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Characters and Artin L-function II

by

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The main aim of this paper is to give some additional explanations about the results of the previous paper [2].*

In [2] we obtained the following theorem (Theorem 5 in [2]);

There exist relative abelian extensions E_1/K_1 and E_2/K_2 of algebraic number fields such that each Hecke L-function on E_1/K_1 coincides with that on E_2/K_2 and yet two Galois groups of E_1/K_1 and E_2/K_2 are not isomorphic.

In Section I, we prove that it is able to take $K_1=K_2$ in the above theorem. (See also Theorem 2 of Addition.)

Using the functional equation of the zeta function, Perlis proved that if algebraic number fields have the same decomposition of all but a finite number of rational primes, then their zeta functions are the same.

Using Artin L-functions and the Čebotarev density theorem, we obtain in Section II that the above statement goes through even if we replace "all but a finite number of rational primes" by "all rational primes in a set of density 1". (cf. Corollary of Theorem 1)

In Section III it is showed that for every positive number N we can find a field to which the number of arithmetically equivalent fields up to conjugate fields is larger than N. (See also Theorem 2 of Addition.)

Notations used here are kept in [2].

Let G_1 be a group of order 16 generated by elements P, Q, R with relations;

$$P^4=1,\ Q^4=1,\ R^2=1,\ Q^{-1}PQ=P^{-1},\ P^2=Q^2,\ R^{-1}QR=Q,\ R^{-1}PR=P\ .$$

Let G_2 be a group of order 16 generated by elements S, T with relations;

$$S^4=1,\ T^4=1,\ T^{-1}ST=S^{-1}\ .$$

I. We prove that it is able to take $K_1=K_2(=K_0)$ in Theorem 5 in [2].

Let the indices of irreducible characters of G_1 and G_2 be as in Section 6 in [2]. The direct product $G_1\times G_2$ is embedded in the sym-

* This paper and [2] form the author's doctoral thesis (Rikkyo University, November 1978).

metric group S_{256} of degree 256 via its regular representation. Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of a group G. It holds

$$\mathrm{Irr}(G_1\times G_2)=\{\chi_i\times\theta_j;\ \chi_i\in\mathrm{Irr}(G_1),\ \theta_j\in\mathrm{Irr}(G_2)\}\ ,$$

where $(\chi_i\times\theta_j)(\sigma,\tau)=\chi_i(\sigma)\theta_j(\tau)$ for $(\sigma,\tau)\in G_1\times G_2$. Put $\tilde{\chi}_i=\chi_i\times\theta_0$ and $\tilde{\theta}_j=\chi_0\times\theta_j$, where χ_0 and θ_0 are the principal characters of G_1 and G_2 respectively. Then characters $\tilde{\chi}_i^*$ and $\tilde{\theta}_j^*$ of S_{256} induced from $\tilde{\chi}_i$ and $\tilde{\theta}_j$, satisfy that

$$\tilde{\chi}_0^*=\tilde{\theta}_0^*,\quad \tilde{\chi}_2^*=\tilde{\theta}_2^*,\quad \tilde{\chi}_4^*=\tilde{\theta}_4^*,\quad \tilde{\chi}_5^*=\tilde{\theta}_5^*\ .$$

Put $H_1=\langle P\rangle\times G_2$ and $H_2=G_1\times\langle S\rangle$. Then H_1 and H_2 are normal subgroups of $G_1\times G_2$. Further the quotient group $(G_1\times G_2)/H_1$ is the Krein's four group and the quotient group $(G_1\times G_2)/H_2$ is a cyclic group of order 4. Here it holds that

$$\mathrm{Irr}((G_1\times G_2)/H_1)=\{\tilde{\chi}_0,\tilde{\chi}_2,\tilde{\chi}_4,\tilde{\chi}_5\}\ ,$$
$$\mathrm{Irr}((G_1\times G_2)/H_2)=\{\tilde{\theta}_0,\tilde{\theta}_2,\tilde{\theta}_4,\tilde{\theta}_5\}\ ,$$

Let E/K be a normal extension of algebraic number fields with $\mathrm{Gal}(E/K)=S_{256}$. Let E_1, E_2 and K_0 be the fixed fields of H_1, H_2 and $G_1\times G_2$ respectively. Similarly to the argument in Section 6 in [2], we see that two abelian extensions E_1/K_0 and E_2/K_0 have the same set of Hecke L-functions and yet their Galois groups are non-isomorphic.

The proceeding comment gives a negative solution to the following problem; Do only the Hecke L-functions determine a finite abelian extension uniquely, when a basic field is fixed? We take an interest in this solution, comparing it with class field theory.

II. We keep notations in Section 2 in [2].

Let K/k be a finite Galois extension of algebraic number fields with $G=\mathrm{Gal}(K/k)$ and ψ be its generalized character. For a positive rational integer m and a prime ideal $\mathfrak{p}$ in k, we put

$$\psi(\mathfrak{p}^m)=|I_{\mathfrak{P}}|^{-1}\sum_{\tau\in I_{\mathfrak{P}}}\psi(\sigma_{\mathfrak{P}}^m\tau)\ ,$$

where $\sigma_{\mathfrak{P}}$ is a Frobenius substitution of an arbitrary prime factor $\mathfrak{P}$ of $\mathfrak{p}$ in K and $I_{\mathfrak{P}}$ is its inertia group. Let p be a rational prime, and define $L_p(s,\psi,K/k)$ by

$$L_p(s,\psi,K/k)=\prod_{\mathfrak{p}|p}\exp\left\{\sum_{m=1}^{\infty}\frac{\psi(\mathfrak{p}^m)}{mN(\mathfrak{p})^{ms}}\right\}$$

for $\mathrm{Re}(s)>1$, where $\mathfrak{p}$ runs over all prime factors of p in k. The following properties of $L_p(s,\psi,K/k)$ are obtained in process of proving the corresponding properties of Artin L-function $L(s,\psi,K/k)$. But they are not usually specified. Therefore we enumerate list here because of reference later on.

(1) $L(s, \psi, K/k)=\prod_p L_p(s, \psi, K/k)$
for $\mathrm{Re}(s)>1$, where p runs over all rational primes.

(2) $L_p(s, \psi_1+\psi_2, K/k)=L_p(s, \psi_1, K/k)L_p(s, \psi_2, K/k)$.

(3) If E is a finite Galois extension field of k including K, then $L_p(s, \psi, K/k)=L_p(s, \psi, E/k)$.

(4) If $K\supset F\supset k$ and ψ is a generalized character of $\mathrm{Gal}(K/F)$, then $L_p(s, \psi, K/F)=L_p(s, \psi^*, K/k)$, where ψ^* is the character of $\mathrm{Gal}(K/k)$ induced from ψ.

Let A be a set of rational primes. If

$$\lim_{s\to 1+0}\Big(\sum_{p\in A}\frac{1}{p^s}\Big)\Big/\log\frac{1}{s-1}$$

exists, then its value is said to be the *Dirichlet density* of A and denoted by $\mu(A)$.

THEOREM 1. *The following conditions are equivalent.*

(i) $L(s, \psi, K/k)=1$.

(ii) $L_p(s, \psi, K/k)=1$ *for every rational prime* p.

(iii) *There exists a set* A *of rational primes such that* $\mu(A)=1$ *and* $L_p(s, \psi, K/k)=1$ *for every* $p\in A$.

Proof. (i)$\Rightarrow$(ii). Let E be the Galois closure of K over $\boldsymbol{Q}$. Applying the independence theorem on Artin L-functions (Theorem 1 in [2]) to $L(s, \psi^*, E/\boldsymbol{Q})=L(s, \psi, K/k)=1$, we have $\psi^*=0$. Therefore it holds for every p

$$L_p(s, \psi, K/k)=L_p(s, \psi^*, E/\boldsymbol{Q})=1\ .$$

(ii)$\Rightarrow$(iii). This is trivial.

(iii)$\Rightarrow$(i) It suffices to check that $L_p(s, \psi, E/\boldsymbol{Q})=1$ for every $p\in A$ implies $\psi=0$ when E is a finite Galois extension field of $\boldsymbol{Q}$ with $G=\mathrm{Gal}(E/\boldsymbol{Q})$ and when ψ is a generalized character of G. Then we obtain $\psi(\sigma_{\mathfrak{P}})=0$ for an unramified prime ideal $\mathfrak{P}$ in E above $p\in A$. For it follows from $\sum_m \psi(p^m)/mp^{ms}=0$ for $R(s)>1$ that $\psi(p)=0$, in particular, $\psi(\sigma_{\mathfrak{P}})=0$ for unramified primes $\mathfrak{P}$. (For the details, cf. Theorem 1 in [2].) Let σ be any element in G and let B_σ be a set of all rational primes p whose prime factors in E correspond to σ as Frobenius substitution. By the Čebotarev density theorem, we get $\mu(B_\sigma)>0$. Hence $A\cap B_\sigma\neq\phi$ and so there exists $p\in A$ such that $\sigma=\sigma_{\mathfrak{P}}$ for an unramified prime factor $\mathfrak{P}$ of p in E. Therefore it holds that $\psi(\sigma)=0$ for all $\sigma\in G$, that is, $\psi=0$. Q.E.D.

The following corollary is detailed more than Cassels-Fröhlich [1, Exercise 6.4]. Compare it with Corollary 3 to Theorem 3 in [2] too.

COROLLARY. *In order that* $\zeta_{K_1}(s)=\zeta_{K_2}(s)$, *it is necessary and sufficient that there exists a set* A *of rational primes with* $\mu(A)=1$ *such* *that for every $p\in A$ the decomposition of p in K_1 and K_2 is the same, in sence that the collection of degrees of the factors of p in K_1 is identical with the collection of degrees of the factors of p in K_2.*

Proof. The necessity follows from Cassels-Fröhlich [1, Exercise 6.4] or Perlis [3, Theorem 1]. (There we can take the set of all rational primes as A.)

Here we give only a proof of the sufficient. It follows from the assumption of the sufficient condition that

$$\prod_{\mathfrak{p}_1|p}\left(1-\frac{1}{N(\mathfrak{p}_1)^s}\right)^{-1}=\prod_{\mathfrak{p}_2|p}\left(1-\frac{1}{N(\mathfrak{p}_2)^s}\right)^{-1}$$

for every $p\in A$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ run over all prime factors of p in K_1 and K_2 respectively. Since

$$\zeta_{K_i}(s)=L(s, 1, K_i/K_i)$$

and

$$L_p(s, 1, K_i/K_i)=\prod_{\mathfrak{p}_i|p}\left(1-\frac{1}{N(\mathfrak{p}_i)^s}\right)^{-1}$$

for $i=1, 2$, we obtain

$$L_p(s, 1, K_1/K_1)=L_p(s, 1, K_2/K_2)\ .$$

Let E be a finite Galois extension field of $\boldsymbol{Q}$ including K_1 and K_2 with $G=\mathrm{Gal}(E/\boldsymbol{Q})$ and let $H_i=\mathrm{Gal}(E/K_i)$ for $i=1, 2$. Now χ_1 and χ_2 denote the characters of G induced from the principal characters of H_1 and H_2 respectively. Then for every $p\in A$,

$$L_p(s, \chi_1-\chi_2, E/Q)=1\ .$$

By Theorem 1, we get $L(s, \chi_1-\chi_2, E/\boldsymbol{Q})=1$, that is,

$$\zeta_{K_1}(s)=\zeta_{K_2}(s)\ . \qquad \text{Q.E.D.}$$

III. Two algebraic number fields K_1, K_2 are said to be *arithmetically equivalent* if $\zeta_{K_1}(s)=\zeta_{K_2}(s)$. Let $A(K)$ be the set of all arithmetically equivalent fields to an algebraic number field K. We define an equivalent relation in $A(K)$ by the following; Fields K_1 and K_2 in $A(K)$ are equivalent if K_1 and K_2 are conjugate. Now $n(K)$ denotes the number of equivalent classes of $A(K)$ with respect to the relation. It follows from Corollary to Theorem 4 in [2] that the number of arithmetically equivalent fields to a given field is always finite. Thus every $n(K)$ is finite. However the following proposition holds.

PROPOSITION. *For any positive number N, there exists a algebraic number K with $n(K)\geqq N$.*

Proof. Let G_1, G_2 be finite groups. Suppose that subgroups H_{11}, H_{12}

of G_1 are Gassmann equivalent in G_1 and subgroups H_{21}, H_{22} are G_2 are Gassmann equivalent in G_2. (Subgroups H_1, H_2 of a finite group G are said to be *Gassmann equivalent* if $|H_1 \cap C| = |H_2 \cap C|$ for every conjugate class C of G. For the details, see [2] or Perlis [3].) Then $H_{11} \times H_{21}$ and $H_{12} \times H_{22}$ are Gassmann equivalent in $G_1 \times G_2$. Groups G_1, G_2 given at the beginning of this paper are Gassmann equivalent and non-conjugate, in fact non-isomorphic, as subgroup of the symmetric group S_{16} of degree 16. (For the details, see Section 6 in [2].) When we regard the direct product G_1^m as subgroup of the direct product S_{16}^m, the number of Gassmann equivalent subgroups of S_{16}^m to G_1^m which are non-conjugate to each other is larger than 2^m. Further there exists a Galois extension E_m/Q with $\mathrm{Gal}(E_m/Q) = S_{16}^m$. Therefore, by Corollary 3 to Theorem 3 in [2], it suffices to take as K the fixed field of G_1^m in E_m for $m \geqq \log_2 N$.
Q.E.D.

Addition. The argument in Section I and III suggests the following theorem. But its proof is omitted here owing to avoidance of repetition. Now $H(L/K)$ denotes the set of all Hecke L-functions attached to an abelian extension L/K.

THEOREM 2. *For any positive number N, there exists an algebraic number field K which has finite abelian extensions $F_1/K, \cdots, F_m/K$ for some $m \geqq N$ such that* $\mathrm{Gal}(F_1/K), \cdots, \mathrm{Gal}(F_m/K)$ *are not isomorphic to each other as abstract group while* $H(F_1/K) = \cdots = H(F_m/K)$.

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