Dissertation

Toric Rings and Toric Ideals Arising from Various Configurations

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Introduction

A purpose of this thesis is to study properties of toric rings and toric ideals associated with various configurations. In particular, we study them targeted at configurations associated with a cut of graphs and matroids.

This thesis is concerned with the strongly Koszul property of the toric ring associated to a cut ideal and a Gröbner basis for a toric ideal of a matroid.

Standard graded algebras R over a field K are said to be Koszul if the R-module $K = R/\mathbf{m}$ has a linear minimal free resolution over R, where $\mathbf{m}$ is the graded maximal ideal of R. Koszul algebras have been introduced by Priddy in 1970 [27]. A strongly Koszul algebra is the stronger notion of Koszulness and was introduced by Herzog, Hibi and Restuccia [13]. For a toric ring R and a toric ideal I, it is known that

$$\begin{array}{c} I \text{ has a quadratic Gröbner basis, or } R \text{ is strongly Koszul} \\ \Downarrow \\ R \text{ is Koszul} \\ \Downarrow \\ I \text{ is generated by quadratic binomials.} \end{array}$$

In general, the converse hierarchy is not true.

The outline of this thesis is as follows.

In Chapter 1, we introduce notation and recall known results about Koszul algebras, Gröbner bases, toric rings, toric fiber products, graphs and matroids.

In Chapter 2, we study properties of the toric ring associated to a cut ideal arising from a graph. A cut ideal was introduced by Sturmfels and Sullivant (see [34]). A cut ideal of a graph records the relations among the cuts of the graph. Cut ideals are used in algebraic statistics to study statistical models defined by graphs.

Let R_G be a toric ring associated to a cut ideal I_G arising from a graph G. The following facts are known for R_G and I_G:

- R_G is compressed if and only if G has no K_5-minor and every induced cycle in G has length 3 or 4 [34];
- If R_G is normal, then G has no K_5-minor [34];
- If G has no $(K_5 \setminus e)$-minor, then R_G is normal [21];

- If I_G is generated by binomials of degree ≤ 4, then G has no K_5-minor [34];
- I_G is generated by quadratic binomials if and only if G has no K_4-minor [11, 19, 34].

As stated above, ring-theoretic properties of R_G and I_G are classified in the class of a graph. Moreover Nagel and Petrović showed that the cut ideal I_G associated with ring graphs has a quadratic Gröbner basis [19]. However we do not know generally when the cut ideal I_G has a quadratic Gröbner basis and when R_G is Koszul except for trivial cases. We give a necessary and sufficient condition for R_G to be strongly Koszul, that is, we characterize the class of graphs such that R_G is strongly Koszul. The following are main results in Chapter 2.

Theorem 1 ([30])**.** *Let G be a finite simple connected graph. If G has no (K_4, C_5)-minor, then I_G has a quadratic Gröbner basis.*

Theorem 2 ([30])**.** *Let G be a finite simple connected graph. Then R_G is strongly Koszul if and only if G has no (K_4, C_5)-minor.*

In Chapter 3, we study a Gröbner basis for a toric ideal associated with bases of a matroid. A matroid was introduced by Whitney in 1935 [39]. A matroid is a structure that captures and generalizes the notion of linear independence in vector spaces. The bases of a matroid M with the ground set $[d] = \{1, \ldots, d\}$ define a standard graded toric ring $R_M \subset K[s_1, \ldots, s_d]$ which is generated by squarefree monomials whose support forms a basis of M. The toric ring R_M is called the *base monomial ring* of M and was introduced by White [37]. White proved that, for any matroid M, the base monomial ring R_M is normal, in particular, Cohen-Macaulay. White conjectured that, for any matroid M on $[d]$, the toric ideal J_M of M is generated by the quadratic binomials $x_i x_j - x_k x_l$ such that the pair of bases B_k, B_l can be obtained from the pair of bases B_i, B_j by a symmetric exchange (see [33, 38]).

Let $\mathcal{M}_{\mathcal{QG}}$ be the class of matroids such that the toric ideal J_M has a Gröbner basis consisting of quadratic binomials and $\mathcal{M}_{\mathcal{Q}}$ be the class of matroids for which J_M is generated by quadratic binomials. Blum defined base-sortable matroids and proved that the class of base-sortable matroids is contained in $\mathcal{M}_{\mathcal{QG}}$ [2]. By using the theories of toric fiber products and combinatorial pure subrings, we have

Theorem 3 ([31])**.** *Classes $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under series and parallel extensions, series and parallel connections and 2-sums.*

Chaourar showed that a matroid M is a minor of 1-sums and 2-sums of uniform matroids if and only if M has no minor isomorphic to any of $M(K_4)$, $\mathcal{W}^3$, Q_6 and P_6 [5]. Since uniform matroids belong to $\mathcal{M}_{\mathcal{QG}}$ [32] and the class $\mathcal{M}_{\mathcal{QG}}$ is closed under 1-sums and taking minors [2], by Theorem 3 and Chaourar's result, we have

Theorem 4 ([31])**.** *Let M be a matroid. If M has no minor isomorphic to any of $M(K_4)$, $\mathcal{W}^3$, Q_6 and P_6, then the toric ideal J_M has a Gröbner basis consisting of quadratic binomials.*

The result in Chapter 2 is scheduled to be published (see [30]). The result in Chapter 3 is submitted (see [31]).

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Chapter 1

Background

In this chapter, we introduce notation and give basic definitions and recall some results. A detailed introduction on the fundamental facts in Section 1.1 and Section 1.2 is in books by Eisenbud [9], and Ene and Herzog [10]. In Section 1.3 and Section 1.4, we consider the powerful tools of Gröbner bases, toric rings and toric ideals (see [14, 32]). Toric fiber products, which we consider in Section 1.5, are introduced by Stullivant [36]. Section 1.6, which we consider the graph theory, is based on Diestel's book [8]. The aim of Section 1.7 is to recall some basic facts about matroid theory. For a detailed introduction to matroid theory, see Oxley's book [26].

1.1 Standard graded algebras

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring with standard grading $\deg(x_i) = 1$ for $1 \le i \le n$. A polynomial f is said to be *homogeneous of degree* i if all monomials appearing in f are of degree i. We write $\deg(f) = i$. Let $f = \sum_{\mathbf{a} \in \mathbb{Z}^n_{\ge 0}} c_{\mathbf{a}} X^{\mathbf{a}}$, where $X^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \dots, a_n)$ and $c_{\mathbf{a}} \in K$, be a polynomial. Then we set

$$f_i = \sum_{\mathbf{a} \in \mathbb{Z}^n_{\ge 0}, |\mathbf{a}| = i} c_{\mathbf{a}} X^{\mathbf{a}},$$

where $|\mathbf{a}| = a_1 + \cdots + a_n$. Then f_i is homogeneous of degree i and called the *i-th homogeneous component* of f. We have $f = \sum_{i \ge 0} f_i$ and this decomposition into homogeneous components is unique. It follows that

$$S = \bigoplus_{j \ge 0} S_j,$$

where S_j is the K-subspace of S consisting of all homogeneous polynomials in S of degree j.

A *graded ideal* is an ideal $I \subset S$ which is generated by homogeneous polynomials. Let I_i denote the K-vector space spanned by all homogeneous polynomials in I of degree i. Then the quotient ring $R = S/I$ has a natural decomposition

$$R = \bigoplus_{i \geq 0} R_i,$$

where $R_i = S_i/I_i$. Each graded component R_i is a finite dimensional K-vector space and $R_0 = K$. We have $R_i R_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}_{\geq 0}$ and R is finitely generated as a K-algebra by elements of R_1.

Definition 1.1.1. A K-algebra R is said to be *standard graded* if it is of the form $R = S/I$, where $I \subset S$ is a graded ideal.

1.2 Koszul algebras

In this section, we introduce the definition of Koszul algebras and strongly Koszul algebras. Let R be a commutative ring. A *maximal ideal* of R is a proper ideal not contained in any other proper ideal.

Definition 1.2.1. Let K be a field and R be a standard graded K-algebra with graded maximal ideal $\mathbf{m}$. The K-algebra R is said to be *Koszul* if the R-module $K = R/\mathbf{m}$ has a linear minimal free resolution over R.

Let R and $R^{'}$ be two standard graded K-algebras. The Segre product we denote with $R * R^{'}$ is defined as the graded algebra

$$R * R^{'} = \bigoplus_{i \geq 0} R_i \otimes_K R^{'}_i.$$

The tensor product $R \otimes_K R^{'}$ is naturally standard graded with components

$$(R \otimes_K R^{'})_i = \bigoplus_{k+l=i} R_k \otimes_K R_l.$$

For $R = K[x_1, \ldots, x_n]/\langle f_1, \ldots, f_r\rangle$ and $R^{'} = K[y_1, \ldots, y_m]/\langle g_1, \ldots, g_s\rangle$, it has a presentation of the form

$$R \otimes_K R^{'} = K[x_1, \ldots, x_n, y_1, \ldots, y_m]/\langle f_1, \ldots, f_r, g_1, \ldots, g_s\rangle.$$

Proposition 1.2.2. *Let R and $R^{'}$ be two K-algebras.*

(1) *If R and $R^{'}$ are Koszul, then $R * R^{'}$ is Koszul.*

(2) *$R \otimes_K R^{'}$ is Koszul if and only if R and $R^{'}$ are Koszul.*

Next, we introduce the following stronger notion of Koszulness given in [13].

Definition 1.2.3 ([13, Definition 1.1]). The homogeneous K-algebra R is said to be *strongly Koszul* if its graded maximal ideal $\mathbf{m}$ admits a minimal system of homogeneous generators $u_1, \dots, u_n$ such that for all subsequences $u_{i_1}, \dots, u_{i_r}$ of $u_1, \dots, u_n$ with $1 \leq i_1 < \cdots < i_r \leq n$, and for all $j = 1, \dots, r$, the colon ideal $\langle u_{i_1}, \dots, u_{i_{j-1}} \rangle : u_{i_j}$ of R is generated by a subset of elements of $\{u_1, \dots, u_n\}$.

Theorem 1.2.4 ([13, Theorem 1.2]). *Let R be strongly Koszul with respect to the minimal homogeneous system $u_1, \dots, u_n$ of generators of the graded maximal ideal $\mathbf{m}$ of R. Then any ideal of the form $\langle u_{i_1}, \dots, u_{i_r} \rangle$ has a linear resolution. In particular, R is Koszul.*

1.3 Gröbner bases

Let Σ be a set. A *partial order* on Σ is a binary relation $\leq$ over Σ such that, for all $x, y, z \in \Sigma$, one has

(1) $x \leq x$ (reflexivity);

(2) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry);

(3) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

We write $x < y$ if $x \leq y$ and $x \neq y$. A partially ordered set is a set Σ with a partial order $\leq$ on Σ. A partial order $\leq$ on Σ is called a *total order* if, for any $x, y \in \Sigma$, one has $x \leq y$ or $y \leq x$.

Let $K[X] = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and $\mathcal{M}_n$ denote the set of all monomials in $K[X]$.

Definition 1.3.1. A *monomial order* on $K[X]$ is a total order $<$ on $\mathcal{M}_n$ such that

- $1 < u$ for all $1 \neq u \in \mathcal{M}_n$;
- if $u, v \in \mathcal{M}_n$ and $u < v$, then $wu < wv$ for all $w \in \mathcal{M}_n$.

We introduce some monomial orders on $K[X]$.

Example 1.3.2. Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be two monomials in $K[X]$. For a fixed order $x_1 > \cdots > x_n$ of the variables, we have

(1) *the lexicographic order* $<_{\text{lex}}$: We set $u <_{\text{lex}} v$ if the leftmost nonzero component of the vector $(b_1 - a_1, \dots, b_n - a_n)$ is positive.

(2) *the reverse lexicographic order* $<_{\text{rev}}$: We set $u <_{\text{rev}} v$ if the rightmost nonzero component of the vector $(b_1 - a_1, \dots, b_n - a_n, |\mathbf{a}| - |\mathbf{b}|)$ is negative, where $|\mathbf{a}| = a_1 + \cdots + a_n$, $|\mathbf{b}| = b_1 + \cdots + b_n$.

For a nonzero polynomial

$$f = \sum_{i=1}^{m} a_i u_i \quad (0 \neq a_i \in K)$$

of $K[X]$, where $u_1, \dots, u_m$ are monomials, the *support* of f is the set of monomials appearing in f. It is written as $\mathrm{supp}(f)$. For any nonzero polynomial f in $K[X]$, the largest monomial $u \in \mathrm{supp}(f)$ with respect to $<$ is called the *initial monomial* of f and written as $\mathrm{in}_<(f)$. Let $I \subset K[X]$ be a nonzero ideal. The *initial ideal* of I is the monomial ideal

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(f) \mid f \in I, f \neq 0 \rangle.$$

If $I = \langle 0 \rangle$, then $\mathrm{in}_<(I) = \langle 0 \rangle$. In general, the initial monomials of a generating set of I do not generate $\mathrm{in}_<(I)$.

Example 1.3.3. Let $K[X] = K[x_1, \dots, x_7]$ and $<$ be the lexicographic order on $K[X]$ with ordering $x_7 < x_6 < \cdots < x_1$. We set $I = \langle f, g \rangle$, where $f = x_1x_4 - x_2x_3$ and $g = x_4x_7 - x_5x_6$. Then $\mathrm{in}_<(f) = x_1x_4$ and $\mathrm{in}_<(g) = x_4x_7$. However $h = x_1x_5x_6 - x_2x_3x_7 = x_7f - x_1g \in I$ and $\mathrm{in}_<(h) = x_1x_5x_6 \notin \langle x_1x_4, x_4x_7 \rangle$. Therefore $\{x_1x_4, x_4x_7\}$ is not a generating set of $\mathrm{in}_<(I)$.

Definition 1.3.4. We fix a monomial order $<$ on $K[X]$. Let I be an ideal of $K[X]$ with $I \neq \langle 0 \rangle$ and let $\mathcal{G} = \{g_1, \dots, g_s\}$ be a finite set of nonzero polynomials belonging to I. We say that $\mathcal{G}$ is a *Gröbner basis* of I with respect to $<$ if $\{\mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_s)\}$ is a generating set of the initial ideal $\mathrm{in}_<(I)$.

Theorem 1.3.5. *Let $K[X]$ be the polynomial ring and I be an ideal of $K[X]$. If $\mathcal{G}$ is a Gröbner basis of I with respect to some monomial order, then $\mathcal{G}$ is a generating set of I.*

However the converse of Theorem 1.3.5 is not true in general.

We say that a Gröbner basis $\mathcal{G} = \{g_1, \dots, g_s\}$ of I is a *minimal Gröbner basis* if the following conditions are satisfied:

- $\{\mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_s)\}$ is a minimal generating set of $\mathrm{in}_<(I)$;
- The coefficient of $\mathrm{in}_<(g_i)$ is equal to 1 for $1 \leq i \leq s$.

A minimal Gröbner basis exists. However a minimal Gröbner basis is not unique.

A Gröbner basis $\mathcal{G} = \{g_1, \dots, g_s\}$ is said to be *reduced* if the following conditions are satisfied:

- The coefficient of $\mathrm{in}_<(g_i)$ is equal to 1 for $1 \leq i \leq s$;
- None of the monomials belonging to $\mathrm{supp}(g_j)$ is divided by $\mathrm{in}_<(g_i)$ for $i \neq j$.

A reduced Gröbner basis exists and is unique.

Let f and g be nonzero polynomials in $K[X]$. Let c_f (resp. c_g) be the coefficient of $\mathrm{in}_<(f)$ (resp. $\mathrm{in}_<(g)$). Then the polynomial

$$S(f,g) = \frac{\mathrm{LCM}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_f \cdot \mathrm{in}_<(f)} f - \frac{\mathrm{LCM}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_g \cdot \mathrm{in}_<(g)} g$$

is called the *S-polynomial* of f and g, where LCM denotes the least common multiple of two monomials in $K[X]$.

Theorem 1.3.6 (Buchberger's Criterion). *Let I be an ideal of $K[X]$ and $\mathcal{G} = \{g_1, \dots, g_s\}$ be a generating set of I. Then $\mathcal{G}$ is a Gröbner basis of I with respect to some monomial order on $K[X]$ if and only if, for all $i \neq j$, the S-polynomial $S(g_i, g_j)$ reduces to 0 with respect to $g_1, \dots, g_s$.*

1.4 Toric rings and toric ideals

Let $\mathbb{Z}^{d \times n}$ denote the set of all $d \times n$ integer matrices. A *configuration* of $\mathbb{R}^d$ is a matrix $A \in \mathbb{Z}^{d \times n}$, for which there exists a hyperplane $\mathcal{H} \subset \mathbb{R}^d$ not passing the origin of $\mathbb{R}^d$ such that each column vector of A lies on $\mathcal{H}$. Let K be a field and $K[T^{\pm 1}] = K[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ the Laurent polynomial ring in d variables over K. For each column vector $\mathbf{a} = {}^t(a_1, \dots, a_d) \in \mathbb{Z}^d$, we denote $T^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$. Let $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$ be a configuration of $\mathbb{R}^d$. The *toric ring* of A is the subalgebra $K[A]$ of $K[T^{\pm 1}]$ that is generated by the Laurent monomials $T^{\mathbf{a}_1}, \dots, T^{\mathbf{a}_n}$. Let $K[X] = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K. Then we define the surjective ring homomorphism

$$\pi \ : \ K[X] \to K[A], \quad x_i \mapsto T^{\mathbf{a}_i} \text{ for } 1 \leq i \leq n.$$

We call the kernel I_A of π the *toric ideal* of A.

Proposition 1.4.1. *Let $A \in \mathbb{Z}^{d \times n}$ be a configuration. Then*

$$I_A = \left\langle \prod_{b_i > 0} x_i^{b_i} - \prod_{b_i < 0} x_i^{-b_i} \;\middle|\; \begin{array}{l} A\mathbf{b} = \mathbf{0} \\ \mathbf{b} = {}^t(b_1, \dots, b_n) \in \mathbb{Z}^n \end{array} \right\rangle.$$

Proposition 1.4.2. *The reduced Gröbner basis of I_A consists of binomials.*

In general, it is not easy to compute a generating set of I_A. In the case of a toric ideal, there exists the following useful result.

Proposition 1.4.3 (See [25, 32]). *Let $A \in \mathbb{Z}^{d \times n}$ be a configuration and $\mathcal{G} = \{g_1, \dots, g_s\} \subset I_A$. Let $\mathcal{M}_n$ denote the set of monomials belonging to $K[X]$ and $\mathrm{in}_<(\mathcal{G}) = \langle \mathrm{in}_<(g_i) \mid 1 \leq i \leq s \rangle$. Then the following conditions are equivalent.*

(1) *$\mathcal{G}$ is a Gröbner basis with respect to $<$;*

(2) $\{\pi(u) \mid u \in \mathcal{M}_n, u \notin \mathrm{in}_<(\mathcal{G})\}$ *is linearly independent over* K;

(3) $\pi(u) \neq \pi(v)$ *for all* $u, v \notin \mathrm{in}_<(\mathcal{G})$ *with* $u \neq v$, *where* $u, v \in \mathcal{M}_n$;

(4) *for any binomial* $u - v \in I_A$, *where* $u, v \in \mathcal{M}_n$, *either* u *or* v *is divided by* $\mathrm{in}_<(g_i)$ *for some* $1 \leq i \leq s$.

In the case of a toric ring, there is the equivalent condition of a strongly Koszul algebra (see [13]).

Proposition 1.4.4 ([13, Proposition 1.4]). *Let* $K[A]$ *be a toric ring generated by* $u_1, \ldots, u_n$. *Then* $K[A]$ *is strongly Koszul if and only if the ideals* $\langle u_i \rangle \cap \langle u_j \rangle$ *are generated in degree* 2 *for all* $i \neq j$.

In general, it is known that, for a toric ring $K[A]$ and a toric ideal I_A,

$$\begin{gathered} I_A \text{ has a quadratic Gröbner basis, or } K[A] \text{ is strongly Koszul} \\ \Downarrow \\ K[A] \text{ is Koszul} \\ \Downarrow \\ I_A \text{ is generated by quadratic binomials.} \end{gathered}$$

The converse hierarchy is not true (for example, see [23, Example 2.1 and 2.2]).

Conjecture 1.4.5 ([13, 7]). *Let* $K[A]$ *be a toric ring and* I_A *be a toric ideal. If* $K[A]$ *is strongly Koszul, then* I_A *has a quadratic Gröbner basis with respect to some monomial order.*

Hibi, Matsuda and Ohsugi showed that Conjecture 1.4.5 is true for edge rings [15].

Proposition 1.4.6. *Let* $K[A]$ *and* $K[A']$ *be toric rings, and* Q *be the tensor product or the Segre product of* $K[A]$ *and* $K[A']$. *Then* Q *is strongly Koszul if and only if both* $K[A]$ *and* $K[A']$ *are strongly Koszul.*

Definition 1.4.7 ([13]). We say that a toric ring $K[A]$ is *trivial* if, starting with polynomial rings, $K[A]$ is obtained by repeated applications of Segre products and tensor products.

It is clear that any trivial toric ring is strongly Koszul. However there exists a non-trivial strongly Koszul toric ring (for example, see [13]).

Let $K[A]$ be a toric ring. Then $K[A]$ is said to be *squarefree* if $K[A]$ is isomorphic to a toric ring generated by squarefree monomials. A toric ring $K[A]$ is said to be *compressed* [35] if the initial ideal of I_A is squarefree with respect to any reverse lexicographic order.

Theorem 1.4.8 ([18]). *Any squarefree strongly Koszul toric ring is compressed.*

Let $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$ be a configuration and $K[A] \subset K[t_1, \ldots, t_d]$ be a toric ring. For a nonempty subset T of $\{1, \ldots, d\}$, we set $K[A_T] = K[A] \cap K[t_j \mid j \in T]$. Then a subring $K[A_T]$ of $K[A]$ is called a *combinatorial pure subring* of $K[A]$ (see [22]). If $A_T = (\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r})$, then we write $K[X_T] = K[x_{i_1}, \ldots, x_{i_r}]$. Thus $I_{A_T} = I_A \cap K[X_T]$ (see [32, Proposition 4.13]).

Proposition 1.4.9 ([20, 22])**.** *If G is a generating set (resp. the reduced Gröbner basis) for I_A, then $G \cap K[X_T]$ is a generating set (resp. the reduced Gröbner basis) for I_{A_T}.*

Proposition 1.4.10 ([22])**.** *Let $K[A_T]$ be a combinatorial pure subring of $K[A]$. If $K[A]$ is normal, Koszul or strongly Koszul, then $K[A_T]$ has this property, too.*

1.5 Toric fiber products

In this section, we introduce the toric fiber product which is defined by Sullivant [36].

Let r be a positive integer and $\alpha, \beta \in \mathbb{Z}^r_{>0}$ be two vectors. Let

$$K[X] = K[x^i_j \mid i \in [r], j \in [\alpha_i]], \quad K[Y] = K[y^i_k \mid i \in [r], k \in [\beta_i]],$$

where α_i (resp. β_i) is the i-th entry of α (resp. β), be multigraded polynomial rings subject to the multigrading

$$\deg(x^i_j) = \deg(y^i_k) = \mathbf{a}^i \in \mathbb{Z}^d.$$

We write $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^r\}$ and assume that there exists a vector $w \in \mathbb{R}^d$ such that $w \cdot \mathbf{a}^i = 1$ for all i, where $w \cdot \mathbf{a}^i$ is the usual inner product of $\mathbb{R}^d$. This means that ideals in $K[X]$ or $K[Y]$ which are homogeneous with respect to the multigrading are homogeneous in the usual sense. If I and J are homogeneous ideals of $K[X]$ and $K[Y]$ with respect to the grading $\mathcal{A}$, then the quotient rings $R_1 = K[X]/I$ and $R_2 = K[Y]/J$ are also multigraded by $\mathcal{A}$. Consider the polynomial ring

$$K[Z] = K[z^i_{jk} \mid i \in [r], j \in [\alpha_i], k \in [\beta_i]]$$

and the ring homomorphism

$$\phi_{I,J} \ : \ K[Z] \to R_1 \otimes_K R_2, \quad z^i_{jk} \mapsto x^i_j \otimes y^i_k.$$

The *toric fiber product* $I \times_{\mathcal{A}} J$ of I and J is the kernel of $\phi_{I,J}$ [36]. The following result is in [36, Theorem 12 and Corollary 14].

Theorem 1.5.1. *Suppose that the set $\mathcal{A}$ of degree vectors is linearly independent. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be homogeneous generating sets for I and J, respectively. Then*

$$N = \mathrm{Lift}(\mathbf{F}_1) \cup \mathrm{Lift}(\mathbf{F}_2) \cup \mathrm{Quad}_{\mathcal{A}}$$

is a homogeneous generating set for $I \times_{\mathcal{A}} J$. Moreover, if $\mathbf{F}_1$ and $\mathbf{F}_2$ are Gröbner bases of I and J, then there exists a monomial order such that N is a Gröbner basis for $I \times_{\mathcal{A}} J$. The sets $\mathrm{Lift}(\mathbf{F}_1)$, $\mathrm{Lift}(\mathbf{F}_2)$ *and* $\mathrm{Quad}_{\mathcal{A}}$ *are defined in* [36].

On the other hand, if I and J are toric ideals, then $I \times_{\mathcal{A}} J$ is also a toric ideal. Let $\mathcal{B} = \{\mathbf{b}_j^i \mid i \in [r], j \in [\alpha_i]\} \subset \mathbb{Z}^{d_1}$ and $\mathcal{D} = \{\mathbf{d}_k^i \mid i \in [r], k \in [\beta_i]\} \subset \mathbb{Z}^{d_2}$ be two vector configurations. Let $I_{\mathcal{B}} \subset K[X]$ and $I_{\mathcal{D}} \subset K[Y]$ be toric ideals of $\mathcal{B}$ and $\mathcal{D}$. Toric ideals $I_{\mathcal{B}}$ and $I_{\mathcal{D}}$ are homogeneous with respect to the grading by $\mathcal{A}$. We consider the following new vector configuration that is the toric fiber product of the vector configurations.

$$\mathcal{B} \times_{\mathcal{A}} \mathcal{D} = \left\{ \begin{pmatrix} \mathbf{b}_j^i \\ \mathbf{d}_k^i \end{pmatrix} \;\middle|\; i \in [r], j \in [\alpha_i], k \in [\beta_i] \right\} \subset \mathbb{Z}^{d_1+d_2}.$$

Then the toric fiber product $I_{\mathcal{B}} \times_{\mathcal{A}} I_{\mathcal{D}}$ is the toric ideal

$$I_{\mathcal{B}} \times_{\mathcal{A}} I_{\mathcal{D}} = I_{\mathcal{B} \times_{\mathcal{A}} \mathcal{D}}.$$

Indeed, if $K[S]$ and $K[T]$ are polynomial rings, and

$$\begin{aligned} \phi : K[X] &\to K[S], \quad x_j^i \mapsto f_j^i(S), \\ \psi : K[Y] &\to K[T], \quad y_k^i \mapsto g_k^i(T) \end{aligned}$$

are ring homomorphism, then we can form the toric fiber product homomorphism

$$\phi \times_{\mathcal{A}} \psi : K[Z] \to K[S,T], \quad z_{jk}^i \mapsto f_j^i(S) g_k^i(T).$$

If $I = \ker(\phi)$ and $J = \ker(\psi)$ and both ideals are homogeneous with respect to the grading by $\mathcal{A}$, then $I \times_{\mathcal{A}} J = \ker(\phi \times_{\mathcal{A}} \psi)$ (see [12]).

1.6 Graphs

In this section, we introduce a graph and its several properties (see [8]).

A *graph* is a pair $G = (V, E)$ of sets such that the elements of E are 2-element subsets of V. The elements of V are called the *vertices* of the graph G and the elements of E are called the *edges* of G. A graph with vertex set V is called a graph *on* V.

We say that two vertices u, v of G are *adjacent* or *neighbours* if uv is an edge of G. Two different edges e, e' of G is said to be *adjacent* if they have an end in common. A graph G is said to be *complete* if all the vertices of G are pairwise adjacent. The complete graph on n vertices is denoted by K_n.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We set $G \cup G' = (V \cup V', E \cup E')$. If $V' \subset V$ and $E' \subset E$, then G' is called a *subgraph* of G. It is written as $G' \subset G$. If $G' \subset G$ and G' contains all edges $uv \in E$ with $u, v \in V'$, then G' is called an *induced subgraph* of G, or G' is *induced* by V'. It is written as $G' = G[V']$. A *clique* in a graph G is a subset V' of V such that $G[V']$ is complete.

A *path* is a non-empty graph $P = (V, E)$ with

$$V = \{u_0, u_1, \ldots, u_k\}, \quad E = \{u_0u_1, u_1u_2, \ldots, u_{k-1}u_k\},$$

where $u_i \neq u_j$ for $i \neq j$. The vertices u_0 and u_k are *linked* by P and are called its *ends*. The number of edges of a path is called *length* of P. If $u_0 = u_k$ and $k \geq 3$, then the graph (V, E) is called a *cycle*. The *length* of a cycle is its number of edges. The cycle of length k is denoted by C_k.

The minimum length of a cycle contained in a graph G is called the *girth* of G and the maximal length of a cycle in G is called the *circumference*. Note that if G does not contain a cycle, then we set the former to ∞, the latter to zero. An edge which joins two vertices of a cycle but is not itself an edge of a cycle is called a *chord* of that cycle. Hence, an *induced cycle* in G, a cycle in G forming an induced subgraph, is one that has no chords.

A non-empty graph G is said to be *connected* if any two vertices of G are linked by a path in G. We say that connected subgraphs $G_1, \ldots, G_s$ of G are *connected component* of G if the following conditions are satisfied:

- $G = G_1 \cup \cdots \cup G_s$;
- If $k \neq l$, then there exists no edge $u_k u_l$ of G such that u_k (resp. u_l) is a vertex of G_k (resp. G_l).

A non-empty graph $G = (V, E)$ is said to be *k-connected*, where $k \in \mathbb{N}$, if $|V| > k$ and $G[V \setminus X]$ is connected for any set $X \subset V$ with $|X| < k$.

A *2-connected component* is a maximal 2-connected subgraph. Any connected graph decomposes into a tree of 2-connected components called the *block tree* of the graph.

A graph that does not contain any cycles is called a *forest*. A connected forest is called a *tree*.

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. We say that $G' \subset G$ is a *spanning* subgraph of G if $V = V'$.

An edge uv of a graph G, where u, v are vertices of G, is called a *loop* if $u = v$. If G has several edges between the same two vertices u, v, then such edges are called *multiedges*. A graph G is said to be *simple* if G has neither loops nor multiedges.

A graph $G = (V, E)$ is said to be *r-partite* if V admits a partition into r classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. We say that an r-partite graph is *complete* if every two vertices from different partition classes are adjacent. We write $K_{l_1,\ldots,l_r}$ for the complete r-partite graph on $V_1 \cup \cdots \cup V_r$, where $|V_i| = l_i$ for $1 \leq i \leq r$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. The complete r-partite graphs for all r together are the *complete multipartite* graphs.

Let $e = uv$ be an edge of a graph $G = (V, E)$. By $G/e = (V', E')$, we denote the graph obtained from G by *contracting* the edge e into a new vertex w_e, which becomes adjacent to all the former neighbours of x and y, that is,

$$\begin{aligned} V' &= (V \setminus \{u, v\}) \cup \{w_e\}, \\ E' &= \{ij \in E \mid \{i, j\} \cap \{u, v\} = \emptyset\} \cup \{w_e k \mid uk \in E \setminus \{e\} \text{ or } vk \in E \setminus \{e\}\}. \end{aligned}$$

By $G \setminus e$, we denote the graph obtained from G by *deleting* the edge e.

A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of deleting and contracting edges of G.

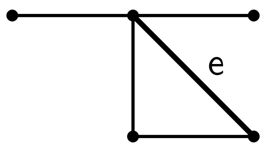

Figure 1.1: G

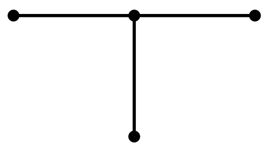

Figure 1.2: G/e

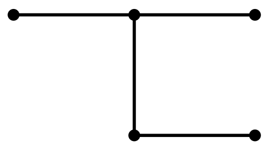

Figure 1.3: $G \setminus e$

1.7 Matroids

In this section, we introduce a matroid and its properties (see [26]).

Definition 1.7.1. A *matroid* is a pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a collection of subsets of E, that satisfies the following conditions:

- $\emptyset \in \mathcal{I}$.
- If $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$.
- If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

We call a member of $\mathcal{I}$ an *independent set* of M. A subset of E that is not contained in $\mathcal{I}$ is said to be *dependent*. A dependent set C is called a *circuit* if any proper subset of C is independent and we write $\mathcal{C}(M)$ for the set of circuits of M.

Example 1.7.2. Let $A = (\mathbf{a}_1, \ldots, \mathbf{a}_5)$ be a 2×5 matrix over the field $\mathbb{R}$, where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{a}_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{a}_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We set $E = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ and $\mathcal{I}$ denotes the collection of subsets X of E such that X is linearly independent in $\mathbb{R}$, i.e.,

$$\mathcal{I} = \{\emptyset, \{\mathbf{a}_1\}, \{\mathbf{a}_2\}, \{\mathbf{a}_4\}, \{\mathbf{a}_5\}, \{\mathbf{a}_1, \mathbf{a}_2\}, \{\mathbf{a}_1, \mathbf{a}_5\}, \{\mathbf{a}_2, \mathbf{a}_4\}, \{\mathbf{a}_2, \mathbf{a}_5\}, \{\mathbf{a}_4, \mathbf{a}_5\}\}.$$

Then a pair $(E, \mathcal{I})$ is a matroid and it is written as $M[A]$. Hence the set of dependent sets of this matroid is

$$\{\{\mathbf{a}_3\}, \{\mathbf{a}_1, \mathbf{a}_3\}, \{\mathbf{a}_1, \mathbf{a}_4\}, \{\mathbf{a}_2, \mathbf{a}_3\}, \{\mathbf{a}_3, \mathbf{a}_4\}, \{\mathbf{a}_3, \mathbf{a}_5\}\} \cup \{X \subset E \mid |X| \geq 3\}.$$

The set of circuits of this matroid is $\{\{\mathbf{a}_3\}, \{\mathbf{a}_1, \mathbf{a}_4\}, \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}, \{\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5\}\}$

Proposition 1.7.3. *Let $\mathcal{C}$ be a collection of subsets of a finite set E. Then $\mathcal{C}$ is the collection of circuits of a matroid on E if and only if $\mathcal{C}$ has the following properties:*

- $\emptyset \notin \mathcal{C}$.
- *If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subset C_2$, then $C_1 = C_2$.*
- *If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subset (C_1 \cup C_2) \setminus \{e\}$.*

An independent set B is said to be *maximal* if there does not exist $x \in E \setminus B$ such that $B \cup \{x\}$ is a member of $\mathcal{I}$. A maximal independent set is called a *basis* of M and we write $\mathcal{B}(M)$ for the collection of bases of M. The collection of bases in Example 1.7.2 is

$$\mathcal{B}(M[A]) = \{\{\mathbf{a}_1, \mathbf{a}_2\}, \{\mathbf{a}_1, \mathbf{a}_5\}, \{\mathbf{a}_2, \mathbf{a}_4\}, \{\mathbf{a}_2, \mathbf{a}_5\}, \{\mathbf{a}_4, \mathbf{a}_5\}\}.$$

Each member of $\mathcal{B}(M[A])$ is a basis of the vector space $\mathbb{R}^2$.

Proposition 1.7.4. *All members of $\mathcal{B}(M)$ have the same cardinality.*

Proposition 1.7.5. *Let M be a matroid on E and $\mathcal{B}$ be a collection of subsets of E. Then $\mathcal{B}$ is the collection of bases of M if and only if $\mathcal{B}$ satisfies the following conditions:*

- *$\mathcal{B}$ is nonempty.*
- *For every $B, B' \in \mathcal{B}$, for any $x \in B \setminus B'$, there exists $y \in B' \setminus B$ such that $(B \cup \{y\}) \setminus \{x\}$ is a member of $\mathcal{B}$.*

Proposition 1.7.5 is called the *exchange axiom*. The exchange axiom is equivalent to the following stronger axiom, known as the *symmetric exchange axiom*.

Proposition 1.7.6. *Let M be a matroid on E and $\mathcal{B}$ be the collection of bases of M. Then*

- *for every B, $B' \in \mathcal{B}$, for any $x \in B$, there exists $y \in B'$ such that $(B \cup \{y\}) \setminus \{x\}$ and $(B' \cup \{x\}) \setminus \{y\}$ are in $\mathcal{B}$.*

Example 1.7.7. We give two examples:

(1) Let r, d be two integers with $0 \leq r \leq d$ and $\mathcal{I}$ be the collection consisting of all subsets with size $\leq r$ of E with $|E| = d$. Then a pair $(E, \mathcal{I})$ is a matroid. This matroid is said to be *uniform* and it is written as $U_{r,d}$. The collection of bases of $U_{r,d}$ consists of all r-element subsets of E and the collection of circuits of $U_{r,d}$ consists of all $(r+1)$-element subsets of E.

(2) Let G be a finite connected graph on the vertex set V with the edge set E. Let $\mathcal{I}$ be the collection consisting of edges of forests in G. Then a pair $(E,\mathcal{I})$ is a matroid. This matroid is said to be *graphic* and it is written as $M(G)$. The collection of bases of $M(G)$ consists of edges of spanning trees in G and the collection of circuits of $M(G)$ consists of edges of cycles in G.

Let $M = (E,\mathcal{I})$ be a matroid and $X \subset E$. Let

$$\mathcal{I}|X = \{I \subset X \mid I \in \mathcal{I}\}.$$

Then $(X,\mathcal{I}|X)$ is a matroid. We call this matroid the *deletion* of $E-X$ from M. It is denoted by $M \setminus (E-X)$. We define the *rank* of X to be the cardinality of a basis of $M \setminus (E-X)$ and it is written as $\mathrm{rk}(X)$. The rank of a matroid M is defined by $\mathrm{rk}(M) = \mathrm{rk}(E)$. The function rk, called the *rank function* of M, maps 2^E to $\mathbb{Z}_{\geq 0}$.

Proposition 1.7.8. *Let E be a finite set. A function $\mathrm{rk} : 2^E \to \mathbb{Z}_{\geq 0}$ is the rank function of a matroid on E if and only if* rk *has the following properties:*

- *If $X \subset E$, then $0 \leq \mathrm{rk}(X) \leq |X|$.*
- *If $X \subset Y \subset E$, then $\mathrm{rk}(X) \leq \mathrm{rk}(Y)$.*
- *If $X, Y \subset E$, then $\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y)$.*

Let K be a field and $K[X] = K[x_1, \ldots, x_n]$ the polynomial ring over K. Let $\mathcal{B}(M) = \{B_1, \ldots, B_n\}$ denote the collection of bases of M on $E = [d] = \{1, \ldots, d\}$. We consider the ring homomorphism

$$\pi_M : K[X] \to K[S] = K[s_1, \ldots, s_d], \quad x_j \mapsto \prod_{l \in B_j} s_l.$$

The toric ideal J_M is the kernel of π_M. The toric ring $R_M = K[X]/J_M$ is called the *bases monomial ring* of M and it was introduced by N. White [37]. White proved that the bases monomial ring R_M is normal, in particular, Cohen-Macaulay for any matroid M (see [37]). White presented the following conjecture.

Conjecture 1.7.9 ([38, 33])**.** For any matroid M, the toric ideal J_M is generated by the quadratic binomials $x_i x_j - x_k x_l$ such that the pair of bases B_k, B_l can be obtained from the pair of bases B_i, B_j by a symmetric exchange.

It is natural to ask whether the following variant of White's conjecture holds.

Conjecture 1.7.10. For any matroid M, the toric ideal J_M has a Gröbner basis consisting of quadratic binomials.

White's conjecture can be posed as two separate conjectures (see [1]).

Conjecture 1.7.11. For any matroid M, the toric ideal J_M is generated by quadratic binomials.

Conjecture 1.7.12. For any matroid M, the quadratic binomials of J_M are in the ideal generated by the binomials $x_i x_j - x_k x_l$ such that the pair of bases B_k, B_l can be obtained from the pair B_i, B_j by a symmetric exchange.

Conjecture 1.7.9 is true for

- graphic matroids [1];
- matroids with rank ≤ 3 [16];
- sparse paving matroids [4]; and
- strongly base orderable matroids [17].

Conjecture 1.7.10 is true for

- uniform matroids [32];
- matroids with rank ≤ 2 [24, 2];
- graphic matroids with no $M(K_4)$-minor [2]; and
- lattice path matroids [29].

In [6], Conca proved that Conjecture 1.7.11 holds for transversal polymatroids.

Let $\mathcal{M}_{\mathcal{QG}}$ be the class of matroids such that J_M has a Gröbner basis consisting of quadratic binomials and $\mathcal{M}_{\mathcal{Q}}$ be the class of matroids for which J_M is generated by quadratic binomials. In Chapter 3, we show that classes $\mathcal{M}_{\mathcal{Q}}$ and $\mathcal{M}_{\mathcal{QG}}$ are closed under the following operations:

- series and parallel extensions;
- series and parallel connections;
- 2-sums.

We prove that Conjecture 1.7.10 and Conjecture 1.7.11 are true if a matroid M has no minor isomorphic to any of $M(K_4)$, $\mathcal{W}^3$, P_6 and Q_6.

Chapter 2

Toric rings associated to cut ideals

A cut ideal of a graph was introduced by Sturmfels and Sullivant [34]. In this chapter, we give a necessary and sufficient condition for toric rings associated to cut ideals to be strongly Koszul. In Section 2.1, we introduce the definition and known results of a cut ideal. In Section 2.2, we show that the set of graphs such that R_G is strongly Koszul is closed under contracting edges, induced subgraphs and 0-sums. In Section 2.3, we compute a Gröbner basis for cut ideals without (K_4, C_5)-minor. In Section 2.4, by using results of Section 2.2 and Section 2.3, we prove that the toric ring R_G is strongly Koszul if and only if G has no (K_4, C_5)-minor.

2.1 Cut ideals

Let G be a finite simple connected graph on the vertex set $V(G) = [n] = \{1, \dots, n\}$ with the edge set $E(G)$. For two subsets A and B of $[n]$ such that $A \cap B = \emptyset$ and $A \cup B = [n]$, the $(0,1)$-vector $\delta_{A|B}(G) \in \mathbb{Z}^{|E(G)|}$ is defined as

$$\delta_{A|B}(G)_{ij} = \begin{cases} 1 & \text{if } |A \cap \{i,j\}| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where ij is an edge of G. Let

$$X_G = \left\{ \begin{pmatrix} \delta_{A_1|B_1}(G) \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} \delta_{A_N|B_N}(G) \\ 1 \end{pmatrix} \right\} \subset \mathbb{Z}^{|E(G)|+1} \quad (N = 2^{n-1}).$$

As necessary, we consider X_G as a collection of vectors or as a matrix. Let K be a field and

$$\begin{aligned} K[q] &= K[q_{A_1|B_1}, \dots, q_{A_N|B_N}], \\ K[s,T] &= K[s, t_{ij} \mid ij \in E(G)] \end{aligned}$$

be two polynomial rings over K. Then a ring homomorphism is defined as follows:

$$\pi_G \ : \ K[q] \to K[s,T], \quad q_{A_l|B_l} \mapsto s \cdot \prod_{\substack{|A_l \cap \{i,j\}|=1 \\ ij \in E(G)}} t_{ij}$$

for $1 \leq l \leq N$. The *cut ideal* I_G of G is the kernel of π_G and the *toric ring* R_G of X_G is the image of π_G. We put $u_{A|B} = \pi_G(q_{A|B})$.

In [34], Sturmfels and Sullivant introduced a cut ideal and posed the problem of relating properties of cut ideals to the class of graphs. For the toric ring R_G and the cut ideal I_G, the following results are known:

Theorem 2.1.1 ([34])**.** *The toric ring R_G is compressed if and only if G has no K_5-minor and every induced cycle in G has length* 3 *or* 4*.*

Theorem 2.1.2 ([11])**.** *The cut ideal I_G is generated by quadratic binomials if and only if G has no K_4-minor.*

Nagel and Petrović showed that the cut ideal I_G associated with ring graphs has a quadratic Gröbner basis [19]. However we do not know generally when the cut ideal I_G has a quadratic Gröbner basis and when R_G is Koszul except for trivial cases.

On the other hand, in [28], Restuccia and Rinaldo gave a sufficient condition for toric rings to be strongly Koszul. In [18], Matsuda and Ohsugi proved that any squarefree strongly Koszul toric ring is compressed.

2.2 Clique sums and strongly Koszul algebras

In this section, we prove that strong Koszulness of the toric ring associated to the cut ideal is closed under the 0-sum, induced subgraphs and contracting edges but is not always closed under the 1-sum.

Recall that a graph H is a *minor* of a graph G if H can be obtained by deleting and contracting edges of G. We say that a subgraph H is an *induced subgraph* of a graph G if H contains all the edges $ij \in E(G)$ with $i, j \in V(H)$.

Proposition 2.2.1. *Let G be a finite simple connected graph. Assume that R_G is strongly Koszul. Then*

(1) *If H_1 is an induced subgraph of G, then R_{H_1} is strongly Koszul.*

(2) *If H_2 is obtained by contracting an edge of G, then R_{H_2} is strongly Koszul.*

Proof. By [20] and [34], R_{H_1} and R_{H_2} are combinatorial pure subrings of R_G. Therefore, by [22, Corollary 1.6], R_{H_1} and R_{H_2} are strongly Koszul. □

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be finite simple connected graphs such that $V_1 \cap V_2$ is a clique of both graphs. The new graph $G = G_1 \# G_2$ with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$ is called the *clique sum* of G_1 and G_2 along $V_1 \cap V_2$. If the cardinality of $V_1 \cap V_2$ is $k + 1$, then this operation is called a *k-sum* of the graphs. It is clear that if $R_{G_1 \# G_2}$ is strongly Koszul, then both R_{G_1} and R_{G_2} are strongly Koszul because G_1 and G_2 are induced subgraphs of $G_1 \# G_2$.

Proposition 2.2.2. *The set of graphs G such that R_G is strongly Koszul is closed under the* 0*-sum.*

Proof. Let G_1 and G_2 be finite simple connected graphs and assume that R_{G_1} and R_{G_2} are strongly Koszul. Then the toric ring $R_{G_1 \# G_2}$, where $G_1 \# G_2$ is the 0-sum of G_1 and G_2, is the usual Segre product of R_{G_1} and R_{G_2}. Thus it follows by Proposition 1.4.6. $\square$

However the set of graphs G such that R_G is strongly Kosuzl is not always closed under the 1-sum.

Recall that K_n denotes the complete graph on n vertices, C_n denotes the cycle of length n and $K_{l_1,\ldots,l_r}$ denotes the complete r-partite graph on the vertex set $V_1 \cup \cdots \cup V_r$, where $|V_i| = l_i$ for $1 \leq i \leq r$ and $V_i \cap V_j = \emptyset$ for $i \neq j$.

Example 2.2.3. Let $G_1 = C_3 \# C_3 (= K_4 \setminus e)$, $G_2 = C_4 \# C_3$ and $G_3 = (K_4 \setminus e) \# C_3$ be graphs shown in Figures 2.1-2.3. All of R_{C_3}, R_{C_4} and R_{G_1} are strongly Koszul because R_{C_3} is isomorphic to the polynomial ring and I_{C_4} and I_{G_1} have quadratic Gröbner bases with respect to any reverse lexicographic order, respectively (see [28, 34]). However neither R_{G_2} nor R_{G_3} is strongly Koszul since an ideal $\langle u_{\emptyset|[5]} \rangle \cap \langle u_{\{1,3,4\}|\{2,5\}} \rangle$ of R_{G_2} is generated by monomials

$$u_{\emptyset|\{1,\ldots,5\}} u_{\{1,3,4\}|\{2,5\}},\ u_{\{1,3,4,5\}|\{2\}} u_{\{1,2,5\}|\{3,4\}} u_{\{1,2,3\}|\{4,5\}},\ u_{\{1\}|\{2,3,4,5\}} u_{\{1,3,4\}|\{2,5\}} u_{\{1,2,3\}|\{4,5\}},$$
$$u_{\{1\}|\{2,3,4,5\}} u_{\{1,2,5\}|\{3,4\}} u_{\{1,2,4\}|\{3,5\}},\ u_{\{1\}|\{2,3,4,5\}} u_{\{1,5\}|\{2,3,4\}} u_{\{1,2,4\}|\{3,5\}}$$

and an ideal $\langle u_{\emptyset|[5]} \rangle \cap \langle u_{\{1,3,4\}|\{2,5\}} \rangle$ of R_{G_3} is generated by monomials

$$u_{\emptyset|\{1,\ldots,5\}} u_{\{1,3,4\}|\{2,5\}},\ u_{\{1,3,4,5\}|\{2\}} u_{\{1,2,5\}|\{3,4\}} u_{\{1,2,3\}|\{4,5\}},\ u_{\{1\}|\{2,3,4,5\}} u_{\{1,3,4\}|\{2,5\}} u_{\{1,2,3\}|\{4,5\}},$$
$$u_{\emptyset|\{1,\ldots,5\}} u_{\{1,3,5\}|\{2,4\}} u_{\{1,2,5\}|\{3,4\}},\ u_{\{1\}|\{2,3,4,5\}} u_{\{1,2,3,5\}|\{4\}} u_{\{1,3,4\}|\{2,5\}}.$$

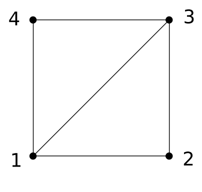

Figure 2.1: $C_3 \# C_3$

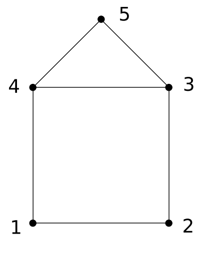

Figure 2.2: $C_4 \# C_3$

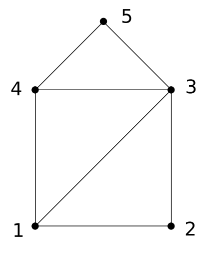

Figure 2.3: $(K_4 \setminus e) \# C_3$

The cut ideal $I_{G_1 \# G_2}$ is the *toric fiber product* of I_{G_1} and I_{G_2} [34]. Therefore, from Example 2.2.3, the set of toric rings R such that R is strongly Koszul is not closed under the toric fiber product.

2.3 Gröbner bases for cut ideals

In this section, we compute a Gröbner basis of I_G such that G has no (K_4, C_5)-minor.

Lemma 2.3.1. *Let G be a finite simple 2-connected graph on the vertex set $V(G)$. Then G has no (K_4, C_5)-minor if and only if G is K_3, $K_{2,n-2}$ or $K_{1,1,n-2}$ for $n \geq 4$.*

Proof. Since G is 2-connected, G contains a cycle. Let C be the longest cycle in G. It follows that $|V(C)| \leq 4$ because G has no C_5-minor. If $|V(C)| = 3$, then $G = K_3$ since G is 2-connected. Suppose that $|V(C)| = 4$. If $|V(G)| = |V(C)|$, then G is either $K_{2,2}$ or $K_{1,1,2}$. Next, we assume that $|V(G)| > |V(C)| = 4$. Consider $v \in V(G) \setminus V(C)$. Let P and Q be two paths each with one end in v and another end in $V(C)$, disjoint except for their common end in v and having no internal vertices in C. Such paths exist since G is 2-connected. If $|V(P)| > 2$, or $|V(Q)| > 2$, or the ends of P and Q in C are consecutive in C, then $P \cup Q$ together with a subpath of C form a cycle of length longer than C. Hence every vertex $v \notin V(C)$ has exactly two neighbors in $V(C)$, which are not consecutive. Moreover, if some two vertices $v_1, v_2 \in V(G) \setminus V(C)$ are adjacent to different pairs of vertices in C, then a cycle of length six is induced in G by $\{v_1, v_2\} \cup V(C)$. Therefore there exist $u_1, u_2 \in V(C)$, which are both adjacent to all vertices in $V(G) \setminus \{u_1, u_2\}$. If two vertices in $V(G) \setminus \{u_1, u_2\}$ are adjacent, then together with $\{u_1, u_2\}$ and any other vertex they induce a cycle in G of length five. Therefore G is either $K_{2,n-2}$ or $K_{1,1,n-2}$. It is easy to see that all of K_3, $K_{2,n-2}$ and $K_{1,1,n-2}$ have no (K_4, C_5)-minor. $\square$

It is already known that the cut ideal $I_{K_{1,n-2}}$ for $n \geq 4$ has a quadratic Gröbner basis since $K_{1,n-2}$ is 0-sums of K_2 and $I_{K_2} = \langle 0 \rangle$ [34, Theorem 2.1]. In this section, to prove Theorem 2.3.3, we compute the reduced Gröbner basis of $I_{K_{1,n-2}}$. Let $<$ be a reverse lexicographic order on $K[q]$ which satisfies $q_{A|B} < q_{C|D}$ with $\min\{|A|, |B|\} < \min\{|C|, |D|\}$.

Lemma 2.3.2. *Let $G = K_{1,n-2}$ be the complete bipartite graph on the vertex set $V_1 \cup V_2$, where $V_1 = \{1\}$ and $V_2 = \{3, \ldots, n\}$ for $n \geq 4$. Then the reduced Gröbner basis of I_G with respect to $<$ consists of*

$$q_{A|B}q_{C|D} - q_{A\cap C|B\cup D}q_{A\cup C|B\cap D} \quad (1 \in A \cap C, A \not\subset C,\ C \not\subset A).$$

The initial monomial of each binomial is the first monomial.

Proof. Let $\mathcal{G}$ be the set of all binomials above. It is easy to see that $\mathcal{G} \subset I_G$. Let $\text{in}(\mathcal{G}) = \langle \text{in}_<(g) \mid g \in \mathcal{G} \rangle$. Let u and v be monomials that do not belong to $\text{in}(\mathcal{G})$:

$$u = \prod_{l=1}^{m} (q_{\{1\}\cup A_l|B_l})^{p_l}, \quad v = \prod_{l=1}^{m'} (q_{\{1\}\cup C_l|D_l})^{p'_l},$$

where $0 < p_l$, $p'_l \in \mathbb{Z}$ for any l. Since neither u nor v is divided by $q_{A|B}q_{C|D}$, it follows that

$$A_1 \subset A_2 \subset \cdots \subset A_m, \quad C_1 \subset C_2 \subset \cdots \subset C_{m'}.$$

Let

$$A_l = A_{l-1} \cup \{b_1^{l-1}, \ldots, b_{\beta_{l-1}}^{l-1}\}, \quad B_k = \bigcup_{i=k}^{m} \{b_1^i, \ldots, b_{\beta_i}^i\},$$
$$C_l = C_{l-1} \cup \{d_1^{l-1}, \ldots, d_{\delta_{l-1}}^{l-1}\}, \quad D_k = \bigcup_{i=k}^{m'} \{d_1^i, \ldots, d_{\delta_i}^i\}$$

for $k \geq 1$ and $l \geq 2$, where $A_1 = V_2 \setminus B_1$, $C_1 = V_2 \setminus D_1$. We suppose that $\pi_G(u) = \pi_G(v)$:

$$\pi_G(u) = s^p \prod_{l=1}^{m} (t_{1b_1^l} \cdots t_{1b_{\beta_l}^l})^{\sum_{k=1}^{l} p_k}, \quad \pi_G(v) = s^{p'} \prod_{l=1}^{m'} (t_{1d_1^l} \cdots t_{1d_{\delta_l}^l})^{\sum_{k=1}^{l} p_k'}.$$

Here we set $p = \sum_{l=1}^{m} p_l$ and $p' = \sum_{l=1}^{m'} p_l'$. Assume that $A_1 \neq C_1$. Then there exists $a \in A_1$ such that $a \notin C_1$. Hence, for some $l_1 \in [m']$, $a \in \{d_1^{l_1}, \ldots, d_{\delta_{l_1}}^{l_1}\}$. However, for any $l \in [m]$, $a \notin \{b_1^l, \ldots, b_{\beta_l}^l\}$. This contradicts that $\pi_G(u) = \pi_G(v)$. Thus $A_1 = C_1$ and $p_1 = p_1'$. By performing this operation repeatedly, it follows that $A_l = C_l$, $B_l = D_l$ and $p_l = p_l'$ for any l. Since $u = v$, $\mathcal{G}$ is a Gröbner basis of I_G. It is trivial that $\mathcal{G}$ is reduced. $\square$

Theorem 2.3.3. *Let $G = K_{2,n-2}$ be the complete bipartite graph on the vertex set $V_1 \cup V_2$, where $V_1 = \{1, 2\}$ and $V_2 = \{3, \ldots, n\}$ for $n \geq 4$. Then a Gröbner basis of I_G consists of*

$$\begin{array}{lcr} q_{A|B}q_{E|F} - q_{\emptyset|[n]}q_{\{1,2\}|\{3,\ldots,n\}} & (1 \in A, 2 \in B), & \text{(i)} \\ q_{A|B}q_{C|D} - q_{A\cap C|B\cup D}q_{A\cup C|B\cap D} & (1 \in A \cap C, 2 \in B \cap D, A \not\subset C, C \not\subset A), & \text{(ii)} \\ q_{A|B}q_{C|D} - q_{A\cap C|B\cup D}q_{A\cup C|B\cap D} & (1, 2 \in A \cap C, A \not\subset C, C \not\subset A), & \text{(iii)} \end{array}$$

where $E = (B \cup \{1\}) \setminus \{2\}$ and $F = (A \cup \{2\}) \setminus \{1\}$. The initial monomial of each binomials is the first binomial.

Proof. Let $\mathcal{G}$ be the set of all binomials above. It is easy to see that $\mathcal{G} \subset I_G$. Let u and v be monomials which do not belong to $\text{in}(\mathcal{G})$:

$$\begin{aligned} u &= \prod_{l=1}^{m_1} (q_{\{1\}\cup A_l|\{2\}\cup B_l})^{p_l} \prod_{l=1}^{m_2} (q_{\{1,2\}\cup C_l|D_l})^{r_l}, \\ v &= \prod_{l=1}^{m_1'} (q_{\{1\}\cup A_l'|\{2\}\cup B_l'})^{p_l'} \prod_{l=1}^{m_2'} (q_{\{1,2\}\cup C_l'|D_l'})^{r_l'}, \end{aligned}$$

where $0 < p_l, r_l, p_l', r_l' \in \mathbb{Z}$ for any l. Since neither u nor v is divided by initial monomials of (ii) and (iii), it follows that

$$A_1 \subset \cdots \subset A_{m_1}, \quad C_1 \subset \cdots \subset C_{m_2},$$
$$A_1' \subset \cdots \subset A_{m_1'}', \quad C_1' \subset \cdots \subset C_{m_2'}'.$$

Suppose that $\pi_G(u)=\pi_G(v)$:

$$\begin{aligned}\pi_G(u) &= \prod_{l=1}^{m_1}(u_{\{1\}\cup A_l|\{2\}\cup B_l})^{p_l}\prod_{l=1}^{m_2}(u_{\{1,2\}\cup C_l|D_l})^{r_l},\\ \pi_G(v) &= \prod_{l=1}^{m_1^{'}}(u_{\{1\}\cup A_l^{'}|\{2\}\cup B_l^{'}})^{p_l^{'}}\prod_{l=1}^{m_2^{'}}(u_{\{1,2\}\cup C_l^{'}|D_l^{'}})^{r_l^{'}}.\end{aligned}$$

Let Y be the matrix consisting of the first $n-2$ rows of $X_{K_{1,n-2}}$. Then X_G is the following matrix:

$$\begin{pmatrix} Y & Y \\ Y & \mathbf{1}_{n-2,2^{n-2}}-Y \\ \mathbf{1} & \mathbf{1}\end{pmatrix},$$

where $\mathbf{1}_{n-2,2^{n-2}}$ is the $(n-2)\times 2^{n-2}$ matrix such that each entry is all ones. Note that

$$\begin{aligned}\begin{pmatrix} Y\\ Y\end{pmatrix} &= \left(\delta_{P_1|Q_1}(K_{2,n-2})\cdots\delta_{P_{2^{n-2}}|Q_{2^{n-2}}}(K_{2,n-2})\right),\\ \begin{pmatrix} Y\\ \mathbf{1}_{n-2,2^{n-2}}-Y\end{pmatrix} &= \left(\delta_{R_1|S_1}(K_{2,n-2})\cdots\delta_{R_{2^{n-2}}|S_{2^{n-2}}}(K_{2,n-2})\right),\end{aligned}$$

where $1,2\in P_l$, $1\in R_l$ and $2\in S_l$ for $1\le l\le 2^{n-2}$. By elementary row operations on X_G, we have

$$X_G^{'}=\begin{pmatrix} 2Y-\mathbf{1}_{n-2,2^{n-2}} & O\\ O & 2Y-\mathbf{1}_{n-2,2^{n-2}}\\ \mathbf{1} & \mathbf{1}\end{pmatrix}.$$

Each column vector of $2Y-\mathbf{1}_{n-2,2^{n-2}}$ is the form ${}^t(\varepsilon_1,\dots,\varepsilon_{n-2})$, where $\varepsilon_i\in\{1,-1\}$ for $1\le i\le n-2$. Let $I_{X_G^{'}}$ denote the toric ideal of $X_G^{'}$.Then $u-v\in I_G$ if and only if $u-v\in I_{X_G^{'}}$. Let $\mathbf{a}_{P|Q}$ denote the column vector of $2Y-\mathbf{1}_{n-2,2^{n-2}}$ in $X_G^{'}$ corresponding to the column vector $\delta_{P|Q}(G)$ of X_G. Then

$$\sum_{l=1}^{m_1}p_l\begin{pmatrix}\mathbf{0}\\ \mathbf{a}_{\{1\}\cup A_l|\{2\}\cup B_l}\\ 1\end{pmatrix}+\sum_{l=1}^{m_2}r_l\begin{pmatrix}\mathbf{a}_{\{1,2\}\cup C_l|D_l}\\ \mathbf{0}\\ 1\end{pmatrix}=\sum_{l=1}^{m_1^{'}}p_l^{'}\begin{pmatrix}\mathbf{0}\\ \mathbf{a}_{\{1\}\cup A_l^{'}|\{2\}\cup B_l^{'}}\\ 1\end{pmatrix}+\sum_{l=1}^{m_2^{'}}r_l^{'}\begin{pmatrix}\mathbf{a}_{\{1,2\}\cup C_l^{'}|D_l^{'}}\\ \mathbf{0}\\ 1\end{pmatrix}.$$

In particular,

$$\sum_{l=1}^{m_1}p_l\mathbf{a}_{\{1\}\cup A_l|\{2\}\cup B_l}=\sum_{l=1}^{m_1^{'}}p_l^{'}\mathbf{a}_{\{1\}\cup A_l^{'}|\{2\}\cup B_l^{'}},\quad \sum_{l=1}^{m_2}r_l\mathbf{a}_{\{1,2\}\cup C_l|D_l}=\sum_{l=1}^{m_2^{'}}r_l^{'}\mathbf{a}_{\{1,2\}\cup C_l^{'}|D_l^{'}}$$

hold. Let $p=\sum_{l=1}^{m_1}p_l$, $r=\sum_{l=1}^{m_2}r_l$, $p^{'}=\sum_{l=1}^{m_1^{'}}p_l^{'}$ and $r^{'}=\sum_{l=1}^{m_2^{'}}r_l^{'}$. Since neither u nor v is divided by initial monomials of (i), it follows that either $A_1\neq\emptyset$ or

$A_{m_1} \neq [n] \setminus \{1,2\}$ (resp. $A_1^{'} \neq \emptyset$ or $A_{m_2^{'}}^{'} \neq [n] \setminus \{1,2\}$). If $A_1 \neq \emptyset$, then there exists $i \in [n] \setminus \{1,2\}$ such that $i \in A_l$ for any $l \in [m_1]$. If $A_{m_1} \neq [n] \setminus \{1,2\}$, that is, $B_{m_1} \neq \emptyset$, then there exists $i \in [n] \setminus \{1,2\}$ such that $i \in B_{m_1}$, and $i \notin A_l$ for any $l \in [m_1]$. Thus either p or $-p$ appears in the entry of $\sum_{l=1}^{m_1} p_l \mathbf{a}_{\{1\}\cup A_l|\{2\}\cup B_l}$. Similarly, either $p^{'}$ or $-p^{'}$ appears in the entry of $\sum_{l=1}^{m_1^{'}} p_l^{'} \mathbf{a}_{\{1\}\cup A_l^{'}|\{2\}\cup B_l^{'}}$. Therefore $p = p^{'}$. Hence

$$\prod_{l=1}^{m_1}(u_{\{1\}\cup A_l|\{2\}\cup B_l})^{p_l} = \prod_{l=1}^{m_1^{'}}(u_{\{1\}\cup A_l^{'}|\{2\}\cup B_l^{'}})^{p_l^{'}}, \quad \prod_{l=1}^{m_2}(u_{\{1,2\}\cup C_l|D_l})^{r_l} = \prod_{l=1}^{m_2^{'}}(u_{\{1,2\}\cup C_l^{'}|D_l^{'}})^{r_l^{'}}$$

hold. Thus

$$\prod_{l=1}^{m_1}(q_{\{1\}\cup A_l|\{2\}\cup B_l})^{p_l} - \prod_{l=1}^{m_1^{'}}(q_{\{1\}\cup A_l^{'}|\{2\}\cup B_l^{'}})^{p_l^{'}} \in I_{Z_1},$$

$$\prod_{l=1}^{m_2}(q_{\{1,2\}\cup C_l|D_l})^{r_l} - \prod_{l=1}^{m_2^{'}}(q_{\{1,2\}\cup C_l^{'}|D_l^{'}})^{r_l^{'}} \in I_{Z_2},$$

where Z_1 (resp. Z_2) is the matrix consisting of the first (resp. last) 2^{n-2} columns of $X_G^{'}$. Here I_{Z_1} and I_{Z_2} are toric ideals of Z_1 and Z_2. By elementary row operations on Z_1 (resp. Z_2), we have

$$\prod_{l=1}^{m_1}(q_{\{1\}\cup A_l|B_l})^{p_l} - \prod_{l=1}^{m_1^{'}}(q_{\{1\}\cup A_l^{'}|B_l^{'}})^{p_l^{'}}, \quad \prod_{l=1}^{m_2}(q_{\{1\}\cup C_l|D_l})^{r_l} - \prod_{l=1}^{m_2^{'}}(q_{\{1\}\cup C_l^{'}|D_l^{'}})^{r_l^{'}} \in I_{K_{1,n-2}}.$$

By Lemma 2.3.2, $u = v$ holds. Therefore $\mathcal{G}$ is a Gröbner basis of I_G. □

Corollary 2.3.4. *If G has no (K_4, C_5)-minor, then I_G has a quadratic Gröbner basis.*

Proof. If G is not 2-connected, then there exist 2-connected components $G_1, \ldots, G_s$ of G such that G is 0-sums of $G_1, \ldots, G_s$. By [34, Theorem 2.1] and Lemma 2.3.1, it is enough to show that I_{K_2}, I_{K_3}, $I_{K_{2,n-2}}$ and $I_{K_{1,1,n-2}}$ have quadratic Gröbner bases. It is trivial that I_{K_2} and I_{K_3} have quadratic Gröbner bases because $I_{K_2} = \langle 0 \rangle$ and $I_{K_3} = \langle 0 \rangle$. Since $K_{1,1,n-2}$ is obtained by 1-sums of K_3, $I_{K_{1,1,n-2}}$ has a quadratic Gröbner basis. Therefore, by Theorem 2.3.3, I_G has a quadratic Gröbner basis. □

2.4 Strongly Koszul toric rings of cut ideals

In this section, we characterize the class of graphs whose toric rings associated to cut ideals are strongly Koszul.

Proposition 2.4.1. *Let $G_1 = K_{1,1,n-2}$ and $G_2 = K_{2,n-2}$ for $n \geq 4$. Then R_{G_1} and R_{G_2} are strongly Koszul.*

Proof. By elementary row operations on X_{G_1}, we have

$$X_{G_1} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ Y & Y \\ Y & \mathbf{1}_{n-2,2^{n-2}} - Y \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \to \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ Y & Y \\ Y & -Y \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \to \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ Y & Y \\ Y & O \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \to \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ O & Y \\ Y & O \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

Hence $R_{G_1} \cong R_{K_{1,n-2}} \otimes_K R_{K_{1,n-2}}$. Since $R_{K_{1,n-2}}$ is Segre products of R_{K_2}, R_{G_1} is strongly Koszul. Next, by the symmetry of $X_{G'}$ in the proof of Theorem 2.3.3, it is enough to consider the following two cases:

(1) $\langle u_{\emptyset|[n]} \rangle \cap \langle u_{\{1\}|\{2,\ldots,n\}} \rangle$,

(2) $\langle u_{\emptyset|[n]} \rangle \cap \langle u_{\{1,2\} \cup A|B} \rangle$.

Since $q_{\emptyset|[n]}$ is the smallest variable and $q_{\{1\}|\{2,\ldots,n\}}$ is the second smallest variable with respect to the reverse lexicographic order $<$, by [18] and Theorem 2.3.3, $\langle u_{\emptyset|[n]} \rangle \cap \langle u_{\{1\}|\{2,\ldots,n\}} \rangle$ is generated in degree 2. Assume that $\langle u_{\emptyset|[n]} \rangle \cap \langle u_{\{1,2\} \cup A|B} \rangle$ is not generated in degree 2. Then there exists a monomial $u_{E_1|F_1} \cdots u_{E_s|F_s}$ belonging to a minimal generating set of $\langle u_{\emptyset|[n]} \rangle \cap \langle u_{\{1,2\} \cup A|B} \rangle$ such that $s \geq 3$. Since $u_{E_1|F_1} \cdots u_{E_s|F_s}$ is in $\langle u_{\emptyset|[n]} \rangle \cap \langle u_{\{1,2\} \cup A|B} \rangle$, it follows that

$$q_{\{1,2\} \cup A|B} \prod_{l=1}^{\alpha} q_{\{1,2\} \cup A_l|B_l} \prod_{l=1}^{\beta} q_{\{1\} \cup C_l|\{2\} \cup D_l} - q_{\emptyset|[n]} \prod_{l=1}^{\gamma} q_{\{1,2\} \cup P_l|Q_l} \prod_{l=1}^{\delta} q_{\{1\} \cup R_l|\{2\} \cup S_l} \in I_{G_2}.$$

If one of the monomials appearing in the above binomial is divided by initial monomials of (i) in Theorem 2.3.3, then $u_{E_1|F_1} \cdots u_{E_s|F_s}$ is divided by $u_{\emptyset|[n]} u_{\{1,2\}|\{3,\ldots,n\}}$. This contradicts that $u_{E_1|F_1} \cdots u_{E_s|F_s}$ belongs to a minimal generating set of $\langle u_{\emptyset|[n]} \rangle \cap \langle u_{\{1,2\} \cup A|B} \rangle$ since for any $u_{A|B}$ and $u_{C|D}$ with $u_{A|B} \neq u_{C|D}$, $u_{\emptyset|[n]} u_{\{1,2\}|\{3,\ldots,n\}}$ belongs to a minimal generating set of $\langle u_{A|B} \rangle \cap \langle u_{C|D} \rangle$. If one of $\prod_{l=1}^{\beta} q_{\{1\} \cup C_l|\{2\} \cup D_l}$ and $\prod_{l=1}^{\delta} q_{\{1\} \cup R_l|\{2\} \cup S_l}$ is divided by initial monomials of (ii) in Theorem 2.3.3, the monomial is reduced to the monomial which is not divided by initial monomials of (ii) with respect to $\mathcal{G}$, where $\mathcal{G}$ is a Gröbner basis of I_{G_2}. Thus we may assume that

$$C_1 \subset \cdots \subset C_\beta, \quad R_1 \subset \cdots \subset R_\delta.$$

Similar to what did in the proof of Theorem 2.3.3, we have

$$\begin{aligned} u_{\{1,2\} \cup A|B} \prod_{l=1}^{\alpha} u_{\{1,2\} \cup A_l|B_l} &= u_{\emptyset|[n]} \prod_{l=1}^{\gamma} u_{\{1,2\} \cup P_l|Q_l}, \\ \prod_{l=1}^{\beta} u_{\{1\} \cup C_l|\{2\} \cup D_l} &= \prod_{l=1}^{\delta} u_{\{1\} \cup R_l|\{2\} \cup S_l}. \end{aligned}$$

It follows that $\alpha = \gamma$, $\beta = \delta$, $C_l = R_l$, $D_l = S_l$ for any l, and

$$q_{\{1\}\cup A|B} \prod_{l=1}^{\alpha} q_{\{1\}\cup A_l|B_l} - q_{\emptyset|[n]\setminus\{2\}} \prod_{l=1}^{\alpha} q_{\{1\}\cup P_l|Q_l} \in I_{K_{1,n-2}}.$$

Hence the ideal $\langle u_{\{1\}\cup A|B}\rangle \cap \langle u_{\emptyset|[n]\setminus\{2\}}\rangle$ of $R_{K_{1,n-2}}$ is not generated in degree 2. However this contradicts that $R_{K_{1,n-2}}$ is strongly Koszul. Therefore R_{G_2} is strongly Koszul. $\square$

Lemma 2.4.2. *Let G be a finite simple 2-connected graph without K_4-minor. If G has C_5-minor, then by only contracting edges of G, we obtain one of C_5, the 1-sum of C_4 and C_3, and the 1-sum of $K_4 \setminus e$ and C_3.*

Proof. Let G be a graph with C_5-minor and C a longest cycle in G. It follows that $|V(C)| \geq 5$. Then, by contracting edges of G, we obtain a graph G' of five vertices such that C_5 is a subgraph of G'. Assume that $G' \neq C_5$. Then there exist $u, v \in V(C_5)$ with $uv \notin E(C_5)$ such that $uv \in E(G')$. Since G has no K_4-minor, there do not exist $\alpha, \beta \in V(C_5) \setminus \{u, v\}$ such that $\alpha\beta \in E(G') \setminus E(C_5)$. Therefore we obtain one of the 1-sum of C_4 and C_3, and the 1-sum of $K_4 \setminus e$ and C_3. $\square$

Theorem 2.4.3. *Let G be a finite simple connected graph. Then R_G is strongly Koszul if and only if G has no (K_4, C_5)-minor.*

Proof. Let G be a graph without (K_4, C_5)-minor. If G is not 2-connected, then there exist 2-connected components $G_1, \ldots, G_s$ of G such that G is 0-sums of $G_1, \ldots, G_s$. By Lemma 2.3.1, it is enough to show that R_{K_2}, R_{K_3}, $R_{K_{2,n-2}}$ and $R_{K_{1,1,n-2}}$ are strongly Koszul. It is clear that R_{K_2} and R_{K_3} are strongly Koszul. By Proposition 2.4.1, $R_{K_{2,n-2}}$ and $R_{K_{1,1,n-2}}$ are strongly Koszul. Next, we suppose that G has K_4-minor. Then the cut ideal I_G is not generated by quadratic binomials [11]. In particular, R_G is not strongly Koszul. Assume that G has no K_4-minor. If G has C_5-minor, then, by Lemma 2.4.2, we obtain one of C_5, $C_4\#C_3$ and $(K_4 \setminus e)\#C_3$ by contracting edges of G. By Example 2.2.3, neither $R_{C_4\#C_3}$ nor $R_{(K_4\setminus e)\#C_3}$ is strongly Koszul. Since R_{C_5} is not compressed [34, Theorem 1.3], R_{C_5} is not strongly Koszul [18, Theorem 2.1]. Therefore, by Proposition 2.2.1, R_G is not strongly Koszul. $\square$

By using above results, we have

Corollary 2.4.4. *The set of graphs G such that R_G is strongly Koszul is minor closed.*

Corollary 2.4.5. *If R_G is strongly Koszul, then I_G has a quadratic Gröbner basis.*

The converse of Corollary 2.4.5 is not true because the cut ideal I_{C_5} has a quadratic Gröbner basis [19], but R_{C_5} is not strongly Koszul.

Chapter 3

Toric ideals associated to matroids

In this chapter, we consider the toric ideal associated to a matroid. In Section 3.1, we introduce known results about properties of toric rings and toric ideals of matroids. In Section 3.2, we prove that the class of matroids such that the toric ideal J_M has a quadratic Gröbner basis is closed under series and parallel extensions. In Section 3.3, we show that the class of matroids such that the toric ideal J_M has a quadratic Gröbner basis is closed under series and parallel connections and 2-sums.

3.1 Operations on matroids

In this section, we introduce several operations on matroids.

Let M be a matroid on $E = [d]$ and $\mathcal{B}(M) = \{B_1, \ldots, B_n\}$ be the collection of bases of M. An element $i \in E$ is called a *loop* of M if it does not belong to any basis of M. Dually, an element $i \in E$ is said to be a *coloop* of M if it is contained in all the bases of M. Let

$$\mathcal{B}^*(M) = \{E \setminus B \mid B \in \mathcal{B}(M)\}.$$

Then a pair $(E, \mathcal{B}^*(M))$ is a matroid. This matroid is called the *dual* of M and denoted as M^*.

Let M and $\mathcal{B}(M)$ be as above, and let $c \in E$. We consider the following collection of subsets of $E \setminus \{c\}$:

$$\mathcal{B}(M) \setminus c = \begin{cases} \{B \setminus \{c\} \mid B \in \mathcal{B}(M)\} & \text{if } c \text{ is a coloop of } M, \\ \{B \mid c \notin B \in \mathcal{B}(M)\} & \text{otherwise.} \end{cases}$$

A pair $(E \setminus \{c\}, \mathcal{B}(M) \setminus c)$ is a matroid. This matroid is called the *deletion* of c from M and denoted as $M \setminus c$. Dually, let M/c, the *contraction* of c from M, be given by $M/c = (M^* \setminus c)^*$. We call a matroid M' a *minor* of a matroid M if M' can be obtained from M by a finite sequence of contractions and deletions.

Let M_1 and M_2 be matroids with $E_1 \cap E_2 = \emptyset$. Let $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ be collections of bases of M_1 and M_2, and let

$$\mathcal{B}(M_1) \oplus \mathcal{B}(M_2) = \{B \cup D \mid B \in \mathcal{B}(M_1), D \in \mathcal{B}(M_2)\}.$$

Then a pair $(E, \mathcal{B}(M_1) \oplus \mathcal{B}(M_2))$, where $E = E_1 \cup E_2$, is a matroid. This matroid is called the 1-*sum* of M_1 and M_2, and it is denoted as $M_1 \oplus M_2$.

Proposition 3.1.1 ([2, 38]). *Classes $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under duality, taking minors and 1-sums.*

Note that

- R_{M^*} is isomorphic to R_M as K-algebra, in particular, $J_M = J_{M^*}$,
- $R_{M/c}$ and $R_{M\setminus c}$ are combinatorial pure subrings of R_M,
- $R_{M_1 \oplus M_2}$ is the Segre product of R_{M_1} and R_{M_2}.

3.2 A series and parallel extension of a matroid

In this section, we introduce a series and parallel extension of a matroid and show that $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under series and parallel extensions.

Let M be a matroid on $E = [d]$ and $\mathcal{B}(M)$ be the collection of bases of M. Then a *series extension* of M at $c \in E$ by $d+1$ is a matroid on $E \cup \{d+1\}$ that has

$$\{B \cup \{d+1\} \mid B \in \mathcal{B}(M)\} \cup \{B \cup \{c\} \mid c \notin B \in \mathcal{B}(M)\}$$

as the collection of bases and is denoted as $M +_c (d+1)$. Dually, we call a matroid $[M^* +_c (d+1)]^*$ a *parallel extension* of M at c by $d+1$. A *series-parallel extension* of M is any matroid derived from M by a finite sequence of series and parallel extensions. We suppose that M does not have $c \in E$ as a coloop. Let $\mathcal{B}(M) = \{B_1, \ldots, B_\gamma, \ldots, B_n\}$ be the collection of bases of M, where $c \notin B_j$ for $j \in [\gamma]$ and $c \in B_j$ for $j \in [n] \setminus [\gamma]$. We renumber the bases of M, if necessary. Let $\mathcal{D}_M = \{\mathbf{b}_j^1 \mid j \in [n]\} \subset \mathbb{Z}^d$ denote a vector configuration satisfying $\mathbf{b}_j^1 = \sum_{l \in B_j} \mathbf{e}_l$, where $\mathbf{e}_l$ is the l-th standard vector. As necessary, we consider $\mathcal{D}_M$ as a collection of vectors or as a matrix.

Now we consider a new vector configuration

$$\widetilde{\mathcal{D}}_M = \left\{ \begin{pmatrix} \mathbf{b}_j^i \\ \mathbf{a}^i \end{pmatrix} \;\middle|\; i = 1, 2,\ j \in [\alpha_i] \right\} \subset \mathbb{Z}^{d+2}$$

that satisfies $\mathbf{b}_j^1 = \mathbf{b}_j^2$ for $j \in [\gamma]$, where $\binom{\alpha_1}{\alpha_2} = \binom{n}{\gamma}$, $\mathbf{a}^1 = \binom{0}{1}$ and $\mathbf{a}^2 = \binom{1}{1}$. We define a ring homomorphism $\widetilde{\pi}_M$ as follows:

$$\begin{array}{rcl} \widetilde{\pi}_M : K[X] = K[x_j^i \mid i = 1, 2, j \in [\alpha_i]] & \to & K[S, W] = K[s_k, w_l \mid k \in [d], l = 1, 2], \\ x_j^i & \mapsto & S^{\mathbf{b}_j^i} W^{\mathbf{a}}. \end{array}$$

Then $J_{\widetilde{\mathcal{D}}_M} = \ker(\widetilde{\pi}_M)$.

Let $\omega \in \mathbb{Z}^n_{\geq 0}$, and let $<$ be an arbitrary monomial order. We define a new monomial order $<_\omega$ as follows:

$$X^{\mathbf{a}} <_\omega X^{\mathbf{b}} \Leftrightarrow \begin{cases} \omega \cdot \mathbf{a} < \omega \cdot \mathbf{b} \ ; \ \text{or} \\ \omega \cdot \mathbf{a} = \omega \cdot \mathbf{b} \text{ and } X^{\mathbf{a}} < X^{\mathbf{b}} \end{cases}$$

for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n_{\geq 0}$. We call a monomial order $<_\omega$ a *weight order* on $K[x_1, \ldots, x_n]$. We use the following useful result:

Proposition 3.2.1 ([32, Proposition 1.11])**.** *For any monomial order $<$ and any ideal $I \subset K[X]$, there exists a vector $\omega \in \mathbb{Z}^n_{\geq 0}$ such that $\mathrm{in}_\omega(I) = \mathrm{in}_<(I)$.*

Let $\mathbf{F}$ be a homogeneous generating set for $J_{\mathcal{D}_M}$, and let

$$f = \underline{\prod_{l=1}^{u_f} x^1_{j_l} \prod_{l=1}^{v_f} x^1_{k_l}} - \prod_{l=1}^{u'_f} x^1_{j'_l} \prod_{l=1}^{v'_f} x^1_{k'_l} \in \mathbf{F},$$

where $j_l, j'_l \in [\gamma], k_l, k'_l \in [n] \setminus [\gamma]$. However, if $u_f \neq u'_f$, then $\pi_M(f) \neq 0$ since the c-th entry of $\sum_{l=1}^{u_f} \mathbf{b}^1_{j_l}$ does not coincide with the c-th entry of $\sum_{l=1}^{u'_f} \mathbf{b}^1_{j'_l}$, and the c-th entries of $\sum_{l=1}^{v_f} \mathbf{b}^1_{k_l}$ and $\sum_{l=1}^{v'_f} \mathbf{b}^1_{k'_l}$ are zero. Therefore $u_f = u'_f$ and $v_f = v'_f$. Now let $I = (i_1, \ldots, i_{u_f}) \in \{1,2\}^{u_f}$ and consider the binomial $f^I \in K[X]$ defined by

$$f^I = \underline{\prod_{l=1}^{u_f} x^{i_l}_{j_l} \prod_{l=1}^{v_f} x^1_{k_l}} - \prod_{l=1}^{u_f} x^{i_l}_{j'_l} \prod_{l=1}^{v_f} x^1_{k'_l}.$$

Since f belongs to $J_{\mathcal{D}_M}$, the new homogeneous binomial f^I belongs to $J_{\widetilde{\mathcal{D}}_M}$. We set

$$\widetilde{\mathbf{F}} = \{f^I \mid f \in \mathbf{F}, I \in \{1,2\}^{u_f}\} \cup \{\underline{x^1_{j_2} x^2_{j_1}} - x^1_{j_1} x^2_{j_2} \mid 1 \leq j_1 < j_2 \leq \gamma\}.$$

Theorem 3.2.2. *Let M be a matroid on E, and let $\mathbf{F}$ be a Gröbner basis for $J_{\mathcal{D}_M}$. Then $\widetilde{\mathbf{F}}$ is a Gröbner basis for $J_{\widetilde{\mathcal{D}}_M}$.*

Proof. First, it is easy to see that $\widetilde{\mathbf{F}} \subset J_{\widetilde{\mathcal{D}}_M}$. Let $\omega = (\omega^1_1, \ldots, \omega^1_n)$ be a weight vector. We denote the underlined monomial of f as the initial monomial of f with respect to a weight order ω. Let $\widetilde{\omega} = (\omega^1_1, \ldots, \omega^1_n, \omega^2_1, \ldots, \omega^2_\gamma)$ denote a weight vector satisfying $\omega^1_j = \omega^2_j$ for $j \in [\gamma]$. Then the underlined monomial of f^I is the initial monomial of f^I with respect to a weight order $<_{\widetilde{\omega}}$. We choose a tie-breaking monomial order on $K[X]$ that makes the monomial $x^1_{j_2} x^2_{j_1}$ for $1 \leq j_1 < j_2 \leq \gamma$ the initial monomial. Let

$\mathrm{in}(\mathbf{F}) = \langle \mathrm{in}_\omega(f) \mid f \in \mathbf{F}\rangle$ and $\mathrm{in}(\widetilde{\mathbf{F}}) = \langle \mathrm{in}_{<_{\widetilde{\omega}}}(f) \mid f \in \widetilde{\mathbf{F}}\rangle$. Let u and v be monomials that are not in $\mathrm{in}(\widetilde{\mathbf{F}})$:

$$\begin{aligned} u &= \prod_{l=1}^{m_1} (x^1_{i_l})^{p_l} \prod_{l=1}^{m_2} (x^2_{j_l})^{q_l} \prod_{l=1}^{m_3} (x^1_{k_l})^{r_l}, \\ v &= \prod_{l=1}^{m_1^{'}} (x^1_{i_l^{'}})^{p_l^{'}} \prod_{l=1}^{m_2^{'}} (x^2_{j_l^{'}})^{q_l^{'}} \prod_{l=1}^{m_3^{'}} (x^1_{k_l^{'}})^{r_l^{'}}, \end{aligned}$$

where p_l, q_l, r_l, $p_l^{'}$, $q_l^{'}$, $r_l^{'} \in \mathbb{Z}_{>0}$ for any l, and $\mathcal{I} = \{i_1, \dots, i_{m_1}\}$, $\mathcal{I}^{'} = \{i_1^{'}, \dots, i^{'}_{m_1^{'}}\}$, $\mathcal{J} = \{j_1, \dots, j_{m_2}\}$, and $\mathcal{J}^{'} = \{j_1^{'}, \dots, j^{'}_{m_2^{'}}\}$ are subsets of $[\gamma]$ with cardinalities m_1, $m_1^{'}$, m_2, and $m_2^{'}$, respectively; and $\mathcal{K} = \{k_1, \dots, k_{m_3}\}$ and $\mathcal{K}^{'} = \{k_1^{'}, \dots, k^{'}_{m_3^{'}}\}$ are subsets of $[n] \setminus [\gamma]$ with cardinalities m_3 and $m_3^{'}$, respectively. Since neither u nor v is divided by $x^1_{j_2} x^2_{j_1}$ for $1 \le j_1 < j_2 \le \gamma$, it follows that $i_l \le j_{l'}$ for $l \in [m_1]$ and $l^{'} \in [m_2]$, and $i_l^{'} \le j^{'}_{l'}$ for $l \in [m_1^{'}]$ and $l^{'} \in [m_2^{'}]$. We suppose that $\widetilde{\pi}_M(u) = \widetilde{\pi}_M(v)$:

$$\begin{aligned} \widetilde{\pi}_M(u) &= w_1^q w_2^{p+q+r} \prod_{l=1}^{m_1} S^{p_l \mathbf{b}^1_{i_l}} \prod_{l=1}^{m_2} S^{q_l \mathbf{b}^2_{j_l}} \prod_{l=1}^{m_3} S^{r_l \mathbf{b}^1_{k_l}}, \\ \widetilde{\pi}_M(v) &= w_1^{q^{'}} w_2^{p^{'}+q^{'}+r^{'}} \prod_{l=1}^{m_1^{'}} S^{p_l^{'} \mathbf{b}^1_{i_l^{'}}} \prod_{l=1}^{m_2^{'}} S^{q_l^{'} \mathbf{b}^2_{j_l^{'}}} \prod_{l=1}^{m_3^{'}} S^{r_l^{'} \mathbf{b}^1_{k_l^{'}}}. \end{aligned}$$

Here we set $p = \sum_{l=1}^{m_1} p_l$, $q = \sum_{l=1}^{m_2} q_l$, $r = \sum_{l=1}^{m_3} r_l$, $p^{'} = \sum_{l=1}^{m_1^{'}} p_l^{'}$, $q^{'} = \sum_{l=1}^{m_2^{'}} q_l^{'}$, and $r^{'} = \sum_{l=1}^{m_3^{'}} r_l^{'}$. Since $\mathbf{b}^1_j = \mathbf{b}^2_j$ for $j \in [\gamma]$, it follows that $\pi_M(u^{'}) = \pi_M(v^{'})$, where

$$\begin{aligned} u^{'} &= \prod_{l=1}^{m_1} (x^1_{i_l})^{p_l} \prod_{l=1}^{m_2} (x^1_{j_l})^{q_l} \prod_{l=1}^{m_3} (x^1_{k_l})^{r_l}, \\ v^{'} &= \prod_{l=1}^{m_1^{'}} (x^1_{i_l^{'}})^{p_l^{'}} \prod_{l=1}^{m_2^{'}} (x^1_{j_l^{'}})^{q_l^{'}} \prod_{l=1}^{m_3^{'}} (x^1_{k_l^{'}})^{r_l^{'}}. \end{aligned}$$

Hence $u^{'} - v^{'}$ belongs to $J_{\mathcal{D}_M}$. If $u^{'}$ and $v^{'}$ belong to $\mathrm{in}(\mathbf{F})$, then $u^{'}$ and $v^{'}$ are in $\mathrm{in}(\widetilde{\mathbf{F}})$. In particular, u and v are in $\mathrm{in}(\widetilde{\mathbf{F}})$. This is a contradiction. Therefore neither $u^{'}$ nor $v^{'}$ belongs to $\mathrm{in}(\mathbf{F})$. Since $\mathbf{F}$ is a Gröbner basis for $J_{\mathcal{D}_M}$, it follows that $u^{'} = v^{'}$. In particular, $\mathcal{I} = \mathcal{I}^{'}$, $\mathcal{J} = \mathcal{J}^{'}$, $\mathcal{K} = \mathcal{K}^{'}$, $p_l = p_l^{'}$, $q_l = q_l^{'}$, and $r_l = r_l^{'}$ for any l. Thus $u = v$. Therefore $\widetilde{\mathbf{F}}$ is a Gröbner basis for $J_{\widetilde{\mathcal{D}}_M}$. □

Corollary 3.2.3. *Let M be a matroid on E. If $\mathbf{F}$ is a homogeneous generating set for $J_{\mathcal{D}_M}$, then $\widetilde{\mathbf{F}}$ is a generating set for $J_{\widetilde{\mathcal{D}}_M}$.*

Proof. We assume that $\mathbf{F}$ and $\mathbf{F}'$ are generating sets for $J_{\mathcal{D}_M}$. Then $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{F}'}$ generate the same ideal. In particular, this holds if $\mathbf{F}'$ is a Gröbner basis for $J_{\mathcal{D}_M}$. Thus $\langle\widetilde{\mathbf{F}}\rangle = \langle\widetilde{\mathbf{F}'}\rangle$. By Theorem 3.2.2, if $\mathbf{F}'$ is a Gröbner basis for $J_{\mathcal{D}_M}$, then $\widetilde{\mathbf{F}'}$ is a generating set for $J_{\widetilde{\mathcal{D}}_M}$, since $\widetilde{\mathbf{F}'}$ is a Gröbner basis for $J_{\widetilde{\mathcal{D}}_M}$. $\square$

Corollary 3.2.4. *Let M be a matroid on E, and let $M +_c (d+1)$ denote a series extension of M at c by $d+1$. Then, by replacing variables, $\widetilde{\mathbf{F}}$ becomes a generating set (resp. a Gröbner basis) for $J_{M+_c(d+1)}$.*

Proof. By elementary row operations on $\widetilde{\mathcal{D}}_M$, we obtain the vector configuration arising from $M +_c (d+1)$. $\square$

Remark 3.2.5. If c is a coloop of M, then $J_{M+_c(d+1)} = J_M$.

Corollary 3.2.6. *Classes $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under series and parallel extensions.*

Example 3.2.7. Let $M = M(K_4)$ and

$$\mathcal{D}_M = \begin{pmatrix} 0&1&1&0&1&1&1&0&0&1&0&1&0&1&0&0\\ 1&1&0&1&0&0&1&1&1&1&0&0&1&0&0&0\\ 1&0&1&1&1&0&0&0&1&0&1&1&0&0&0&1\\ 1&0&1&0&0&1&1&1&0&0&1&0&0&1&1&0\\ 0&1&0&1&1&1&0&1&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix}.$$

Let $<$ be the lexicographic order on $K[x_1^1, \ldots, x_{16}^1]$ with ordering

$$\begin{aligned} &x_9^1 > x_{10}^1 > x_{11}^1 > x_{12}^1 > x_1^1 > x_{13}^1 > x_2^1 > x_3^1 > x_{14}^1 > x_4^1 \\ &> x_5^1 > x_{15}^1 > x_6^1 > x_{16}^1 > x_7^1 > x_8^1. \end{aligned}$$

From [3], J_M has a quadratic Gröbner basis with respect to $<$. Then

$$\widetilde{\mathcal{D}}_M = \left(\begin{array}{cccccccccccccccc|cccccccc} 0&1&1&0&1&1&1&0&0&1&0&1&0&1&0&0&0&1&1&0&1&1&1&0\\ 1&1&0&1&0&0&1&1&1&1&0&0&1&0&0&0&1&1&0&1&0&0&1&1\\ 1&0&1&1&1&0&0&0&1&0&1&1&0&0&0&1&1&0&1&1&1&0&0&0\\ 1&0&1&0&0&1&1&1&0&0&1&0&0&1&1&0&1&0&1&0&0&1&1&1\\ 0&1&0&1&1&1&0&1&0&0&0&0&1&0&1&1&0&1&0&1&1&1&0&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{array}\right).$$

By elementary row operations on $\widetilde{\mathcal{D}}_M$, we have

$$\left(\begin{array}{cccccccccccccccc|cccccccc}
0&1&1&0&1&1&1&0&0&1&0&1&0&1&0&0&0&1&1&0&1&1&1&0\\
1&1&0&1&0&0&1&1&1&1&0&0&1&0&0&0&1&1&0&1&0&0&1&1\\
1&0&1&1&1&0&0&0&1&0&1&1&0&0&0&1&1&0&1&1&1&0&0&0\\
1&0&1&0&0&1&1&1&0&0&1&0&0&1&1&0&1&0&1&0&0&1&1&1\\
0&1&0&1&1&1&0&1&0&0&0&0&1&0&1&1&0&1&0&1&1&1&0&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
\hline
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{array}\right).$$

Therefore $J_{\widetilde{\mathcal{D}}_M}$ has a quadratic Gröbner basis with respect to the lexicographic order with ordering

$$\begin{aligned}x_9^1 > x_{10}^1 > x_{11}^1 > x_{12}^1 > x_1^2 > x_1^1 > x_{13}^1 > x_2^2 > x_2^1 > x_3^2 > x_3^1 > x_{14}^1 > x_4^2 > x_4^1 \\ > x_5^2 > x_5^1 > x_{15}^1 > x_6^2 > x_6^1 > x_{16}^1 > x_7^2 > x_7^1 > x_8^2 > x_8^1.\end{aligned}$$

3.3 A series and parallel connection of matroids

Let M_1 and M_2 be matroids with $E_1 \cap E_2 = \{c\}$ and $E = E_1 \cup E_2$. Suppose that for both M_1 and M_2, c is neither a loop nor a coloop. Let

$$\begin{aligned}\mathcal{B}_{\mathcal{S}} &= \{B \cup D \mid B \in \mathcal{B}(M_1), D \in \mathcal{B}(M_2), B \cap D = \emptyset\},\\ \mathcal{B}_{\mathcal{P}} &= \{B \cup D \mid B \in \mathcal{B}(M_1), D \in \mathcal{B}(M_2), c \in B \cap D\}\\ &\quad \cup \{(B \cup D) \setminus \{c\} \mid B \in \mathcal{B}(M_1), D \in \mathcal{B}(M_2), c \text{ is in exactly one of } B \text{ and } D\}.\end{aligned}$$

Then pairs $(E, \mathcal{B}_{\mathcal{S}})$ and $(E, \mathcal{B}_{\mathcal{P}})$ are matroids. These matroids are said to be the *series* and *parallel* connections of M_1 and M_2 with respect to the basepoint c. We denote them as $S((M_1; c), (M_2; c))$ and $P((M_1; c), (M_2; c))$, or briefly, $S(M_1, M_2)$ and $P(M_1, M_2)$ [26, Proposition 7.1.13].

On the other hand, when c is a loop of M_1, then we define

$$P(M_1, M_2) = M_1 \oplus (M_2/c) \quad \text{and} \quad S(M_1, M_2) = (M_1/c) \oplus M_2.$$

When c is a coloop of M_1, then we define

$$P(M_1, M_2) = (M_1 \setminus c) \oplus M_2 \quad \text{and} \quad S(M_1, M_2) = M_1 \oplus (M_2 \setminus c)$$

(see [26, 7.1.5 - 7.1.8]). Moreover, the 2-*sum* $M_1 \oplus_2 M_2$ of M_1 and M_2 is $S(M_1, M_2)/c$, or equivalently, $P(M_1, M_2) \setminus c$, where c is neither a loop nor a coloop of either M_1 or M_2.

Let M_1 and M_2 be matroids on $E_1 = [d_1]$ and $E_2 = [d_2]$. We identify the set $[d_2]$ with the set $\{d_1 + 1, \ldots, d_1 + d_2\}$. Assume that $c_i \in E_i$ is not a coloop of M_i for $i = 1, 2$. Let

$$\mathcal{B}(M_1) = \{B_1, \ldots, B_{\gamma_1}, \ldots, B_{n_1}\} \quad \text{and} \quad \mathcal{B}(M_2) = \{D_1, \ldots, D_{\gamma_2}, \ldots, D_{n_2}\}$$

be collections of bases of M_1 and M_2, where $c_1 \notin B_j$ for $j \in [\gamma_1]$ and $c_2 \notin D_k$ for $k \in [\gamma_2]$. Let $\mathcal{D}_{M_1} = \{\mathbf{b}_j^1 \mid j \in [n_1]\} \subset \mathbb{Z}^{d_1}$ and $\mathcal{D}_{M_2} = \{\mathbf{d}_k^2 \mid k \in [n_2]\} \subset \mathbb{Z}^{d_2}$ be two vector configurations satisfying $\mathbf{b}_j^1 = \sum_{l \in B_j} \mathbf{e}_l$ and $\mathbf{d}_k^2 = \sum_{l \in D_k} \mathbf{e}_l$. We define ring homomorphisms π_{M_1} and π_{M_2} by setting

$$\begin{array}{rclcl} \pi_{M_1} : K[x_j^1 \mid j \in [n_1]] & \to & K[S], & & x_j^1 \mapsto S^{\mathbf{b}_j^1}, \\ \pi_{M_2} : K[y_k^2 \mid k \in [n_2]] & \to & K[T], & & y_k^2 \mapsto T^{\mathbf{d}_k^2}. \end{array}$$

Similar to what we did in Section 3.2, we consider two new vector configurations

$$\begin{array}{rcl} \widetilde{\mathcal{D}}_{M_1} & = & \left\{ \begin{pmatrix} \mathbf{b}_j^i \\ \mathbf{a}^i \end{pmatrix} \;\middle|\; i = 1, 2,\ j \in [\alpha_i] \right\} \subset \mathbb{Z}^{d_1+2}, \\ \widetilde{\mathcal{D}}_{M_2} & = & \left\{ \begin{pmatrix} \mathbf{d}_k^i \\ \mathbf{a}^i \end{pmatrix} \;\middle|\; i = 1, 2,\ k \in [\beta_i] \right\} \subset \mathbb{Z}^{d_2+2} \end{array}$$

such that $\mathbf{b}_j^1 = \mathbf{b}_j^2$ for $j \in [\gamma_1]$ and $\mathbf{d}_k^1 = \mathbf{d}_k^2$ for $k \in [\gamma_2]$, where $\binom{\alpha_1}{\alpha_2} = \binom{n_1}{\gamma_1}$, $\binom{\beta_1}{\beta_2} = \binom{\gamma_2}{n_2}$, $\mathbf{a}^1 = \binom{0}{1}$, and $\mathbf{a}^2 = \binom{1}{1}$. We define ring homomorphisms $\widetilde{\pi}_{M_1}$ and $\widetilde{\pi}_{M_2}$ as follows:

$$\begin{array}{rclcl} \widetilde{\pi}_{M_1} : K[X] = K[x_j^i \mid i = 1, 2,\ j \in [\alpha_i]] & \to & K[S, W], & & x_j^i \mapsto S^{\mathbf{b}_j^i} W^{\mathbf{a}^i}, \\ \widetilde{\pi}_{M_2} : K[Y] = K[y_k^i \mid i = 1, 2,\ k \in [\beta_i]] & \to & K[T, W], & & y_k^i \mapsto T^{\mathbf{d}_k^i} W^{\mathbf{a}^i}. \end{array}$$

Then $J_{\widetilde{\mathcal{D}}_{M_i}} = \ker(\widetilde{\pi}_{M_i})$ for $i = 1, 2$. Moreover, we consider the vector configuration

$$\widetilde{\mathcal{D}} = \left\{ \begin{pmatrix} \mathbf{b}_j^i \\ \mathbf{d}_k^i \\ \mathbf{a}^i \end{pmatrix} \;\middle|\; i = 1, 2,\ j \in [\alpha_i], k \in [\beta_i] \right\} \subset \mathbb{Z}^{d_1+d_2+2}.$$

Let $K[Z] = K[z_{jk}^i \mid i = 1, 2,\ j \in [\alpha_i], k \in [\beta_i]]$ be the polynomial ring over K. The ring homomorphism $\widetilde{\pi}$ is defined by

$$\widetilde{\pi} : K[Z] \to K[S, T, W], \qquad z_{jk}^i \mapsto S^{\mathbf{b}_j^i} T^{\mathbf{d}_k^i} W^{\mathbf{a}^i}.$$

Then $J_{\widetilde{\mathcal{D}}} = \ker(\widetilde{\pi})$.

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be homogeneous generating sets for $J_{\mathcal{D}_{M_1}}$ and $J_{\mathcal{D}_{M_2}}$, respectively. Then we define $\widetilde{\mathbf{F}}_1$ and $\widetilde{\mathbf{F}}_2$ in a way analogous to what we did in Section 3.2. Let

$$f = \prod_{l=1}^{u_f} x_{j_l^1}^{i_l} - \prod_{l=1}^{u_f} x_{j_l^2}^{i_l} \in \widetilde{\mathbf{F}}_1,$$

and let $k = (k_1, \ldots, k_{u_f})$ with $k_l \in [\beta_{i_l}]$ for $1 \le l \le u_f$. We consider the binomial $f_k \in K[Z]$ defined by

$$f_k = \prod_{l=1}^{u_f} z_{j_l^1 k_l}^{i_l} - \prod_{l=1}^{u_f} z_{j_l^2 k_l}^{i_l}.$$

Since f belongs to $J_{\widetilde{\mathcal{D}}_{M_1}}$, the new homogeneous binomial f_k belongs to $J_{\widetilde{\mathcal{D}}}$. If $\widetilde{\mathbf{F}}_1$ is any set of binomials in $J_{\widetilde{\mathcal{D}}_{M_1}}$, then

$$\mathrm{Lift}(\widetilde{\mathbf{F}}_1) = \left\{ f_k \;\middle|\; f \in \widetilde{\mathbf{F}}_1, k \in \prod_{l=1}^{u_f} [\beta_{i_l}] \right\}.$$

We define $\mathrm{Lift}(\widetilde{\mathbf{F}}_2)$ in an analogous way. Furthermore, the quadratic binomial set $\mathrm{Quad}(\widetilde{\mathcal{D}}_{M_1}, \widetilde{\mathcal{D}}_{M_2})$ is defined by

$$\mathrm{Quad}(\widetilde{\mathcal{D}}_{M_1}, \widetilde{\mathcal{D}}_{M_2}) = \left\{ z^i_{j_1k_2} z^i_{j_2k_1} - z^i_{j_1k_1} z^i_{j_2k_2} \;\middle|\; i = 1, 2, \;\; \begin{array}{l} 1 \le j_1 < j_2 \le \alpha_i, \\ 1 \le k_1 < k_2 \le \beta_i \end{array} \right\}.$$

We set $\widetilde{\mathrm{N}} = \mathrm{Lift}(\widetilde{\mathbf{F}}_1) \cup \mathrm{Lift}(\widetilde{\mathbf{F}}_2) \cup \mathrm{Quad}(\widetilde{\mathcal{D}}_{M_1}, \widetilde{\mathcal{D}}_{M_2})$.

Theorem 3.3.1. *Let M_1 and M_2 be matroids on $E_1 = [d_1]$ and $E_2 = [d_2]$, respectively; and assume that $c_i \in E_i$ is not a coloop of M_i for $i = 1, 2$. Let $S(M_1, M_2)$ be a series connection of M_1 and M_2 with respect to the basepoint $c = c_1 = c_2$. Then, by replacing variables,*

$$N = \widetilde{\mathrm{N}} \cap K[\widehat{Z}]$$

is a generating set for $J_{S(M_1,M_2)}$. Here we set $K[\widehat{Z}] = K[z^i_{jk} \mid i = 1, 2, \; j \in [\alpha_i], k \in V_i]$, where $V_1 = [\gamma_2]$ and $V_2 = [n_2] \setminus [\gamma_2]$. Moreover, if $\mathbf{F}_1$ and $\mathbf{F}_2$ are Gröbner bases for $J_{\mathcal{D}_{M_1}}$ and $J_{\mathcal{D}_{M_2}}$, then there exists a monomial order such that N is a Gröbner basis for $J_{S(M_1,M_2)}$.

Proof. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be generating sets (resp. Gröbner bases) for $J_{\mathcal{D}_{M_1}}$ and $J_{\mathcal{D}_{M_2}}$. From Theorem 1.5.1, Theorem 3.2.2 and Corollary 3.2.3, $\widetilde{\mathrm{N}}$ is a generating set (resp. a Gröbner basis) for $J_{\widetilde{\mathcal{D}}}$. Now we consider two vector configurations

$$\begin{aligned} \widetilde{\mathcal{D}}' &= \left\{ \begin{pmatrix} \mathbf{b}^i_j \\ \mathbf{d}^i_k \\ \mathbf{c}^i_{jk} \end{pmatrix} \;\middle|\; i = 1, 2, \; j \in [\alpha_i], k \in [\beta_i] \right\} \subset \mathbb{Z}^{d_1+d_2+2}, \\ \mathcal{D} &= \left\{ \begin{pmatrix} \mathbf{b}^i_j \\ \mathbf{d}^i_k \\ \mathbf{a}^i \end{pmatrix} \;\middle|\; i = 1, 2, \; j \in [\alpha_i], k \in V_i \right\} \subset \mathbb{Z}^{d_1+d_2+2}, \end{aligned}$$

where $\mathbf{c}^1_{jk} = \mathbf{a}^1$ and

$$\mathbf{c}^2_{jk} = \begin{cases} \mathbf{a}^2 & \text{if } k \in [\gamma_2], \\ \mathbf{a}^1 & \text{otherwise.} \end{cases}$$

Then $J_{\widetilde{\mathcal{D}}'} = J_{\widetilde{\mathcal{D}}}$ because $\widetilde{\mathcal{D}}'$ can be obtained by an elementary row operation on $\widetilde{\mathcal{D}}$. Let $\delta = (0, \ldots, 0, -1, 0) \in \mathbb{Z}^{d_1+d_2+2}$. Since the usual inner product $\delta \cdot (\mathbf{b}^i_j, \mathbf{d}^i_k, \mathbf{c}^i_{jk})$ equals

$$\begin{cases} -1 & \text{if } i = 2 \text{ and } k \in [\gamma_2], \\ 0 & \text{otherwise,} \end{cases}$$

it follows that a subring $K[\widehat{Z}]/J_{\mathcal{D}}$ of $K[Z]/J_{\widetilde{\mathcal{D}}'}$ is a combinatorial pure subring of $K[Z]/J_{\widetilde{\mathcal{D}}'}$ (see [20]). Thus $J_{\mathcal{D}} = J_{\widetilde{\mathcal{D}}'} \cap K[\widehat{Z}]$. In particular, N is a generating set (resp. a Gröbner basis) for $J_{\mathcal{D}}$. Furthermore, by elementary row operations on $\mathcal{D}$, we can obtain the vector configuration arising from $S(M_1, M_2)$ with respect to the basepoint c. Therefore, by replacing variables, N is a generating set (resp. a Gröbner basis) for $J_{S(M_1,M_2)}$. □

Corollary 3.3.2. *Classes $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under series and parallel connections and 2-sums.*

Proof. Let M_1 and M_2 be matroids with $E_1 \cap E_2 = \{c\}$. Let $S(M_1, M_2)$ (resp. $P(M_1, M_2)$) denote a series (resp. parallel) connection of M_1 and M_2 with respect to the basepoint c.

In the case of series and parallel connections, if c is a loop or a coloop of M_1, then $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under series and parallel connections. Suppose that neither M_1 nor M_2 has c as a loop or a coloop. Then by Theorem 3.2.2 and Theorem 3.3.1, $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under series connections. Also, $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under parallel connections from Proposition 3.1.1, and $P(M_1, M_2) = [S(M_1^*, M_2^*)]^*$ for any matroids M_1 and M_2 [26, Proposition 7.1.14].

In the case of the 2-sum, since $M_1 \oplus_2 M_2 = S(M_1, M_2)/c$, $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_{\mathcal{Q}}$ are closed under 2-sums. □

Example 3.3.3. Let $U_{2,4}$ be the uniform matroid on $E = \{1, 2, 3, 4\}$ with rank 2. Then $J_{U_{2,4}}$ has a quadratic Gröbner basis [32]. By Corollary 3.3.2, the toric ideal $J_{U_{2,4} \oplus_2 U_{2,4}}$ has a quadratic Gröbner basis. Moreover, it is known that $U_{2,4} \oplus_2 U_{2,4}$ is isomorphic to R_6 (see [26]). Therefore it follows that J_{R_6} has a quadratic Gröbner basis.

Using the above results, we have

Theorem 3.3.4. *Let M be a matroid. If M has no minor isomorphic to any of $M(K_4)$, $\mathcal{W}^3$, P_6 and Q_6, then the toric ideal J_M has a Gröbner basis consisting of quadratic binomials.*

Since uniform matroids belong to $\mathcal{M}_{\mathcal{QG}}$ [32] and $\mathcal{M}_{\mathcal{QG}}$ is closed under 1-sums and taking minors by Proposition 3.1.1 [2, 38], Theorem 3.3.4 holds from the following result:

Theorem 3.3.5 ([5, Corollary 3.1])**.** *A matroid M is a minor of 1-sums and 2-sums of uniform matroids if and only if M has no minor isomorphic to any of $M(K_4)$, $\mathcal{W}^3$, P_6 and Q_6.*

Let rk be the rank function of a matroid M and let $\lambda_M(X) = \mathrm{rk}(X) + \mathrm{rk}(E - X) - \mathrm{rk}(M)$ for $X \subset E$. We call λ_M the *connectivity function* of M. For $X \subset E$, if $\lambda_M(X) < k$, where k is a positive integer, then both X and $(X, E - X)$ are called *k-separating.* A k-separating pair $(X, E - X)$ for which $\min\{|X|, |E - X|\} \geq k$ is called a *k-separation* of M with *sides* X and $E - X$. For all $n \geq 2$, we say that M is n-connected if, for any $k < n$, it has no k-separation.

Any matroid that is not 3-connected can be constructed from 3-connected proper minors of itself by a sequence of the operations of 1-sums and 2-sums. Therefore, in order to prove Conjecture 1.7.10 and Conjecture 1.7.11, it is enough to prove the following conjecture:

Conjecture 3.3.6. *The class of all 3-connected matroids belongs to $\mathcal{M}_{\mathcal{Q}}$ and $\mathcal{M}_{\mathcal{QG}}$.*

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