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Approximate Functional Equations for the Hurwitz and Lerch Zeta-functions

by

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Abstract. As one of the asymptotic formulas for the zeta-function, Hardy and Littlewood gave asymptotic formulas called the approximate functional equation. In 2003, R. Garunkštis, A. Laurinčikas, and J. Steuding (in [1]) proved the Riemann-Siegel type of the approximate functional equation for the Lerch zeta-function $\zeta_L(s,\alpha,\lambda)=\sum_{n=0}^{\infty} e^{2\pi i n\lambda}(n+\alpha)^{-s}$. In this paper, we prove another type of approximate functional equations for the Hurwitz and Lerch zeta-functions. R. Garunkštis, A. Laurinčikas, and J. Steuding (in [2]) obtained the results on the mean square values of $\zeta_L(\sigma+it,\alpha,\lambda)$ with respect to t. We obtain the main term of the mean square values of $\zeta_L(1/2+it,\alpha,\lambda)$ using a simpler method than their method in [2].

1. Introduction and the statement of results

Let $s=\sigma+it$ be a complex variable, and let $0<\alpha\le 1, 0<\lambda\le 1$ be real parameters. The Hurwitz zeta-function $\zeta_H(s,\alpha)$ and the Lerch zeta-function $\zeta_L(s,\alpha,\lambda)$ are defined by

$$\zeta_H(s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}\,, \tag{1.1}$$

$$\zeta_L(s,\alpha,\lambda)=\sum_{n=0}^{\infty}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}\,, \tag{1.2}$$

respectively. There series are absolutey convergent for $\sigma>1$. Also, if $0<\lambda<1$, then the series (1.2) is convergent even for $\sigma>0$.

As a classical asymptotic formula for the Riemann zeta-function, the following was proved by Hardy and Littlewood (§4 in [5]); we suppose that $\sigma_0>0$, $x\ge 1$, then

$$\zeta(s)=\sum_{n\le x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}+O(x^{-\sigma})$$

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uniformly for $\sigma \geq \sigma_0$, $|t| < 2\pi x/C$, where $C > 1$ is a constant. Also, Hardy and Littlewood proved the following asymptotic formula (§4 in [5]); we suppose that $0 \leq \sigma \leq 1$, $x \geq 1$, $y \geq 1$ and $2\pi xy = |t|$, then

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \chi(s) \sum_{n\leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{1/2-\sigma} y^{\sigma-1}), \tag{1.3}$$

where $\chi(s) = 2\Gamma(1-s)\sin(\pi s/2)(2\pi)^{s-1}$ and note that $\zeta(s) = \chi(s)\zeta(1-s)$ holds. This is called approximate functional equation.

Further, there is a Riemann-Siegel type of the approximate functional equation for $\zeta(s)$; suppose that $0 \leq \sigma \leq 1$, $x = \sqrt{t/2\pi}$, and $N < Ct$ with a sufficiently small constant C. Then

$$\begin{aligned}\zeta(s) = &\sum_{n\leq x} \frac{1}{n^s} + \chi(s) \sum_{n\leq x} \frac{1}{n^{1-s}} + (-1)^{[x]-1} e^{\pi i(1-s)/2} (2\pi t)^{s/2-1/2} e^{it/2-i\pi/8} \\ &\times \Gamma(1-s) \left(S_N + O\left(\left(\frac{CN}{t} \right)^{N/6} \right) + O(e^{-Ct}) \right), \end{aligned}\tag{1.4}$$

where

$$S_N = \sum_{n=0}^{N-1} \sum_{\nu\leq n/2} \frac{n! i^{\nu-n}}{\nu!(n-2\nu)! 2^n} \left(\frac{2}{\pi} \right)^{n/2-\nu} a_n \psi^{(n-2\nu)} \left(\sqrt{\frac{2t}{\pi}} - 2[x] \right),$$

with a_n defined by

$$\exp\left((s-1)\log\left(1 + \frac{z}{\sqrt{t}}\right) - iz\sqrt{t} + \frac{1}{2} i z^2 \right) = \sum_{n=0}^{\infty} a_n z^n,$$

with $a_0 = 1$, $a_n \ll t^{-n/2+[n/3]}$. R. Garunkštis, A. Laurinčikas, and J. Steuding proved an analogue of (1.4) for the Lerch zeta-function as follows;

THEOREM 1 (R. Garunkštis, A. Laurinčikas, and J. Steuding [1]). *Suppose that* $0 < \alpha \leq 1$, $0 < \lambda < 1$ *and* $0 \leq \sigma \leq 1$. *Suppose that* $t \geq 1$, $x = \sqrt{t/2\pi}$, $N = [x]$, $M = [x-\alpha]$ *and* $\beta = N - M$. *Then*

$$\begin{aligned}\zeta_L(s,\alpha,\lambda) = &\sum_{m=0}^{M} \frac{e^{2\pi i m\lambda}}{(m+\alpha)^s} + \left(\frac{2\pi}{t} \right)^{\sigma-1/2+it} e^{it+\pi i/4-2\pi i\{\lambda\}\alpha} \sum_{n=0}^{N} \frac{e^{-2\pi i\alpha n}}{(n+\lambda)^{1-s}} \\ &+ \left(\frac{2\pi}{t} \right)^{\sigma/2} e^{\pi i f(\lambda,\alpha,\sigma,t)} \phi(2x-2N+\beta-\{\lambda\}-\alpha) + O(t^{(\sigma-2)/2}), \end{aligned}\tag{1.5}$$

where

$$\begin{aligned} f(\lambda,\alpha,\sigma,t) = &-\frac{t}{2\pi}\log\frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(\alpha^2 - \{\lambda\}^2) \\ &-\alpha\beta + 2x(\beta + \{\lambda\} - \alpha) - \frac{1}{2}(N+M) - \{\lambda\}(\beta+\alpha). \end{aligned}$$

We prove an analogue of the approximate functional equation (1.3) for (1.1) and (1.2) (in Theorem 2), and gave another proof of the mean square formula for $\zeta_L(1/2+it,\alpha,\lambda)$ with respect to t (in Theorem 3).

THEOREM 2. *Let* $0<\alpha\le 1$ *and* $0<\lambda<1$. *Suppose that* $0\le\sigma\le 1,\ x\ge 1,\ y\ge 1$ *and* $2\pi xy=|t|$. *Then*

$$\zeta_L(s,\alpha,\lambda)=\sum_{0\le n\le x}\frac{e^{2\pi in\lambda}}{(n+\alpha)^s}$$
$$+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\sum_{0\le n\le y}\frac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1-s}}\right.$$
$$\left.+e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\sum_{0\le n\le y}\frac{e^{2\pi in\alpha}}{(n+1-\lambda)^{1-s}}\right\}$$
$$+O(x^{-\sigma})+O(|t|^{1/2-\sigma}y^{\sigma-1}). \tag{1.6}$$

Also, in the case $\lambda=1$ *that is* $\zeta_H(s,\alpha)$ *it follows that*

$$\zeta_H(s,\alpha)=\sum_{0\le n\le x}\frac{1}{(n+\alpha)^s}$$
$$+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{e^{\frac{\pi i}{2}(1-s)}\sum_{1\le n\le y}\frac{e^{2\pi in(1-\alpha)}}{n^{1-s}}+e^{-\frac{\pi i}{2}(1-s)}\sum_{1\le n\le y}\frac{e^{2\pi in\alpha}}{n^{1-s}}\right\}$$
$$+O(x^{-\sigma})+O(|t|^{1-\sigma}y^{\sigma-1}). \tag{1.7}$$

REMARK 1. Theorem 2 can be proved by the method similar to the proof of Theorem 1, but results of Theorem 2 has advantage of choosing parameters x and y freely, only under the condition $2\pi xy=|t|$ as compared with the result of Theorem 1. Also for approximate functional equations (1.6) and (1.7), $\zeta_L(s,\alpha,\lambda)$ is a generalization of $\zeta_H(s,\alpha)$, but (1.6) in Theorem 2 does not include (1.7).

THEOREM 3. *Let* $0<\alpha\le 1,\ 0<\lambda\le 1$. *Then,*

$$\int_1^T\left|\zeta_L\left(\frac{1}{2}+it,\alpha,\lambda\right)\right|^2dt=T\log\frac{T}{2\pi}+\begin{cases}O(T(\log T)^{1/2}) & (0<\alpha<1),\\ O(T(\log T)^{3/4}) & (\alpha=1),\end{cases} \tag{1.8}$$

as $T\to\infty$.

REMARK 2. The result of Theorem 3 has larger error term than the result already proved by R. Garunkštis, A. Laurinčikas and J. Steuding [2], and they proved using Theorem 1 (see [2]). However, the main term on the right-hand side of (1.8) can be obtained more simply than the method of [2] by using Theorem 2. We will describe the proof of Theorem 3 in Section 3.

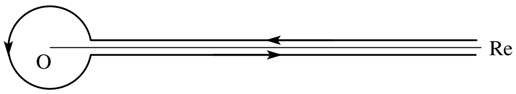
2. Proof of Theorem 2

In this section, we prove Theorem 2. The basic tool of the proof is the same as the approximate functional equation for the Riemann zeta-function (1.3), that is the saddle point method.

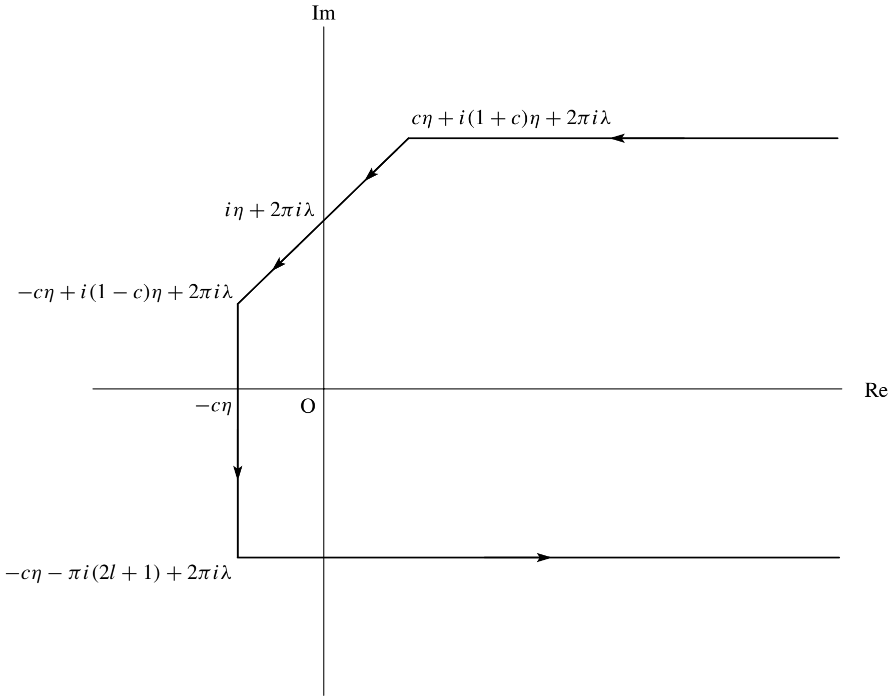
PROOF OF THEOREM 2. Let $M \in \mathbb{N}$ be sufficiently large. We have

$$\begin{aligned}\zeta_L(s,\alpha,\lambda) &= \sum_{n=0}^{M} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s} + \sum_{n=M+1}^{\infty} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s} \\ &= \sum_{n=0}^{M} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s} + \frac{e^{2\pi i\lambda M}}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(M+\alpha)t}}{e^{t-2\pi i\lambda}-1}dt \\ &= \sum_{n=0}^{M} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s} + \frac{e^{2\pi i\lambda M}\Gamma(1-s)}{2\pi i e^{\pi i s}} \int_C \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}dz\,, \end{aligned} \tag{2.1}$$

where C is the contour integral path that comes from $+\infty$ to ε along the real axis, then goes along the circle of radius ε counter clockwise, and finally goes from ε to $+\infty$.



Let $t > 0$ and $x \leq y$, so that $1 \leq x \leq \sqrt{t/2\pi}$. Let $\sigma \leq 1, M = [x], N = [y], \eta = 2\pi y$. We deform the contour integral path C to the combination of the straight lines C_1, C_2, C_3, C_4 joining ∞, $c\eta + i(1+c)\eta + 2\pi i\lambda$, $-c\eta + i(1-c)\eta + 2\pi i\lambda$, $-c\eta - \pi i(2l+1) + 2\pi i\lambda$, ∞, where c is an absolute constant, $0 < c \leq 1/2$.



We calculate the residue of integrand of (2.1). Since

$$\begin{aligned}
&\lim_{z\to 2\pi i(\lambda+n)}\{z-2\pi i(\lambda+n)\}\cdot\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}\\
&\qquad=\lim_{z\to 2\pi i(\lambda+n)}\left(\frac{e^{z-2\pi i\lambda}-1}{z-2\pi i(\lambda+n)}\right)^{-1}e^{-(M+\alpha)z}\cdot z^{s-1}\\
&\qquad=e^{-2\pi i(M+\alpha)(\lambda+n)}(2\pi i(n+\lambda))^{s-1},
\end{aligned}$$

we have

$$\begin{aligned}
&\operatorname*{Res}_{z=2\pi i(\lambda+n)}\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}\\
&\quad=e^{-2\pi i(M+\alpha)(\lambda+n)}(2\pi(n+\lambda)i)^{s-1}\\
&\quad=\begin{cases}e^{-2\pi i(M+\alpha)(\lambda+n)}(2\pi(n+\lambda)e^{\pi i/2})^{s-1} & (n\geq 0)\\ e^{2\pi i(M+\alpha)(|n|-\lambda)}(2\pi(|n|-\lambda)e^{3\pi i/2})^{s-1} & (n\leq -1)\end{cases}\\
&\quad=\begin{cases}-\dfrac{e^{\pi is}}{(2\pi)^{1-s}}\cdot e^{\{(1-s)/2-2(M+\alpha)\lambda\}\pi i}\cdot\dfrac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1-s}} & (n\geq 0)\\ -\dfrac{e^{\pi is}}{(2\pi)^{1-s}}\cdot e^{-\{(1-s)/2+2(M+\alpha)(1-\lambda)\}\pi i}\cdot\dfrac{e^{2\pi i(-n)\alpha}}{(|n|-\lambda)^{1-s}} & (n\leq -1)\end{cases}
\end{aligned}$$

and we have

$$\sum_{n=-N+1}^{N} \operatorname*{Res}_{z=2\pi in} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}$$

$$= -\frac{e^{\pi is}}{(2\pi)^{s-1}}\left\{e^{\{(1-s)/2-2(M+\alpha)\lambda\}\pi i}\sum_{n=0}^{N}\frac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1-s}} + e^{-\{(1-s)/2+2(M+\alpha)(1-\lambda)\}\pi i}\sum_{n=0}^{N}\frac{e^{2\pi in\alpha}}{(n+1-\lambda)^{1-s}}\right\}.$$

Therefore we obtain

$$\begin{aligned}\zeta_L(s,\alpha,\lambda) &= \sum_{n=0}^{M}\frac{e^{2\pi in\lambda}}{(n+\alpha)^s} \\ &\quad+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\sum_{n=0}^{N}\frac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1-s}} + e^{-\{(1-s)/2+2\alpha(1-\lambda)\}\pi i}\sum_{n=0}^{N}\frac{e^{2\pi in\alpha}}{(n+1-\lambda)^{1-s}}\right\} \\ &\quad+\frac{e^{2\pi i\lambda M}\Gamma(1-s)}{2\pi ie^{\pi is}}\left(\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right)\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}dz\,. \quad (2.2)\end{aligned}$$

From here, we consider the order of integral terms on right-hand side of (2.2).

First, we consider the integral path C_4. Let $z=u+iv=re^{i\theta}$ then $|z^{s-1}|=r^{\sigma-1}$, and since $\theta \geq 5\pi/4,\ r \gg \eta,\ |e^{z-2\pi i\lambda}-1| \gg 1$, we have

$$\begin{aligned}\int_{C_4}\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}dz &= \int_{C_4}\frac{(re^{i\theta})^{\sigma+it-1}e^{-(M+\alpha)(u+iv)}}{e^{z-2\pi i\lambda}-1}dz \\ &\ll \eta^{\sigma-1}e^{-5\pi t/4}\int_{c\eta}^{\infty}e^{-(M+\alpha)u}du \\ &= \eta^{\sigma-1}(M+\alpha)^{-1}e^{(M+\alpha)c\eta-5\pi t/4} \\ &\ll e^{(c-5\pi/4)t}. \quad (2.3)\end{aligned}$$

Secondly, we consider the order of integral on C_3 of (2.2). Noting

$$\arctan\varphi = \int_0^{\varphi}\frac{d\mu}{1+\mu^2} > \int_0^{\varphi}\frac{d\mu}{(1+\mu)^2} = \frac{\varphi}{1+\varphi}$$

for $\varphi > 0$, we can write

$$\theta = \arg z = \frac{\pi}{2} + \arctan\frac{c}{1-c} = \frac{\pi}{2} + c + A(c)$$

on C_3, where $A(c)$ is a constant depending on c. Then we have

$$\begin{aligned}|z^{s-1}e^{-(M+\alpha)z}| &= r^{\sigma}e^{-t\theta+(M+\alpha)c\eta} \\ &\ll \eta^{\sigma-1}e^{-(\pi/2+c+A(c))t+(M+\alpha)\eta}\end{aligned}$$

$$\ll \eta^{\sigma-1} e^{-(\pi/2+A(c)} t.$$

Therefore, since $|e^{z-2\pi i\lambda}-1| \gg 1$, we have

$$\int_{C_3} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} dz \ll \eta^{\sigma} e^{-(\pi/2+A(c))t}\,. \tag{2.4}$$

Thirdly, since $|e^{z-2\pi i\lambda}-1| \gg e^{u}$ on C_1, we have

$$\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \ll \eta^{\sigma-1} \exp\left(-t \arctan \frac{(1+c)\eta+2\pi\lambda}{u} - (M+\alpha+1)u\right).$$

Since $M+\alpha+1 \geq x = t/\eta$, the term $(M+\alpha+1)u$ on the right-hand side of the above may be replaced by tu/η. Also, since

$$\frac{d}{du}\left(\arctan \frac{(1+c)\eta+2\pi\lambda}{u} + \frac{u}{\eta}\right) = -\frac{(1+c)\eta+2\pi\lambda}{u^2+((1+c)\eta+2\pi\lambda)^2} + \frac{1}{\eta} > 0$$

and

$$\arctan\varphi = \int_0^{\varphi} \frac{d\mu}{1+\mu^2} < \int_0^{\varphi} d\mu = \varphi\,.$$

for $0 < \varphi < \pi/2$, we have

$$\begin{aligned}\arctan \frac{(1+c)\eta+2\pi i}{u} + \frac{u}{\eta} &\geq \arctan\left(\frac{1+c}{c} + \frac{2\pi\lambda}{\eta}\right) + c \\ &= \frac{\pi}{2} - \arctan \frac{c}{1+c+2\pi c\lambda/\eta} + c > \frac{\pi}{2} + B(c)\end{aligned}$$

in $u \geq c\eta$, where $B(c) = (\eta + c\eta + 2\pi\lambda c)C/\{\eta + (2\pi\lambda+\eta)c\}$. Then we have

$$\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \ll \eta^{\sigma-1} \exp\left(-\left(\frac{\pi}{2}+B(c)\right)t\right).$$

Since

$$\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \ll \begin{cases} \eta^{\sigma-1} \exp\left(-\left(\frac{\pi}{2}+B(c)\right)t\right) & (c\eta \leq u \leq \pi\eta)\,, \\ \eta^{\sigma-1} \exp(-xu) & (u \geq \pi\eta)\,, \end{cases}$$

we obtain

$$\begin{aligned}\int_{C_1} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} &\ll \eta^{\sigma-1}\left\{\int_{c\eta}^{\pi\eta} e^{-(\pi/2+B(c))t} du + \int_{\pi\eta}^{\infty} e^{-xu} du\right\} \\ &\ll \eta^{\sigma} e^{-(\pi/2+B(c))t} + \eta^{\sigma-1} e^{-\pi\eta x} \\ &\ll \eta^{\sigma} e^{-(\pi/2+B(c))t}.\end{aligned} \tag{2.5}$$

Finally, we describe the evaluation of the integral on C_2. Rewriting $z = i(\eta+2\pi\lambda) + \xi e^{\pi i/4}$ (where $\xi \in \mathbb{R}$ and $|\xi| \leq \sqrt{2}c\eta$), we have

$$\begin{aligned} z^{s-1} &= \exp\left\{(s-1)\left(\log\left(i(\eta+2\pi\lambda)+\xi e^{\pi i/4}\right)\right)\right\} \\ &= \exp\left\{(s-1)\left(\frac{\pi i}{2} + \log\left(\eta+2\pi\lambda+\xi e^{-\pi i/4}\right)\right)\right\} \\ &= \exp\left\{(s-1)\left(\frac{\pi i}{2} + \log(\eta+2\pi\lambda) + \frac{\xi}{\eta+2\pi\lambda} e^{-\pi i/4}\right.\right.\end{aligned}$$

$$-\frac{\xi^2}{2(\eta+2\pi\lambda)^2}e^{-\pi i/2}+O\left(\frac{\xi^3}{\eta^3}\right)\bigg)\bigg\}$$

$$\ll(\eta+2\pi\lambda)^{\sigma-1}\exp\left\{\left(-\frac{\pi}{2}+\frac{\xi}{\sqrt{2}(\eta+2\pi\lambda)}-\frac{\xi^2}{2(\eta+2\pi\lambda)^2}+O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}$$

as $\eta\to\infty$. Also, since

$$\frac{e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}=\frac{e^{-(M+\alpha-x)z}}{e^{z-2\pi i\lambda}-1}\cdot e^{-xz}$$

and

$$\frac{e^{-(M+\alpha-x)z}}{e^{z-2\pi i\lambda}-1}\ll\begin{cases}e^{(x-M-\alpha-1)u} & \left(u>\dfrac{\pi}{2}\right)\\ e^{(x-M-\alpha)u} & \left(u<-\dfrac{\pi}{2}\right)\end{cases},$$

we have

$$\frac{e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}\ll|e^{-xz}|=e^{-\xi t/\sqrt{2}\eta}\quad\left(|u|>\frac{\pi}{2}\right).$$

Hence

$$\begin{aligned}
&\int_{C_2\cap\{z\,|\,|u|>\pi/2\}}\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}dz\\
&\ll\int_{C_2\cap\{z\,|\,|u|>\pi/2\}}(\eta+2\pi\lambda)^{\sigma-1}\\
&\quad\times\exp\left\{\left(-\frac{\pi}{2}+\frac{\xi}{\sqrt{2}(\eta+2\pi\lambda)}-\frac{\xi^2}{2(\eta+2\pi\lambda)^2}+O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}\exp\left(-\frac{\xi t}{\sqrt{2}\eta}\right)d\xi\\
&\ll\int_{-\sqrt{2}c\eta}^{\sqrt{2}c\eta}(\eta+2\pi\lambda)^{\sigma-1}e^{-\pi t/2}\exp\left\{\left(-\frac{\xi^2}{2(\eta+2\pi\lambda)^2}+O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}d\xi\\
&\ll\int_{-\infty}^{\infty}(\eta+2\pi\lambda)^{\sigma-1}e^{-\pi t/2}\exp\left\{\left(-\frac{\xi^2}{2(\eta+2\pi\lambda)^2}+O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}d\xi\\
&\ll\eta^{\sigma-1}e^{-\pi t/2}\int_{-\infty}^{\infty}\exp\left\{-\frac{D(c)\xi^2t}{\eta^2}\right\}d\xi\\
&\ll\eta^{\sigma}t^{-1/2}e^{\pi t/2},
\end{aligned}\tag{2.6}$$

where $D(c)$ is a constant depending on c. The argument can also be applied to the part $|u|\le\pi/2$ if $|e^{z-2\pi i\lambda}|>A$. If not, that is the case when the contour goes too near to the pole at $z=2\pi iN+2\pi i\lambda$, we take an arc of the circle $|z-2\pi iN-2\pi i\lambda|=\pi/2$. On this arc we can write to $z=2\pi iN+2\pi i\lambda+(\pi/2)e^{i\beta}$, and

$$\begin{aligned}
\log(z^{s-1})&=(s-1)\log\left(2\pi iN+2\pi i\lambda+\frac{\pi}{2}e^{i\beta}\right)\\
&=(s-1)\log e^{\pi i/2}\left(2\pi N+2\pi\lambda+\frac{\pi}{2}\cdot\frac{e^{i\beta}}{i}\right)\\
&=(s-1)\left\{\frac{\pi i}{2}+\log(2\pi(N+\lambda))+\log\left(1+\frac{e^{i\beta}}{4(N+\lambda)i}\right)\right\}
\end{aligned}$$

$$= -\frac{\pi t}{2} + (s-1)\log(2\pi(N+\lambda)) + \frac{te^{i\beta}}{4(N+\lambda)} + O(1)\,.$$

On the last line of the above calculations, we used $N^2 \gg t$ which follows from the assumption $x \leq y$. Then

$$z^{s-1}e^{-(M+\alpha)z}$$
$$= \exp\left(-\frac{\pi t}{2} + (s-1)\log(2\pi(N+\lambda)) + \frac{te^{i\beta}}{4(N+\lambda)} - \frac{\pi}{2}(M+\alpha)e^{i\beta} + O(1)\right),$$

and since

$$\frac{te^{i\beta}}{4(N+\lambda)} - \frac{\pi}{2}(M+\alpha)e^{i\beta} = \frac{2\pi xy - 2\pi([x]+\alpha)([y]+\lambda)}{4(N+\lambda)}e^{i\beta} = O(1)$$

we have

$$z^{s-1}e^{-(M+\alpha)z} \ll \exp\left(-\frac{\pi t}{2} + (s-1)\log(2\pi(N+\lambda)) + O(1)\right)$$
$$\ll N^{\sigma-1}e^{-\pi t/2}\,.$$

Hence, the integral on the small semicircle can be evaluated as $O(\eta^{\sigma-1}e^{-\pi t/2})$. Therefore together with (2.6), we have

$$\int_{C_2} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}dz \ll \eta^{\sigma}t^{-1/2}e^{-\pi t/2} + \eta^{\sigma-1}e^{-\pi t/2}. \tag{2.7}$$

Now, evaluation of all the integrals was done. Using the results (2.3), (2.4), (2.5), (2.7) and $e^{2\pi i(\lambda N - s/2)}\Gamma(1-s) \ll t^{1/2-\sigma}e^{\pi t/2}$, we see that the integral term of (2.2) is

$$\ll t^{1/2-\sigma}e^{\pi t/2}\{\eta^{\sigma}e^{-(\pi/2+B(c))t} + \eta^{\sigma}t^{-1/2}e^{-\pi t/2} + \eta^{\sigma-1}e^{\pi t/2}$$
$$+\eta^{\sigma}e^{-(\pi/2+A(c))t} + e^{(c-5\pi/4)t}\}$$
$$\ll t^{1/2}\left(\frac{\eta}{t}\right)^{\sigma}e^{-(A(c)+B(c))t} + \left(\frac{\eta}{t}\right)^{\sigma} + t^{-1/2}\left(\frac{\eta}{t}\right)^{\sigma-1} + t^{1/2-\sigma}e^{(c-3\pi/4)t}$$
$$\ll e^{-\delta t} + x^{-\sigma} + t^{-1/2}x^{1-\sigma} \ll x^{-\sigma}\,,$$

where δ is a small positive real number. Therefore we have

$$\zeta_L(s,\alpha,\lambda) = \sum_{0\leq n\leq x}\frac{e^{2\pi in\lambda}}{(n+\alpha)^s}$$
$$+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\sum_{0\leq n\leq y}\frac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1-s}}\right.$$
$$\left.+e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\sum_{0\leq n\leq y}\frac{e^{2\pi in\alpha}}{(n+1-\lambda)^{1-s}}\right\}$$
$$+O(x^{-\sigma})\,, \tag{2.8}$$

that is, Theorem 2 in the case of $x \leq y$ has been proved.

To prove Theorem 2 in the case $x \geq y$, we use the following functional equation of the Lerch zeta-function;

$$\zeta_L(s,\alpha,\lambda) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\zeta_L(1-s,\lambda,1-\alpha) + e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\zeta_L(1-s,1-\lambda,\alpha)\}\,. \tag{2.9}$$

Applying (2.8) to $\zeta_L(1-s,\lambda,1-\alpha)$ and $\zeta_L(1-s,1-\lambda,\alpha)$, and substitute these into (2.9), we have

$$\begin{aligned}
&\zeta_L(s,\alpha,\lambda)\\
&= \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Bigg[e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\Bigg\{\sum_{0\leq n\leq x}\frac{e^{2\pi i n\lambda}}{(n+\lambda)^{1-s}}\\
&\quad+\frac{\Gamma(s)}{(2\pi)^s}\Bigg(e^{\{s/2-2\lambda(1-\lambda)\}\pi i}\sum_{0\leq n\leq y}\frac{e^{2\pi i n(1-\lambda)}}{(n+1-\alpha)^s}+e^{\{-s/2+2\alpha\lambda\}\pi i}\sum_{0\leq n\leq y}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}\Bigg)\Bigg\}\\
&\quad+e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\Bigg\{\sum_{0\leq n\leq x}\frac{e^{2\pi i n\alpha}}{(n+1-\lambda)^{1-s}}\\
&\quad+\frac{\Gamma(s)}{(2\pi)^s}\Bigg(e^{\{(s/2-2(1-\lambda)\alpha\}\pi i}\sum_{0\leq n\leq y}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}\\
&\quad+e^{\{-s/2+2(1-\lambda)(1-\alpha)\}\pi i}\sum_{0\leq n\leq y}\frac{e^{2\pi i n(1-\lambda)}}{(n+\alpha)^s}\Bigg)\Bigg\}\Bigg]\\
&\quad+O(\Gamma(\sigma-1)(2\pi)^{-\sigma}(e^{\pi t/2}+e^{-\pi t/2})x^{\sigma-1})\\
&=\sum_{0\leq n\leq y}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Bigg\{e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\sum_{0\leq n\leq x}\frac{e^{2\pi i n(1-\alpha)}}{(n+\lambda)^{1-s}}\\
&\quad+e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\sum_{0\leq n\leq x}\frac{e^{2\pi i n\alpha}}{(n+1-\lambda)^{1-s}}\Bigg\}\\
&\quad+O(t^{1/2-\sigma}x^{\sigma-1})\,.
\end{aligned}$$

Interchanging x and y, we obtain the theorem with $x \geq y$. Combining this equation with (2.8), we obtain the proof of (1.6).

The proof of (1.7) is similar. However, the four integral path C_1, C_2, C_3 and C_4 are different from the proof of (1.6), that is, as follows; The straight lines C_1, C_2, C_3, C_4 joining ∞, $c\eta + i\eta(1+c)$, $-c\eta + i\eta(1-c)$, $-c\eta - (2L+1)\pi i$, ∞, where c is an absolute constant, $0 < c \leq 1/2$. Also, in the proof for the case $x \geq y$, we use the functional equation

$$\zeta_H(s,\alpha) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{e^{(1-s)\pi i/2}\zeta_L(1-s,1,1-\alpha) + e^{-(1-s)\pi i/2}\zeta_L(1-s,1,\alpha)\}\,,$$

but this equation is not included in the functional equation (2.9). Noticing these points, we can prove (1.7) by a similar method. This completes the proof of Theorem 2. □

3. Proof of Theorem 3

In this section, using Theorem 2, we give the proof of Theorem 3.

PROOF OF THEOREM 3. Let

$$x = \frac{t}{2\pi\sqrt{\log t}}, \quad y = \sqrt{\log t}$$

and we assume $t > 0$ satisfies $x \geq 1$ and $y \geq 1$. Use the Stirling formula

$$\Gamma(1-s)e^{\{(1-s)/2-2\alpha\lambda\}\pi i} \ll 1, \ \Gamma(1-s)e^{\{-(1-s)/2-2\alpha(1-\lambda)\}\pi i} \ll 1.$$

Then if $0 < \lambda < 1$, using (1.6) we have

$$\begin{aligned}\zeta_L\left(\frac{1}{2}+it,\alpha,\lambda\right) &= \sum_{0\leq n\leq x}\frac{e^{2\pi in\lambda}}{(n+\alpha)^{1/2+it}} \\ &+O\left(\sum_{0\leq n\leq y}\frac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1/2-it}}+\sum_{0\leq n\leq y}\frac{e^{2\pi in\alpha}}{(n+1-\lambda)^{1/2-it}}\right) \\ &+O(t^{-1/2}(\log t)^{1/4})+O((\log t)^{-1/4}), \qquad (3.1)\end{aligned}$$

and if $\lambda = 1$, using (1.7) we have

$$\begin{aligned}\zeta_H\left(\frac{1}{2}+it,\alpha\right) &= \sum_{0\leq n\leq x}\frac{1}{(n+\alpha)^{1/2+it}}+O\left(\sum_{1\leq n\leq y}\frac{e^{2\pi in(1-\alpha)}}{n^{1/2-it}}+\sum_{1\leq n\leq y}\frac{e^{2\pi in\alpha}}{n^{1/2-it}}\right) \\ &+O(t^{-1/2}(\log t)^{1/4})+O((\log t)^{-1/4}). \qquad (3.2)\end{aligned}$$

(i) In the case $0 < \lambda < 1$ and $0 < \alpha < 1$, since

$$\sum_{n=0}^{\infty}\frac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1/2}}, \quad \sum_{n=0}^{\infty}\frac{e^{2\pi in\alpha}}{(n+1-\lambda)^{1/2}}$$

are convergent, and $t^{-1/2}(\log t)^{1/4} = o(1)$, $(\log t)^{-1/4} = o(1)$, we have

$$\zeta_L\left(\frac{1}{2}+it,\alpha,\lambda\right) = \sum_{0\leq n\leq x}\frac{e^{2\pi in\lambda}}{(n+\alpha)^{1/2+it}}+O(1).$$

(ii) In the case $0 < \lambda < 1$ and $\alpha = 1$, the second term on right-hand side of (1.6) is

$$\ll \int_0^y \frac{1}{(u+\lambda)^{1/2}}du = O(\sqrt{y}) = O((\log t)^{1/4}),$$

so we have

$$\zeta_L\left(\frac{1}{2}+it,1,\lambda\right) = \sum_{1\leq n\leq x}\frac{e^{2\pi in\lambda}}{n^{1/2+it}}+O((\log t)^{1/4}).$$

(iii) In the case $\lambda = 1$ and $0 < \alpha < 1$, consider similarly as in the case of (i) to obtain

$$\zeta_L\left(\frac{1}{2} + it, \alpha, 1\right) = \sum_{0 \le n \le x} \frac{1}{(n+\alpha)^{1/2+it}} + O(1)\,.$$

(iv) In the case $\lambda = 1$ and $\alpha = 1$, since $\zeta_L(s, 1, 1) = \zeta(s)$ we obtain

$$\zeta_L\left(\frac{1}{2} + it, 1, 1\right) = \sum_{1 \le n \le x} \frac{1}{n^{1/2+it}} + O((\log t)^{1/4})$$

(see Chap. VII in [5]).

Let

$$\Sigma(\alpha, \lambda) = \sum_{0 \le n \le x} \frac{e^{2\pi i n \lambda}}{(n+\alpha)^{1/2+it}}\,,$$

and calculate as

$$\begin{aligned} |\Sigma(\alpha, \lambda)|^2 &= \sum\sum_{0 \le m, n \le x} \frac{e^{2\pi i (m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left(\frac{n+\alpha}{m+\alpha}\right)^{it} \\ &= \sum_{0 \le n \le x} \frac{1}{n+\alpha} + \sum\sum_{\substack{0 \le m, n \le x \\ m \ne n}} \frac{e^{2\pi i (m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left(\frac{n+\alpha}{m+\alpha}\right)^{it}\,. \end{aligned}$$

Also $T_1 = T_1(m, n)$ is a function in m, n satisfying

$$\max\{m, n\} = \frac{T_1}{2\pi\sqrt{\log T_1}}\,.$$

Let $X = T/2\pi\sqrt{\log T}$, then

$$\begin{aligned} \int_1^T |\Sigma(\alpha, \lambda)|^2 dt &= \sum_{0 \le n \le X} \frac{1}{n+\alpha}\{T - T_1(n, n)\} \\ &\quad + O\left(\sum\sum_{0 \le m < n \le X} \frac{e^{2\pi i (m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left(\log \frac{n+\alpha}{m+\alpha}\right)^{-1}\right)\,. \end{aligned} \tag{3.3}$$

Here, since

$$n\sqrt{\log n} = \frac{T_1}{2\pi\sqrt{\log T_1}} \left(\log \frac{T_1}{2\pi\sqrt{\log T_1}}\right)^{1/2} \sim \frac{1}{2\pi} T_1(n, n)$$

and

$$\sum\sum_{0 \le m < n \le X} \frac{e^{2\pi i (m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left(\log \frac{n+\alpha}{m+\alpha}\right)^{-1} \ll X \log X \ll T(\log T)^{1/2}$$

(see Lemma 3 in [2] or Lemma 2.6 in [4]), (3.3) can be rewritten as

$$\int_1^T |\Sigma(\alpha, \lambda)|^2 dt = T \log \frac{T}{2\pi} + O(T(\log T)^{1/2})\,. \tag{3.4}$$

Therefore from (i), (ii), (iii), (iv) and (3.4) , and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_1^T \left|\zeta_L\left(\frac{1}{2}+it,\alpha,\lambda\right)\right|^2 dt \\
&\quad = \int_1^T |\Sigma(\alpha,\lambda)|^2 dt + \begin{cases} O(T^{1/2}(\log T)^{1/4})) + O(T) & (0<\alpha<1)\,, \\ O(T(\log T)^{3/4}) + O(T(\log T)^{1/2}) & (\alpha=1) \end{cases} \\
&\quad = T\log\frac{T}{2\pi} + \begin{cases} O(T(\log T)^{1/2}) & (0<\alpha<1)\,, \\ O(T(\log T)^{3/4}) & (\alpha=1)\,. \end{cases}
\end{aligned}$$

Thus we obtain the proof of Theorem 3. □

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