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A Stable Harmonic Map of a Manifold with Boundary

by

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The purpose of this paper is to give sufficient conditions for a harmonic map of a manifold with boundary to be stable (Theorems 1 and 2), and also give a property of a harmonic map without boundary (Proposition 4).

1. Introduction

Let (M, g) and (N, h) be smooth Riemannian manifolds. A smooth map $f: M\to N$ is said to be *harmonic* if it is a critical point of *the energy integral*

$$E(f)=\int_M e(f)dx\,, \qquad e(f)(x)=\frac{1}{2}|df(x)|^2\,,$$

where $|df(x)|$ is the norm of the differential of f at $x\in M$, dx the volume element of M, and $e(f)$ is called *the energy density* of f. A harmonic map is called *stable* if it has positive second variations, namely if

$$d^2E(f_t)/dt^2\,|_{t=0}>0$$

for any one-parameter family f_t with $f_0=f$. Otherwise a harmonic map is called *unstable*. We are interested in knowing whether a harmonic map is stable or not.

If N has negative sectional curvature, then it is easy to see any harmonic map is stable. Eells and Sampson [3] showed that any smooth map f: $S^n\to S^n$ with degree $\neq 0$ does not have minimizing energy for $n>2$, hence the identity map f: $S^n\to S^n$ is unstable. In fact, Smith [5] showed that its index is $n+1$. Xin [6] generalized this result showing that there is no nonconstant stable harmonic map from S^n with $n>2$ to any Riemannian manifold. Suppose that M, N are two Kaehler manifolds. Then a holomorphic map f: $M\to N$ is harmonic and stable [2].

2. Statement of results

THEOREM 1. *Let (M, g) be a compact Riemannian manifold with piecewise smooth boundary ∂M, $\dim M=m$. Let (N, h) be a complete Riemannian manifold whose sectional curvature is bounded from above by a constant $C\geqslant 0$. Let f: $M\to N$ be harmonic and $\lambda_1(M)$ the first eigenvalue of the Laplacian acting on functions. Assume furthermore that*

$$2\cdot C\cdot\sup_{x\in M} e(f)(x)<\lambda_1(M)\,. \tag{2.1}$$

Then f is stable.

$\lambda_1(M)$ is the smallest positive real number such that there exists a solution $\varphi\in H(M)$ of $\Delta\varphi+\lambda\varphi=0$, where $H(M)$ is the space of C^2-functions on M which are not identically zero and vanish on ∂M. It is well-known that

$$\lambda_1(M)=\inf_{\varphi\in H_0'(M)}\frac{\int_M|\nabla\varphi|^2dx}{\int_M\varphi^2dx}, \tag{2.2}$$

where $H_0'(M)$ is the space of functions φ on M such that they are not identically zero and vanish on ∂M, and φ and φ and $|\nabla\varphi|$ are integrable on M. We denote by $\nabla\varphi$ the gradient of function φ.

Next we get Theorem 2 in which a condition is a form of integral of energy density $e(f)$.

THEOREM 2. *Let (M, g), (N, h) and f be as in Theorem* 1. *Assume furthermore that*

$$\sqrt{C}\cdot\|e(f)^{1/2}\|_{L^m}<\frac{D\cdot(m-2)}{2\cdot\sqrt{2}\,(m-1)}\qquad \textit{if}\quad m\geqslant 3 \tag{2.3}$$

$$\sqrt{C}\cdot\|e(f)^{1/2}\|_{L^4}<\frac{D}{2\cdot\sqrt{2}\,\mathrm{vol}(M)}\qquad \textit{if}\quad m=2\,. \tag{2.4}$$

Then f is stable.

Here D is the Sobolev constant depending only on manifold M, and satisfies

$$\|\nabla\varphi\|_{L^1}\geqslant D\cdot\|\varphi\|_{L^{m/(m-1)}}\qquad \text{for any}\quad \varphi\in H_0'(M)\,, \tag{2.5}$$

$\mathrm{vol}(M)$ being the volume of M, and $\|\cdot\|_{L^p}$ denoting the L^p norm.

COROLLARY 3. *Let (M, g) be a compact Riemannian manifold with piecewise smooth boundary ∂M. Then a harmonic map $f\colon M\to \boldsymbol{R}^n$ $(n\geqslant 1)$ is stable.*

Next we show a property of a harmonic map similar to that of a minimal submanifold.

PROPOSITION 4. *Let (M, g) be a compact Riemannian manifold without boundary, and $S^n(1)$ a unit sphere with the standard metric. Then for a nonconstant harmonic map $f\colon M\to S^n(1)$ $(\subset \boldsymbol{R}^{n+1})$ $f(M)$ is not contained in an open hemisphere.*

3. Preliminaries

Let (M, g) be a compact Riemannian manifold with boundary (possibly without boundary), $\dim M = m$, and (N, h) a complete Riemannian manifold. Let $f\colon M \to N$ be a smooth map. The differential $df\colon TM \to TN$ of f is a map of the tangent bundle of M into that of N. We denote by ∇, $\bar{\nabla}$, $\bar{\bar{\nabla}}$ the canonical Riemannian connections of Riemannian vector boundles $TM, f^{-1}TN$, TN. $f^{-1}TN$ is the induced vector bundle over M by f. For any vector field V along f, let f_t be one-parameter family with $f_0 = f$ and $\partial f/\partial t\,|_{t=0} = V$. The first variation formula is

$$\frac{d}{dt}E(f_t)\bigg|_{t=0} = -\int_M \langle V, \Delta f\rangle dx = \int_M \Sigma\langle \bar{\nabla}_{e_i}V, dfe_i\rangle dx\,, \tag{3.1}$$

where

$$\Delta f\colon = \sum_{i=1}^{m} (\bar{\nabla}_{e_i}(df(e_i)) - df(\nabla_{e_i}e_i))$$

is the tension field defined as a section of $f^{-1}TN$, $\{e_i\}$ an orthonormal frame in M, and $\langle,\rangle$ the Riemannian metric in N. Hence the Euler-Lagrange equation for a harmonic map is $\Delta f = 0$. For any given harmonic map f, the second variation formula is [5]:

$$\frac{d^2}{dt^2}E(f_t)\bigg|_{t=0} = \int_M \left\langle -\bar{\Delta}V - \sum_{i=1}^{m} {}^{N}R(V, dfe_i)dfe_i, V\right\rangle dx\,, \tag{3.2}$$

where ${}^{N}R$ is the curvature tensor of the manifold N; i.e.

$${}^{N}R(X, Y) = [\bar{\bar{\nabla}}_X, \bar{\bar{\nabla}}_Y] - \bar{\bar{\nabla}}_{[X,Y]}\,,$$

and

$$\bar{\Delta}V\colon = \mathrm{trace}\,\bar{\nabla}^2 V = (\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}V - \bar{\nabla}_{\nabla_{e_i}e_i}V)\,.$$

The index form for a harmonic map is

$$I(V, W) = \int_M \left\langle -\bar{\Delta}V - \sum_i {}^{N}R(V, dfe_i)dfe_i, W\right\rangle dx$$

$$= \int_M \langle J_f V, W\rangle dx\,,$$

where

$$J_f V = -\bar{\Delta}V - \sum_i {}^{N}R(V, dfe_i)dfe_i\,.$$

Therefore a harmonic map is stable if $I(V, V) > 0$ for any vector field V along f.

4. Proof of Theorem 1

We need a lemma

LEMMA. *Let (M, g) be a compact Riemannian manifold with boundary, (N, h) a* *complete Riemannian manifold whose sectional curvature is bounded from above by a constant $C \geqslant 0$. Let f: $M \to N$ be a harmonic map. If for any $\varphi \in H_0'(M)$*

$$\int_M |\nabla\varphi|^2 dx > 2C \int_M \varphi^2 e(f) dx, \tag{4.1}$$

then f is stable.

Proof of Lemma. We set a variation vector field V along f as follows:

$$V = \varphi X \qquad \varphi \in H_0'(M), \tag{4.2}$$

X being a unit variation vector field along f; i.e.

$$\langle X(f(y)), X(f(y)) \rangle = 1 .$$

Let $\{e_i\}$ be orthonormal frame on M. We calculate $\bar{\Delta} V$.

$$\begin{aligned} \bar{\Delta} V = \operatorname{trace} \bar{\nabla}^2 V &= \sum_i (\bar{\nabla}_{e_i} \bar{\nabla}_{e_i}(\varphi X)) - \bar{\nabla}_{\nabla_{e_i} e_i}(\varphi X)) \\ &= (\Delta\varphi) X + \varphi \bar{\Delta} X + 2 \sum_i e_i(\varphi) \bar{\nabla}_{e_i} X , \end{aligned} \tag{4.3}$$

where $\Delta\varphi$ is the Laplacian of φ. Hence

$$\langle \bar{\Delta} V, V \rangle = \varphi \Delta \varphi + \varphi^2 \langle X, \bar{\Delta} X \rangle , \qquad \text{since} \quad \langle \bar{\nabla}_{e_i} X, X \rangle = 0$$

From (3.8) we have the formula

$$I(V, V) = \int_M \left[-\varphi\Delta\varphi - \varphi^2 \langle X, \bar{\Delta} X \rangle - \varphi^2 \sum_i \langle {}^N R(X, df e_i) df e_i, X \rangle \right] dx . \tag{4.4}$$

We denote by $k_N(X \wedge Y)$ the sectional curvature of N spanned by two independent vectors X and Y.

$$k_N(X \wedge df e_i) = \frac{\langle {}^N R(X, df e_i) df e_i, X \rangle}{|df e_i|^2 |X|^2 - \langle df e_i, X \rangle^2} \tag{4.5}$$

By the condition of lemma

$$\sum_i \langle {}^N R(X, df e_i) df e_i, X \rangle \leqslant C \sum_i \{ |df e_i|^2 |X|^2 - \langle df e_i, X \rangle^2 \} \leq 2C e(f) . \tag{4.6}$$

On the other hand,

$$\begin{aligned} \langle X, \bar{\Delta} X \rangle &= \sum_i \langle X, \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} X - \bar{\nabla}_{\nabla_{e_i} e_i} X \rangle \\ &= \sum_i \langle X, \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} X \rangle = -\sum_i |\bar{\nabla}_{e_i} X|^2 \leq 0 . \end{aligned} \tag{4.7}$$

Here we used the identities

$$\langle X, \bar{\nabla}_Y X \rangle = 0 \quad \text{and} \quad 0 = Y \langle X, \bar{\nabla}_Y X \rangle = |\bar{\nabla}_Y X|^2 + \langle X, \bar{\nabla}_Y \bar{\nabla}_Y X \rangle$$

for each vector field Y on M. Therefore from (4.6) and (4.7) we get the following formula

$$I(V, V) \geqslant \int_M \{-\varphi\Delta\varphi - 2C\varphi^2 e(f)\}\, dx \tag{4.8}$$

$$= \int_M \{|\nabla\varphi|^2 - 2C\varphi^2 e(f)\}\, dx .$$

We used Green's formula

$$\int_M (\varphi\Delta\varphi + \langle\nabla\varphi, \nabla\varphi\rangle)\, dx = \int_{\partial M} \langle\varphi\nabla\varphi, \nu\rangle\, dx ,$$

where ν is a unit outer normal vector. The lemma follows under its condition.

Proof of Theorem 1. For any $\varphi \in H_0'(M)$

$$\lambda_1(M) < \frac{\int_M |\nabla\varphi|^2 dx}{\int_M \varphi^2 dx}, \tag{4.9}$$

hence

$$2\cdot C\int_M \varphi^2 e(f) dx \leqslant 2\cdot C \sup_{x\in M} e(f)(x) \int_M \varphi^2 dx < \int_M |\nabla\varphi|^2 dx$$

This implies the condition of lemma. Therefore the theorem follows.

5. Proof of Theorem 2

(a) when $m \geqslant 3$, replacing φ by $|\varphi|^{2(m-1)/(m-2)}$ in the Sobolev inequality, we get

$$\int_M \frac{2(m-1)}{m-2} |\varphi|^{m/(m-2)} |\nabla\varphi|\, dx \geqslant D\cdot\left(\int_M |\varphi|^{2m/(m-2)} dx\right)^{(m-1)/m},$$

using the Hörder inequality

$$\left(\int_M |\varphi|^{2m/(m-2)} dx\right)^{(m-2)/2m} \leqslant \frac{2(m-1)}{D\cdot(m-2)}\left(\int_M |\nabla\varphi|^2 dx\right)^{1/2}. \tag{5.1}$$

Also by the Hörder inequality and (5.1)

$$\left(\int_M \varphi^2 e(f) dx\right)^{1/2} \leq \left(\int_M e(f)^{m/2} dx\right)^{1/m} \left(\int_M |\varphi|^{2m/(m-2)} dx\right)^{(m-2)/2m}$$

$$\leq \frac{2(m-1)}{D\cdot(m-2)}\left(\int_M e(f)^{m/2} dx\right)^{1/m} \left(\int_M |\nabla\varphi|^2 dx\right)^{1/2},$$

hence by the hypothesis of Theorem 2,

$$\left(2C\int_M \varphi^2 e(f)\,dx\right)^{1/2} \leq \left(\int_M |\nabla\varphi|^2 dx\right)^{1/2}.$$

We get the stable condition of lemma by squaring both sides.

(b) When $m=2$, replacing φ by φ^2 in the Sobolev inequality, and using the Hörder inequality,

$$D\left(\int_M \varphi^4 dx\right)^{1/2} \leqslant \int_M 2|\varphi||\nabla\varphi|\,dx \leqslant 2\ \mathrm{vol}(M)^{1/4}\left(\int_M \varphi^4 dx\right)^{1/4}\left(\int_M |\nabla\varphi|^2 dx\right)^{1/2}.$$

Hence

$$\left(\int_M \varphi^4 dx\right)^{1/4} \leqslant \frac{2}{D}\mathrm{vol}(M)^{1/4}\left(\int_M |\nabla\varphi|^2 dx\right)^{1/2}. \tag{5.2}$$

Again by the Hörder inequality and (5.2)

$$\begin{aligned}\left(2C\int_M \varphi^2 e(f)dx\right)^{1/2} &\leq \sqrt{2C}\left(\int_M e(f)^2 dx\right)^{1/4}\left(\int_M \varphi^4 dx\right)^{1/4}\\ &\leq \|e(f)^{1/2}\|_{L^4}\frac{2\sqrt{2C}}{D}\mathrm{vol}(M)^{1/4}\left(\int_M |\nabla\varphi|^2 dx\right)^{1/2},\end{aligned}$$

the hypothesis of lemma follows, and theorefore the theorem follows.

6. Proof of Proposition 4

Choose any $\varepsilon \in \boldsymbol{R}^{n+1}$, $(\varepsilon, \varepsilon)=1$. Let $(\,,)$ be Riemannian metric in $\boldsymbol{R}^{n+1}$. We use the first variation formula (3.6). We put a variation vector field V along f

$$V(x)=\varepsilon-(\varepsilon, y(f(x)))y(f(x))\,, \tag{6.1}$$

where y is the position vector of $f(M)$ in R

$$\begin{aligned}\bar{\nabla}_{e_i}V=\bar{\bar{\nabla}}_{dfe_i}V &= (\tilde{\nabla}_{dfe_i}[\varepsilon-(\varepsilon, y)y])^T\\ &=(-\varepsilon, dfe_i)y-(\varepsilon, y)dfe_i)^T\\ &=-(\varepsilon, y)dfe_i\,,\end{aligned}$$

where $\bar{\nabla}$, $\bar{\bar{\nabla}}$, $\tilde{\nabla}$ are the Riemannian connection of $f^{-1}TS^n(1)$, $T\boldsymbol{R}^{n+1}$, $TS^n(1)$ respectively. $(\ \ )^T$ the tangent component of $S^n(1)$. It follows

$$-\int_M (\varepsilon, y)\sum_i \langle dfe_i, dfe_i\rangle dx = -\int_M (\varepsilon, y)e(f)dx=0\,.$$

By $e(f)\neq 0$ (nonconstant map). It is impossible to be $(\varepsilon, y)>0$ or $(\varepsilon, y)<0$.

7. Appendix

7.1. *An example of stable harmonic maps*

Define M, N, and f as follows.

$M := \{(x^1, x^2, \cdots, x^n) \in \boldsymbol{R}^n : \sum_i (x^i)^2 = 1,\ x^n \geqslant c > 0,\ n \geqslant 2\}$

$N := S^n$ ($\subset \boldsymbol{R}^{n+1}$): a n-dimension unit sphere with the standard metric

$f \colon M \to N$; $f(x^1, x^2, \cdots, x^n) = (x^1, x^2, \cdots, x^n, 0)$: totally geodesic map.

Then using the same notation as in Theorem 1, it is easy to see that $e(f) = (n-1)/2$, $C = 1$, $\lambda_1(M) > n-1$ (see [1] for $\lambda_1(M)$.) Therefore it is a stable harmonic map.

7.2. *A relation with a minimal submanifold*

Let $f:(M, g) \to (N, h)$ be a Reimannian immersion ($f^*h = g$). Then f is harmonic if only if it is minimal [3]. Let f_t be a one-parameter family of Riemannian immersions with $f_0 = f$. Put

$$V(t) := \int_M \det(f_t^* h) dx .$$

Then f is minimal if it is a critical point of $V(t)$. When we identify M with $f(M)$, the second variation formula of minimal submanifold is [4];

$$d^2V(t)/dt^2 \Big|_{t=0} = \int_M \Big[-\langle E, \Delta E\rangle - \sum_{i,j} (\langle B(e_i, e_j), E\rangle) - \sum_j \langle R(E, e_j)e_j, E\rangle \Big] dx , \tag{7.1}$$

where $\{e_i\}$ is an orthonormal frame, E the variation vector field along f vanishing on boundary ∂M. B denotes the second fundamental form of M in N. By comparing (3.7) with (7.1), a stable minimal submanifold yields a stable harmonic map.

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