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Interpolation between Arakawa–Kaneko and Kaneko–Tsumura Multiple Zeta Functions

by

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Abstract. t-Arakawa–Kaneko multiple zeta functions which interpolate Arakawa–Kaneko and Kaneko–Tsumura multiple zeta functions are defined, and general formulas of their values as t-functions for any integer s are given. For any positive integer s, the values are written in terms of t-multiple zeta values which was introduced by Yamamoto, and for any non-positive integer s, they are t-polynomials whose coefficients are multi-poly-Bernoulli numbers.

1. Introduction

Classically, two types of Bernoulli numbers are known. They are given by the following generating functions

$$\frac{z}{1-e^{-z}}=\sum_{n=1}^{\infty}B_n\frac{z^n}{n!}\,,\qquad \text{and}\qquad \frac{z}{e^z-1}=\sum_{n=1}^{\infty}C_n\frac{z^n}{n!}\,,$$

and thus $B_1=\frac{1}{2}$ and $C_1=-\frac{1}{2}$. Based on this difference, two types of multi-poly-Bernoulli numbers $B_n^{(\mathbf{k})}$ and $C_n^{(\mathbf{k})}$ are defined in [4] by the generating series

$$\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-z})}{1-e^{-z}}=\sum_{n=0}^{\infty}B_n^{(\mathbf{k})}\frac{z^n}{n!}\,,\qquad \text{and}\qquad \frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-z})}{e^z-1}=\sum_{n=0}^{\infty}C_n^{(\mathbf{k})}\frac{z^n}{n!}\,,$$

for any r-tuple $\mathbf{k}=(k_1,\ldots,k_r)$ of positive integers. Here, $\mathrm{Li}_{\mathbf{k}}(z)$ is the multi-polylogarithm defined by

$$\mathrm{Li}_{\mathbf{k}}(z)=\sum_{0<m_1<\cdots<m_r}\frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}},\quad (|z|<1).$$

By using $\mathrm{Li}_{\mathbf{k}}(z)$, two types of generalization of the Riemann zeta function are defined in [1, 8] as*

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*Though $\xi(\mathbf{k};s)$ is defined for $\mathrm{Re}(s)>0$ in [1], it actually converges in $\mathrm{Re}(s)>1-r$.

$$\xi(\mathbf{k};s)=\frac{1}{\Gamma(s)}\int_0^\infty z^{s-1}\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-z})}{e^z-1}\,dz \qquad (\mathrm{Re}(s)>1-r)$$

and

$$\eta(\mathbf{k};s)=\frac{1}{\Gamma(s)}\int_0^\infty z^{s-1}\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{z})}{1-e^z}\,dz \qquad (\mathrm{Re}(s)>1-r)\,.$$

Both $\xi(\mathbf{k};s)$ and $\eta(\mathbf{k};s)$ are holomorphically continued to the whole complex s-plane, and called Arakawa–Kaneko and Kaneko–Tsumura multiple zeta functions (AKZs and KTZs for short), respectively.

These functions satisfy $\xi(1;s)=\eta(1;s)=s\zeta(s+1)$, thus they are considered to be a type of natural generalizations of Riemann zeta function $\zeta(s)$. AKZs and KTZs are used to prove many linear relations among multiple zeta(-star) values (see §3 for the definition), including duality like formula ([7, Theorem 1.3]) and a kind of shuffle relations ([1, Corollary 11]), since the values of AKZs and KTZs at positive integral points are linear combinations of multiple zeta(-star) values. On the other hand, the values of AKZs and KTZs at non-positive integral points are multi-poly-Bernoulli numbers $C_n^{(\mathbf{k})}$ and $B_n^{(\mathbf{k})}$, respectively, which have many number theoretical or combinatorial properties similar to Bernoulli numbers (See [2, 4, 5, 6] for examples).

According to Landen connection formula for multi-polylogarithms ([14])

$$\mathrm{Li}_{\mathbf{k}}\left(\frac{z}{z-1}\right)=(-1)^r\sum_{\mathbf{k}\preceq\mathbf{k}'}\mathrm{Li}_{\mathbf{k}'}(z)\,, \tag{1.1}$$

an AKZ can be expressed as a linear combination of KTZs, and vice versa ([8]). Indeed, we have

$$\xi(\mathbf{k};s)=(-1)^{r-1}\sum_{\mathbf{k}\preceq\mathbf{k}'}\eta(\mathbf{k}';s) \tag{1.2}$$

and

$$\eta(\mathbf{k};s)=(-1)^{r-1}\sum_{\mathbf{k}\preceq\mathbf{k}'}\xi(\mathbf{k}';s)\,, \tag{1.3}$$

where $\mathbf{k}\preceq\mathbf{k}'$ denotes that $\mathbf{k}$ is obtaind from $\mathbf{k}'$ by replacing some commas (,) by pluses (+). For example, $(4)=(2+2)\preceq(2,2)=(2,1+1)\preceq(2,1,1)$, etc. The above story is in the direction of generalizing the Riemann zeta function by using multi-polylogarithms.

In the other direction, a t-interpolation between the multiple zeta and zeta-star values is known as the t-multiple zeta values defined by Yamamoto ([15]), who gave t-interpolated version of the sum and the cyclic sum formulas ([3, 12]). Yamamoto also used the t-multiple zeta values to give a well-proportioned expression of the two-one formula ([13, 18]).

Based on these situations, for any index $\mathbf{k}=(k_1,k_2,\ldots,k_r)\in(\mathbb{Z}_{>0})^r$, we introduce a function

$$\xi^t(\mathbf{k};s)=\frac{1}{\Gamma(s)}\int_0^\infty z^{s-1}\frac{\mathrm{Li}^t_{\mathbf{k}}(1-e^{-z})}{e^z-1}\,dz \quad (\mathrm{Re}(s)>1-r)\,. \tag{1.4}$$

Here the t-multi-polylogarithm $\mathrm{Li}^t_{\mathbf{k}}(z)$ is a t-polynomial defined for $|z| < 1$ by

$$\mathrm{Li}^t_{\mathbf{k}}(z) = \sum_{\mathbf{k} \preceq \mathbf{k}'} t^{\mathrm{dep}(\mathbf{k}')-r}\, \mathrm{Li}_{\mathbf{k}'}(z)\,, \tag{1.5}$$

and $\mathrm{dep}(\mathbf{k}) = r$ denotes the length of $\mathbf{k}$ and is called the depth of $\mathbf{k}$.† From (1.3), we see that $\xi^0(\mathbf{k}; s) = \xi(\mathbf{k}; s)$ and $\xi^1(\mathbf{k}; s) = (-1)^{r-1}\, \eta(\mathbf{k}; s)$, thus the function $\xi^t(\mathbf{k}; s)$ interpolates AKZs and KTZs.

In this paper, we investigate the values $\xi^t(\mathbf{k}; m)$ at $m \in \mathbb{Z}$ and give formulas for both positive and non-positive integer m. They naturally include the formulas given in [1, Theorem 6] and [8, Theorem 2.3, Remark 2.4, and Theorem 2.5], as the cases when $t = 0$ and 1. In particular, we see that the values at positive integer m wonderfully fits to Yamamoto's t-multiple zeta values.

The paper is organized as follows. In §2, we study some fundamental properties of both t-multi-polylogarithms and related t-Bernoulli numbers. In §3, we state our main results, after reviewing the t-multiple zeta values.

2. t-polylogarithms and t-multi-poly-Bernoulli numbers

We shall give some fundamental properties of both t-polylogarithms and a kind of t-multi-poly-Bernoulli numbers, including a differential formula of t-polylogarithms.

First, we introduce some notation. Let r be a positive integer. For an index $\mathbf{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{>0})^r$, we put $\mathbf{k}_+ = (k_1, \ldots, k_{r-1}, k_r + 1)$. Especially if $k_r > 1$ holds, we call $\mathbf{k}$ an admissible index, and for such $\mathbf{k}$, we put $\mathbf{k}_- = (k_1, \ldots, k_{r-1}, k_r - 1)$. For $\mathbf{j} = (j_1, \ldots, j_r) \in (\mathbb{Z}_{\geq 0})^r$, we define the depth and the weight of $\mathbf{j}$ by $\mathrm{dep}(\mathbf{j}) = r$ and $\mathrm{wt}(\mathbf{j}) = j_1 + \cdots + j_r$, respectively.

The t-multi-polylogarithm $\mathrm{Li}^t_{\mathbf{k}}(z)$, defined as (1.5), satisfies the following properties.

LEMMA 2.1. *Let* $\mathbf{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{>0})^r$ *and* $k = \mathrm{wt}(\mathbf{k})$.
(i) *We have*

$$\frac{d}{dz}\mathrm{Li}^t_{\mathbf{k}}(z) = \begin{cases} \left(\dfrac{t}{1-z} + \dfrac{1}{z}\right)\mathrm{Li}^t_{\mathbf{k}_-}(z) & \text{if } \mathbf{k} \text{ is admissible} \\ \dfrac{1}{1-z}\,\mathrm{Li}^t_{k_1,\ldots,k_{r-1}}(z) & \text{if } \mathbf{k} \text{ is non-admissible and } \mathbf{k} \neq (1) \\ \dfrac{1}{1-z} & \text{if } \mathbf{k} = (1)\,. \end{cases}$$

(ii) *We have*

$$\mathrm{Li}^t_{\mathbf{k}}(z) = \int \cdots \int_{0<u_1<\cdots<u_k<z} \prod_{i=1}^{k} f_i(u_i)\, du_i$$

†Note that the definition of $\mathrm{Li}^t_{\mathbf{k}}(z)$ is different from that of $\mathrm{Li}_{\mathbf{k}}(t, z)$ in [11].

with

$$f_i(u)=\begin{cases}\dfrac{1}{1-u} & \text{if } i\in\{1,\ 1+k_1,\ 1+k_1+k_2,\dots,\ 1+k_1+\cdots+k_{r-1}\}\,,\\ \dfrac{1}{u}+\dfrac{t}{1-u} & \text{otherwise.}\end{cases}$$

Proof. (i) If $\mathbf{k}$ is admissible, then

$$\begin{aligned}\frac{d}{dz}\mathrm{Li}^t_{\mathbf{k}}(z)&=\sum_{\mathbf{k}_-\preceq\mathbf{k}'}\left(t^{\mathrm{dep}(\mathbf{k}'_+)-r}\frac{d}{dz}\mathrm{Li}_{\mathbf{k}'_+}(z)+t^{\mathrm{dep}((\mathbf{k}',1))-r}\frac{d}{dz}\mathrm{Li}_{(\mathbf{k}',1)}(z)\right)\\&=\sum_{\mathbf{k}_-\preceq\mathbf{k}'}\left(t^{\mathrm{dep}(\mathbf{k}')-r}\frac{1}{z}+t^{\mathrm{dep}(\mathbf{k}')+1-r}\frac{1}{1-z}\right)\mathrm{Li}_{\mathbf{k}'}(z)\\&=\left(\frac{1}{z}+\frac{t}{1-z}\right)\mathrm{Li}_{\mathbf{k}_-}(z)\,.\end{aligned}$$

Non-admissible cases are obtained by usual computations. (ii) is derived from (i). □

PROPOSITION 2.2. *For any r-tuple $\mathbf{k}=(k_1,\dots,k_r)$ of positive integers and variables t_1, t_2, we have*

$$\mathrm{Li}^{t_1+t_2}_{\mathbf{k}}(z)=\sum_{\mathbf{k}\preceq\mathbf{k}'}t_2^{\mathrm{dep}(\mathbf{k}')-r}\,\mathrm{Li}^{t_1}_{\mathbf{k}'}(z)\,.$$

Proof. Note that any index $\mathbf{k}'$ satisfying $\mathbf{k}\preceq\mathbf{k}'\preceq\mathbf{k}''$ and $\mathrm{dep}(\mathbf{k}')=\mathrm{dep}(\mathbf{k}'')-l$ is obtained by replacing l of $(\mathrm{dep}(\mathbf{k}'')-\mathrm{dep}(\mathbf{k}))$ commas of $\mathbf{k}''$ by +'s. Then we can rewrite the right-hand side of the assertion as

$$\begin{aligned}\sum_{\mathbf{k}\preceq\mathbf{k}'}t_2^{\mathrm{dep}(\mathbf{k}')-r}\,\mathrm{Li}^{t_1}_{\mathbf{k}'}(z)&=\sum_{\mathbf{k}\preceq\mathbf{k}'\preceq\mathbf{k}''}t_1^{\mathrm{dep}(\mathbf{k}'')-\mathrm{dep}(\mathbf{k}')}t_2^{\mathrm{dep}(\mathbf{k}')-r}\,\mathrm{Li}_{\mathbf{k}''}(z)\\&=\sum_{\mathbf{k}\preceq\mathbf{k}''}\left(\sum_{l=0}^{\mathrm{dep}(\mathbf{k}'')-r}\binom{\mathrm{dep}(\mathbf{k}'')-r}{l}t_1^l t_2^{\mathrm{dep}(\mathbf{k}'')-r-l}\right)\mathrm{Li}_{\mathbf{k}''}(z)\\&=\sum_{\mathbf{k}\preceq\mathbf{k}''}(t_1+t_2)^{\mathrm{dep}(\mathbf{k}'')-r}\mathrm{Li}_{\mathbf{k}''}(z)\\&=\mathrm{Li}^{t_1+t_2}_{\mathbf{k}}(z)\,,\end{aligned}$$

which equals to the left-hand side. □

Next, we define t-multi-poly-Bernoulli numbers related with the t-multi-polylogarithms.

DEFINITION 2.3. We define the polynomials $C^{(\mathbf{k})}_n(t)\in\mathbb{R}[t]$ by

$$\frac{\mathrm{Li}^t_{\mathbf{k}}(1-e^{-z})}{e^z-1}=\sum_{n=0}^{\infty}C^{(\mathbf{k})}_n(t)\,\frac{z^n}{n!}\,.$$

It is clear from the definition that each $C_n^{(\mathbf{k})}(t)$ can be written as a polynomial whose coefficients are multi-poly-Bernoulli numbers:

$$C_n^{(\mathbf{k})}(t) = \sum_{\mathbf{k} \preceq \mathbf{k}'} t^{\mathrm{dep}(\mathbf{k}')-r}\, C_n^{(\mathbf{k}')} .$$

We also see by (1.1) that the following equations hold.

$$C_n^{(\mathbf{k})}(0) = C_n^{(\mathbf{k})}, \quad C_n^{(\mathbf{k})}(1) = (-1)^{n+r-1} B_n^{(\mathbf{k})} .$$

As a consequence of Proposition 2.2, we have the following property.

PROPOSITION 2.4. *For any r-tuple $\mathbf{k} = (k_1, \dots, k_r)$ of positive integers, for non-negative integer n, and for variables t_1, t_2, we have*

$$C_n^{(\mathbf{k})}(t_1 + t_2) = \sum_{\mathbf{k} \preceq \mathbf{k}'} t_2^{\mathrm{dep}(\mathbf{k}')-r}\, C_n^{(\mathbf{k}')}(t_1) .$$

Proof. It is obvious from Proposition 2.2. □

3. t-Arakawa–Kaneko multiple zeta functions

In this section, we study the special values of t-Arakawa–Kaneko multiple zeta functions in s.

For any admissible index $\mathbf{k}$, Yamamoto ([15]) defined the polynomial

$$\zeta^t(\mathbf{k}) = \sum_{\mathbf{k}' \preceq \mathbf{k}} t^{r-\mathrm{dep}(\mathbf{k}')}\, \zeta(\mathbf{k}') .$$

It interpolates multiple zeta and zeta-star values defined by the convergent series

$$\zeta^0(\mathbf{k}) = \zeta(\mathbf{k}) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

and

$$\zeta^1(\mathbf{k}) = \zeta^\star(\mathbf{k}) = \sum_{0<m_1\le\cdots\le m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} ,$$

respectively.

The iterated integral representation of ζ^t is as follows.

LEMMA 3.1. *For any admissible index $\mathbf{k} = (k_1, k_2, \dots, k_r)$ with $k = \mathrm{wt}(\mathbf{k})$, we have*

$$\zeta^t(\mathbf{k}) = \int \cdots \int_{0<u_1<\cdots<u_k<1} \frac{du_1}{1-u_1} \frac{du_2}{u_2} \cdots \frac{du_{k_1}}{u_{k_1}} \Big(\frac{du_{k_1+1}}{1-u_{k_1+1}} + \frac{t\, du_{k_1+1}}{u_{k_1+1}} \Big) \frac{du_{k_1+2}}{u_{k_1+2}} \cdots \frac{du_{k_1+k_2}}{u_{k_1+k_2}}$$
$$\cdots \Big(\frac{du_{k-k_r+1}}{1-u_{k-k_r+1}} + \frac{t\, du_{k-k_r+1}}{u_{k-k_r+1}} \Big) \frac{du_{k-k_r+2}}{u_{k-k_r+2}} \cdots \frac{du_k}{u_k} .$$

Proof. It is clear from the definition of $\zeta^t(\mathbf{k})$ and the iterated integral representation of MZVs (cf. [17, p. 510]). □

Now we define the interpolated version of AKZs and KTZs.

DEFINITION 3.2. For $\mathrm{Re}(s) > 1 - r$,

$$\xi^t(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^\infty z^{s-1} \frac{\mathrm{Li}^t_{\mathbf{k}}(1 - e^{-z})}{e^z - 1} \, dz \,.$$

Note that this function can be written as

$$\xi^t(\mathbf{k}; s) = \sum_{\mathbf{k} \preceq \mathbf{k}'} t^{\mathrm{dep}(\mathbf{k}') - r} \, \xi(\mathbf{k}'; s) \tag{3.1}$$

for $\mathbf{k} \in (\mathbb{Z}_{>0})^r$, and from (1.3) it satisfies

$$\xi^0(\mathbf{k}; s) = \xi(\mathbf{k}; s) \quad \text{and} \quad \xi^1(\mathbf{k}; s) = (-1)^{r-1} \, \eta(\mathbf{k}; s) \,.$$

For $\mathbf{j} = (j_1, \ldots, j_r) \in (\mathbb{Z}_{\geq 0})^r$, we set $\mathbf{k} + \mathbf{j} = (k_1 + j_1, \ldots, k_r + j_r)$ and

$$b(\mathbf{k}; \mathbf{j}) = \prod_{i=1}^{r} \binom{k_i + j_i - 1}{j_i} .$$

The following theorem is the t-interpolated version of [8, Theorem 2.3, Remark 2.4, and Theorem 2.5] (and also [9, Theorem 2.2]).

THEOREM 3.3. (i) *The function $\xi^t(\mathbf{k}; s)$ is holomorphically continued to the whole s-plane, and for $m \in \mathbb{Z}_{\geq 0}$,*

$$\xi^t(\mathbf{k}; -m) = (-1)^m C_m^{(\mathbf{k})}(t) \,.$$

(ii) *For $m \in \mathbb{Z}_{>0}$,*

$$\xi^t(\mathbf{k}; m) = \sum_{\mathrm{wt}(\mathbf{j}) = m-1, \mathrm{dep}(\mathbf{j}) = n} b((\mathbf{k}_+)^*; \mathbf{j}) \, \zeta^t((\mathbf{k}_+)^* + \mathbf{j}) \,,$$

where $n = \mathrm{dep}((\mathbf{k}_+)^*)$ and $\mathbf{h}^*$ is the usual dual of an admissible index $\mathbf{h}$ (i.e.,

$$\mathbf{h}^* = (\{1\}^{b_h - 1}, a_h + 1, \ldots, \{1\}^{b_1 - 1}, a_1 + 1)$$

for

$$\mathbf{h} = (\{1\}^{a_1 - 1}, b_1 + 1, \ldots, \{1\}^{a_h - 1}, b_h + 1) \,,$$

where $a_1, \ldots, a_h, b_1, \ldots, b_h \in \mathbb{Z}_{>0}$, and $\{1\}^c = \underbrace{1, \ldots, 1}_{c}$).

Proof. (i) is clear from (3.1) and [8, Remark 2.4]. To prove (ii), we write the index $\mathbf{k}$ as

$$\mathbf{k} = (\{1\}^{a_1 - 1}, b_1 + 1, \ldots, \{1\}^{a_h - 1}, b_h + 1)$$

with (uniquely determined) integers $h \in \mathbb{Z}_{>0}$, $a_1, \ldots, a_h, b_1, \ldots, b_{h-1} \in \mathbb{Z}_{>0}$ and $b_h \in \mathbb{Z}_{\geq 0}$. Put $c_i = (a_1 + b_1) + \cdots + (a_i + b_i) \quad (1 \leq i \leq h)$, then

$$\xi^t(\mathbf{k}; m) = \int_{0<x<1} (-\log(1 - x))^{m-1} \mathrm{Li}^t_{\mathbf{k}}(x) \frac{dx}{x}$$

$$= \int\cdots\int_{\substack{0<x<1\\0<v_1,v_2,\dots,v_{m-1}<x\\0<u_1<\cdots<u_{c_h}<x}} \frac{dv_1}{1-v_1}\cdots\frac{dv_{m-1}}{1-v_{m-1}}\prod_{j=1}^{a_1}\frac{du_j}{1-u_j}\prod_{j=a_1+1}^{c_1}\left(\frac{t\,du_j}{1-u_j}+\frac{du_j}{u_j}\right)$$

$$\cdots\prod_{j=c_{h-1}+1}^{c_{h-1}+a_h}\frac{du_j}{1-u_j}\prod_{j=c_{h-1}+a_h+1}^{c_h}\left(\frac{t\,du_j}{1-u_j}+\frac{du_j}{u_j}\right)\frac{dx}{x}$$

The change of variables $x\mapsto 1-x$, $v_i\mapsto 1-v_i$, and $u_j\mapsto 1-u_j$ leads us to

$$\xi^t(\mathbf{k};m) = \int\cdots\int_{\substack{0<x<1\\x<v_1,v_2,\dots,v_{m-1}<1\\x<u_{c_h}<\cdots<u_1<1}} \frac{dv_1}{v_1}\cdots\frac{dv_{m-1}}{v_{m-1}}\prod_{j=1}^{a_1}\frac{du_j}{u_j}\prod_{j=a_1+1}^{c_1}\left(\frac{t\,du_j}{u_j}+\frac{du_j}{1-u_j}\right)$$

$$\cdots\prod_{j=c_{h-1}+1}^{c_{h-1}+a_h}\frac{du_j}{u_j}\prod_{j=c_{h-1}+a_h+1}^{c_h}\left(\frac{t\,du_j}{u_j}+\frac{du_j}{1-u_j}\right)\frac{dx}{1-x}$$

By counting shuffles concretely, in the same manner to the combinatorial proof of [9, Theorem 2.2], we obtain the theorem by Lemma 3.1. □

The following corollary to the theorem is the interpolated version of [1, Corollary 10 (i)] and [8, Corollary 2.8].

COROLLARY 3.4. *For $k, m \geq 1$, we have*

$$\xi^t(k;m) = \sum_{\substack{a_1+\cdots+a_k=m-1\\\forall a_j\geq 0}} (a_k+1)\zeta^t(a_1+1,\dots,a_{k-1}+1,a_k+2) \tag{3.2}$$

The value $\xi^t(\mathbf{k};m)$ can also be written in an extra manner. Define $S(\mathbf{k};m)$ for $\mathbf{k}=(k_1,\dots,k_r)\in(\mathbb{Z}_{>0})^r$ and $m\in\mathbb{Z}_{>0}$ as

$$S(\mathbf{k};m) = \sum_{0<a_1<\cdots<a_r=b_m\geq\cdots\geq b_1>0}\frac{1}{a_1^{k_1}\cdots a_r^{k_r}b_1\cdots b_m},$$

then it is known by [10, Theorem 2] (and also [16, Lemma A.2]) that

$$\xi(\mathbf{k};m) = S(\mathbf{k};m),$$

and thus we obtain the following proposition.

PROPOSITION 3.5. For $\mathbf{k}=(k_1,\dots,k_r)\in(\mathbb{Z}_{>0})^r$ and $m\in\mathbb{Z}_{>0}$,

$$\xi^t(\mathbf{k};m) = \sum_{\mathbf{k}\preceq\mathbf{k}'} t^{\mathrm{dep}(\mathbf{k}')-r}S(\mathbf{k}';m). \tag{3.3}$$

Especially for $r=1$ and $k,m\geq 1$, we have

$$\xi^t(k;m) = \sum_{0<a_1\leq\cdots\leq a_k=b_m\geq\cdots\geq b_1>0}\frac{t^{\sum_{i=1}^{k-1}(1-\delta(a_i,a_{i+1}))}}{a_1\cdots a_k b_1\cdots b_m}, \tag{3.4}$$

where $\delta(x, y)$ is the Kronecker delta.

By putting $t = 1$ in (3.4), we easily obtain the duality

$$\eta(k; m) = \xi^1(k; m) = \sum_{0<a_1\leq\cdots\leq a_k=b_m\geq\cdots\geq b_1>0} \frac{1}{a_1\cdots a_k b_1\cdots b_m} = \xi^1(m; k) = \eta(m; k)\,,$$

which was conjectured by Kaneko and Tsumura in [8], and firstly proved by Yamamoto in [16] and secondly in [9].

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