\kappa}$, then $2^\kappa\leq\lambda^{+\lambda}$.*
(c) *If $2^\nu=\kappa^{+(\kappa+\kappa)}$, then $2^\kappa\leq\lambda^{+(\lambda+\lambda)}$.*
(d) *If $2^\nu=\kappa^{+\kappa\cdot\kappa}$, then $2^\kappa\leq\lambda^{+\lambda\cdot\lambda}$.*
(e) *If $2^\nu=\kappa^{+(\kappa^{+\kappa})}$, then $2^\kappa\leq\lambda^{+(\lambda^{+\lambda})}$.*

Proof. Let E and f be as before, and F the functional with uniform degree λ. Since $\nu \leq E < \nu^+$, $f \circ E \simeq f(\nu)$.

a) $f(\nu) = \gamma$ and $\deg(\gamma) = \gamma$.

b) $\deg(\kappa) = \deg(E^+) \leq \lambda$.

c) $\deg(\kappa + \kappa) \leq \deg(F + F) = \lambda + \lambda$.

d) $\deg(\kappa \cdot \kappa) \leq \deg(F \cdot F) = \lambda \cdot \lambda$.

e) $\deg(\kappa^{+\kappa}) \leq \deg(F^{+F}) \leq \lambda^{+\lambda}$. □

This proof is an elementary proof of Theorems 3.3.1 and 3.3.2 in [6].

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