COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 31, No. 2, 1982

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

Some Properties on a Set of Ideals

by

Nobuo KUBOTA

(Received May 18, 1982)

Saturation property is a property of ideals. In [2], A. Taylor extended a saturation property to sets of ideals. In this paper we shall propose a certain property of ideals, and shall prove the property implies a saturation property. Further we shall extend the property to sets of ideals.

Let κ be a regular uncountable cardinal number. An ideal over κ is a set I of subsets of κ satisfying the following conditions:

(1) $\phi \in I$,

(2) if $X \in I$ and $Y \subseteq X$ then $Y \in I$,

(3) if $X \in I$ and $Y \in I$ then $X \cup Y \in I$.

An ideal I over κ is proper, if $\kappa \notin I$. And an ideal I over κ is called uniform, if I satisfies the following condition:

If $X \subseteq \kappa$ and $|X| < \kappa$, then $X \in I$. ($|X|$ denotes the cardinality of X.)

Let ν be a cardinal. An ideal I over κ is said to be ν-complete, if I satisfies the following condition:

$$\text{If} \quad \lambda < \nu \quad \text{and} \quad \{X_\alpha \mid \alpha < \lambda\} \subseteq I, \quad \text{then} \quad \bigcup_{\alpha < \lambda} X_\alpha \in I.$$

An ideal I over κ is nontrivial, if $\{\alpha\} \in I$, for all $\alpha < \kappa$. Hence if I is a nontrivial κ-complete ideal over κ, I is a uniform ideal.

Throughout this paper, an ideal means a proper uniform ideal over κ. Let I be an ideal. We set

$$I^+ = \{X \mid X \subseteq \kappa \text{ and } X \notin I\}.$$

And if $A \in I^+$, then the ideal

$$I \mid A = \{X \mid X \subseteq \kappa \text{ and } X \cap A \in I\}$$

is an ideal generated by $I \cup \{\kappa - A\}$. It is easy to know that if I is ν-complete, then $I \mid A$ is also ν-complete. Let I be an ideal over κ. Then a function f is called I-function if $\mathrm{dom}(f) \in I^+$. Let λ be a cardinal number, and $\mathscr{I} = \{I_\alpha \mid \alpha < \lambda\}$ be a set of ideals over κ. We set

$$\mathscr{I}^+ = \bigcap_{\alpha < \lambda} I_\alpha^+ .$$

And a function f is said to be $\mathscr{I}$-function if $\mathrm{dom}(f) \in \mathscr{I}^+$.

PROPOSITION 1. *Let I be a κ-complete ideal over $\kappa=\mu^+$. Then if there is a collection of I-functions $F=\{f_\alpha \mid \alpha<\kappa\}$ such that for all $\alpha<\kappa$ $\operatorname{range}(f_\alpha)\subseteq\mu$, and $\{\delta \mid f_\alpha(\delta)=f_\beta(\delta)\}\in I$ for $\alpha<\beta<\kappa$, then there is a collection $\{X_\alpha \mid \alpha<\kappa\}\subseteq I^+$ such that $X_\alpha\cap X_\beta\in I$ for $\alpha<\beta<\kappa$.*

Proof. Let $F=\{f_\alpha \mid \alpha<\kappa\}$ be a set of I-functions satisfying the hypothesis. Let $\{Y_\alpha^\beta \mid \beta<\mu\}$ be a collection of subsets of $\operatorname{dom}(f_\alpha)$ such that $f_\alpha^{-1}(\beta)=Y_\alpha^\beta$ for all $\beta<\mu$. If $Y_\alpha^\beta\in I$ for all $\beta<\mu$, then

$$\operatorname{dom}(f_\alpha)=\bigcup_{\beta<\mu} Y_\alpha^\beta\in I\,,$$

because $\mu<\kappa$ and I is a κ-complete ideal over κ. This contradicts f_α is an I-function. Hence we have $\gamma_\alpha<\mu$ and $Y_\alpha\subseteq\operatorname{dom}(f_\alpha)$ such that

$$Y_\alpha\in I^+ \quad\text{and}\quad f_\alpha''(Y_\alpha)=\gamma_\alpha\,, \qquad \text{for all}\quad \alpha<\kappa\,.$$

Since κ is regular and $\mu<\kappa$, there is a collection $\{X_\alpha \mid \alpha<\kappa\}$ and $\gamma<\mu$ such that

$$\{X_\alpha \mid \alpha<\kappa\}\subseteq\{Y_\alpha \mid \alpha<\kappa\} \quad\text{and}\quad f_\alpha''(X_\alpha)=\gamma\,, \qquad \text{for all}\quad \alpha<\kappa\,.$$

Thus if $X_\alpha\cap X_\beta\in I$ for $\alpha<\beta<\kappa$, then the proof is complete.

Let $X_\alpha\cap X_\beta\in I^+$ for some $\alpha<\beta<\kappa$. Then we have

$$\{\delta \mid f_\alpha(\delta)=f_\beta(\delta)=\gamma\}\supseteq X_\alpha\cap X_\beta\in I^+\,,$$

a contradiction.

COROLLARY 2. *Let I be a κ-complete ideal over $\kappa=\mu^+$. Then if there is a collection of I-functions $F=\{f_\alpha \mid \alpha<\kappa^+\}$ such that for all $\alpha<\kappa^+$ $\operatorname{range}(f_\alpha)\subseteq\mu$, and $\{\delta \mid f_\alpha(\delta)=f_\beta(\delta)\}\in I$ for $\alpha<\beta<\kappa^+$, then there is a collection $\{X_\alpha \mid \alpha<\kappa^+\}\subseteq I^+$ such that $X_\alpha\cap X_\beta\in I$ for $\alpha<\beta<\kappa^+$.*

Let ν be a cardinal. An ideal I over κ is said to be not ν-saturated, if there is a collection $\{X_\alpha \mid \alpha<\nu\}\subseteq I^+$ such that $X_\alpha\cap X_\beta\in I$ for $\alpha<\beta<\nu$.

Hence Proposition 1 (Corollary 2) means that if I is a κ-complete ideal over $\kappa=\mu^+$ such that there is a collection of I-functions $F=\{f_\alpha \mid \alpha<\kappa\}$ ($F=(f_\alpha \mid \alpha<\kappa^+\}$) with $\operatorname{range}(f_\alpha)\subseteq\mu$ for all $\alpha<\kappa$ ($\alpha<\kappa^+$) and $\{\delta \mid f_\alpha(\delta)=f_\beta(\delta)\}\in I$ for $\alpha<\beta<\kappa$ ($\alpha<\beta<\kappa^+$), then I is not κ-saturated (κ^+-saturated).

In [2] Taylor showed that some set of ideals has a certain saturation property, if a single ideal has the same one. That is,

THEOREM 3 ([2]). *Let κ be a regular uncountable cardinal number and ν be a cardinal number with $\nu<\kappa$. Then the following two assertions are equivalent.*

(1) *If I is at least ν^+-complete ideal over κ, then I is not ν^+-saturated.*

(2) *If $\mathscr{I}=\{I_\eta \mid \eta<\nu\}$ is a set of at least ν^+-complete ideal over κ, then there is a collection $\{X_\alpha \mid \alpha<\nu^+\}\subseteq\mathscr{I}^+$ such that*

$$X_\alpha\cap X_\beta\in\bigcap_{\eta<\nu} I_\eta \qquad \text{for}\quad \alpha<\beta<\nu^+\,.$$

Taylor proved this theorem in order to modify the technique used by Baumgartner, Hajnal and Máté in [1].

Now we shall show that some set of ideals has the property in the hypothesis of Proposition 1, if a single ideal has the same one using the technique of Baumgartner, Hajnal and Máté.

THEOREM 4. *Let ν be a cardinal number with $\nu<\kappa=\mu^+$. Then the following two assertions are equivalent.*

(1) *If I is a κ-complete ideal over κ, then there is a collection of I-functions $F=\{f_\alpha \mid \alpha<\kappa\}$ such that* $\operatorname{range}(f_\alpha)\subseteq\mu$ *for all $\alpha<\kappa$ and $\{\delta \mid f_\alpha(\delta)=f_\beta(\delta)\}\in I$ for $\alpha<\beta<\kappa$.*

(2) *If $\mathscr{I}=\{I_\eta \mid \eta<\nu\}$ is a set of κ-complete ideals over κ, then there is a collection of $\mathscr{I}$-functions $G=\{g_\alpha \mid \alpha<\kappa\}$ such that* $\operatorname{range}(g_\alpha)\subseteq\mu$ *for all $\alpha<\kappa$ and*

$$\{\delta \mid g_\alpha(\delta)=g_\beta(\delta)\}\in\bigcap_{\eta<\nu} I_\eta \qquad \text{for} \quad \alpha<\beta<\kappa .$$

Proof. From (2) to (1) is obvious. We shall prove (1) to (2).

Let $\mathscr{I}=\{I_\xi \mid \xi<\nu\}$ be a set of κ-complete ideals over κ. For each $\xi<\nu$ I_ξ is κ-complete over κ. So by (1) and Proposition 1 there is a collection $\{X^\xi_\alpha \mid \alpha<\kappa\}\subseteq I_\xi{}^+$ such that $X^\xi_\alpha\cap X^\xi_\beta\in I_\xi$ for $\alpha<\beta<\kappa$. Hence we can construct a collection $\{Y^\xi_\alpha \mid \alpha<\kappa\}\subseteq I_\xi{}^+$ of pairwise disjoint sets from $\{X^\xi_\alpha \mid \alpha<\kappa\}$, because I_ξ is κ-complete. Now we deine a function h: $\nu-\nu$ by

$$h(\eta)=\inf\{\xi<\nu \mid |\{\alpha<\kappa \mid Y^\xi_\alpha\in I^+_\eta\}|=\kappa\} \qquad \text{for} \quad \eta<\nu .$$

If $\gamma<h(\eta)$, then $|\{\alpha<\kappa \mid Y^\gamma_\alpha\in I^+_\eta\}|<\kappa$. Hence there is a $\beta_\gamma<\kappa$ such that, if $Y^\gamma_\alpha\in I_\eta{}^+$ then $\alpha<\beta_\gamma$. So if we set $\delta_\eta=\bigcup_{\gamma<h(\eta)}\beta_\gamma$, we know that for every $\gamma<h(\eta)$ if $Y^\gamma_\alpha\in I^+_\eta$ then $\alpha<\delta_\eta<\kappa$, because κ is a regular cardinal number. Now set

$$\delta=\bigcup_{\eta<\nu}\delta_\eta \qquad (<\kappa) .$$

Then we get that for every $\eta<\nu$ if $\gamma<h(\eta)$ and $Y^\gamma_\alpha\in I_\eta{}^+$ then $\alpha<\delta$. Of course $|\{\alpha<\kappa \mid Y^{h(\eta)}_\alpha\in I_\eta{}^+\}|=\kappa$, so $|\{Y^{h(\eta)}_\alpha \mid Y^{h(\eta)}_\alpha\in I_\eta{}^+ \text{ and } \delta<\alpha<\kappa\}|=\kappa$. Thus we can get a collection $\{W_\eta \mid \eta<\nu\}$ of distinct sets such that $W_\eta\in\{Y^{h(\eta)}_\alpha \mid Y^{h(\eta)}_\alpha\in I_\eta{}^+ \text{ and } \delta<\alpha<\kappa\}$ for each $\eta<\nu$. Define T_η for every $\eta<\nu$ by

$$T_\eta=W_\eta-\bigcup_{h(\lambda)<h(\eta)}W_\lambda .$$

We show that $\{T_\eta \mid \eta<\nu\}$ is a collection of pairwise disjoint sets such that $T_\eta\in I_\eta{}^+$ for every $\eta<\nu$. First we show $T_\eta\in I_\eta{}^+$. If we can prove $W_\lambda\in I_\eta$ for all $h(\lambda)<h(\eta)$ we have $T_\eta\in I_\eta{}^+$, because $W_\eta\in I_\eta{}^+$, $|\{\lambda \mid h(\lambda)<h(\eta)\}|\le\nu<\kappa$ and I_η is κ-complete. As we show above if $h(\lambda)<h(\eta)$ and $W_\lambda=Y^{h(\lambda)}_\alpha\in I^+_\eta$ then $\alpha<\delta$. But this contradicts the definition of W_η. Second we show $\{T_\eta \mid \eta<\nu\}$ is a collection of pairwise disjoint sets. If $h(\lambda)\neq h(\eta)$, T_λ and T_η are clearly disjoint. Hence let $h(\lambda)=h(\eta)=\xi$. Then we have $W_\lambda=Y^\xi_\sigma$ and $W_\eta=Y^\xi_\tau$ for some σ and τ. But $\{Y^\xi_\alpha \mid \alpha<\kappa\}$ is a collection of pairwise disjoint sets and W_λ and W_η are distinct, so we get W_λ and W_η are disjoint. Therefore T_λ and T_η are disjoint. Since $T_\eta\in I^+_\eta$, $I_\eta \mid T_\eta$ is a κ-complete ideal over κ. Then from (1) there is a collection of $I_\eta \mid T_\eta$-functions $F_\eta = \{f^\eta_\alpha \mid \alpha < \kappa\}$ such that range$(f^\eta_\alpha) \subseteq \mu$ and $\{\delta \mid f^\eta_\alpha(\delta) = f^\eta_\beta(\delta)\} \in I_\eta \mid T_\eta$ for $\alpha < \beta < \kappa$. Hence by the proof of Proposition 1, there is a collection $\{C^\eta_\alpha \mid \alpha < \kappa\} \subseteq (I_\eta \mid T_\eta)^+$ and a $\gamma_\eta < \mu$ such that $C^\eta_\alpha \cap C^\eta_\beta \in I_\eta \mid T_\eta$ for $\alpha < \beta < \kappa$ and $f^{\eta\,\prime\prime}_\alpha(C^\eta_\alpha) = \gamma_\eta$ for all $\alpha < \kappa$. Let $D^\eta_\alpha = C^\eta_\alpha - \bigcup_{\beta < \alpha} C^\eta_\beta$. Then $\{D^\eta_\alpha \mid \alpha < \kappa\} \subseteq (I_\eta \mid T_\eta)^+$ is a collection of pairwise disjoint sets, because $I_\eta \mid T_\eta$ is κ-complete. Set

$$E^\eta_\alpha = D^\eta_\alpha \cap T_\eta \qquad \text{for all} \quad \alpha < \kappa\,.$$

Then $\{E^\eta_\alpha \mid \alpha < \kappa \text{ and } \eta < \nu\}$ is a collection of pairwise disjoint sets such that

$$\{E^\eta_\alpha \mid \alpha < \kappa\} \subseteq I_\eta{}^+ \qquad \text{for all} \quad \eta < \nu \quad \text{and} \quad f^{\eta\,\prime\prime}_\alpha(E^\eta_\alpha) = \gamma_\eta \qquad \text{for all} \quad \alpha < \kappa\,.$$

Now set

$$Z_\alpha = \bigcup_{\eta < \nu} E^\eta_\alpha \qquad \text{for all} \quad \alpha < \kappa.$$

Then we get $Z_\alpha \in \bigcap_{\eta < \nu} I_\eta{}^+ = \mathscr{I}^+$ and $\{Z_\alpha \mid \alpha < \kappa\}$ is a collection of pairwise disjoint sets.

Define functions g_α for all $\alpha < \kappa$ such that

$$\mathrm{dom}\,(g_\alpha) = Z_\alpha \quad \text{and} \quad g_\alpha(\delta) = f^\eta_\alpha(\delta) = \gamma_\eta \qquad \text{if} \quad \delta \in E^\eta_\alpha\,.$$

Because $\{E^\eta_\alpha \mid \eta < \nu\}$ is a collection of pairwise disjoint sets, each g_α is well defined and of course g_α is an $\mathscr{I}$-function. Let $\alpha < \beta < \kappa$, then dom $(g_\alpha) \cap$ dom $(g_\beta) = \varnothing$. So we have

$$\{\delta \mid g_\alpha(\delta) = g_\beta(\delta)\} = \varnothing \in \bigcap_{\eta < \nu} I_\eta\,.$$

Hence we get a collection of $\mathscr{I}$-functions $G = \{g_\alpha \mid \alpha < \kappa\}$ such that range$(g_\alpha) \subseteq \mu$ for all $\alpha < \kappa$ and

$$\{\delta \mid g_\alpha(\delta) = g_\beta(\delta)\} \in \bigcap_{\eta < \nu} I_\eta \qquad \text{for} \quad \alpha < \beta < \kappa\,.$$

This completes the proof.

References

[1] BAUMGARTNER, J., HAJNAL, A. and MÁTÉ, A.; Weak saturation properties of ideals, Colloquia Mathematica Societatis János Bolyai 10, *Infinte and Finite Sets*, Keszthely, 1973, pp. 137–158.

[2] TAYLOR, A.; On saturated sets of ideals and Ulam's problem, *Fundamenta Mathematicae*, **CIX** (1980), 37–53.

Department of Mathematics
Rikkyo University
Ikebukuro, Tokyo
Japan