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Parshin's Conjecture and Motivic Cohomology with Compact Support

by

Thomas GEISSER

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Abstract. We discuss Parshin's conjecture on rational K-theory over finite fields and its implications for motivic cohomology with compact support.

1. Introduction

Parshin's conjecture states that higher algebraic K-groups of smooth projective schemes over finite fields are torsion. In [7], we studied the properties that Parshin's conjecture would imply for rational higher Chow groups. We compared rational higher Chow groups $H_i^c(X, \mathbb{Q}(n)) = CH_n(X, i-2n)_{\mathbb{Q}}$ to weight homology $H_i^W(X, \mathbb{Q}(n))$, defined by Jannsen [11] based on the work of Gillet-Soulé [9], and obtained a diagram

$$\begin{array}{ccc} H_i^c(X, \mathbb{Q}(n)) & \xrightarrow{\pi} & H_i^W(X, \mathbb{Q}(n)) \\ \alpha\big\downarrow & & \gamma\big\uparrow \\ \tilde{H}_i^c(X, \mathbb{Q}(n)) & \xrightarrow{\beta} & \tilde{H}_i^W(X, \mathbb{Q}(n)) . \end{array} \tag{1}$$

The terms with the tilde are the homology of the first non-vanishing E^1-line of the niveau spectral sequence for $H_i^c(X, \mathbb{Q}(n))$ and $H_i^W(X, \mathbb{Q}(n))$, respectively. Parshin's conjecture in weight n is equivalent to π being an isomorphism for all X and i. We showed that π is an isomorphism if and only if α, β and γ are isomorphisms, and gave criteria for this to happen.

In this article, we take the cohomological point of view and examine the properties that Parshin's conjecture implies for motivic cohomology with compact support. Surprisingly, the properties obtained are not dual to the properties for higher Chow groups, but have a different flavor. The method to study motivic cohomology with compact support is to use the coniveau filtration. To avoid the problems arising from the covariance of motivic cohomology with compact support for open embeddings (for example, one gets large groups by taking inverse limits, and has to deal with derived inverse limits), we consider the dual groups $H^i_c(X, \mathbb{Q}(n))^*$. We obtain a niveau spectral sequence, and compare it with the spectral sequence for the dual of weight cohomology $H^i_W(X, \mathbb{Q}(n))^*$ as in [7] to obtain

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a diagram

$$\begin{array}{ccc} \tilde{H}^i_W(X,\mathbb{Q}(n))^\sharp & = & \tilde{H}^i_c(X,\mathbb{Q}(n))^\sharp \\ \gamma^*\downarrow & & \alpha^*\downarrow \\ H^i_W(X,\mathbb{Q}(n))^* & \xrightarrow{\pi^*} & H^i_c(X,\mathbb{Q}(n))^* . \end{array} \tag{2}$$

Again, the upper terms are given by the first non-vanishing row of E^1-terms in the niveau spectral sequence. The map π^* is an isomorphism for all X if and only if Parshin's conjecture holds. In contrast to the homological situation, α^* being an isomorphism is stronger than Parshin's conjecture. We go on to examine the relationship between diagrams (1) and (2). Not surprisingly, this is related to Beilinson's conjecture that rational equivalence and numerical equivalence of algebraic cycles agree up to torsion over finite fields. Finally we relate bounds for all four rational motivic theories to Parshin's conjecture.

Throughout the paper we assume the existence of resolution of singularities in order to refer to Friendlander and Voevoesky [2] for basic properties of motivic cohomology with compact support, and to have smooth and proper models for varieties over finite fields at our disposal. The first use can be avoided if properties (3), (4) below can be shown rationally without assumptions. The assumption on the existence of smooth and proper models is necessary to do devissage and refer to Jannsen [11] for properties of weight homology. Its precise use is discussed in Remark 4.2.

Throughout this paper, the category of schemes over k, written Sch/k, means the category of separated schemes of finite type over k, and Sm/k the category of smooth schemes over k. For an abelian group A, A^* denotes its $\mathbb{Q}$-dual $\mathrm{Hom}(A,\mathbb{Q})$.

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2. Motivic cohomology with compact support

2.1. Definition and basic properties

For a scheme X over a perfect field k, motivic cohomology with compact support is defined as

$$H^i_c(X,\mathbb{Z}(n)) = \mathrm{Hom}_{DM^-}(M^c(X),\mathbb{Z}(n)[i]),$$

where DM^- is Voevodsky's triangulated category of homotopy invariant Nisnevich sheaves with transfers.

A concrete description is given as follows [2, §3]: Let $\rho : (Sch/k)_{cdh} \to (Sm/k)_{Nis}$ be the map from the large cdh-site of k to the smooth site with the Nisnevich topology. Let $\mathbb{Z}(n)$ be the motivic complex on $(Sm/k)_{Nis}$, and consider an injective resolution $\rho^*\mathbb{Z}(n) \to I^\cdot$ on $(Sch/k)_{cdh}$ (we need resolution of singularites to ensure that ρ^* is exact). Let $\mathbb{Z}^c(X)$ be the cdh-sheafification of the presheaf which associates to U the free abelian group generated by those subschemes $Z \subseteq X \times U$ whose projection to U induces an open embedding $Z \subseteq U$. Then $H^i_c(X,\mathbb{Z}(n)) = \mathrm{Hom}_{D(Shv_{cdh})}(\mathbb{Z}^c(X), I^\cdot[i])$, where $D(Shv_{cdh})$

is the derived catagory of complexes of cdh-sheaves on Sch/k. This satisfies the following properties:

a) Contravariance for proper maps.
b) Covariance for flat quasi-finite maps.
c) For a closed subscheme Z of X with open complement U, there is a localization sequence

$$\cdots \to H^i_c(U, \mathbb{Z}(n)) \to H^i_c(X, \mathbb{Z}(n)) \to H^i_c(Z, \mathbb{Z}(n)) \to \cdots . \tag{3}$$

If X is proper, then since $\mathbb{Z}^c(X) = \rho^*\mathbb{Z}(X)$, motivic cohomology with compact support agrees with motivic cohomology $H^i(X, \mathbb{Z}(n)) := H^i_{cdh}(X, \mathbb{Z}(n)) = \mathrm{Hom}_{D(Shv_{cdh})}(\rho^*\mathbb{Z}(X), I^\cdot[i])$. Moreover, under resolution of singularities, we get for smooth X isomorphisms [13, Thm. 5.14], [14]

$$H^i_{cdh}(X, \mathbb{Z}(n)) \cong H^i_{Nis}(X, \mathbb{Z}(n)) \cong CH^n(X, 2n - i) . \tag{4}$$

PROPOSITION 2.1. a) *We have* $H^i_c(X, \mathbb{Z}(n)) = 0$ *for* $i > n + \dim X$.

b) *If k is finite and X is smooth of dimension d, then* $H^{n+d}(X, \mathbb{Q}(n)) = 0$ *unless* $n = d$.

c) *If k is finite and if* $n > \dim X$, *then* $H^i_c(X, \mathbb{Q}(n)) = 0$ *for* $i \geq n + \dim X$.

Proof. a) Using the localization sequence and induction on the dimension, the statement is easily reduced to the case where X is proper. Then we use that the complex $\mathbb{Z}(n)$ is concentrated in degrees at most n, and X has cdh-cohomological dimension equal to $\dim X$.

b) If $n < d$, then this follows by comparing to higher Chow groups. If $n > d$, consider the spectral sequence

$$E_1^{s,t} = \bigoplus_{x \in X^{(s)}} H^{t-s}(k(x), \mathbb{Z}(n - s)) \Rightarrow H^{s+t}(X, \mathbb{Z}(n)) . \tag{5}$$

In order for the $E_1^{s,t}$-terms not to vanish, we need $t \leq n$ and $s \leq d$, hence to have $s+t = n+d$ we need $s = d$ and $t = n$. But $E_1^{d,n}$ is a sum of terms of the form $H^{n-d}(k(x), \mathbb{Z}(n-d))$, for $k(x)$ a finite field. The latter agrees with Milnor K-theory $K^M_{n-d}(k(x))$ by the Theorem of Nesterenko-Suslin and Tataro, and we can conclude because higher Milnor K-theory of finite fields is torsion.

c) This was proved in [8, Prop.6.3]. The only case not covered by a) is $i = n+d$, $n > d$. Using localization and induction on the dimension one reduces to the case X smooth. Using de Jong's theorem on alteration and a trace argument one then reduces the problem to X smooth and proper where the result follows from b). □

2.2. The niveau spectral sequence

In order not to deal with derived inverse limits and to get smaller groups, we work with the dual of motivic cohomology with compact support

$$H^i_c(X, \mathbb{Q}(n))^* := \mathrm{Hom}(H^i_c(X, \mathbb{Z}(n)), \mathbb{Q}) .$$

These groups are covariant for proper maps and contravariant for quasi-finite flat maps. Let Z_s be the set of closed subschemes of X of dimension at most s and let Z_s/Z_{s-1} be the set of ordered pairs $(Z, Z') \in Z_s \times Z_{s-1}$ such that $Z' \subseteq Z$. Then Z_s as well as Z_s/Z_{s-1} are ordered by inclusion, and we obtain a filtration $Z_0 \subseteq Z_1 \subseteq \cdots$. We use covariance for proper maps to define

$$H_c^i(Z_s, \mathbb{Q}(n))^\sharp := \operatorname*{colim}_{Z \in Z_s} H_c^i(Z, \mathbb{Q}(n))^* .$$

For a point $x \in X$ we write $x \in Z_s$ if $\overline{\{x\}} \in Z_s$, and using contravariance for open embeddings define

$$H_c^i(k(x), \mathbb{Q}(n))^\sharp := \operatorname*{colim}_{U \cap \overline{\{x\}} \neq \emptyset} H_c^i(U \cap \overline{\{x\}}, \mathbb{Q}(n))^* ,$$

where U runs through those open subsets of X whose intersection with $\overline{\{x\}}$ in non-empty. We use the symbol $A^\sharp$ instead of A^* to indicate that this may not be the dual of any group. For example, for the function field $k(C)$ of a smooth and proper curve C we have $\lim_U H_c^1(U, \mathbb{Q}(0)) = (\prod_{C_{(0)}} \mathbb{Q})/\mathbb{Q}$, $C_{(0)}$ the closed points of C, whereas taking duals allows us to work with the countable "predual" group $H_c^1(k(C), \mathbb{Q}(0))^\sharp = \operatorname{colim}_U H_c^1(U, \mathbb{Q}(0))^* = \ker\big(\oplus_{C_{(0)}} \mathbb{Q} \to \mathbb{Q}\big)$.

From the localization sequence we obtain

$$H_c^i(Z_s/Z_{s-1}, \mathbb{Q}(n))^\sharp := \operatorname*{colim}_{(Z,Z') \in Z_s/Z_{s-1}} H_c^i(Z - Z', \mathbb{Q}(n))^* = \bigoplus_{x \in Z_s} H_c^i(k(x), \mathbb{Q}(n))^\sharp .$$

The usual yoga with exact couples gives

PROPOSITION 2.2. *There is a homological spectral sequence*

$$E^1_{s,t} = \bigoplus_{x \in X_{(s)}} H_c^{s+t}(k(x), \mathbb{Q}(n))^\sharp \Rightarrow H_c^{s+t}(X, \mathbb{Q}(n))^* . \tag{6}$$

Here $X_{(s)}$ denotes the set of points of X whose closure has dimension s.

The d^1-differential is given by

$$H_c^{i+1}(Z_{s+1}/Z_s, \mathbb{Q}(n))^\sharp \to H_c^i(Z_s, \mathbb{Q}(n))^\sharp \to H_c^i(Z_s/Z_{s-1}, \mathbb{Q}(n))^\sharp .$$

By Proposition 2.1a), we know that $H_c^i(k(x), \mathbb{Q}(n))^\sharp = 0$ for $i > n + s$ if $x \in X_{(s)}$, i.e. $E^1_{s,t}$ vanishes for $t > n$, so that the spectral sequence (6) is concentrated below and on the line $t = n$. On the line $t = n$, the terms $E^1_{s,n}$ vanish for $s < n$ by Proposition 2.1c). We define $\tilde{H}_c^j(X, \mathbb{Q}(n))^\sharp$ to be the cohomology of the line $E^1_{*,n}$

$$\bigoplus_{x \in X_{(n)}} H_c^{2n}(k(x), \mathbb{Q}(n))^\sharp \leftarrow \cdots \leftarrow \bigoplus_{x \in X_{(d)}} H_c^{n+d}(k(x), \mathbb{Q}(n))^\sharp , \tag{7}$$

where we put the term indexed by $X_{(i)}$ in degree $n + i$. It is easy to check that we obtain canonical maps

$$\tilde{H}_c^i(X, \mathbb{Q}(n))^\sharp \xrightarrow{\alpha^*} H_c^i(X, \mathbb{Q}(n))^* . \tag{8}$$

3. Parshin's conjecture

Parshin's conjecture states that for all smooth and projective X over $\mathbb{F}_q$, the groups $K_i(X)$ are torsion for $i > 0$. In [3] we showed that Parshin's conjecture is implied by Tate's conjecture and Beilinson's conjecture that rational and numerical equivalence agree up to torsion. Since $K_i(X)_{\mathbb{Q}} = \oplus_n H^{2n-i}(X, \mathbb{Q}(n))$, it follows that Parshin's conjecture is equivalent to the following conjecture for all n.

Conjecture P^n: *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, and all $i \neq 2n$, the group $H^i(X, \mathbb{Z}(n))$ is torsion.*

Conjecture P^n is known for $n = 0, 1$ and is trivial for $n < 0$. In [7], we considered the homological analog (it was denoted $P(m)$ in loc.cit.):

Conjecture P_m: *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, and all $i \neq 2m$, the group $H_i^c(X, \mathbb{Z}(m))$ is torsion.*

This conjecture is not known for any m. One can also consider the restrictions $P^n(d)$ and $P_m(d)$ of the above conjectures to varieties of dimension at most d. By the projective bundle formula one gets $P^n(d) \Rightarrow P^{n-1}(d-1)$ and $P_m(d) \Rightarrow P_{m-1}(d-1)$, hence $P^n \Rightarrow P^{n-1}$ and $P_m \Rightarrow P_{m-1}$.

LEMMA 3.1. *Conjectures $P^n(d)$ and $P_{d-n}(d)$ are equivalent.*

Proof. Suppose $P_n(d)$. Then conjecture P^{n-d+e} holds for any smooth and projective X of dimension $e \leq d$. Hence the formula $H^i(X, \mathbb{Z}(a)) \cong H^c_{2e-i}(X, \mathbb{Z}(e-a))$ implies conjecture P_{d-n} for X. The converse is proved the same way. □

LEMMA 3.2. *If conjecture P_{-1} holds, then $H^i_c(X, \mathbb{Q}(n)) = 0$ for any X and $n > d = \dim X$. In particular, the terms $E^1_{s,t}$ vanish for $s < n$ in the spectral sequence* (6).

Proof. By induction on the dimension of X and the sequence (3) we can assume that X is smooth and proper. Then $H^i_c(X, \mathbb{Q}(n)) = H^i(X, \mathbb{Q}(n)) = H^c_{2d-i}(X, \mathbb{Q}(d-n))$ which vanishes by conjecture P_{-1}. □

Since conjecture P^{-1} is trivially true, the previous Lemma explains why the spectral sequence for motivic homology with compact support in [7] is concentrated in degrees $s \geq n$, whereas (6) a priori is not.

LEMMA 3.3. *The following statements are equivalent*:

a) *Conjecture P^n.*
b) *For all schemes X over $\mathbb{F}_q$, we have $H^i_c(X, \mathbb{Q}(n)) = 0$ for $i < 2n$.*
c) *For all finitely generated fields $K/\mathbb{F}_q$, we have $H^i_c(K, \mathbb{Q}(n))^\sharp = 0$ for $i < 2n$.*

Proof. a) $\Rightarrow$ b) follows by induction on the dimension, localization and a trace argument using alterations to reduce to the smooth and proper case. b) $\Rightarrow$ c) follows by taking colimits, and c) $\Rightarrow$ a) follows with the spectral sequence (6). □

It is not a priori clear if the terms $H^i_c(k(x), \mathbb{Q}(n))^\sharp$ with $2n \le i < \operatorname{trdeg} k(x)+n$ should vanish or not. Thus the following statement is possibly stronger than Parshin's conjecture (but see Proposition 5.2):

PROPOSITION 3.4. *The following statements are equivalent*:

a) *Conjecture P^n holds*, *and for smooth and projective X we have*

$$\tilde{H}^i_c(X, \mathbb{Q}(n))^\sharp \cong \begin{cases} CH^n(X)^* & i = 2n; \\ 0 & else. \end{cases}$$

b) *The groups $H^i_c(K, \mathbb{Q}(n))^\sharp$ vanish for $i \neq n + \operatorname{trdeg} K$.*

c) *The map α^* is an isomorphism for all X and i.*

Proof. a) $\Rightarrow$ b): We proceed by induction on the transcendence degree. By a trace argument it suffices to show the vanishing for a finite extension of K, and by de Jong we can choose a smooth and projective model X of such a finite extension of K. Since $H^i_c(X, \mathbb{Q}(n))$ is $CH^n(X)_\mathbb{Q}$ for $i = 2n$ and vanishes for $i \neq 2n$, an inspection of the spectral sequence (6) shows the vanishing.

b) $\Leftrightarrow$ c) is clear.

c) $\Rightarrow$ a): Conjecture P^n follows because $\tilde{H}^i_c(X, \mathbb{Q}(n))^\sharp$ vanishes for $i < 2n$, and the sequence is exact because for smooth and proper X, $H^i_c(X, \mathbb{Q}(n))^*$ vanishes for $i > 2n$ and is isomorphic to $CH^n(X)_\mathbb{Q}$ for $i = 2n$. $\square$

The statements of this Proposition are non-trivial even in the case $n = 0$ (but they can be proven with methods similar to [11, Thm.5.10] in this case).

4. Weight cohomology

4.1. Definition

Let $\mathcal{C}$ be the category of correspondences with objects smooth projective varieties $[X]$ over the field k, $\operatorname{Hom}_\mathcal{C}([X], [Y]) = \oplus CH^{\dim Y_i}(X \times Y_i)_\mathbb{Q}$ for $Y = \coprod Y_i$ the decomposition into connected components, and the usual composition of correspondences. In [9], Gillet and Soulé defined, for every separated scheme of finite type, a weight complex $W(X)$ in the homotopy category of bounded complexes in $\mathcal{C}$, satisfying the following properties [9, Thm. 2]:

a) $W(X)$ is represented by a bounded complex

$$[X_0] \leftarrow [X_1] \leftarrow \cdots \leftarrow [X_k]$$

with $\dim X_i \le \dim X - i$.

b) $W(-)$ is covariantly functorial for proper maps.

c) $W(-)$ is contravariantly functorial for open embeddings.

d) If $T \to X$ is a closed embedding with open complement U, then there is a distinguished triangle

$$W(T) \xrightarrow{i_*} W(X) \xrightarrow{j^*} W(U)\,.$$

Our notation differs from loc.cit. in variance. In loc.cit., resolution of singularities is used, but in [10], it is shown that the same statement holds with rational coefficients without the assumption on resolution of singularities. Moreover by [10, p.3139]:

c') $W(-)$ is contravariantly functorial for flat maps.

It is easy to check that if $f : X \to Y$ is a finite etale map of smooth schemes of degree d, then f_*f^* induces multiplication by d on $W(Y)$.

We define dual weight cohomology (with compact support) $H^i_W(X, \mathbb{Q}(n))^*$ to be the ith homology of the complex

$$CH^n(X_0)^* \leftarrow CH^n(X_1)^* \leftarrow CH^n(X_2)^* \leftarrow \cdots ,$$

induced by contravariance of CH^n, and with $CH^n(X_i)^*$ placed in degree $2n+i$. This is the contravariant analog of [11, Thm.5.13]. It is clear from the definition that $H^i_W(X, \mathbb{Q}(n))^*$ is covariant for proper maps, and since the weight complex for an open subscheme is the cone of the weight complexes of the complement and the ambient space, the groups $H^i_W(X, \mathbb{Q}(n))^*$ are contravariant for open embeddings. We define dual weight cohomology of a field to be

$$H^i_W(K, \mathbb{Q}(n))^\sharp := \operatorname*{colim}_U H^i_W(U, \mathbb{Q}(n))^* ,$$

where U runs through smooth schemes with function field K.

LEMMA 4.1. *We have* $H^i_W(X, \mathbb{Q}(n))^* = 0$ *unless* $2n \le i \le \dim X + n$. *In particular,* $H^i_W(K, \mathbb{Q}(n))^\sharp = 0$ *for every finitely generated field* K/k *unless* $2n \le i \le \operatorname{trdeg}_k K + n$.

Proof. This follows from the first property of weight complexes together with $CH^n(T)_{\mathbb{Q}} = 0$ for $n > \dim T$. □

It follows from Lemma 4.1 that the niveau spectral sequence

$$E^1_{s,t} = \bigoplus_{x \in X_{(s)}} H^{s+t}_W(k(x), \mathbb{Q}(n))^\sharp \Rightarrow H^{s+t}_W(X, \mathbb{Q}(n))^* \tag{9}$$

is concentrated on and below the line $t = n$ and on and above the line $s + t = 2n$. If we let $\tilde{H}^i_W(X, \mathbb{Q}(n))^\sharp = E^2_{i-n,n}(X)$ be the homology of the complex

$$\bigoplus_{x \in X_{(n)}} H^{2n}_W(k(x), \mathbb{Q}(n))^\sharp \leftarrow \cdots \leftarrow \bigoplus_{x \in X_{(d)}} H^{n+d}_W(k(x), \mathbb{Q}(n))^\sharp , \tag{10}$$

where $d = \dim X$, then we obtain a canonical and natural map

$$\gamma^* : \tilde{H}^i_W(X, \mathbb{Q}(n))^\sharp \to H^i_W(X, \mathbb{Q}(n))^* .$$

4.2. Comparison

We are going to check the hypothesis of [11, Prop.5.16] to construct a natural transformation between motivic cohomology with compact support and weight cohomology. Recall that motivic cohomology with compact support is defined as the cohomology of $C'(X) = \operatorname{Hom}_{D(Shv_{cdh})}(\mathbb{Z}^c(X), I^\cdot)$, where $\rho^*\mathbb{Z}(n) \to I^\cdot$ is an injective resolution on the cdh-site. Then C' is a covariant functor from the category of schemes over k with proper maps to the category of complexes with bounded above cohomology, which is contravariant for open embeddings. Moreover, for proper X we have $C'(X) = I^{\cdot}(X)$, and a closed embedding $i : Y \to X$ with open complement $j : U \to X$ gives a short exact sequence

$$0 \to C'(U) \to C'(X) \to C'(Y) \to 0\,.$$

Restricting C' to smooth and proper X, we have $H^i C'(X) = 0$ for $i > 2n$, and a functorial isomorphism

$$H^{2n} C'(X) = H^{2n} I^{\cdot}(X) \cong \tau_{\geq 2n} I^{\cdot}(X) \cong CH^n(X)$$

by (4). We obtain a morphism of functors on the category of smooth and proper schemes,

$$C' = I^{\cdot} \to \tau_{\geq 2n} I^{\cdot} \xleftarrow{\sim} H^{2n}(I^{\cdot})[-2n] = CH^n(-)[-2n]\,.$$

Reversing all the arrows induced by arrows between schemes, but not by arrows between cohomology theories in the proof of [11, Prop.5.16] gives a natural transformation $H^i_c(X, \mathbb{Z}(n)) \to H^i_W(X, \mathbb{Z}(n))$, hence a natural transformation

$$\pi^* : H^i_W(X, \mathbb{Q}(n))^* \to H^i_c(X, \mathbb{Q}(n))^*\,.$$

This transformation is compatible with covariance for proper maps and contravariance for open embeddings, as well as localization sequences.

From now on we return to the situation k finite. For the remainder of the paper we assume that all finitely generated fields over a finite field admit a smooth proper model.

REMARK 4.2. In order to remove the hypothesis on the existence of smooth and proper models for fields over a finite field, it would suffice, by a standard argument using alterations [4, Lemma 4.1], to show the following statements:

a) *Weight cohomology groups are contravariant for finite etale maps* $f : X \to Y$ *between smooth schemes.*
b) *The composition* $f_* f^*$ *is an isomorphism on* $H^i_W(Y, \mathbb{Q}(n))^*$.
c) *This contravariance is compatible with the maps* π^*.

PROPOSITION 4.3. *Let* K/k *be finitely generated of transcendence degree* d.
a) The map π^* *induces isomorphisms*

$$H^{n+d}_W(K, \mathbb{Q}(n))^\sharp \xrightarrow{\sim} H^{n+d}_c(K, \mathbb{Q}(n))^\sharp$$

for all n. *In particular, we have* $\tilde{H}^i_W(X, \mathbb{Q}(n))^\sharp \cong \tilde{H}^i_c(X, \mathbb{Q}(n))^\sharp$ *for all* X *and* i.
b) If $d > n$, *then* π^* *induces isomorphisms*

$$H^{n+d-1}_W(K, \mathbb{Q}(n))^\sharp \xrightarrow{\sim} H^{n+d-1}_c(K, \mathbb{Q}(n))^\sharp\,.$$

Proof. We proceed by induction on d. Choose a smooth and projective model X of a finite extension K of the field and compare (6) and (9).

a) If $d < n$, then both terms vanish by Proposition 2.1c) and Lemma 4.1. For $d = n$ we obtain $CH^n(X)_\mathbb{Q} \cong H^{n+d}_c(K, \mathbb{Q}(n))^\sharp \cong H^{n+d}_W(K, \mathbb{Q}(n))^\sharp$. For $d > n$, we obtain from

$H_c^{n+d}(X,\mathbb{Q}(n)) = H_W^{n+d}(X,\mathbb{Q}(n)) = 0$ a commutative diagram with exact rows

$$\begin{array}{ccccccc} \cdots & \longleftarrow & \bigoplus_{x\in X_{(d-1)}} H_W^{n+d-1}(k(x),\mathbb{Q}(n))^\sharp & \longleftarrow & H_W^{n+d}(K,\mathbb{Q}(n))^\sharp & \longleftarrow & 0 \\ & & \| & & \downarrow & & \\ \cdots & \longleftarrow & \bigoplus_{x\in X_{(d-1)}} H_c^{n+d-1}(k(x),\mathbb{Q}(n))^\sharp & \longleftarrow & H_c^{n+d}(K,\mathbb{Q}(n))^\sharp & \longleftarrow & 0\,. \end{array}$$

b) follows by a similar argument, noting that the d_2-differentials originating from the terms in question end in terms considered in a), and there are no higher differentials. □

We obtain a commutative diagram

$$\begin{array}{ccc} \tilde{H}_W^i(X,\mathbb{Q}(n))^\sharp & = & \tilde{H}_c^i(X,\mathbb{Q}(n))^\sharp \\ {\scriptstyle\gamma^*}\downarrow & & {\scriptstyle\alpha^*}\downarrow \\ H_W^i(X,\mathbb{Q}(n))^* & \xrightarrow{\pi^*} & H_c^i(X,\mathbb{Q}(n))^*\,. \end{array} \tag{11}$$

PROPOSITION 4.4. *The following statements are equivalent*:

a) *Conjecture P^n.*
b) *The map π^* is isomorphisms for all X and i.*
c) *We have $H_W^i(K,\mathbb{Q}(n))^\sharp \cong H_c^i(K,\mathbb{Q}(n))^\sharp$ for all i and all finitely generated extensions K/k.*

Proof. a) ⇔ b): For smooth and proper X this is clear. In general, one does induction on the dimension and uses localization sequences.

b) ⇔ c): One direction follows by taking colimits, and the other by comparing the spectral sequences (6) and (9). □

The following Proposition is analogous to Proposition 3.4 and dual to [7, Prop.3.4]:

PROPOSITION 4.5. *The following statements are equivalent and follow from α^* being an isomorphism*:

a) *For smooth and projective X, we have*

$$\tilde{H}_W^i(X,\mathbb{Q}(n))^* \cong \begin{cases} CH^n(X)^* & i = 2n; \\ 0 & else. \end{cases}$$

b) *The groups $H_W^i(K,\mathbb{Q}(n))^\sharp$ vanish for $i \neq n + \operatorname{trdeg} K$.*
c) *The map γ^* is an isomorphism for all X and i.*

Proof. The proof is similar to Proposition 3.4.

a) ⇒ b): We proceed by induction on the transcendence degree. Choose a smooth and projective model X of K. Since $H_W^i(X,\mathbb{Q}(n))$ is $CH^n(X)_{\mathbb{Q}}$ for $i = 2n$ and vanishes for $i \neq 2n$, an inspection of the spectral sequence (9) gives the result.

b) ⇒ c) ⇒ a) are clear. If α^* is an isomorphism, then so is π^*, and hence γ^*. □

5. Beilinson's conjecture and duality

Beilinson conjectured that over a finite field, rational equivalence and numerical equivalence agree up to torsion. This can be reformulated to the following:

Conjecture $D(n)$**:** *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, the intersection pairing gives a functorial isomorphism*

$$CH^n(X)_{\mathbb{Q}} \cong \mathrm{Hom}(CH_n(X), \mathbb{Q})\,.$$

Note that since both sides are countable, this implies finite dimensionality. By the projection formula, the intersection pairing induces a map of complexes

$$\begin{array}{ccccccc}
CH_n(X_0)_{\mathbb{Q}} & \longleftarrow & CH_n(X_1)_{\mathbb{Q}} & \longleftarrow & CH_n(X_2)_{\mathbb{Q}} & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
CH^n(X_0)^* & \longleftarrow & CH^n(X_1)^* & \longleftarrow & CH^n(X_2)^* & \longleftarrow & \cdots\,.
\end{array}$$

Taking homology, we obtain a map

$$\delta : H_i^W(X, \mathbb{Q}(n)) \to H^i_W(X, \mathbb{Q}(n))^*\,.$$

Taking the limit over decreasing open sets with function field K, δ induces a map $H_i^W(K, \mathbb{Q}(n)) \to H^i_W(K, \mathbb{Q}(n))^\sharp$. This in turn induces a map of complexes

$$\begin{array}{ccccc}
\bigoplus_{x\in X_{(n)}} H_{2n}^W(k(x), \mathbb{Q}(n)) & \longleftarrow & \bigoplus_{x\in X_{(n+1)}} H_{2n+1}^W(k(x), \mathbb{Q}(n)) & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \\
\bigoplus_{x\in X_{(n)}} H_W^{2n}(k(x), \mathbb{Q}(n))^\sharp & \longleftarrow & \bigoplus_{x\in X_{(n+1)}} H_W^{2n+1}(k(x), \mathbb{Q}(n))^\sharp & \longleftarrow & \cdots\,,
\end{array}$$

which gives the map τ making the following diagram commutative

$$\begin{array}{ccccccc}
H_i^c(X, \mathbb{Q}(n)) & \xrightarrow{\pi} & H_i^W(X, \mathbb{Q}(n)) & \xrightarrow{\delta} & H^i_W(X, \mathbb{Q}(n))^* & \xrightarrow{\pi^*} & H^i_c(X, \mathbb{Q}(n))^* \\
\alpha\downarrow & & \gamma\uparrow & & \gamma^*\uparrow & & \alpha^*\uparrow \\
\tilde{H}_i^c(X, \mathbb{Q}(n)) & \xrightarrow{\beta} & \tilde{H}_i^W(X, \mathbb{Q}(n)) & \xrightarrow{\tau} & \tilde{H}^i_W(X, \mathbb{Q}(n))^\sharp & = & \tilde{H}^i_c(X, \mathbb{Q}(n))^\sharp\,.
\end{array} \tag{12}$$

Here the left square is the diagram obtained in [7] by a similar method as above.

LEMMA 5.1. *Conjecture $D(n)$ is equivalent to δ being an isomorphism for all i and X, and implies that τ is an isomorphism for all i and X.*

Proof. The equivalence follows from the definition of δ, and the statement about τ follows by a colimit argument. □

Parshin's conjecture and Beilinson's conjecture can be combined into the following

Conjecture $BP(n)$**:** *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, the cup product pairing*

$$H^i(X, \mathbb{Q}(n)) \times H^{2d-i}(X, \mathbb{Q}(d-n)) \to \mathbb{Q}$$

is perfect.

PROPOSITION 5.2. *For fixed n, the following statements are equivalent*:

a) *Conjecture $BP(n)$.*
b) *Conjectures $D(n)$, P^n and P_n.*
c) *There are perfect pairings of finite dimensional vector spaces*

$$H_i^c(X, \mathbb{Q}(n)) \times H_c^i(X, \mathbb{Q}(n)) \to \mathbb{Q}$$

for all smooth projective X.

d) *The same as* c), *but for all X.*
e) *All maps in* (12) *are isomorphisms for all smooth and proper X.*
f) *The same as* e), *but for all X.*

Proof. a) $\Rightarrow$ b): If $i > 2n$, then the left hand side in $BP(n)$ vanishes, hence perfectness is equivalent to the vanishing of $H^{2d-i}(X, \mathbb{Q}(d-n)) \cong H_i^c(X, \mathbb{Q}(n))$ for $i > 2n$, i.e. conjecture P_n of [7]. If $i < 2n$, then the right hand side in $BP(n)$ vanishes, so perfectness is equivalent to P^n. For $i = 2n$, we recover conjecture $D(n)$.

b) $\Rightarrow$ d): It suffices to construct a functorial map $H_i^c(X, \mathbb{Q}(n)) \to H_c^i(X, \mathbb{Q}(n))^*$ which is the intersection pairing for smooth and projective X, and which is functorial, and compatible with localization sequences on both sides. Indeed having such a map one can use the usual devissage and de Jong's theorem to reduce to the case that X is smooth and projective. One way to construct such a map is to write $H_i^c(X, \mathbb{Z}(n)) \cong \mathrm{Hom}_{DM^-}(\mathbb{Z}(n)[i], M^c(X))$, $H_c^i(X, \mathbb{Z}(n)) \cong \mathrm{Hom}_{DM^-}(M^c(X), \mathbb{Z}(n)[i])$. Then the pairing is given by the composition

$$\begin{aligned}\mathrm{Hom}_{DM^-}(\mathbb{Z}(n)[i], M^c(X)) \times \mathrm{Hom}_{DM^-}(M^c(X), \mathbb{Z}(n)[i]) \to \\ \mathrm{Hom}_{DM^-}(\mathbb{Z}(n), \mathbb{Z}(n)) \cong \mathrm{Hom}_{DM^-}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z},\end{aligned}$$

using the cancellation theorem.

d) $\Rightarrow$ c) $\Rightarrow$ a) is clear.

b) $\Rightarrow$ f): Conjecture P_n, $D(n)$ and P^n imply that the left square, middle horizontal maps, and right horizontal maps of (12) are isomorphisms for all X.

f) $\Rightarrow$ d) $\Rightarrow$ b): If the three upper maps of (12) are isomorphisms for smooth and proper X, then P_n, $D(n)$, and P^n hold, respectively. $\square$

6. Parshin's conjecture and the four motivic theories

Recall from [2] that we have four motivic theories: Motivic cohomology, motivic cohomology with compact support, motivic homology and motivic homology with compact support. All four theories are homotopy invariant and satisfy a projective bundle formula.

Motivic cohomology is contravariant, has a Mayer-Vietoris long exact sequence for Zariski covers, and a long exact sequence for abstract blow-ups. Motivic cohomology with compact support is contravariant for proper maps, covariant for quasi-finite flat maps, and satisfies a localization long exact sequence (which implies in particular Mayer-Vietoris and abstract blow-up long exact sequences). Motivic homology and motivic homology with compact support satisfy the dual properties. The theories are related by the following diagram

$$\begin{array}{ccc} H^i_c(X,\mathbb{Q}(n)) & \xrightarrow[\cong]{proper} & H^i(X,\mathbb{Q}(n)) \\ {\scriptstyle smooth}\downarrow\cong & & {\scriptstyle smooth}\downarrow\cong \\ H_j(X,\mathbb{Q}(m)) & \xrightarrow[\cong]{proper} & H^c_j(X,\mathbb{Q}(m)) \end{array}$$

The horizontal maps are isomorphisms for proper X, and the vertical maps are isomorphisms if X is smooth of pure dimension d, and $m+n=d$ and $j+i=2d$, see [2, 14]. The functorialities suggest that groups diagonally opposite should be in some form of duality; we saw that with rational coefficients, this is equivalent to deep conjectures, for a result with torsion coefficients see [6] and [12].

The following diagram describes the range where these groups can be non-zero, and where they can be non-zero assuming Parshin's conjecture. The last two rows give improved estimates for smooth X assuming Parshin's conjecture, and for proper X assuming Parshin's conjecture, respectively. The bold faced inequalities indicate that they are strong enough to recover Parshin's conjecture.

	$H^i_c(X,\mathbb{Q}(n))$	$H^i(X,\mathbb{Q}(n))$	$H_j(X,\mathbb{Q}(m))$	$H^c_j(X,\mathbb{Q}(m))$
always	$i\le n+d$	$i\le n+d$	$j\ge m$	$j\ge 2m$
		$i\le 2n$ X smooth	$j\ge 2m$ X proper	
Parshin $\Rightarrow$	$\mathbf{2n}\le\mathbf{i}\le n+d$	$n\le i\le n+d$	$m\le j\le m+d$	$2m\le j\le \mathfrak{m}+d$
P+smooth		$n\le i\le 2n$	$m\le\mathbf{j}\le\mathbf{2m}$	
P+proper		$\mathbf{2n}\le\mathbf{i}\le n+d$	$2m\le j\le m+d$	

Proof. The first row follows from the definitions (and that the cdh-cohomological dimension agrees with the dimension). Since motivic homology with compact support $H^c_j(X,\mathbb{Q}(m))$ is isomorphic to higher Chow groups $CH_m(X,j-2m)_{\mathbb{Q}}$, they can only be non-zero for $j\ge 2m$. The second row is the translation of this fact into a statement for motivic cohomology for smooth X, and for motivic homology for proper X.

The bounds under Parshin's conjecture for motivic homology with compact support and motivic cohomology with compact support can be obtained by using induction on the dimension and the localization sequences. To obtain bounds for motivic homology and cohomology, one uses the isomorphisms $H_i(X,\mathbb{Z}(n))\cong H^{2d-i}_c(X,\mathbb{Z}(d-n))$ and $H^i(X,\mathbb{Z}(n))\cong H^c_{2d-i}(X,\mathbb{Z}(d-n))$ for a smooth scheme X of dimension d to first obtain the bounds for smooth schemes. Then induction on the dimension and the blow-up long exact sequences gives bounds for all schemes.

The extra information for the smooth and proper case in case of homology and cohomology is obtained by comparing to the other theories. □

The bold faced inequalities were a motivation to write this paper: It might be difficult to prove a statement which only holds for smooth and proper X, as in the case of higher Chow groups. It might be easier to prove a statement which holds for all smooth schemes (motivic homology), or all proper schemes (motivic cohomology), or all schemes (motivic cohomology with compact support).

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Department of Mathematics
Rikkyo University
3–34–1 Nishi-Ikebukuro, Toshima-ku,
Tokyo, 171-8501, Japan
E-mail: geisser@rikkyo.ac.jp