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Poly-Bernoulli Numbers with One Parameter and Their Generating Functions

by

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Abstract. Poly-Bernoulli numbers with one parameter are introduced by using a generalization of multi-polylogarithm functions. These numbers interpolate poly-Bernoulli numbers and polycosecant numbers. We prove a functional equation of the ordinary generating function of them, and in the negative index case, we give an explicit representation of the exponential generating function and a symmetric formula. We also consider an analogue of the Arakawa-Kaneko zeta function related to poly-Bernoulli numbers and multiple T-values with one parameter.

1. Introduction

Bernoulli numbers B_n ($n \geq 0$) are rational numbers defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Two kinds of poly-Bernoulli numbers, which are generalizations of B_n, are defined as follows:

$$\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_n^{(\mathbf{k})} \frac{t^n}{n!} \quad \text{and} \quad \frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(\mathbf{k})} \frac{t^n}{n!},$$

where $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ is a multi-index and $\mathrm{Li}_{\mathbf{k}}(z)$ is the multi-polylogarithm function defined by

$$\mathrm{Li}_{\mathbf{k}}(z) := \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \in \mathbb{Z}}} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \in \mathbb{Q}[[z]].$$

These numbers for $r = 1$ were first introduced by Kaneko [7] and Arakawa-Kaneko [1] (see e.g. [5][6] for general $r \geq 1$). Since $\mathrm{Li}_1(z) = -\log(1-z)$, we have $C_n^{(1)} = B_n$ and $B_n^{(1)} = (-1)^n B_n$ for $n \geq 0$.

As a level two analogue of $C_n^{(\mathbf{k})}$, Kaneko-Pallewatta-Tsumura (the earlier version of [9], also see [16]) introduced polycosecant numbers $D_n^{(\mathbf{k})}$ as follows:

$$\frac{\mathrm{A}(\mathbf{k};\tanh\frac{t}{2})}{\sinh t}=\sum_{n=0}^{\infty}D_n^{(\mathbf{k})}\frac{t^n}{n!}\,,$$

where

$$\mathrm{A}(\mathbf{k};z):=2^r\sum_{\substack{0<m_1<\cdots<m_r\\ m_i\equiv i \pmod 2\\ 1\le i\le r}}\frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}\in\mathbb{Q}[[z]]\,. \tag{1}$$

When $\mathbf{k}=(1)$, the numbers $D_n^{(1)}=D_n$ are called cosecant numbers (cf. [14]).

For an index $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}^r$ $(r\ge 1)$, r is called the depth of $\boldsymbol{k}$. An index $\mathbf{k}=(k_1,\dots,k_r)$ is said to be admissible if $k_1,\dots,k_{r-1}\ge 1$ and $k_r\ge 2$. If $\mathbf{k}=(k_1,\dots,k_r)$ is admissible, then $\mathrm{Li}_{\mathbf{k}}(z)$ and $\mathrm{A}(\mathbf{k};z)$ are convergent at $z=1$. The values $\zeta(\mathbf{k}):=\mathrm{Li}_{\mathbf{k}}(1)$ and $T(\mathbf{k}):=\mathrm{A}(\mathbf{k};1)$ are called multiple zeta values (MZVs) and multiple T-values (MTVs), respectively. MTVs were recently introduced by Kaneko-Tsumura [11] and have some interesting similarities to MZVs (cf. [11] [12] [18]). We remark that the concept of MTVs was essentially first given by Sasaki (see [17, Definition 4]).

For a parameter $c<1$ and an admissible index $\mathbf{k}=(k_1,\dots,k_r)$, Chapoton [2] introduced MTVs with one parameter by iterated integrals. Let

$$\omega_0(t):=\frac{dt}{t},\quad \omega_1(t):=\frac{dt}{1-t}-\frac{cdt}{1-ct}$$

$$\text{and}\quad I(\varepsilon_1,\dots,\varepsilon_k):=\int\cdots\int_{0<t_1<\cdots<t_k<1}\omega_{\varepsilon_1}(t_1)\cdots\omega_{\varepsilon_k}(t_k)\,,$$

where each $\varepsilon_i=0$ or 1. For an admissible index $\mathbf{k}=(k_1,\dots,k_r)$, MTVs with one parameter are defined as

$$Z_c(\mathbf{k}):=I(1,\{0\}^{k_1-1},\dots,1,\{0\}^{k_r-1}).$$

When $c=0$ it coincides with an integral representation of MZVs, i.e., $Z_0(\mathbf{k})=\zeta(\mathbf{k})$. When $c=-1$, we have $Z_{-1}(\mathbf{k})=T(\mathbf{k})$ because $\omega_1(t)$ becomes $2\,dt/(1-t^2)$. Therefore $Z_c(\mathbf{k})$ can be considered as an interpolation of $\zeta(\mathbf{k})$ and $T(\mathbf{k})$.

An admissible index $\mathbf{k}$ can be written in the form

$$\mathbf{k}=(\underbrace{1,\dots,1}_{a_1-1},b_1+1,\underbrace{1,\dots,1}_{a_2-1},b_2+1,\dots,\underbrace{1,\dots,1}_{a_m-1},b_m+1)$$

for some $a_i,b_i\in\mathbb{Z}_{>0}$ $(1\le i\le m)$. Then the dual index $\mathbf{k}^\dagger$ of $\mathbf{k}$ is defined as

$$\mathbf{k}^\dagger=(\underbrace{1,\dots,1}_{b_m-1},a_m+1,\underbrace{1,\dots,1}_{b_{m-1}-1},a_{m-1}+1,\dots,\underbrace{1,\dots,1}_{b_1-1},a_1+1)\,.$$

Chapoton proved a duality formula $Z_c(\mathbf{k})=Z_c(\mathbf{k}^\dagger)$, which is a natural generalization of the classical duality formulas for MZVs and MTVs. In [2], some numerical observations of the graded dimensions of the $\mathbb{Q}$-vector spaces spanned by MTVs with one parameter were also reported.

For an admissible index $\mathbf{k}$ and $-1 \le c < 1$, it is easily checked that $Z_c(\mathbf{k})$ has the following series representation:

$$Z_c(\mathbf{k}) = \sum_{m_1,\ldots,m_r \ge 1} \frac{(1-c^{m_1})\cdots(1-c^{m_r})}{m_1^{k_1}(m_1+m_2)^{k_2}\cdots(m_1+\cdots+m_r)^{k_r}}\,.$$

REMARK 1.1. *Yamamoto [20] indicated that a duality formula proved in [4] is equivalent to that for the following generalized polylogarithms:*

$$\tilde{\mathrm{Li}}(\tilde{\mathbf{k}}; z) := \sum_{0=m_0<m_1<\cdots<m_r} \prod_{i=1}^{r} \frac{\mu_i + (-1)^{\mu_i} z^{m_i - m_{i-1}}}{m_i^{k_i}} \quad (|z|<1)\,,$$

where $\tilde{\mathbf{k}} = ((k_1,\mu_1),\ldots,(k_r,\mu_r)) \in (\mathbb{Z}_{>0}\times\{0,1\})^r$ *with* $(k_r,\mu_r) \ne (1,1)$. *For an admissible index* $\mathbf{k} = (k_1,\ldots,k_r)$, *the function* $Z_c(\mathbf{k})$ *can be expressed as* $\tilde{\mathrm{Li}}(\tilde{\mathbf{k}}; c)$ *with* $\mu_i = 1$ $(1 \le i \le r)$ *in Yamamoto's notation.*

For a parameter $c \in \mathbb{R}\setminus\{1\}$ and an index $\mathbf{k} = (k_1,\ldots,k_r) \in \mathbb{Z}^r$ (not necessarily admissible), we define multi-polylogarithm functions with one parameter as

$$\mathrm{Li}_c(\mathbf{k}; z) := \sum_{m_1,\ldots,m_r \ge 1} \frac{(1-c^{m_1})\cdots(1-c^{m_r}) z^{m_1+\cdots+m_r}}{m_1^{k_1}(m_1+m_2)^{k_2}\cdots(m_1+\cdots+m_r)^{k_r}} \in z^r\mathbb{R}[[z]]\,. \tag{2}$$

Remark that $\mathrm{Li}_0(\mathbf{k}; z) = \mathrm{Li}_{\mathbf{k}}(z)$ and $\mathrm{Li}_{-1}(\mathbf{k}; z) = \mathrm{A}(\mathbf{k}; z)$. When $\mathbf{k}$ is an admissible index, $-1 \le c < 1$ and $z = 1$, the infinite series (2) converges and it coincides with Chapoton's $Z_c(\mathbf{k})$.

Throughout the paper, we assume that c is a fixed real number not equal to 1. For such c and $\mathbf{k} \in \mathbb{Z}^r$, we introduce poly-Bernoulli numbers $B_n^{(\mathbf{k};c)}$ with one parameter by the following generating function:

$$\left(\frac{1}{e^t-1} - \frac{c}{e^t-c}\right)\mathrm{Li}_c\left(\mathbf{k}; \frac{e^t-1}{e^t-c}\right) = \sum_{n=0}^{\infty} B_n^{(\mathbf{k};c)} \frac{t^n}{n!}\,. \tag{3}$$

One can see that $B_n^{(\mathbf{k};0)} = C_n^{(\mathbf{k})}$ and $B_n^{(\mathbf{k};-1)} = D_n^{(\mathbf{k})}$, hence the number $B_n^{(\mathbf{k};c)}$ interpolates $C_n^{(\mathbf{k})}$ and $D_n^{(\mathbf{k})}$. When $\mathbf{k} = (0)$, we have $\mathrm{Li}_c(0; z) = z/(1-z) - cz/(1-cz)$ and the left-hand side of (3) becomes 1. Hence we have

$$B_n^{(0;c)} = \begin{cases} 1 & (n = 0)\,, \\ 0 & (n \ge 1)\,, \end{cases} \tag{4}$$

for any $c \in \mathbb{R}\setminus\{1\}$. In general, as we will see in the next section, the numbers $B_n^{(\mathbf{k};c)}$ can be expressed as a polynomial in $(1+c)/(1-c)$.

This paper is organized as follows. In Section 2 we give recurrence relations of poly-Bernoulli numbers $B_n^{(\mathbf{k};c)}$ with one parameter. In Section 3 we treat the ordinary generating function of $B_n^{(\mathbf{k};c)}$. It is known that an ordinary generating function of the classical Bernoulli numbers satisfies a simple functional equation. We give a generalization of this result and prove that this functional equation determines $B_n^{(\mathbf{k};c)}$ inductively. In Section 4 we focus on $B_n^{(\mathbf{k};c)}$ for negative indices and give an explicit representation of a generating function of them. Moreover, in the case $\mathbf{k} = -k$ ($k \in \mathbb{Z}_{\geq 0}$), we give a kind of duality formula. In the last Section 5 we give short remarks on an analogue of the Arakawa-Kaneko zeta function. This function can be analytically continued to an entire function, and its values at non-positive integers are expressed by poly-Bernoulli numbers with one parameter. Moreover, its values at positive integers are expressed in terms of multiple T-values with one parameter.

2. Recurrence relations

For an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ and an integer $i \geq 1$, we set $\mathbf{k} \oplus i := (k_1, \dots, k_{r-1}, k_r + 1, \overbrace{1, \dots, 1}^{i-1})$. For any integer j, we also denote the index $(k_1, \dots, k_r, j)$ by $\mathbf{k}, j$ if there is no risk of confusion. For example, $\mathrm{Li}_c\,(\mathbf{k}, 1; z)$ means $\mathrm{Li}_c\,((3, 2, 1); z)$ for $\mathbf{k} = (3, 2)$.

In this section we give some fundamental properties of $B_n^{(\mathbf{k};c)}$. Let us start with the following proposition.

PROPOSITION 2.1. *For any index $\mathbf{k} \in \mathbb{Z}^r$ and $c \neq 0$, we have $B_n^{(\mathbf{k};1/c)} = (-1)^{r+n-1} B_n^{(\mathbf{k},c)}$ $(n \geq 0)$.*

Proof. By definition, the equation $\mathrm{Li}_{1/c}(\mathbf{k}; z) = (-1)^r \mathrm{Li}_c(\mathbf{k}; z/c)$ holds for $c \neq 0$. Hence we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(\mathbf{k};1/c)} \frac{t^n}{n!} &= \frac{(1-\frac{1}{c})e^t}{(e^t-1)(e^t-\frac{1}{c})} \mathrm{Li}_{1/c}\left(\mathbf{k};\, \frac{e^t-1}{e^t-\frac{1}{c}}\right) \\ &= \frac{-(1-c)e^t}{(e^t-1)(ce^t-1)} (-1)^r \mathrm{Li}_c\left(\mathbf{k};\, \frac{e^t-1}{ce^t-1}\right) \\ &= \frac{(-1)^{r-1}(1-c)e^{-t}}{(e^{-t}-1)(e^{-t}-c)} \mathrm{Li}_c\left(\mathbf{k};\, \frac{e^{-t}-1}{e^{-t}-c}\right) \\ &= (-1)^{r-1} \sum_{n=0}^{\infty} (-1)^n B_n^{(\mathbf{k};c)} \frac{t^n}{n!}\end{aligned}$$

and this completes the proof. □

For any index $\mathbf{k} \in \mathbb{Z}^r$ $(r \geq 1)$, we have

$$\begin{cases} \dfrac{d}{dz}\mathrm{Li}_c(\mathbf{k} \oplus 1; z) = \dfrac{1}{z}\mathrm{Li}_c(\mathbf{k}; z), \\ \dfrac{d}{dz}\mathrm{Li}_c(\mathbf{k}, 1; z) = \left(\dfrac{1}{1-z} - \dfrac{c}{1-cz}\right)\mathrm{Li}_c(\mathbf{k}; z) \end{cases}$$

by straightforward calculation. Thus we have

$$(5)\qquad \begin{cases} \dfrac{d}{dt}\mathrm{Li}_c\left(\mathbf{k}\oplus 1;\, \dfrac{e^t-1}{e^t-c}\right) = \dfrac{(1-c)e^t}{(e^t-1)(e^t-c)}\mathrm{Li}_c\left(\mathbf{k};\, \dfrac{e^t-1}{e^t-c}\right), \\ \dfrac{d}{dt}\mathrm{Li}_c\left(\mathbf{k},1;\, \dfrac{e^t-1}{e^t-c}\right) = \mathrm{Li}_c\left(\mathbf{k};\, \dfrac{e^t-1}{e^t-c}\right). \end{cases}$$

By using these equations, we get the following recurrence relations for poly-Bernoulli numbers with one parameter.

PROPOSITION 2.2. *For an index* $\mathbf{k}\in\mathbb{Z}^r$ *(*$r\geq 1$*), the following equalities hold:*

(i)

$$(6)\qquad B_n^{(\mathbf{k};c)} = \frac{1}{1-c}\sum_{i=0}^{n}\binom{n+1}{i+1}B_{n-i}^{(\mathbf{k}\oplus 1;c)}(1-c(-1)^i) \quad (n\geq 0).$$

(ii)

$$(7)\qquad B_n^{(\mathbf{k},0;c)} = \frac{1}{1-c}\sum_{i=1}^{n}\binom{n}{i}B_{n-i}^{(\mathbf{k};c)}(1+c(-1)^i) \quad (n\geq 0).$$

Proof. (i) By the first equation of (5), we have

$$\mathrm{Li}_c\left(\mathbf{k}\oplus 1;\, \frac{e^t-1}{e^t-c}\right) = \sum_{n=1}^{\infty}B_{n-1}^{(\mathbf{k};c)}\frac{t^n}{n!}.$$

Here we used the fact $\mathrm{Li}_c\left(\mathbf{k}\oplus 1;\, \dfrac{e^t-1}{e^t-c}\right)$ has no constant term as an element of $\mathbb{Q}[[t]]$.

On the other hand, by definition, we have

$$\mathrm{Li}_c\left(\mathbf{k}\oplus 1;\, \frac{e^t-1}{e^t-c}\right) = \frac{(e^t-1)(e^t-c)}{(1-c)e^t}\sum_{n=0}^{\infty}B_n^{(\mathbf{k}\oplus 1;c)}\frac{t^n}{n!}.$$

Therefore we have

$$\sum_{n=1}^{\infty}B_{n-1}^{(\mathbf{k};c)}\frac{t^n}{n!} = \frac{1}{1-c}(e^t-1+c(e^{-t}-1))\sum_{n=0}^{\infty}B_n^{(\mathbf{k}\oplus 1;c)}\frac{t^n}{n!}.$$

By comparing the coefficients of both sides, we have

$$B_{n-1}^{(\mathbf{k};c)} = \frac{1}{1-c}\sum_{i=1}^{n}\binom{n}{i}(1+c(-1)^i)B_{n-i}^{(\mathbf{k}\oplus 1;c)}.$$

By shifting n to $n+1$ and i to $i+1$, we get (6).

(ii) By definition, it holds that $\mathrm{Li}_c\,(\mathbf{k},0;z) = \left(\dfrac{z}{1-z} - \dfrac{cz}{1-cz}\right)\mathrm{Li}_c\,(\mathbf{k};z)$ and

$$\mathrm{Li}_c\left(\mathbf{k},0;\, \frac{e^t-1}{e^t-c}\right) = \left(\frac{\frac{e^t-1}{e^t-c}}{1-\frac{e^t-1}{e^t-c}} - \frac{c\frac{e^t-1}{e^t-c}}{1-c\frac{e^t-1}{e^t-c}}\right)\mathrm{Li}_c\left(\mathbf{k};\, \frac{e^t-1}{e^t-c}\right)$$

$$= \frac{1}{1-c}(e^t - 1 + c(e^{-t} - 1))\mathrm{Li}_c\left(\mathbf{k}; \frac{e^t-1}{e^t-c}\right).$$

Therefore we have

$$\sum_{n=0}^{\infty} B_n^{(\mathbf{k},0;c)} \frac{t^n}{n!} = \frac{1}{1-c}(e^t - 1 + c(e^{-t} - 1)) \sum_{n=0}^{\infty} B_n^{(\mathbf{k};c)} \frac{t^n}{n!}.$$

By comparing the coefficients of both sides, we obtain (7). □

REMARK 2.3. *By applying $c = -1$ in Proposition 2.2, we have*

$$D_n^{(\mathbf{k})} = \sum_{\substack{i=0 \\ i:\ even}}^{n} \binom{n+1}{i+1} D_{n-i}^{(\mathbf{k}\oplus 1)},$$

$$D_n^{(\mathbf{k},0)} = \sum_{\substack{i=1 \\ i:\ odd}}^{n} \binom{n}{i} D_{n-i}^{(\mathbf{k})}$$

for any index $\mathbf{k} \in \mathbb{Z}^r$. These formulas were given by Pallewatta [16] (see Prop. 3.7 and its proof).

PROPOSITION 2.4. *Let $\mathbf{k} \in \mathbb{Z}^r$ be an index and $n \geq 0$ an integer. There exists a polynomial $f_{\mathbf{k},n}(X) \in \mathbb{Q}[X]$ not depending on c such that*

$$B_n^{(\mathbf{k};c)} = f_{\mathbf{k},n}\left(\frac{1+c}{1-c}\right).$$

Moreover, the polynomial $f_{\mathbf{k},n}(X)$ is even if $n \not\equiv r \pmod 2$ and odd if $n \equiv r \pmod 2$.

Proof. By (4), the statement is true for $\mathbf{k} = (0)$. Hence we only have to prove that if the statement is true for $\mathbf{k} = (k_1, \ldots, k_r)$ then it is also true for $(k_1, \ldots, k_r \pm 1)$ and $(k_1, \ldots, k_r, 0)$.

Assume that the statement is true for $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$. By (6), we have

$$B_n^{(\mathbf{k};c)} = \sum_{\substack{0 \leq i \leq n \\ i:\ \text{even}}} \binom{n+1}{i+1} B_{n-i}^{(\mathbf{k}\oplus 1;c)} + \sum_{\substack{0 \leq i \leq n \\ i:\ \text{odd}}} \binom{n+1}{i+1} B_{n-i}^{(\mathbf{k}\oplus 1;c)} \frac{1+c}{1-c},$$

or equivalently,

$$B_n^{(\mathbf{k}\oplus 1;c)} = \frac{1}{n+1} B_n^{(\mathbf{k};c)} - \frac{1}{n+1}\left(\sum_{\substack{1 \leq i \leq n \\ i:\ \text{even}}} \binom{n+1}{i+1} B_{n-i}^{(\mathbf{k}\oplus 1;c)} + \sum_{\substack{1 \leq i \leq n \\ i:\ \text{odd}}} \binom{n+1}{i+1} B_{n-i}^{(\mathbf{k}\oplus 1;c)} \frac{1+c}{1-c}\right).$$

By replacing $\mathbf{k} \oplus 1$ with $\mathbf{k}$, the first equation proves the statement is true for $(k_1, \ldots, k_r - 1)$. From the second equation and initial values

$$B_0^{(\mathbf{k}\oplus 1,c)} = \begin{cases} 1 & (r = 1), \\ 0 & (r \geq 2), \end{cases}$$

we can prove that the statement is also true for $(k_1, \ldots, k_r + 1)$ by induction on n. Finally, by using (7), we can prove the statement is true for $(k_1, \ldots, k_r, 0)$. □

As an example, we see the case $\mathbf{k} = (1)$. By using (4) and (6), we have

$$\begin{aligned}
B_0^{(1;c)} &= 1, \\
B_1^{(1;c)} &= -\frac{1}{2}\left(\frac{1+c}{1-c}\right), \\
B_2^{(1;c)} &= \frac{1}{2}\left(\frac{1+c}{1-c}\right)^2 - \frac{1}{3}, \\
B_3^{(1;c)} &= -\frac{3}{4}\left(\frac{1+c}{1-c}\right)^3 + \frac{3}{4}\left(\frac{1+c}{1-c}\right), \\
B_4^{(1;c)} &= \frac{3}{2}\left(\frac{1+c}{1-c}\right)^4 - 2\left(\frac{1+c}{1-c}\right)^2 + \frac{7}{15}.
\end{aligned}$$

REMARK 2.5. 1. *The constant term of $f_{\mathbf{k},n}(X)$ is the polycosecant number $D_n^{(\mathbf{k})}$ $(n \geq 0)$ because of $B_n^{(\mathbf{k};-1)} = D_n^{(\mathbf{k})}$.*

2. *For an index $\mathbf{k} = (-k_1, \ldots, -k_r)$ with $k_i \geq 0$ $(0 \leq i \leq r)$, the polynomial $f_{\mathbf{k},n}(X)$ is an element of $\mathbb{Z}[X]$.*

3. The ordinary generating function

We consider the ordinary generating function of Bernoulli numbers, i.e.,

$$\beta(t) := \sum_{n=0}^{\infty} B_n t^{n+1} \in \mathbb{Q}[[t]].$$

The radius of convergence of this series is zero, so we consider these types of generating functions as a formal power series in t.

It is known that the series $\beta(t)$ satisfies a simple functional equation and the sequence $\{B_n\}_{n\geq 0}$ of Bernoulli numbers is characterized by this functional equation.

THEOREM 3.1 (e.g., Zagier [21], Chen [3, Cor. 4.6]). *$\beta(t)$ is the unique solution in $\mathbb{Q}[[t]]$ of the equation*

$$\beta\left(\frac{t}{1-t}\right) - \beta(t) = t^2. \tag{8}$$

We define the ordinary generating function of poly-Bernoulli numbers with one parameter as

$$\beta^{(\mathbf{k};c)}(t) := \sum_{n=0}^{\infty} B_n^{(\mathbf{k};c)} t^{n+1}.$$

It is clear that $\beta^{(1;0)}(t) = \beta(t)$, and by (4), we have $\beta^{(0;c)}(t) = t$. We can generalize Theorem 3.1 to a result on a generating function of poly-Bernoulli numbers with one parameter.

THEOREM 3.2. (i) *For an index* $\mathbf{k} \in \mathbb{Z}^r$ *and an integer* $i \geq 1$, *we have*

$$(1-c)t^i \beta^{(\mathbf{k};c)}(t) = \beta^{(\mathbf{k}\oplus i;c)}\left(\frac{t}{1-t}\right) - (1+c)\beta^{(\mathbf{k}\oplus i;c)}(t) + c\beta^{(\mathbf{k}\oplus i;c)}\left(\frac{t}{1+t}\right). \tag{9}$$

(ii) *All* $\beta^{(\mathbf{k};c)}(t)$ ($\mathbf{k} \in \mathbb{Z}^r$) *are characterized by functional equations* (9) *and the initial condition* $\beta^{(0;c)}(t) = t$.

REMARK 3.3. (i) *Applying* $c = 0$, $\mathbf{k} = (0)$ *and* $i = 1$ *in* (9), *we obtain the functional equation* (8).

(ii) *Applying* $c = -1$, $\mathbf{k} = (0)$ *and* $i = 1$ *in* (9), *we obtain a functional equation*

$$\delta\left(\frac{t}{1-t}\right) - \delta\left(\frac{t}{1+t}\right) = 2t^2, \tag{10}$$

where $\delta(t)$ *is the ordinary generating function of cosecant numbers:*

$$\delta(t) := \sum_{n=0}^{\infty} D_n t^{n+1}.$$

This functional equation was given by Chen [3, Theorem 4.4].

To prove Theorem 3.2, we need the following lemma.

LEMMA 3.4. *For sequences* $(p_n)_{n\geq 0}$ *and* $(q_n)_{n\geq 0}$ *(each* p_n *and* $q_n \in \mathbb{R}$*), let*

$$P(t) := \sum_{n=0}^{\infty} p_n t^{n+1}, \quad \text{and} \quad Q(t) := \sum_{n=0}^{\infty} q_n t^{n+1}.$$

These series satisfy

$$P(t) = c_1 Q\left(\frac{t}{1-\lambda_1 t}\right) + \cdots + c_r Q\left(\frac{t}{1-\lambda_r t}\right) \tag{11}$$

if and only if

$$\sum_{n=0}^{\infty} \frac{p_n}{n!} t^n = (c_1 e^{\lambda_1 t} + \cdots + c_r e^{\lambda_r t}) \sum_{n=0}^{\infty} \frac{q_n}{n!} t^n. \tag{12}$$

Proof. Since

$$\begin{aligned}
Q\left(\frac{t}{1-\lambda t}\right) &= \sum_{i=0}^{\infty} q_i \left(\frac{t}{1-\lambda t}\right)^{i+1} \\
&= \sum_{i=0}^{\infty} q_i \sum_{n=i}^{\infty} \binom{n}{i} (\lambda t)^{n+1} \lambda^{-i-1} \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} q_i \lambda^{n-i} t^{n+1},
\end{aligned}$$

the right-hand side of (11) is equal to

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}q_i(c_1\lambda_1^{n-i}+\cdots+c_r\lambda_r^{n-i})t^{n+1}\,.$$

Because the condition (12) is equivalent to

$$p_n=\sum_{i=0}^{n}\binom{n}{i}q_i(c_1\lambda_1^{n-i}+\cdots+c_r\lambda_r^{n-i})$$

for any $n\geq 0$, we obtain the desired result. □

Proof of Theorem 3.2. By the same argument of the proof of Proposition 2.2 (i), we have

$$\sum_{n=i}^{\infty}B_{n-i}^{(\mathbf{k};c)}\frac{t^n}{n!}=\frac{1}{1-c}(e^t-(1+c)+ce^{-t})\sum_{n=0}^{\infty}B_n^{(\mathbf{k}\oplus i;c)}\frac{t^n}{n!}$$

for any $i\geq 1$. By applying

$$p_n=\begin{cases}0 & (0\leq n<i)\\ B_{n-i}^{(\mathbf{k};c)} & (n\geq i)\end{cases}$$

and $q_n=B_n^{(\mathbf{k}\oplus i;c)}$ in Lemma 3.4, we get the proof of (i).

By the functional equation (9), the function $\beta^{(\mathbf{k};c)}(t)$ is determined from $\beta^{(\mathbf{k}\oplus i;c)}(t)$. In Lemma 3.4, under the condition $(c_1,\ldots,c_r)\neq(0,\ldots,0)$, the equation (12) means that a sequence $\{p_n\}_{n\geq 1}$ is determined from $\{q_n\}_{n\geq 1}$ and vice versa. Since the functional equation (9) is the form of (11), the function $\beta^{(\mathbf{k}\oplus i;c)}(t)$ is also determined from $\beta^{(\mathbf{k};c)}(t)$. Any index can be obtained from the initial index (0) by repeating procedures $\mathbf{k}\mapsto\mathbf{k}\oplus i$ or $\mathbf{k}\oplus i\mapsto\mathbf{k}$ $(i=1,2,\ldots)$. Therefore the statement (ii) follows. □

4. The case of non-positive indices

In this section we investigate poly-Bernoulli numbers of non-positive indices, that is, $B_n^{(\mathbf{k};c)}$ for $\mathbf{k}=(-k_1,\ldots,-k_r)$ with $k_1,\ldots,k_r\geq 0$.

In the case $c=0$ and $r=1$, it is known that the numbers $C_n^{(-k)}$ have the following simple generating function:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}=\frac{1}{1-e^x+e^{x-y}} \tag{13}$$

(for general $r\geq 1$, see [6, Prop. 5]). For an integer $r\geq 1$, let

$$N_c(x,y_1,\ldots,y_r):=\sum_{n,k_1,\ldots,k_r\geq 0}B_n^{((-k_1,\ldots,-k_r);c)}\frac{x^n}{n!}\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_r^{k_r}}{k_r!}\,.$$

We can prove the following theorem which is a generalization of (13).

THEOREM 4.1. *We have*

$$N_c(x, y_1, \dots, y_r) = \frac{(1-c)e^x(e^x-1)^{r-1}}{e^x-c} \prod_{i=1}^{r} \left(\frac{1}{e^{x-Y_i} - ce^{-Y_i} - e^x + 1} - \frac{c}{e^{x-Y_i} - ce^{-Y_i} - ce^x + c} \right), \tag{14}$$

where $Y_i := \sum_{m=i}^{r} y_m$ $(1 \le i \le r)$.

Proof. We have

$$\begin{aligned}
&\sum_{k_1,\dots,k_r \ge 0} \mathrm{Li}_c\left((-k_1,\dots,-k_r); \frac{e^x-1}{e^x-c}\right) \frac{y_1^{k_1}}{k_1!} \cdots \frac{y_r^{k_r}}{k_r!} \\
&= \sum_{m_1,\dots,m_r \ge 1} \sum_{k_1,\dots,k_r \ge 0} m_1^{k_1} \cdots (m_1+\cdots+m_r)^{k_r} (1-c^{m_1}) \\
&\qquad \cdots (1-c^{m_r}) \left(\frac{e^x-1}{e^x-c}\right)^{m_1+\cdots+m_r} \\
&\qquad \frac{y_1^{k_1}}{k_1!} \cdots \frac{y_r^{k_r}}{k_r!} \\
&= \sum_{m_1,\dots,m_r \ge 1} e^{m_1 y_1} \cdots e^{(m_1+\cdots+m_r)y_r} (1-c^{m_1}) \cdots (1-c^{m_r}) \left(\frac{e^x-1}{e^x-c}\right)^{m_1+\cdots+m_r} \\
&= \sum_{m_1,\dots,m_r \ge 1} \left(e^{Y_1}\frac{e^x-1}{e^x-c}\right)^{m_1} (1-c^{m_1}) \cdots \left(e^{Y_r}\frac{e^x-1}{e^x-c}\right)^{m_r} (1-c^{m_r}) \\
&= \prod_{i=1}^{r} \left(\frac{e^{Y_i}\frac{e^x-1}{e^x-c}}{1-e^{Y_i}\frac{e^x-1}{e^x-c}} - \frac{ce^{Y_i}\frac{e^x-1}{e^x-c}}{1-ce^{Y_i}\frac{e^x-1}{e^x-c}} \right) \\
&= (e^x-1)^r \prod_{i=1}^{r} \left(\frac{1}{e^{x-Y_i} - ce^{-Y_i} - e^x + 1} - \frac{c}{e^{x-Y_i} - ce^{-Y_i} - ce^x + c} \right).
\end{aligned}$$

By multiplying $(1-c)e^x/((e^x-1)(e^x-c))$ to both sides, we get (14). □

When $r = 1$, Theorem 4.1 deduces

$$\begin{aligned}
N_c(x, y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B_n^{(-k;c)} \frac{x^n}{n!}\frac{y^k}{k!} \\
&= \frac{(1-c)e^x}{e^x-c} \left(\frac{1}{e^{x-y} - ce^{-y} - e^x + 1} - \frac{c}{e^{x-y} - ce^{-y} - ce^x + c} \right).
\end{aligned} \tag{15}$$

In particular, the function $N_0(x, y) = \dfrac{1}{1 - e^x + e^{x-y}}$ gives the generating function (13) of $C_n^{(-k)}$.

In the remainder of this section we discuss the numbers $B_n^{(-k;c)}$ $(k \in \mathbb{Z}_{\geq 0})$ having depth 1. When $c = -1$, a simple symmetric formula

$$D_{2n}^{(-2k-1)} = D_{2k}^{(-2n-1)} \quad (n, k \geq 0) \tag{16}$$

is known ([9, Theorem 4]). Since $D_{2n+1}^{(-k)} = 0$ for all $n, k \geq 0$, we can state a symmetric formula in the form of

$$D_n^{(-k-1)} = D_k^{(-n-1)} \quad (n, k \geq 0,\ n + k\text{: even}) . \tag{17}$$

The following theorem states that this formula also holds for poly-Bernoulli numbers with one parameter.

THEOREM 4.2. *The following equation holds:*

$$B_n^{(-k-1;c)} = B_k^{(-n-1;c)} \quad (n, k \geq 0,\ n + k\text{: even}) .$$

Proof. First we remember $N_c(x, y)$ is a double series defined by (15). Let $g_c(x, y) := \frac{\partial}{\partial y} N_c(x, y)$ and $G(x, y) := g_c(x, y) + g_c(-x, -y)$. Because

$$G(x, y) = 2 \sum_{\substack{n,k \geq 0 \\ n+k\text{: even}}} B_n^{(-k-1,c)} \frac{x^n}{n!} \frac{y^k}{k!} ,$$

we only have to prove that $G(x, y) = G(y, x)$. By (15), we have

$$\begin{aligned} g_c(x, y) &= \frac{(1-c)e^x}{e^x - c} \left(\frac{-(-e^{x-y} + ce^{-y})}{(e^{x-y} - ce^{-y} - e^x + 1)^2} + \frac{c(-e^{x-y} + ce^{-y})}{(e^{x-y} - ce^{-y} - ce^x + c)^2} \right) \\ &= (1-c)e^x \left(\frac{e^{-y}}{(e^{x-y} - ce^{-y} - e^x + 1)^2} + \frac{-ce^{-y}}{(e^{x-y} - ce^{-y} - ce^x + c)^2} \right) \\ &= \frac{(1-c)e^{x+y}}{(e^x - c - e^{x+y} + e^y)^2} + \frac{-c(1-c)e^{x+y}}{(e^x - c - ce^{x+y} + ce^y)^2} . \end{aligned} \tag{18}$$

Let the first part of the last line of (18) be $I(x, y)$ and the second part $J(x, y)$. It is easily showed that $I(x, y) = I(y, x)$ and $J(x, y) = J(-y, -x)$. Then we have

$$\begin{aligned} G(x, y) &= I(x, y) + J(x, y) + I(-x, -y) + J(-x, -y) \\ &= I(y, x) + J(-y, -x) + I(-y, -x) + J(y, x) \\ &= g_c(y, x) + g_c(-y, -x) \\ &= G(y, x) \end{aligned}$$

and this completes the proof. □

REMARK 4.3. *In the case $c = 0$, the function $g_0(x, y)$ satisfies a simple relation $g_0(x, y) = g_0(y, x)$. From this equation, we obtain the relation*

$$C_n^{(-k-1)} = C_k^{(-n-1)} \quad (n, k \geq 0)$$

(see e.g., [8, Section 2]).

5. Arakawa-Kaneko zeta functions with one prameter

In this section we consider an analogue of the Arakawa-Kaneko zeta function. We first see a sufficient condition for convergence of the function $\mathrm{Li}_c(\mathbf{k}; z)$.

PROPOSITION 5.1. *For an index* $\mathbf{k} \in \mathbb{Z}^r$*, the function* $\mathrm{Li}_c(\mathbf{k}; z)$ *is absolutely convergent if* $|z| < \begin{cases} 1 & (|c| \leq 1), \\ 1/|c| & (|c| > 1). \end{cases}$

Proof. When $|c| \leq 1$, we have $|1 - c^m| \leq 1 + |c|^m \leq 2$ for any positive integer m. Therefore

$$|\mathrm{Li}_c(\mathbf{k}; z)| \leq \sum_{m_1,\dots,m_r \geq 1} \frac{2^r |z|^{m_1+\cdots+m_r}}{m_1^{k_1}(m_1+m_2)^{k_2}\cdots(m_1+\cdots+m_r)^{k_r}}$$

and this series converges if $|z| < 1$.

When $|c| > 1$, we have $|1 - c^m| \leq 2|c|^m$ for any positive integer m. Therefore

$$|\mathrm{Li}_c(\mathbf{k}; z)| \leq \sum_{m_1,\dots,m_r \geq 1} \frac{2^r |cz|^{m_1+\cdots+m_r}}{m_1^{k_1}(m_1+m_2)^{k_2}\cdots(m_1+\cdots+m_r)^{k_r}}$$

and this series converges if $|z| < 1/|c|$. □

Assume that $-1 \leq c < 1$. For an index $\mathbf{k} \in \mathbb{Z}^r$ and $s \in \mathbb{C}$ with $\Re(s) > 1 - r$, we define a generalization of the Arakawa-Kaneko zeta function as

$$\xi_c(\mathbf{k}; s) := \frac{1}{\Gamma(s)} \int_0^\infty z^{s-1} \frac{(1-c)e^z}{(e^z-1)(e^z-c)} \mathrm{Li}_c\left(\mathbf{k}; \frac{e^z-1}{e^z-c}\right) dz\,. \tag{19}$$

By

$$0 \leq \frac{e^z-1}{e^z-c} < 1 \quad (z \in [0, \infty))$$

and

$$\mathrm{Li}_c\left(\mathbf{k}; \frac{e^z-1}{e^z-c}\right) = O(z^r) \text{ as } z \to +0\,,$$

the integral (19) is convergent for $\Re(s) > 1 - r$. When $c = 0$, the function $\xi_0(\mathbf{k}; s)$ coincides with the original Arakawa-Kaneko zeta function $\xi(\mathbf{k}; s)$ ([1]). When $c = -1$, the function $\xi_{-1}(\mathbf{k}; s)$ coincides with a level two analogue of the Arakawa-Kaneko zeta function, denoted by $\psi(\mathbf{k}; s)$ in [11].

By the well-known method using contour integrals, the function $\xi_c(\mathbf{k}; s)$ can be continued to an entire function, and its values at non-positive integers are given by

$$\xi_c(\mathbf{k}; -m) = (-1)^m B_m^{(\mathbf{k};c)} \quad (m = 0, 1, 2, \dots)$$

(cf. [10]).

For two indices $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{>0}$ and $\mathbf{j} = (j_1, \dots, j_r) \in \mathbb{Z}^r_{\geq 0}$, we use the notation

$$b(\mathbf{k}; \mathbf{j}) := \prod_{i=1}^r \binom{k_i + j_i - 1}{j_i}.$$

Then we obtain the following theorem and the results for $c = 0$ and $c = -1$ are known [10, Theorem 2.5] [19, Theorem 4.3]. This theorem can be proved in parallel with the proof in [13] [15] and we omit its proof.

THEOREM 5.2. *For* $\mathbf{k} \in \mathbb{Z}^r_{>0}$ *and* $m \in \mathbb{Z}_{>0}$, *it holds that*

$$\xi_c(\mathbf{k}; m) = \sum_{\mathbf{j}} b\left((\mathbf{k} \oplus 1)^\dagger; \mathbf{j}\right) Z_c((\mathbf{k} \oplus 1)^\dagger + \mathbf{j}),$$

where the sum runs over all $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geq 0}$ *with* $j_1 + \cdots + j_n = m - 1$ *and* n *is the depth of* $(\mathbf{k} \oplus 1)^\dagger$. *Here* $\mathbf{k} + \mathbf{j}$ *means the componentwise sum of* $\mathbf{k}$ *and* $\mathbf{j}$.

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References

[1] T. Arakawa and M. Kaneko: Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, Nagoya Math. J. **153** (1999), 189–209.

[2] F. Chapoton: Multiple T-values with one parameter, Tsukuba J. Math. **46** (2022), 153–163.

[3] K.-W. Chen: A summation on Bernoulli numbers, J. Number Theory **111** (2005), 372–391.

[4] M. Hirose, K. Iwaki, N. Sato, K. Tasaka: Duality/sum formulas for iterated integrals and their application to multiple zeta values, J. Number Theory **195** (2019), 72–83.

[5] K. Imatomi: Multiple zeta values and multi-poly-Bernoulli numbers, Doctoral thesis, Kyushu University (2014).

[6] K. Imatomi, M. Kaneko and E. Takeda: Multi-poly-Bernoulli numbers and finite multiple zeta values, J. Integer Seq. **17** (2014), Article 14.4.5.

[7] M. Kaneko: Poly-Bernoulli numbers, J. Théor. Nombres Bordeaux **9** (1997), 199–206.

[8] M. Kaneko: Poly-Bernoulli numbers and related zeta functions, Algebraic and Analytic Aspects of Zeta Functions and L-functions, MSJ Mem. **21** (2010), 73–85.

[9] M. Kaneko, M. Pallewatta and H. Tsumura: On polycosecant numbers, J. Integer Sequences **23** (2020), Article 20.6.4 (the earlier version is arXiv:1907.13441).

[10] M. Kaneko and H. Tsumura: Multi-poly-bernoulli numbers and related zeta functions, Nagoya Math. J. **232** (2018), 19–54.

[11] M. Kaneko and H. Tsumura: Zeta functions connecting multiple zeta values and poly-Bernoulli numbers, Advanced Studies in Pure Math. **84** (2020), 181–204.

[12] M. Kaneko and H. Tsumura: On multiple zeta values of level two, Tsukuba J. Math. **44** (2020), 213–234.

[13] N. Kawasaki and Y. Ohno, Combinatorial proofs of identities for special values of Arakawa-Kaneko multiple zeta functions, Kyushu J. Math. **72** (2018), 215–222.

[14] V. Kowalenko: Applications of the cosecant and related numbers, Acta Appl. Math. **114** (2011), 15–134.

[15] Y. Ohno and H. Wayama: Interpolation between Arakawa-Kaneko and Kaneko-Tsumura multiple zeta functions, Comment. Math. Univ. St. Pauli **68** (2020), 83–91.

[16] M. Pallewatta: On polycosecant numbers and level two generalization of Arakawa-Kaneko zeta functions, Doctoral thesis, Kyushu University (2020).

[17] Y. Sasaki: On generalized poly-Bernoulli numbers and related L-functions, J. Number Theory **132** (2012), 156–170.

[18] Ce Xu: Duality of weighted sum formulas of alternating multiple T-values, Bull. Korean Math. Soc. **58** (2021), 1261–1278.

[19] Ce Xu and J. Zhao: Variants of multiple zeta values with even and odd summation indices, Math. Z. **300** (2022), 3109–3142.
[20] S. Yamamoto: Duality of one-variable multiple polylogarithms and their q-analogues, arXiv:2010.05505.
[21] D. Zagier: A modified Bernoulli number, Nieuw Arch. Wisk. (4) **16** (1998), 63–72.

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