Doctoral thesis

Primordial Non-Gaussianities from General Models of Inflation and Bounce

一般化されたインフレーション及びバウンスモデル起源の原始非ガウス性

Shingo Akama

Department of Physics, Graduate School of Science,
Rikkyo University

Abstract

Inflation is the most successful model of the early universe since that not only can resolve the various problems in the standard Big-Bang cosmology but also can be compatible with anisotropies of the Cosmic Microwave Background (CMB). However, even the successful model faces the initial singularity and Trans-Planckian problem. In light of this, the alternatives to inflation evading the singularity have been also explored so far. Therefore, when the non-singular models can resolve all the problems, it would be worth investigating the consistency with CMB observations and also finding how to discriminate between inflation and alternatives.

One of the familiar alternative models is the so-called matter bounce cosmology which can generate the scale-invariant curvature perturbation. However, it has been pointed out that the matter bounce model in the k-essence theory cannot satisfy the observational constraints on the tensor-to-scalar ratio and the non-Gaussianity of the curvature perturbation simultaneously. In this thesis, we show that this is not the case in more general models of bounce. To do so, we calculate the primordial power spectra and the primordial bispectra of scalar and tensor perturbations in general bounce cosmology. We also investigate how to discriminate contracting models from inflation based on the non-Gaussian signatures of tensor perturbations. As a result, we show that the non-Gaussian amplitudes and shapes in general bounce models have different properties compared to those in general models of single-field inflation.

In order to distinguish inflation from the alternatives observationally, it is important to find specific features of inflation. It has been shown that the amplitude of the primordial non-Gaussianity of the curvature perturbation in the case of non-Bunch-Davies initial states can be enhanced compared with that in the case of the Bunch-Davies one due to the interactions caused by the subhorizon perturbations. The enhancement results from the fact which the physical wavelengths of the inflationary perturbations become Planckian lengths in the far past. Therefore, we can anticipate that the signatures due to the non-Bunch-Davies effects are peculiar to inflation. The purpose of the present paper is to see whether or not the primoridial non-Gaussianities of the tensor perturbations are enhanced as well. In doing so, we consider some general theory of gravity including an inflaton and calculate the tensor auto-bispectrum and the cross-bispectrum involving one tensor and two scalar modes with the non-Bunch-Davies initial states. In particular, we prove that the

amplitude of the primordial cross-bispectrum can be enhanced at non-trivial configurations characterized by not only the wave numbers but also the propagation speeds of the perturbations.

Contents

Chapter 1

Introduction

Inflation [1, 2, 3] is the successful model of the early universe owing to the facts that inflation can resolve the problems in the standard Big Bang cosmology and be consistent with Planck and other data [4]. Then, one may ask how inflation can be conclusive observationally, but the answer is still unknown. To find the answer, we would necessarily clarify the observational predictions of not only inflation but also alternatives.

Inflation suffers from the initial singularity [5] and Trans-Planckian problem [6]. In light of this, the nonsingular alternatives, e.g., bounce models (see, e.g., Ref [7] for a review), have been also explored so far. If the alternatives are consistent with CMB observations, we should have no reason to exclude those at the present stage. However, in some cases, a large tensor-to-scalar ratio or a large scalar non-Gaussianity has been found. In this thesis, we focus on general bounce cosmology, and investigate how the alternatives are viable by evaluating the primordial power spectrum and the primordial non-Gaussianity.

Although inflation also faces some problems, it is definite that the paradigm is successful and representative. In this thesis, we also focus on the tensor non-Gaussianity from inflation in the context of the non-Bunch-Davies initial states. It has been previously shown that the scalar non-Gaussianity is enhanced due to the non-Bunch-Davies effect [8]. It can be anticipated that the non-Gaussian amplitudes of the tensor non-Gaussianities can be enhanced due to the non-Bunch-Davies effects as well as the scalar non-Gaussianity. There have been several studies about the tensor non-Gaussianities from standard models of inflation with the non-Bunch-Davies states [9, 10]. In this thesis, we see whether or not the non-Gaussian amplitude in the non-Bunch-Davies states can be enhanced compared with that in the

Bunch-Davies one in general single-field inflation models.

This thesis is organized as follows.

•Chapter 2.

In this chapter, we introduce the standard Big Bang cosmology and its problems. We then review how the problems can be resolved in inflation and bouncing cosmology. In light of the recent developments of inflation models in extended gravitational theories, we introduce a general scalar-tensor theory.

•Chapter 3.

In this chapter, we first study the basics of the cosmological perturbation theory. Then, we calculate the primordial power spectra and the primordial non-Gaussianities of the scalar and tensor perturbations in the single-field slow-roll inflation model. We also investigate the problems in alternatives to inflation and also review how those can be resolved.

•Chapter 4.

In this chapter, we first introduce the general contracting background. Then, we evaluate the primordial power spectra of curvature and tensor perturbations and derive the conditions in order for the power spectra to be scale-invariant. Also, we calculate the primordial non-Gaussianities and investigate whether or not a small tensor-to scalar ratio and small scalar non-Gaussianity can be obtained simultaneously in the Horndeski theory. Last, we also discuss how to distinguish bounce from inflation through the tensor non-Gaussianities.

•Chapter 5.

In the chapter, we first consider the quadratic and cubic actions for the tensor perturbations in general scalar-tensor theory and introduce non-Bunch-Davies initial states which are the states obtained by performing a Bogoliubov transformation to the usual Bunch-Davies states. We then calculate the auto-bispectrum of the tensor perturbations, and investigate whether the enhanced non-Gaussian amplitudes can be obtained or not. We also compute the cross-bispectrum generated by one tensor and two scalar modes and investigate the signatures which have not been predicted in the case of the Bunch-Davies initial state.

•Chapter 6.

The conclusion of this thesis is drawn in this chapter.

Conventions

In this thesis, we use the unit, $c = 1 = \hbar$. The metric signature is $(-, +, +, +)$.

Chapter 2

Standard Big Bang cosmology and the Early Universe

According to the standard Big Bang cosmology, the decelerating expansion of our universe started from the hot Big Bang universe in which each of matter components is characterized by some high temperature and high density. (See, e.g., [11, 12].) However, the paradigm faces various problems regarding fine-tuning, i.e., the horizon and flatness problems. Furthermore, the origin of the density fluctuations leading to the large scale structure and the anisotropies of CMB cannot be explained within the paradigm. In this chapter, we briefly review the dynamics of the universe in the standard Big-Bang cosmology and explain the problems. Then, we introduce some early universe models which can resolve the problems.

2.1 Standard Big Bang cosmology and its problems

2.1.1 Homogeneous and isotropic universe

By assuming that the spacetime is homogeneous and isotropic on cosmological scales, the metric of the spacetime without a spatial curvature is written by the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j, \tag{2.1}$$

where $a(t)$ is a scale factor. One can investigate the time evolution of the universe by solving the Einstein equation,

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}^2}T_{\mu\nu}, \tag{2.2}$$

where $G_{\mu\nu}$ is the Einstein tensor. The Ricci tensor, $R_{\mu\nu}$, and the Ricci scalar, R, are defined by

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\lambda\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\rho\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda}, \tag{2.3}$$

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{2.4}$$

with

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}), \tag{2.5}$$

which is the Christoffel symbol. Each component of the Einstein tensor is given by

$$G_{00} = 3H^2,\ G_{ij} = -a^2(3H^2 + 2\dot{H})\delta_{ij},\ G_{0i} = G_{i0} = 0, \tag{2.6}$$

where $H := \dot{a}/a$ is the Hubble parameter, and a dot represents a differentiation with respect to the cosmic time, t. The energy momentum tensor, $T_{\mu\nu}$, can be written in terms of the energy density, ρ, and pressure, P, of matter as $T_{\mu\nu} = \mathrm{diag}(\rho, a^2P, a^2P, a^2P)$. In particular, the (00)- and (ij) components of the Einstein equation, respectively, read

$$3M_{\rm Pl}^2 H^2 = \rho, \tag{2.7}$$

$$M_{\rm Pl}^2(3H^2 + 2\dot{H}) = -P, \tag{2.8}$$

where the first and second equations are the Friedmann and evolution equations, respectively. By combining Eqs. (2.7) and (2.8), we obtain the following equation

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{2.9}$$

Given a matter parametrized by $P = w\rho$ with w being the equation of state (EoS) parameter, we have

$$\rho \propto a^{-3(1+w)}. \tag{2.10}$$

In the universe with the pressure-less matter ($w = 0$) and radiation ($w = 1/3$), the Friedmann equation can be written as

$$3M_{\rm Pl}^2 H^2 = \frac{\rho_{r,0}}{a^4} + \frac{\rho_{m,0}}{a^3}, \tag{2.11}$$

where $\rho_{r,0}$ and $\rho_{m,0}$ are the present values of both the energy densities, and we normalized the scale factor as $a(t_0) = 1$ with t_0 being the present time. We also obtain the another equation by combining the Friedmann and evolution equations as

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm Pl}^2}(\rho + 3P). \tag{2.12}$$

For the standard matter enjoying $\rho \geq 0$ and $P \geq 0$, the expansion of the universe decelerates. However, it has been found that the expansion of the current universe is accelerating [13] by the so-called dark energy. The simplest and consistent candidate to realize the accelerating expansion is the cosmological constant which satisfies $P = -\rho (< 0)$ implying $\ddot{a} > 0$ when the cosmological constant is the dominant component in the total energy densities. (See, however, Ref. [14].) Although we do not go deeply into the topics on the dark energy, various candidates have been explored so far in the literature.

2.1.2 Problems

Let us get back to a topic on the universe in the past. At least during some durations after the Big Bang, the expansion of the universe has decelerated. This fact gives us fine-tuning problems about the initial conditions of our universe. In this thesis, we particularly deal with the horizon and flatness problems.

•Flatness problem

In the FLRW spacetime with a constant spatial curvature, $\mathcal{K}$, the curvature contributes to the Friedmann equation as $\rho_{\mathcal{K}} = \mathcal{O}(M_{\rm Pl}^2 \mathcal{K}/a^2)$. In particular, the spatially flatness of the universe can be evaluated by the curvature parameter, $\Omega_{\mathcal{K}}$, defined by

$$\Omega_{\mathcal{K}} := -\frac{\mathcal{K}}{a^2 H^2}. \tag{2.13}$$

According to Planck's observations [4, 15], *Planck* TT,TE,EE+lowE+lensing has put a constraint as

$$\Omega_{\mathcal{K}} = -0.011^{+0.013}_{-0.012} \quad (95\%\ \mathrm{CL}), \tag{2.14}$$

and *Planck* TT,TE,EE+lowE+lensing+BAO has done

$$\Omega_{\mathcal{K}} = 0.0007 \pm 0.0037 \quad (95\%\ \mathrm{CL}). \tag{2.15}$$

In the decelerating universe, the curvature parameter decreases as the time goes back to the past since we have $\Omega_{\mathcal{K}} \propto \dot{a}^{-2}$. This indicates that the universe is highly spatially flat, $|\Omega_{\mathcal{K}}| \ll \mathcal{O}(10^{-2})$ at the Big-Bang, and thus one must detune the spatially flatness.

•Horizon problem

The CMB observation indicates that the CMB-temperature is nearly homogeneous over the CMB sky, $\Delta T_{\rm CMB}/T_{\rm CMB} = \mathcal{O}(10^{-5})$. To understand the homogeneity which seems to be mysterious at a glance, let us consider the causal structure by focusing on the path of light. The light propagates along the null geodesic, $\mathrm{d}s^2 = 0$, and the maximum path which the light can propagate from t_0 to t is

$$\eta(t) := \int_{t_0}^{t} \frac{\mathrm{d}t'}{a(t')}, \tag{2.16}$$

where t_0 now corresponds to the time of the Big Bang, and η determines causally-(dis)connected regions at t. In a decelerating universe, $a \propto t^n$ with $0 < n < 1$, the size of the comoving particle horizon decreases as the time goes back to the past. In particular, the horizon scale at the last scattering is much smaller than that at the present time, (i.e., $\eta(t_{\rm CMB}) \ll \eta(t_0)$ with $t_{\rm CMB}$ being the time of the last scattering). This implies that we observe the radiations which were inside the causally disconnected regions at the last scattering surface. Nevertheless, those share almost the same temperature. Therefore, we need to detune the small anisotropy of the CMB temperature.

2.2 Early universe models

The problems introduced in the previous section may be evaded by assuming that such was our universe. However, it would be reasonable to find how the "initially" detuned universe was realized. This can be accomplished by considering the early universe, the phase before the Big Bang. The representative model which can resolve the problems is inflation. Whereas alternatives to inflation have been also explored. In this chapter, we introduce both models and briefly explain how the problems can be evaded.

2.2.1 Inflation

Inflation [1, 2, 3] is an early universe model in which there was a phase of the exponentially accelerating expansion of the universe before the Big Bang. By recalling Eq. (2.12), it can be seen that the accelerating universe can be realized by imposing $\rho + 3P < 0$, i.e., violating the strong energy condition, and the violation can be accomplished even by a canonical scalar field.

Let us consider the Einstein gravity with a canonical scalar field whose action is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right), \tag{2.17}$$

where we denoted the potential term of ϕ as $V(\phi)$. The Friedmann and evolution equations read

$$3M_{\mathrm{Pl}}^2H^2 = \frac{\dot{\phi}^2}{2} + V(=\rho), \tag{2.18}$$

$$M_{\mathrm{Pl}}^2(3H^2 + 2\dot{H}) = -\frac{\dot{\phi}^2}{2} + V(=-P). \tag{2.19}$$

We also have the evolution of equation for ϕ,

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0, \tag{2.20}$$

where we defined $V_\phi := \mathrm{d}V/\mathrm{d}\phi$. In this case, we have

$$\rho + 3P = -6M_{\mathrm{Pl}}^2H^2(1-\epsilon), \tag{2.21}$$

where $\epsilon := -\dot{H}/H^2$. The universe is accelerating (i.e., $\ddot{a} > 0$) if $\epsilon < 1$ holds (the strong energy condition is violated).

Then, we deal with the potential-driven slow-roll inflation model. Let us characterize the epoch in which a scalar field is slowly rolling on its potential in terms of two conditions as

$$\epsilon_1 := M_{\mathrm{Pl}}^2\left(\frac{V_\phi}{V}\right)^2 \ll 1, \tag{2.22}$$

$$\epsilon_2 := M_{\mathrm{Pl}}^2\frac{V_{\phi\phi}}{V} \ll 1, \tag{2.23}$$

implying that

$$3M_{\mathrm{Pl}}^2H^2 \simeq V, \tag{2.24}$$

$$3H\dot{\phi} + V \simeq 0. \tag{2.25}$$

In this regime, the spacetime is approximated by the quasi-de Sitter spacetime, i.e., $\epsilon \ll 1$.

2.2.2 Resolutions for the problems in the standard Big-Bang cosmology

The density parameter of the spatial curvature are diluted away as $\Omega_{\mathcal{K}} \propto e^{-2Ht}$ during the quasi-de Sitter expansion. Hence, the spatially flatness can be resolved.

As for the horizon problem, we recall the comoving particle horizon, η. In accelerating universes, e.g., $a \propto e^{Ht}$ (de Sitter) and $a \propto t^n$ with $n > 1$, the horizon scale η becomes bigger as the time goes back unlike the decelerating universe. This indicates that the would-be causally disconnected lights can be causally connected in the far past, and hence there is no longer any problem regarding the causal structure if one requires the sufficiently long period of the quasi-de Sitter.

2.2.3 Models of inflation

A lot of models have been constructed so far by choosing the explicit form of the potential, e.g., large inflation models ($V(\phi) \propto \phi^n$) [16], by adding higher curvature terms, e.g., Starobinsky's model [2], by driving inflation by a kinetic term of a scalar field, the so-called k-inflation [17], and also by invoking a scalar field having higher-derivatives, e.g., G-inflation [18], and so on. In light of the recent developments of inflation, the numbers of inflation models are anymore countless. To find viable models from a lot of inflation models, those need to be observationally distinguished. Here, it is quite convenient to use some unified framework and formulate the general inflation model. The familiar framework of the slow-roll inflation models has been constructed in the Generalized Galileon theory whose action is [19, 20]

$$S = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}, \tag{2.26}$$

with

$$\begin{aligned}\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\phi\nabla_\nu\phi)^2\right] \\ + G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right],\end{aligned} \tag{2.27}$$

where we denoted $\partial_X G$ as G_X, and G_i is an arbitrary function of ϕ and X. By taking the arbitrary functions appropriately, the models proposed before can be reproduced. In particular, GR with a canonical scalar field model can be reproduced by taking $G_2 = X - V$, $G_4 = M_{\rm Pl}^2/2$, and $G_3 = 0 = G_5$. One may want to know how general this theory is. The generality has been found through the equivalence between Generalized Galileon theory and the Horndeski theory, the most general single scalar-tensor theory in $4D$ having second-order field equations [21]. The equivalence has been proven in Ref. [20]. Therefore, the action of Eq. (2.27) is conventionally called the Horndeski theory. Based on this theory, the observational signatures in slow-roll inflation models have been classified.

2.2.4 Bouncing cosmology

The bounce models are characterized by three phases. The first phase is the contracting one. The simple example is the matter-dominated contracting phase, and the model with the phase has been considered in the context of the matter bounce cosmology [22]. As we will see later, the advantage of this model is to be able to generate the scale-invariant curvature perturbation [22]. Whereas this model suffers from the growth of anisotropy. In general relativity with a matter field, σ, the Friedmann equation can be written as

$$3M_{\rm Pl}^2 H^2 = \rho_\sigma + \rho_{\rm aniso}, \tag{2.28}$$

where

$$\rho_\sigma \propto a^{-3(1+w)}, \tag{2.29}$$

$$\rho_{\rm aniso} \propto a^{-6}, \tag{2.30}$$

with w being the EoS parameter of σ, $P_\sigma = w\rho_\sigma$. Therefore, in the case of the matter-dominated contracting universe, i.e., $w = 0$, the anisotropy gradually becomes the dominant energy component. This indicates that the spacetime is no longer isotropic. The simple strategy to evade the growth of the anisotropy is to drive the slow contraction by the matter field enjoying $w \ll 1$ which is invoked in the so-called Ekpyrotic cosmology, see, e.g., [23].

The second phase is the bouncing one. At somewhere in this phase, the scale factor is minimized, and hence the Trans-Planckian problem is avoidable only if the physical wavenumber of the perturbation satisfies $k_{\rm phys} := k/a \leq k/a_{\rm min} < M_{\rm Pl}$ for a given k. Similarly, the initial singularity predicted in the inflationary paradigm can be also evaded. In general, one needs to violate the null-energy-condition (NEC), $T_{\mu\nu}k^\mu k^\nu \geq 0$ with k^μ being an arbitrary null vector, to realize the non-singular bounce [24]. By using the Einstein equation, it can be seen that the energy condition in a spatially flat FLRW spacetime is equivalent to[*1]

$$\dot{H} > 0. \tag{2.31}$$

To investigate the relation between a model of a scalar field and the violation of NEC, we first focus on a canonical scalar field model with the Einsten-Hilbert action,

[*1] In this thesis, I denote the condition, $\dot{H} \geq 0$, as NEC for convention though, exactly speaking, that is equivalent to the Null Convergence Condition, $R_{\mu\nu}k^\mu k^\nu \geq 0$.

Eq. (2.17). By combining the Friedmann and evolution equations, we have

$$\dot{H} = -\frac{X}{M_{\rm Pl}^2} \leq 0, \tag{2.32}$$

where we denoted the kinetic term of ϕ by $X := -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Eq. (2.32) states that the violation of the NEC is not allowed at the background level. The situation can change by invoking a scalar field having a non-canonical kinetic term, e.g., X^2, or a higher derivative term, e.g., $X\Box\phi$. To illustrate how the scalar field can violate the NEC, we use the k-essence field as the matter field,

$$S = \int {\rm d}^4x\sqrt{-g}\Big(R + K(\phi, X)\Big). \tag{2.33}$$

In this theory, Eq. (2.32) is modified as

$$\dot{H} = -\frac{XK_X}{M_{\rm Pl}^2}, \tag{2.34}$$

whose r.h.s. can be positive depending on the sign of K_X. Similarly, the higher derivative term can also make it possible to violate the NEC. The dynamics of the non-standard scalar field can connect the bouncing phase and the subsequent expanding phase which is the last phase in bouncing cosmology. One of examples of bounce models has been constructed within the cubic galileon theory [25],

$$\mathcal{L} = \frac{M_{\rm Pl}^2}{2}R + K(\phi, X) + G(\phi, X)\Box\phi, \tag{2.35}$$

where

$$K(\phi, X) = (1 - g(\phi))X + \beta X^2 - V(\phi), \tag{2.36}$$

$$G(\phi, X) = \gamma X, \tag{2.37}$$

with

$$g(\phi) = 2g_0\left[e^{-\sqrt{(2/p)}\phi} + e^{b_g\sqrt{(2/p)}\phi}\right]^{-1}, \tag{2.38}$$

$$V(\phi) = -2V_0\left[e^{-\sqrt{(2/q)}\phi} + e^{b_V\sqrt{(2/q)}\phi}\right]^{-1}, \tag{2.39}$$

and β, γ, g_0, p, b_g, V_0, q and b_V are constant. Based on this Lagrangian, one can obtain the ekpyrotic, bouncing, and expanding phases.

This solution does not suffer from the growth of anisotropy due to the ekpyrotic contraction and the bouncing phase can occur by violating the NEC. However, the solution and also other ones constructed within Eq. (2.35) generally develop some problem at a perturbation level. We will consider this point later.

2.2.5 Resolutions for the problems in the standard Big-Bang cosmology

Let us consider the general contracting universe, $a(t) \propto (-t)^n$ for $0 < n < 1$ and $-\infty < t < 0$ to discuss the flatness problem. In this universe, the curvature parameter behaves as

$$\Omega_{\mathcal{K}} \propto (-t)^{2(1-n)}. \tag{2.40}$$

Therefore, even though there is a phase such that $|\Omega_{\mathcal{K}}| = \mathcal{O}(1)$, the effect of the spatial curvature on the energy components is diluted away during the contracting phase. This is due to the fact that the scale of the spatial curvature, $\sqrt{\mathcal{K}}/a$, becomes grows as the time passes while the scale of the spacetime curvature, H, does faster. Last, we move to the argument on the horizon problem. Similarly to the case of inflation, the size of the (comoving particle) horizon scale can be infinite in the past infinity, and thus the horizon problem can be also resolved.

Chapter 3

Cosmological Perturbations

In this chapter, we first study the basics of the cosmological perturbation theory, including the definition of the perturbations, the quantization of those, and the primordial power spectrum and primordial non-Gaussianity. Then, we calculate the observable quantities in the single-field slow-roll inflation model, and explain challenges in alternatives to inflation.

3.1 Perturbations in FLRW spacetime

In this chapter, we define the cosmological perturbations in FLRW spacetime. (See, e.g., Ref. [26] for the cosmological perturbation theory.) We denote the metric including the perturbations as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{3.1}$$

where the first and second terms express the background and perturbed metrics, respectively. The explicit forms of the perturbations around the spatially flat FLRW spacetime can be written as

$$\mathrm{d}s^2 = -(1+2A)\mathrm{d}t^2 + 2aB_i\mathrm{d}x^i\mathrm{d}t + (\delta_{ij} + C_{ij})\mathrm{d}x^i\mathrm{d}x^j. \tag{3.2}$$

Then, we decompose the perturbations into a scalar, vector, and tensor as

$$B_i = \partial_i B - S_i, \tag{3.3}$$

$$C_{ij} = -C\delta_{ij} + \partial_i\partial_j E + \partial_{(i}F_{j)} + \frac{1}{2}h_{ij}, \tag{3.4}$$

where the vector and tensor modes satisfy

$$\partial^i F_i = 0,\ \partial^i h_{ij} = 0,\ \delta^{ij}h_{ij} = 0. \tag{3.5}$$

Each mode independently evolves in time in linear perturbation theory. Here, the perturbation C is associated with the Ricci scalar on a t-const hypersurface as

$$^{(3)}R = \frac{4}{a^2}\partial^2 C, \tag{3.6}$$

and hence C is called curvature perturbation. The metric (3.2)-(3.4) includes some gauge degrees of freedom which can be eliminated by using some gauge-invariant variables or fixing the gauge. To eliminate those, we consider the infinitesimal transformation,

$$x^\mu \to x^\mu + \xi^\mu, \tag{3.7}$$

where $\xi^\mu = (\delta t, \delta x^i)$ with $\delta x^i = \delta^{ij}\partial_j \delta x$ for the scalar mode. The perturbed metric changes via the gauge transformation Eq. (3.7) as

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} - \bar{g}_{\mu\nu,\sigma}\xi^\sigma - \bar{g}_{\mu\sigma}\xi^\sigma_{,\nu} - \bar{g}_{\nu\sigma}\xi^\sigma_{,\mu}. \tag{3.8}$$

First, the scalar perturbations obey

$$A \to A - \frac{\mathrm{d}\delta t}{\mathrm{d}t}, \tag{3.9}$$

$$B \to B + \frac{\delta t}{a} - a\frac{\mathrm{d}\delta x}{\mathrm{d}t}, \tag{3.10}$$

$$\psi \to \psi + H\delta t, \tag{3.11}$$

$$E \to E - \delta x. \tag{3.12}$$

For example, the gauge degrees of freedom can be elimintaed as

$$E = 0,\ \ B = 0,\ \ \text{(Newtonian gauge)} \tag{3.13}$$

$$E = 0,\ \ C = 0,\ \ \text{(Spatially flat gauge)}. \tag{3.14}$$

This gauge fixing purely depends on only the perturbations of the metric. Whereas one can fix the gauge by imposing conditions for the perturbation associated with a matter field. In a similar way with the metric perturbations, the perturbed energy momentum tensor can be obtained as

$$T_{\mu\nu} \to T_{\mu\nu} + \delta T_{\mu\nu}. \tag{3.15}$$

In particular, the $(0,i)$-component of $\delta T_{\mu\nu} := \partial_i \delta q$, changes under the gauge transformation as

$$\delta q \to \delta q + (\rho + P)\delta t. \tag{3.16}$$

This gives us the way to fix the gauge associated with δt,

$$\delta T_{0i} = 0 \ \rightarrow \ \delta q = 0, \ \text{(comoving gauge)}. \tag{3.17}$$

Also the perturbation of the scalar field behaves as

$$\delta\phi \rightarrow \delta\phi + \dot{\phi}\delta t. \tag{3.18}$$

Based on this transformation, the gauge can be fixed as

$$\delta\phi = 0, \ \text{(unitary gauge)}. \tag{3.19}$$

Throughout this paper, we fix the gauge under the unitary gauge.

Next, the vector modes obey

$$S_i \rightarrow S_i + a\frac{\mathrm{d}\delta x_i}{\mathrm{d}t}, \tag{3.20}$$

$$F_i \rightarrow F_i - \delta x_i. \tag{3.21}$$

and the gauge degree of freedom δx^i can be completely fixed by choosing $F_i = 0$. In scalar-tensor theories including 1 scalar and 2 tensor degrees of freedom, the vector modes are non-dynamical.

By construction, the tensor modes are gauge invariant, $h_{ij} \rightarrow h_{ij}$.

3.1.1 Perturbations in GR with a canonical scalar field

In the context of the early universe models, the perturbed metric under the unitary gauge is often written by employing the ADM formalism [27] as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + g_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right), \tag{3.22}$$

where

$$N = 1 + \delta n, \ N_i = \partial_i\chi, \tag{3.23}$$

$$g_{ij} = a^2 e^{2\zeta}\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_i^k h_{kj} + \cdots\right), \tag{3.24}$$

with ζ being the curvature perturbation.

To discuss the behavior of the dynamical perturbations, we need to derive the evolution equations for perturbations. We can derive the equations from the quadratic actions for the perturbations. For simplicity, we derive the quadratic actions in GR with a canonical scalar field whose action is given by Eq. (2.17). Substituting the perturbed metric into the action and expanding it up to quadratic order in perturbations, we obtain

$$S^{(2)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \Big[-3M_{\rm Pl}^2\dot{\zeta}^2 + \frac{M_{\rm Pl}^2}{a^2}(\partial_i\zeta)^2 + (X - 3M_{\rm Pl}^2H^2)\delta n^2 - \frac{2M_{\rm Pl}^2 H}{a^2}\delta n\partial^2\chi + \frac{2M_{\rm Pl}^2}{a^2}\dot{\zeta}\partial^2\chi + 6M_{\rm Pl}^2 H\delta n\dot{\zeta} - \frac{2M_{\rm Pl}^2}{a^2}\delta n\partial^2\zeta + \frac{M_{\rm Pl}^2}{8}(\dot{h}_{ij})^2 - \frac{M_{\rm Pl}^2}{8a^2}(\partial_k h_{ij})^2 \Big], \tag{3.25}$$

where δn and χ are auxiliary fields. By varying the perturbed action with respect to δn and χ, we have the constraint equations,

$$(X - 3M_{\rm Pl}^2H^2)\delta n - \frac{M_{\rm Pl}^2 H}{a^2}\partial^2\chi + 3M_{\rm Pl}^2 H\dot{\zeta} - \frac{M_{\rm Pl}^2}{a^2}\partial^2\zeta = 0, \tag{3.26}$$

$$H\delta n - \dot{\zeta} = 0. \tag{3.27}$$

Then, by eliminating the auxiliary fields by use of the constraint equations, we obtain the quadratic actions of ζ and h_{ij} as

$$S_\zeta^{(2)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \left[\mathcal{G}_S\dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2}(\partial_i\zeta)^2 \right], \tag{3.28}$$

$$S_h^{(2)} = \frac{1}{8}\int \mathrm{d}t\mathrm{d}^3x a^3 \left[\mathcal{G}_T\dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2}(\partial_k h_{ij})^2 \right], \tag{3.29}$$

where

$$\mathcal{G}_S = \frac{X}{H^2},\ \mathcal{F}_S = M_{\rm Pl}^2\left(-\frac{\dot{H}}{H^2}\right),\ \mathcal{G}_T = M_{\rm Pl}^2,\ \mathcal{F}_T = M_{\rm Pl}^2. \tag{3.30}$$

The functions, $\mathcal{G}_S$, $\mathcal{F}_S$, $\mathcal{G}_T$, and $\mathcal{F}_T$, must be positive to avoid the ghost and gradient instabilities.(See, e.g., Ref. [28].) In this theory, those are indeed positive when $\dot{H} < 0$ holds.

With a similar procedure, one can obtain the quadratic actions in more general theories. In particular, the actions in the Horndeski theory take the same forms with Eqs. (3.28) and (3.29).

3.2 Quantization in a curved spacetime

In this section, we quantize the perturbations in a curved spacetime [29].

In scalar-tensor theories having second-order field equations, the quadratic action of the perturbation, ψ (should be understood as ζ and h_{ij}), takes the form of [20]

$$S_\psi^{(2)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \left[\mathcal{G}\dot{\psi}^2 - \frac{\mathcal{F}}{a^2}(\partial_i\psi)^2 \right]. \tag{3.31}$$

Note that $\mathcal{G}$ and $\mathcal{F}$ are not always constant unlike the coefficients in the quadratic action of the tensor perturbations in GR, and thus in this section we keep $\mathcal{G}$ and $\mathcal{F}$ functions of t.

For the purpose of the quantization, one needs to introduce canonically normalized variables as [20]

$$\mathrm{d}y := \frac{\mathcal{F}^{1/2}}{a\mathcal{G}^{1/2}}\mathrm{d}t,\ u := z\psi,\ z := \sqrt{2}a(\mathcal{G}\mathcal{F})^{1/4}, \tag{3.32}$$

where u is the so-called Mukhanov-Sasaki variable, and the above variables make the original perturbed action canonically normalized one as

$$S_u^{(2)} = \frac{1}{2}\int \mathrm{d}y\mathrm{d}^3x\left[\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 - (\partial_i u)^2 + \frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}y^2}u^2\right]. \tag{3.33}$$

We quantize the perturbations by promoting u to $\hat{u}$ and expanding the perturbation in terms of the creation and annihilation operators. First, the canonically normalized scalar perturbation, $\hat{u}$, is

$$\hat{u}(y,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left(u_k(y)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(y)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right), \tag{3.34}$$

where u_k is the mode function, and the creation and annihilation operators satisfy

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}'), \tag{3.35}$$

$$\text{others} = 0. \tag{3.36}$$

Next, the canonically normalized tensor perturbation, $\hat{v}_{ij}$, is

$$\hat{v}_{ij}(y,\mathbf{x}) = \sum_s \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left(v_k(y)e_{ij}^{(s)}\hat{a}_{\mathbf{k}}^{(s)}e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(y)e_{ij}^{(s)*}\hat{a}_{\mathbf{k}}^{(s)\dagger}e^{-i\mathbf{k}\cdot\mathbf{x}}\right), \tag{3.37}$$

where v_k is the mode function, s denotes the two polarization modes of the gravitational waves as $s = \pm 1$, and the creation and annihilation operators satisfy

$$[\hat{a}_{\mathbf{k}}^{(s)}, \hat{a}_{\mathbf{k}'}^{(s)\dagger}] = (2\pi)^3\delta_{ss'}\delta(\mathbf{k}-\mathbf{k}'), \tag{3.38}$$

$$\text{others} = 0. \tag{3.39}$$

The polarization tensor satisfies

$$\delta_{ij}e_{ij}^{(s)}(\mathbf{k}) = 0 = k^i e_{ij}^{(s)}(\mathbf{k}),\ e_{ij}^{(s)}(\mathbf{k})e_{ij}^{(s')*}(\mathbf{k}) = \delta_{ss'},\ e_{ij}^{(s)*}(\mathbf{k}) = e_{ij}^{(-s)}(\mathbf{k}) = e_{ij}^{(s)}(-\mathbf{k}). \tag{3.40}$$

Both mode functions share the same equation as

$$\frac{\mathrm{d}^2u_k}{\mathrm{d}y^2} + \omega^2(y)u_k = 0, \tag{3.41}$$

where

$$\omega^2 := k^2 - m_\psi^2(y),\ m_\psi^2(y) := \frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}y^2}. \tag{3.42}$$

We hereafter deal with only the case of the scalar perturbation, but the discussion below can straightforwardly apply to the case of the tensor perturbation.

The mode function is never uniquely determined. This is due to the fact that there is no unique choice of the vacuum state in a curved spacetime. Therefore, we need a requirement for the mode function. It is often assumed that the mode function in the subhorizon region, $|k/m_\psi| \to \infty$ which is approximately equivalent to $|ky| \to \infty$, coincides with that in Minkowski spacetime since the mode function in such the region is not affected by gravity (spacetime curvature). In fact, by taking the subhorizon limit, the approximated solution of the mode function can be obtained as

$$u_k \simeq \frac{\alpha_k}{\sqrt{2k}} e^{-iky} + \frac{\beta_k}{\sqrt{2k}} e^{iky}, \tag{3.43}$$

where the coefficients, α_k and β_k, are normalized as

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{3.44}$$

This normalization results from the Wronskian condition,

$$u_k \frac{\mathrm{d}u_k^*}{\mathrm{d}y} - u_k^* \frac{\mathrm{d}u_k}{\mathrm{d}y} = \mathrm{const.} = i, \tag{3.45}$$

where the factor of the right hand side is due to the canonical commutation relations, Eqs. (3.35) and (3.39). As the first step to determine α_k and β_k, we reminder the quantization in Minkowski spacetime.

In the Minkowski spacetime, the (initial) vacuum state which fixes the mode function at the initial time can be uniquely determined by requiring that the Hamiltonian at the state is minimized. The Hamiltonian can be written as

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3 x \left[\hat{\pi}^2 + \delta_{ij}\partial_i \hat{u} \partial_j \hat{u}\right], \tag{3.46}$$

where π is the conjugate momentum of u defined by $\pi := \delta S_u^{(2)}/\delta u = \mathrm{d}u/\mathrm{d}y$. Then, by moving to Fourier space, we have

$$\hat{H} = \frac{1}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} F_k^* + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger F_k + \left(2\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \delta(0)\right) E_k\right], \tag{3.47}$$

where

$$F_k := \left(\frac{\mathrm{d}u_k}{\mathrm{d}y}\right)^2 + k^2 u_k^2, \tag{3.48}$$

$$E_k := \left|\frac{\mathrm{d}u_k}{\mathrm{d}y}\right|^2 + k^2 |u_k|^2. \tag{3.49}$$

For a vacuum, $|0\rangle$, annihilated by $\hat{a}_{\mathbf{k}}$, the vacuum expectation value of the Hamiltonian is given by

$$\langle 0|\hat{H}|0\rangle = \frac{\delta(0)}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3} E_k. \tag{3.50}$$

In particular, by parametrizing the mode function as $u_k = r_k \exp(i\theta_k)$ with $r_k^2(\mathrm{d}\theta_k/\mathrm{d}y) = -1/2$ (i.e., the Wronskian condition) being imposed, E_k can be rewritten as

$$E_k = \left(\frac{\mathrm{d}r_k}{\mathrm{d}y}\right)^2 + \frac{1}{4r_k^2} + k^2 r_k^2, \tag{3.51}$$

which can be minimized for $\mathrm{d}r_k/\mathrm{d}y = 0$ and $r_k = 1/\sqrt{2k}$, and hence $\theta_k = -ky$. We thus have

$$u_k = \frac{1}{\sqrt{2k}} e^{-iky}, \tag{3.52}$$

in Minkowski spacetime. At the present stage, it should be emphasized that $\mathrm{d}E_k/\mathrm{d}y = 0$ holds in Minkowski while does not in a curved spacetime. In the FLRW spacetime, the coefficient in front of the second term in Eq. (3.49) changes from k^2 to $\omega^2(y)$. The time-dependence of ω^2 makes E_k time-dependent function. This makes it difficult to choose the vacuum state in a curved spacetime. However, as far as the modes are deep inside the horizon, one can use the WKB (adiabatic) approximation as

$$u_k \simeq \frac{1}{\sqrt{2k}} e^{-iky}, \tag{3.53}$$

even in the FLRW spacetime, and the above fixes α_k and β_k as $\alpha_k = 1$ and $\beta_k = 0$. In particular, the initial state with the positive frequency mode is called the Bunch-Davies vacuum state in de Sitter spacetime [30]. As different choices, the non-Bunch-Davies vacuum states have been also studied. The simplest example is given by Eq. (3.43) with non-vanishing β_k, and we will also study the case later.

Before moving to the explicit calculation, we consider the behaviors of the perturbations in different scales. The perturbations are deep inside the horizon (in subhorizon scales) in the far past. Then, those amplitudes are stretched due to the cosmic expansion, and finally the perturbations are outside the horizon (in superhorizon scales). Especially, the superhorizon modes evolve in time as

$$\psi_k \sim \text{const.} + \int^y \frac{\mathrm{d}y}{z^2} \sim \int^t \frac{\mathrm{d}t'}{a^3\mathcal{G}}, \tag{3.54}$$

which determines the power spectrum. In the case of the standard inflation model, the second term exponentially dumps and thus is responsible for the decaying mode. As a result, the amplitudes of the perturbations freeze out. Whereas in non-standard inflation models, e.g., the non-attractor inflation model in which $\mathcal{G}$ has a non-trivial time-dependence such that the second term in Eq. (3.54) increases with a time (see, e.g., Ref [31] as for the tensor perturbations), and also in alternatives to inflation, e.g., contracting models in which a decreases with a time, the would-be decaying mode grows [22]. In the following subsection, we deal with the standard inflation model and will do the alternatives to inflation in the next Chapter.

3.3 Primordial power spectrum

The Fourier transformation is defined by

$$\psi(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\psi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.55}$$

The two-point correlation function of the curvature perturbation is defined by[*1]

$$\langle\hat{\zeta}(\mathbf{k})\hat{\zeta}(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\mathcal{P}_\zeta, \tag{3.57}$$

where $\mathcal{P}_\zeta$ is the power spectrum of the curvature perturbation, and we have

$$\mathcal{P}_\zeta = \frac{k^3}{2\pi^2}|\zeta_k|^2 = \frac{k^3}{2\pi^2}\frac{|u_k|^2}{z^2}, \tag{3.58}$$

with ζ_k and z being time-dependent functions. The power spectrum is evaluated at the time of the end of the early epoch of the universe (e.g., the time of the end of inflation) when the perturbations are in superhorizon scales.

[*1] More explicitly, the correlation should be understood as the vacuum expectation value, that is

$$\langle\hat{\zeta}(\mathbf{k})\hat{\zeta}(\mathbf{k}')\rangle = \langle 0|\hat{\zeta}(\mathbf{k})\hat{\zeta}(\mathbf{k}')|0\rangle. \tag{3.56}$$

Similarly, the two-point correlation function of the tensor perturbation is defined by

$$\langle \hat{h}_{ij}(\mathbf{k})\hat{h}_{kl}(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\mathcal{P}_{ij,kl}, \tag{3.59}$$

where

$$\mathcal{P}_{ij,kl}(\mathbf{k}) = |h_k|^2\Pi_{ij,kl}(\mathbf{k}) = \frac{|v_k|^2}{z^2}\Pi_{ij,kl}(\mathbf{k}), \tag{3.60}$$

with

$$\Pi_{ij,kl} = \sum_s e^{(s)}_{ij}(\mathbf{k})e^{*(s)}_{kl}(\mathbf{k}). \tag{3.61}$$

By using the above, we can obtain the power spectrum of the tensor perturbations as

$$\mathcal{P}_h := \mathcal{P}_{ij,ij} = 2\frac{k^3}{2\pi^2}|h_k|^2 = \frac{k^3}{\pi^2}\frac{|v_k|^2}{z^2}. \tag{3.62}$$

This power spectrum is evaluated at the superhorizon scales.

In particular, the constraint on the primordial power spectrum of the tensor perturbations has been obtained from that on the tensor-to-scalar ratio as [4]

$$r := \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} < 0.064, \ (95\% \text{ CL}, \ \textit{Planck}\mathrm{TT,TE,EE+lowE+lensing+BK14}). \tag{3.63}$$

3.4 Primordial non-Gaussianity

The non-Gaussianity, the deviation from the Gaussian distribution, is characterized by the n-point correlation function with $n > 2$. For the Gaussian perturbation, the statistical property is completely determined by the two-point correlation function. In the linear perturbation theory, each mode completely decouples each other, but the modes interact with each other at a non-linear level, and thus initially Gaussian perturbations inevitably change the statistical property. In the following subsection, we introduce the method of the computation of the non-Gaussianity.

3.4.1 In-in formalism

First, we briefly review the in-in formalism [32] which is the way to compute the non-Gaussianity. In the interaction picture, the three-point correlation functions of the perturbations, e.g., the scalar and tensor perturbations, (denoted by $\psi(t,\mathbf{k})$ together) can be computed as

$$\langle \hat{\psi}(t_f,\mathbf{k}_1)\hat{\psi}(t_f,\mathbf{k}_2)\hat{\psi}(t_f,\mathbf{k}_3)\rangle = \langle \Omega|\hat{\psi}(t_f,\mathbf{k}_1)\hat{\psi}(t_f,\mathbf{k}_2)\hat{\psi}(t_f,\mathbf{k}_3)|\Omega\rangle, \tag{3.64}$$

where

$$|\Omega\rangle = T\exp\left(-i\int_{t_i}^{t_f} \mathrm{d}t H_{\mathrm{int}}(t)\right)|0\rangle, \tag{3.65}$$

with T being the time ordering operator, and H_{int} being the interaction Hamiltonian. In the interaction picture, the Hamiltonian, H, is composed of a free part H_0, and non-linear parts, H_{int}. The former decides the time evolution of the perturbation while the latter does that of the state. In particular, H_0 and H_{int} are derived from the quadratic and cubic (or higher-order) actions, respectively, and we define the interaction Hamiltonian by $H_{\mathrm{int}} := -\int \mathrm{d}^3x\mathcal{L}_\psi^{(3)}$ which is thus of $\mathcal{O}(\psi^3)$. Here, we labeled i and f such that the perturbations are deep inside the horizon at $t=t_i$ and the early epoch of the universe ends at t_f. To first order in H_{int}, we have

$$\langle \hat{\psi}(t_f,\mathbf{k}_1)\hat{\psi}(t_f,\mathbf{k}_2)\hat{\psi}(t_f,\mathbf{k}_3)\rangle \simeq -i\int_{t_i}^{t_f} \mathrm{d}t\langle 0|[\hat{\psi}(t_f,\mathbf{k}_1)\hat{\psi}(t_f,\mathbf{k}_2)\hat{\psi}(t_f,\mathbf{k}_3),H_{\mathrm{int}}(t')]|0\rangle, \tag{3.66}$$

where we supposed that the perturbations at $t=t_i$ to be Gaussian. We can know by computing Eq. (3.66) that how the statistical properties deviate from the Gaussian distribution due to the non-linear interactions among the perturbations from $t=t_i$ (subhorizon scales) to $t=t_f$ (superhozion scales).

We explain how to calculate the above by focusing on the cubic operator of the form, $\mathcal{L}_\zeta^{(3)} \sim C\dot{\zeta}^3$ with C being a constant. The quantized curvature perturbation in Fourier space can be written as

$$\hat{\zeta}(t,\mathbf{k}) = \zeta_k\hat{a}_{\mathbf{k}} + \zeta_k^*\hat{a}^\dagger_{-\mathbf{k}}. \tag{3.67}$$

It is instructive to first deal with the two-point correlation function. By using the symbol of the Wick's contraction, the product of the operators can be decomposed into two parts in which the each vacuum expectation values of the first and another one vanishes and does not, respectively, as

$$\hat{\zeta}(t,\mathbf{k})\hat{\zeta}(t,\mathbf{k}') = \contraction{}{\hat{\zeta}}{(t,\mathbf{k})}{\hat{\zeta}}\hat{\zeta}(t,\mathbf{k})\hat{\zeta}(t,\mathbf{k}') + (\cdots), \tag{3.68}$$

where $\langle(\cdots)\rangle$ vanishes by construction. By using this symbol, one can compute the two-point correlation function as

$$\langle\hat{\zeta}(t,\mathbf{k})\hat{\zeta}(t,\mathbf{k}')\rangle = \zeta_k(t)\zeta^*_{k'}(t)\langle 0|\hat{a}_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}'}|0\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')|\zeta_k(t)|^2. \tag{3.69}$$

Then, we move to the computation of Eq. (3.66). Similarly, one can write the r.h.s. of Eq. (3.66) as

$$\begin{aligned}
&\langle\hat{\zeta}(t_f,\mathbf{k}_1)\hat{\zeta}(t_f,\mathbf{k}_2)\hat{\zeta}(t_f,\mathbf{k}_3)H_{\rm int}(t')\rangle \\
&= C\int \mathrm{d}^3x \frac{\mathrm{d}^3\tilde{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\tilde{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\tilde{k}_3}{(2\pi)^3}\langle\hat{\zeta}(t_f,\mathbf{k}_1)\hat{\zeta}(t_f,\mathbf{k}_2)\hat{\zeta}(t_f,\mathbf{k}_3)\dot{\hat{\zeta}}(t,\tilde{\mathbf{k}}_1)\dot{\hat{\zeta}}(t,\tilde{\mathbf{k}}_2)\dot{\hat{\zeta}}(t,\tilde{\mathbf{k}}_3)\rangle \\
&\quad + (\text{sym. of } 1,2,3) \qquad (3.70)\\
&= (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\times 3!C\zeta_{k_1}(t_f)\zeta_{k_2}(t_f)\zeta_{k_3}(t_f)\dot{\zeta}^*_{k_1}(t)\dot{\zeta}^*_{k_2}(t)\dot{\zeta}^*_{k_3}(t), \qquad (3.71)
\end{aligned}$$

where we discarded some terms contracted by different combinations which are canceled out with the conjugated themselves in Eq. (3.66) of which the integrand in the r.h.s. is composed of the above and its conjugated term. Thus those do not contribute to the resultant non-Gaussianity. Finally, one can obtain the explicit form of Eq. (3.66) regarding the cubic operator of the form $\dot{\zeta}^3$ as

$$\begin{aligned}
&\langle\hat{\zeta}(t_f,\mathbf{k}_1)\hat{\zeta}(t_f,\mathbf{k}_2)\hat{\zeta}(t_f,\mathbf{k}_3)\rangle \\
&\simeq (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\mathrm{Im}\left[12\zeta_{k_1}(t_f)\zeta_{k_2}(t_f)\zeta_{k_3}(t_f)\int_{t_i}^{t_f}\mathrm{d}t' C\dot{\zeta}^*_{k_1}(t)\dot{\zeta}^*_{k_2}(t)\dot{\zeta}^*_{k_3}(t)\right].
\end{aligned} \tag{3.72}$$

Similarly, one can derive the concrete expressions for the other cubic operators.

3.4.2 Amplitudes and Shapes of non-Gaussianities

In this subsection, we investigate the amplitudes and shapes of the primordial non-Gaussianities. We first focus on the amplitudes.

The curvature perturbation is composed of both the linear and non-linear parts, and the explicit form first introduced in Ref. [33] is*2

$$\zeta(\mathbf{x}) = \zeta_{\rm L}(\mathbf{x}) + \frac{3}{5}f_{\rm NL}^{\rm local}\left[\zeta_{\rm L}^2(\mathbf{x}) - \langle\zeta_{\rm L}^2(\mathbf{x})\rangle\right]. \tag{3.74}$$

*2 The original definition of the non-linearity parameter is [33]

$$\Phi(\mathbf{x}) = \Phi_{\rm L}(\mathbf{x}) + f_{\rm NL}^{\rm local}\left[\Phi_{\rm L}^2(\mathbf{x}) - \langle\Phi_{\rm L}^2(\mathbf{x})\rangle\right], \tag{3.73}$$

where Φ is the gravitational potential, and the factor $3/5$ in Eq. (3.74) is due to the fact that ζ is related with Ψ as $\Phi \simeq (3/5)\zeta$ during the matter-dominated era [26].

The above expression is local in real space, and hence the corresponding non-linearity parameter, $f_{\rm NL}$, is conventionally labeled by local. To first order in the non-linear correction, one can calculate the three-point correlation function as

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{6}{5}(2\pi^2)^2\mathcal{P}_\zeta^2\frac{\sum_i k_i^3}{k_1^3k_2^3k_3^3}f_{\rm NL}^{\rm local}, \tag{3.75}$$

where we assumed that the power spectrum is scale invariant. In light of this result, the general form of the non-linearity parameter is defined by

$$f_{\rm NL}(k_1,k_2,k_3) := \frac{5}{6}\frac{\mathcal{B}_\zeta(k_1,k_2,k_3)}{(2\pi^2)^2\mathcal{P}_\zeta^2}\frac{k_1^3k_2^3k_3^3}{\sum_i k_i^3}, \tag{3.76}$$

where $\mathcal{B}_\zeta$ is the bispectrum of the curvature perturbation defined by

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle =: (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\mathcal{B}_\zeta. \tag{3.77}$$

Next, we focus on the shape dependence of the bispectrum which is characterized by the shape function, $S(k_1,k_2,k_3)$, defined by [34, 35]

$$S(k_1,k_2,k_3) := N(k_1k_2k_3)^2\mathcal{B}_\zeta(k_1,k_2,k_3), \tag{3.78}$$

where N is some normalization factor, and usually normalized such that $S(k,k,k) = 1$. This shape function can be written in terms of the non-dimensional quantities, $y := k_2/k_1$ and $z := k_3/k_1$, and thus one can express it as $S = S(1,y,z)$. For example, the bispectrum of the local type has a peak at the squeezed configuration which is characterized by one long-wavelength and two short-wavelength modes, i.e., $k_1 \ll k_2 = k_3$, and the explicit shape function is plotted in Fig.3.1. In addition to this, there are several types which have been well studied, e.g., the equilateral, $k_1 = k_2 = k_3$, and flattened, $k_1 = k_2 + k_3$, configurations. The bispectra peaked at the equilateral and flattened configurations are plotted in Fig.3.2 and Fig.3.3. As we will see later, the shape dependence of the non-Gaussianity is useful to discriminate among the early universe models.

Here, the current CMB observations have put constraints on the non-linearity parameter at the squeezed, $k_1 \ll k_2 = k_3$, and equilateral, $k_1 = k_2 = k_3$, configurations as [36]

$$f_{\rm NL}^{\rm local} = -0.9 \pm 5.1, \tag{3.79}$$
$$f_{\rm NL}^{\rm eq} = -26 \pm 47. \tag{3.80}$$

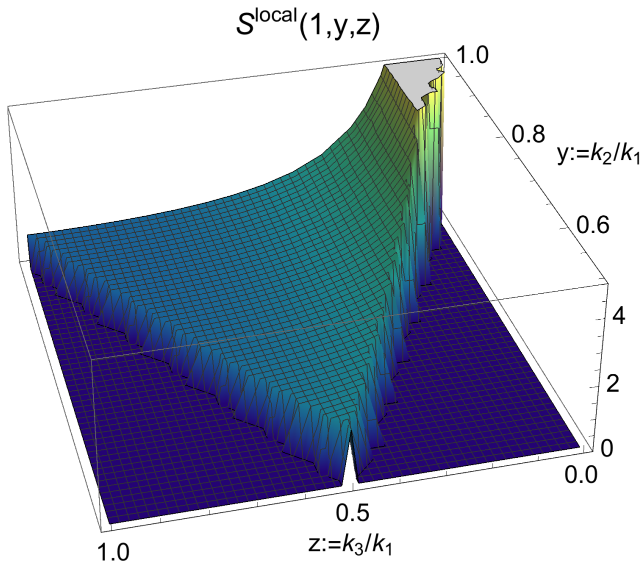

Fig.3.1 The shape function of the local type, $S(k_1, k_2, k_3) \propto (\sum_i k_i^3)/(k_1 k_2 k_3)$, as a function of $y = k_2/k_1$ and $z = k_3/k_1$. The plot is normalized to 1 for the equilateral configuration, $y = 1 = z$.

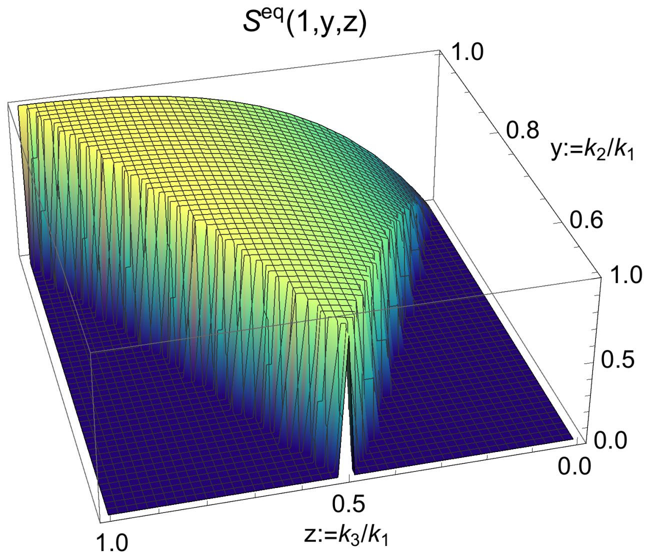

Fig.3.2 The shape function of the equilateral type, $S(k_1, k_2, k_3) \propto k_1 k_2 k_3/K^3$, as a function of $y = k_2/k_1$ and $z = k_3/k_1$. The plot is normalized to 1 for the equilateral configuration, $y = 1 = z$.

3.5 Application 1: Inflation

We use the results in the previous section, i.e., we impose $\epsilon(= \text{const}) \ll 1$ and $X \ll V$. Note also that the new time coordinate, y, coincides with the conformal

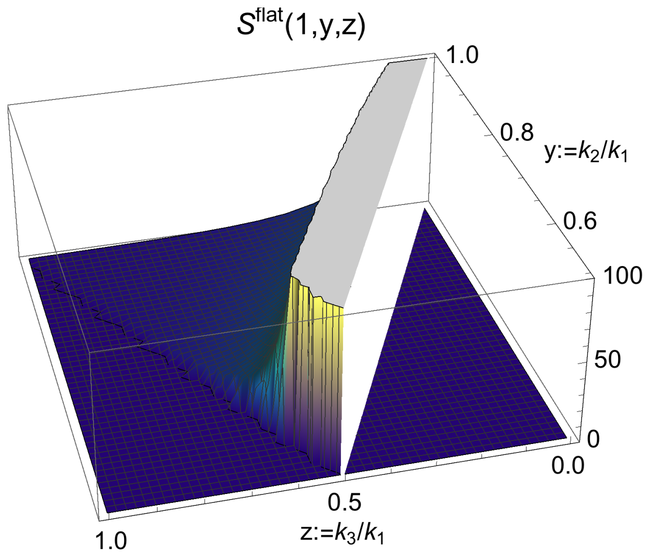

Fig.3.3 The shape function of the flattened type, $S(k_1,k_2,k_3) \propto k_1k_2k_3[(-k_1+k_2+k_3)^{-1}+(1\leftrightarrow 2)+(1\leftrightarrow 3)]$ as a function of $y=k_2/k_1$ and $z=k_3/k_1$. The plot is normalized to 1 for the equilateral configuration, $y=1=z$.

time, η, whose region is $-\infty<\eta<0$.

3.5.1 Power spectrum of the curvature perturbation

We denote the mode function of the curvature perturbation as ζ_k, and that obeys

$$\zeta_k''+\left(k^2-\frac{\nu_s^2-1/4}{\eta^2}\right)\zeta_k=0, \tag{3.81}$$

where $\nu_s=3/2+2\epsilon-\tilde{\epsilon}+\mathcal{O}(\epsilon^2,\tilde{\epsilon}^2)$ with $\tilde{\epsilon}:=-\ddot{H}/(2H\dot{H})\ll 1$. Then, the solution with the Bunch-Davies initial state can be derived as

$$\zeta_k=\frac{\sqrt{\pi}}{2}\sqrt{-\eta}H_{\nu_s}^{(1)}(-k\eta), \tag{3.82}$$

where $H_{\nu_s}^{(1)}$ is the Hankel function of the first kind. The modes are in subhorizon scales in the far past, $\eta\to-\infty$ ($|k\eta|\gg 1$), while are in superhorizon scales at the end of inflation, $\eta\to 0$ ($|k\eta|\ll 1$). Using the solution in the superhorizon scales, one can obtain the power spectrum and the spectral index as

$$\mathcal{P}_\zeta=\left.\frac{|\zeta_k|^2}{z^2}\right|_{|k\eta|\ll 1}\simeq\frac{1}{2M_{\rm Pl}^2\epsilon}\frac{H^2}{4\pi^2}, \tag{3.83}$$

$$n_s-1=3-2\nu_s\simeq 2\tilde{\epsilon}-4\epsilon. \tag{3.84}$$

3.5.2 Power spectrum of the tensor perturbations

Similarly to the case of the curvature perturbation, the mode function can be derived as

$$v_k = \frac{\sqrt{\pi}}{2}\sqrt{-\eta}H^{(1)}_{\nu_t}(-k\eta), \tag{3.85}$$

where $\nu_t = 3/2 + \epsilon + \mathcal{O}(\epsilon^2)$. As a result, the power spectrum and the spectral index read

$$\mathcal{P}_h \simeq \frac{8}{M_{\rm Pl}^2}\frac{H^2}{4\pi^2}, \tag{3.86}$$

$$n_t = 3 - 2\nu_t \simeq -2\epsilon, \tag{3.87}$$

where we parametrized the power spectrum as $\mathcal{P}_h \propto k^{n_t}$. Combining both results, we have the tensor-to-scalar ratio,

$$r \simeq 16\epsilon \simeq -8n_t. \tag{3.88}$$

which is the so-called consistency relation. We thus find that the small deviation from the exactly scale-invariant power spectrum of ζ and the small tensor-to-scalar ratio can be represented by the deviation from the exact-de Sitter spacetime.

As seen previously, it is one of the successful points of inflation that the observable quantities are related to the symmetry of the spacetime. Whereas in the case of the bounce models, it is difficult to construct the models based on some symmetry argument, and observable quantities are not associated with the violation of spacetime symmetry. In addition, the models are often incompatible with the observational constraints. Therefore, we do not deal with the bounce models here, and explain the details of the perturbations in the models in the next section.

3.5.3 Non-Gaussianities

• Scalar non-Gaussianity

From now on, we introduce some inflation models which predict the bispectra of the local, equilateral, and flattened configurations.

(1) Canonical scalar field

The shape function is [37]

$$S_{\rm canonical} \propto \frac{1}{k_1k_2k_3}\left[\epsilon\left(-\frac{1}{8}\sum_i k_i^3 + \frac{1}{8}\sum_{i\neq j} k_ik_j^2 + \frac{1}{K}\sum_{i>j} k_i^2k_j^2\right) + \frac{\tilde{\epsilon}}{8}\sum_i k_i^3\right]. \tag{3.89}$$

Both terms multiplied by ϵ and $\tilde{\epsilon}$ have peaks at the squeezed configurations. Then, the non-linear parameter of the local type is of $\mathcal{O}(\epsilon, \tilde{\epsilon}) \ll 1$. This is consistent with the current upper bound on $f_{\rm NL}^{\rm local}$. We thus find that the standard inflation models predict a small scalar non-Gaussianity having a peak at the squeezed configuration.

In subhorizon scales, the integrand in the time integral in Eq. (3.66) is proportional to $e^{-iK\eta}$ with $K := k_1 + k_2 + k_3$. (We denote the sum of k_i as K apart from the k-essence field, $K(\phi, X)$, without any confusion.) This indicates that the interactions in the scales do not contribute to the generation of the non-Gaussianity since the rapid oscillation, $|K\eta| = |K/(aH)| \to \infty$, makes the integral exponentially suppressed. Therefore, it can be anticipated that the non-Gaussianity is generated by the interactions among the scalar perturbations whose physical wavenumbers, $k_{i,\rm phys} = k_i/a$, satisfy $H \lesssim k_{i,\rm phys}$. Therefore, the long-wavelength mode in the definition of the squeezed configuration is in superhorizon scales sufficiently before the other modes cross the horizon. In the standard inflation models, the amplitude of the perturbation freezes out in the superhorizon scales, and thus one can naively anticipate that the sizable squeezed non-Gaussianity is not generated because the operators are differentiated with respect to time and/or spatial derivatives. The example to generate the squeezed non-Gaussianity (not slow-roll suppressed) is the early universe model in which the would-be decaying mode grows in time and thus the growing mode dominates the constant one. Below we will introduce two examples which predict different shapes.

(2) Higher-derivative operators

Let us change the Lagrangian of the canonical scalar field to that of the k-essence field. In this theory, the higher-derivative terms can be included, e.g., $\mathcal{L} \supset c(\nabla\phi)^4$ with c being a constant. Then, the shape function is modified as [37]

$$\begin{aligned} S_{\rm h.d.} \propto \frac{1}{k_1 k_2 k_3} \Bigg[& \left(\frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma}\right) \frac{3k_1^2 k_2^2 k_3^2}{2K^3} \\ & + \left(\frac{1}{c_s^2} - 1\right) \left(-\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3\right) \\ & + \frac{\epsilon}{c_s^2} \left(-\frac{1}{8} \sum_i k_i^3 + \frac{1}{8} \sum_{i \neq j} k_i k_j^2 + \frac{1}{K} \sum_{i>j} k_i^2 k_j^2\right) + \frac{\tilde{\epsilon}}{8c_s^2} \sum_i k_i^3 \\ & + \frac{f_s}{c_s^2} \left(-\frac{1}{4} \sum_i k_i^3 - \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3\right) \Bigg]. \end{aligned} \tag{3.90}$$

with $f_c := \dot{c}_s/(Hc_s)$. The shapes in the first two lines have peaks at the equilateral configurations while the others do at the squeezed ones. The non-linearity parameter at the equilateral configuration associated with the first two terms reads

$$f_{\rm NL}^{\rm eq} = \mathcal{O}\left(\frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma}\right) + \mathcal{O}\left(\frac{1}{c_s^2} - 1\right). \tag{3.91}$$

Here, we consider the effects of the higher-derivative terms on the non-Gaussianity. For example, we find from the form of c_s^2 that the higher-derivative terms can realize $c_s^2 = \mathcal{O}(1)$ (but deviates from unity) and also $c_s^2 \ll 1$ only if $K_X \ll XK_{XX}$ is satisfied. In this case, the magnitudes of the interactions among the perturbations can be enhanced since the perturbed Lagrangian includes some cubic operators multiplied by negative powers of c_s such as $\Sigma/(c_s^2 H^3)\dot{\zeta}^3$ and $\epsilon/(c_s^4)\zeta\dot{\zeta}^2$. However, the case of the small c_s^2 is excluded due to the constraint on $f_{\rm NL}^{\rm eq}$. Therefore, the viable slow-roll inflation models within the k-essence theory are restricted to the cases of $c_s^2 = \mathcal{O}(1)$. Even in such cases, the presence of the equilateral shape and the non-linearity parameter of $\mathcal{O}(1)$ would be useful to discriminate between the standard model and the non-standard one.

(3) Non-Bunch-Davies effects

So far we have focused on the case of the Bunch-Davies initial state. In principle, one can also consider the non-Bunch-Davies initial states. Now let me first introduce the resultant non-Gaussian shapes from the non-Bunch-Davies states. In Ref. [8], it has been found that the scalar non-Gaussianities from inflation models with the non-Bunch-Davies initial states can have peaks at the flattened configuration. This result can be easily understood. As opposed to the Bunch-Davies state, both positive and negative frequency modes exist in the cases of the non-Bunch-Davies states. This indicates that the integrand in the time integral in Eq. (3.66) includes the term proportional to $e^{-i(-k_1+k_2+k_3)\eta}$ which does not oscillate at the flattened configuration even in the far past, $|k_i\eta| \gg 1$. This results in the generation of the non-Gaussianity in subhorizon scales, and finally the flattened non-Gaussianity is predicted.

- Tensor non-Gaussianity

The cubic action of the tensor perturbations is

$$S_h^{(3)} = \int {\rm d}t{\rm d}^3x a^3 \left[\frac{M_{\rm Pl}^2}{4a^2}\left(h_{ik}h_{jl} - \frac{1}{2}h_{ij}h_{kl}\right)\partial_k\partial_l h_{ij}\right] =: -\int {\rm d}t H_{\rm int}^h. \tag{3.92}$$

By using the interaction Hamiltonian, $H_{\rm int}^h$, and the solution of the mode function (3.85), one can compute the three-point correlation function which is defined by

$$\langle \xi^{s_1}(t_f,\mathbf{k}_1)\xi^{s_2}(t_f,\mathbf{k}_2)\xi^{s_3}(t_f,\mathbf{k}_3)\rangle := (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\mathcal{B}_h^{s_1s_2s_3}, \tag{3.93}$$

where $\mathcal{B}_h^{s_1s_2s_3}$ is the bispectrum. The bispectrum can be obtained as

$$\begin{aligned}\mathcal{B}_h^{s_1s_2s_3} = \frac{(2\pi)^4\mathcal{P}_h^2}{k_1^3k_2^3k_3^3}\mathcal{A}\Big\{&\Pi_{i_1j_1,ik}(\mathbf{k}_1)\Pi_{i_2j_2,jl}\left[k_{3k}k_{3l}\Pi_{i_3j_3,ij}-\frac{1}{2}k_{3i}k_{3k}\Pi_{i_3j_3,jl}\right]\\ &+5\text{ perms of }1,2,3\Big\}e^{*(s_1)}_{i_1j_1}(\mathbf{k}_1)e^{*(s_2)}_{i_2j_2}(\mathbf{k}_2)e^{*(s_3)}_{i_3j_3}(\mathbf{k}_3),\end{aligned} \tag{3.94}$$

where

$$\mathcal{A} := -\frac{K}{16}\left[1-\frac{1}{K^3}\sum_{i\neq j}k_i^2k_j-4\frac{k_1k_2k_3}{K^3}\right]. \tag{3.95}$$

Then, we derive the explicit form of the polarization tensor. (See, e.g., Ref [38].) First, one can set the direction of the propagation of the graviton, $\mathbf{k}_1$, as z-axis. In light of this, one can parametrize the wave vectors as

$$\mathbf{k}_1 = k_1(1,0,0),\ \mathbf{k}_2 = k_2(\cos\theta,\sin\theta,0),\ \mathbf{k}_3 = k_3(\cos\phi,\sin\phi,0), \tag{3.96}$$

where θ (ϕ) is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ ($\mathbf{k}_3$), and the angles satisfy $0\le\theta\le\pi$ and $\pi\le\phi\le 2\pi$. Here, by using $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{0}$ which is the consequence of the spatial rotational invariance (or equivalently the conservation law of the momentum), one can relate the angles with the magnitudes of the wave vectors as

$$\cos\theta = \frac{k_3^2-k_1^2-k_2^2}{2k_1k_2},\ \sin\theta = \frac{\sqrt{K(-k_1+k_2+k_3)(k_1-k_2+k_3)(k_1+k_2-k_3)}}{2k_1k_2}, \tag{3.97}$$

$$\cos\phi = \frac{k_2^2-k_3^2-k_1^2}{2k_1k_3},\ \sin\phi = -\frac{\sqrt{K(-k_1+k_2+k_3)(k_1-k_2+k_3)(k_1+k_2-k_3)}}{2k_1k_3}. \tag{3.98}$$

Based on these setup, one can derive the explicit form of the polarization tensor as

$$\mathbf{e}^{s_i}(\mathbf{k}_i) = \frac{1}{2}\begin{pmatrix}\sin^2\alpha & -\sin\alpha\cos\alpha & -is_i\sin\alpha\\ -\sin\alpha\cos\alpha & \cos^2\alpha & is_i\cos\alpha\\ -is_i\sin\alpha & is_i\cos\alpha & -1\end{pmatrix}, \tag{3.99}$$

where α takes $\alpha=0,\theta,\phi$ for $i=1,2,3$, respectively, and the polarization tensor for s_2 (s_3) can be obtained by rotating the $e^{(s_1)}_{ij}(\mathbf{k}_1)$ along θ (ϕ). By substituting Eq. (3.99) into Eq. (3.95), we can derive the resultant bispectrum as

$$B_h^{s_1s_2s_3} = \frac{(2\pi)^4\mathcal{P}_h^2}{k_1^3k_2^3k_3^3}\frac{\mathcal{A}}{2}(s_1k_1+s_2k_2+s_3k_3)^2F(s_1k_1,s_2k_2,s_3k_3), \tag{3.100}$$

where

$$F(x,y,z) := \frac{1}{64x^2y^2z^2}(x+y+z)^3(x-y+z)(x+y-z)(x-y-z). \qquad (3.101)$$

Interestingly, the amplitude of the tensor non-Gaussianity does not depend on the potential term of ϕ. This result can be straightforwardly extended to a subclass of the Horndeski theory such that $G_{5X} = 0$. In the subclass, the amplitude of the tensor non-Gaussianity does not depend on ϕ and X.

As well as the scalar non-Gaussianity, one may explore the other non-Gaussianities having different shapes. In the Horndeski theory with $G_{5X} \neq 0$, the another cubic interaction term of the form, $\dot{h}_{ij}^3$ can appear [39]. In Ref. [39], it has been found that the non-Gaussian amplitude depends on the functions of ϕ, X, and H, and also the non-Gaussianity of the equilateral shape can be generated.

In light of these results, one can find that the non-Gaussian amplitudes and shapes are powerful quantities to obtain rich information on the detail of the theory (form of the Lagrangian), initial conditions for the perturbations, and so on. In addition, it would be useful to classify the inflation models and discriminate among those. Also, it is important to investigate the non-Gaussian signatures from alternatives to inflation to distinguish inflation with alternatives. Before doing so, let us introduce some problems in alternatives to inflation in the following section.

3.6 Application 2: Alternatives to inflation

Alternatives to inflation not only resolve the various problems in the standard Big-Bang cosmology but also evade the initial singularity and Trans-Planckian problems. However, those suffer from some problems which are not problematic in inflation in general, and the problems should be resolved in order for the models to be viable. In this chapter, we review the problems and also recent progress on those.

3.6.1 Gradient instabilities

In Chapter 2, we have seen that one can construct the non-singular cosmological solutions within the k-essence theory at a background level. However, some problem occurs at a perturbation level. In the k-essence theory, the quadratic action of the curvature perturbation takes the form,

$$S_\zeta^{(2)} = \int \mathrm{d}t\mathrm{d}^3x a^3\left[\mathcal{G}_S\dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2}(\partial_i\zeta)^2\right], \qquad (3.102)$$

where

$$\mathcal{G}_S \simeq \frac{XK_X + 2X^2K_{XX}}{H^2}, \ \mathcal{F}_S \simeq M_{\rm Pl}^2\left(-\frac{\dot{H}}{H^2}\right). \tag{3.103}$$

In particular, the sign of $\mathcal{F}_S$ is negative if the NEC is violated (i.e., $\dot{H} > 0$), and thus the gradient instability is inevitable[*3].

Cubic Galileon theory

Next, we consider the cubic Galileon theory in which the second-order derivative, $\partial^2\phi$, is included and the action is

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big(R + K(\phi, X) - G_3(\phi, X)\Box\phi\Big). \tag{3.105}$$

In this theory, $\mathcal{F}_S$ takes the form,

$$\mathcal{F}_S = M_{\rm Pl}^2\left[\frac{M_{\rm Pl}^2}{a}\frac{\mathrm{d}}{\mathrm{d}t}\left(a\Theta^{-1}\right) - 1\right], \tag{3.106}$$

where $\Theta := M_{\rm Pl}^2H - \dot{\phi}XG_{3X}$. It can be easily seen that the signs of $\mathcal{F}_S$ and $\dot{H}$ can be positive simultaneously thanks to the non-vanishing G_{3X}.

This result may motivate one to construct the alternative models to inflation within the cubic galileon theory. However, it has been found that some non-singular cosmological solutions constructed within the cubic galileon theory are plagued with the gradient instability at somewhere even though the stability conditions hold in the NEC-violating phase. (See e.g., Refs [40, 41].) Based on these results, whether the instability is dependent on concrete models or not has been investigated in Ref [42]. One can rewrite the stability condition $\mathcal{F}_S > 0$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{aM_{\rm Pl}^4}{\Theta}\right) - aM_{\rm Pl}^2 > 0 \ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{aM_{\rm Pl}^4}{\Theta}\right) > aM_{\rm Pl}^2 > 0, \tag{3.107}$$

where we used $a > 0$, and the above indicates that $aM_{\rm Pl}^4/\Theta$ is a monotonically increasing function of t. Then by assuming that $\mathcal{F}_S$ is a continuous function of t,

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*3 For the high-frequency modes ($k \to \infty$), the solution of the curvature perturbation exponentially grows in time as

$$\zeta \sim \exp\left(k\int_{\eta_-}^{\eta_+}|c_s|\mathrm{d}\eta\right), \tag{3.104}$$

where $\Delta\eta = \eta_+ - \eta_-$ is the duration during which the instability is occurring.

i.e., Θ does not cross 0, it can be seen that

$$\int_{-\infty}^{t} a(t')M_{\rm Pl}^2 {\rm d}t' < \infty, \text{ or } \int_{t}^{\infty} a(t')M_{\rm Pl}^2 {\rm d}t < \infty, \tag{3.108}$$

which prohibits the non-singular cosmological solutions. This is the reason why even the solutions in the cubic galileon theory are plagued the gradient instability.

Horndeski theory

In Ref. [43], a no-go theorem stating that the gradient instability is inevitable within the Horndeski theory has been proven. Let us illustrate the no-go argument.

In the Horndeski theory, $\mathcal{F}_S$ can be obtained as

$$\mathcal{F}_S = \frac{1}{a}\frac{\rm d}{{\rm d}t}\left(\frac{a\mathcal{G}_T^2}{\Theta}\right) - \mathcal{F}_T, \tag{3.109}$$

where

$$\mathcal{G}_T := 2\left[G_4 - X\left(\ddot{\phi}G_{5X} + G_{5\phi}\right)\right], \tag{3.110}$$

$$\mathcal{F}_T := 2\left[G_4 - 2XG_{4X} - X\left(H\dot{\phi}G_{5X} - G_{5\phi}\right)\right], \tag{3.111}$$

which are the coefficients in the quadratic action of the tensor perturbations and thus we require that $\mathcal{G}_T,\ \mathcal{F}_T > 0$ to avoid the ghost and gradient instabilities.

Here, we impose $a > 0$ to characterize the non-singular cosmologies. Then, we can rewrite the condition, $\mathcal{F}_S > 0$, as

$$\frac{{\rm d}\xi}{{\rm d}t} > a\mathcal{F}_T > 0,\ \xi := \frac{a\mathcal{G}_T^2}{\Theta}. \tag{3.112}$$

indicating that

$$\int_{-\infty}^{t} a\mathcal{F}_T {\rm d}t', \text{ or } \int_{t}^{\infty} a\mathcal{F}_T {\rm d}t' \tag{3.113}$$

should converge. Thanks to the non-minimal coupling between a scalar field and gravity, the convergent condition now can hold even in non-singular cosmologies. The concrete example was presented in Ref. [43].

Here, we consider the meaning of the convergent integral. Via the disformal transformation [44], one can move to the Einstein frame for tensor perturbations, that is $\mathcal{G}_T$ and $\mathcal{F}_T$ are constant. The transformation is given by[*4]

$$\tilde{a} = M_{\rm Pl}^{-1}\mathcal{F}_T^{1/4}\mathcal{G}_T^{1/4}a, \tag{3.118}$$

$$d\tilde{t} = M_{\rm Pl}^{-1}\mathcal{F}_T^{3/4}\mathcal{G}_T^{-1/4}dt. \tag{3.119}$$

In Ref. [45], by using the fact that we have

$$\tilde{a}d\tilde{t} = a\mathcal{F}_T dt, \tag{3.120}$$

which is the affine parameter of the null geodesics in the Einstein frame, it has been found that the convergence of the integral (3.113) indicates past (or future) incompleteness of geodesics of graviton. (See also Ref. [46]).

Here, we also consider the pathology of gravitons based on the another approach proposed in Ref. [47]. The tensor perturbation h_{ij} obeys

$$Z^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu h_{ij} = 0, \tag{3.121}$$

where

$$Z_{\mu\nu}dx^\mu dx^\nu = -\frac{\mathcal{F}_T^{3/2}}{\mathcal{G}_T^{1/2}}dt^2 + a^2\left(\mathcal{F}_T\mathcal{G}_T\right)^{1/2}\delta_{ij}dx^i dx^j, \tag{3.122}$$

and $\mathcal{D}_\mu$ is the covariant derivative defined in the new geometry characterized by $Z_{\mu\nu}$. One can thus find from Eq. (3.121) that the paths of gravitons can be regarded as the null geodesics in the geometry defined by the effective metric, and the affine parameter λ of null geodesics is then given by $d\lambda = a\mathcal{F}_T dt$. Hence, it can be seen that the geodesic incompleteness of gravitons can be proven without performing the disformal transformation.

3.6.2 Evading the no-go theorem

Before closing this section, we introduce how to circumvent the no-go theorem.

[*4] When the quadratic action of the perturbation can be written as

$$S_\psi^{(2)} = \int dt d^3x a^3\left[\mathcal{G}\dot{\psi}^2 - \frac{\mathcal{F}}{a^2}(\partial_i\psi)^2\right], \tag{3.114}$$

the disformal transformation such that $\mathcal{G}, \mathcal{F} = \text{const}$ is

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\partial_\mu\phi\partial_\nu\phi. \tag{3.115}$$

where

$$A = M^{-2}\sqrt{\mathcal{G}\mathcal{F}}, \tag{3.116}$$

$$B = \frac{M^{-2}\sqrt{\mathcal{G}\mathcal{F}}}{2X}\left(1 - \frac{\mathcal{F}}{\mathcal{G}}\right) = \frac{A}{2X}\left(1 - \frac{\mathcal{F}}{\mathcal{G}}\right). \tag{3.117}$$

Positive spatial curvature

In a non-flat FLRW spacetime, the condition to avoid the gradient instability within the Horndeski theory is modified as [48]

$$\mathcal{F}_S := \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a\mathcal{G}_T^2}{\Theta_\mathcal{K}}\right) - \mathcal{F}_T + \frac{\mathcal{G}_T^3}{\Theta_\mathcal{K}^2}\frac{\mathcal{K}}{a^2} > 0, \tag{3.123}$$

where

$$\Theta_\mathcal{K} := \Theta - \dot{\phi}XG_{5X}\frac{\mathcal{K}}{a^2}. \tag{3.124}$$

The stability condition can be written as

$$\frac{\mathrm{d}\xi_\mathcal{K}}{\mathrm{d}t} > a\left(\mathcal{F}_T - \frac{\mathcal{G}_T^3}{\Theta_\mathcal{K}^2}\frac{\mathcal{K}}{a^2}\right),\ \xi_\mathcal{K} := \frac{a\mathcal{G}_T^2}{\Theta_\mathcal{K}}. \tag{3.125}$$

In open universes, the convergence of $\int a\mathcal{F}_T\mathrm{d}t$ is required and hence the no-go theorem holds. Whereas in closed universes, the convergence is not always required, and the no-go argument cannot be applied to the universes. The simplest example which evades the no-go theorem is the quasi-de Sitter solution accomplished by a canonical scalar field in a closed FLRW spacetime.

Beyond Horndeski

One of strategies to circumvent the instability is invoking beyond Horndeski terms which change the dispersion relation of the curvature perturbation such that k^4-term or higher-order terms are included. For example, when the dispersion relation is modified as

$$\omega^2 = c_s^2k^2 + \alpha k^4, \tag{3.126}$$

even though c_s^2 becomes negative, the stability for high frequency modes can be guaranteed if and only if α is positive. Also, when the higher-order terms are invoked, the positivity of the coefficient in front of the highest-order term in the dispersion relation can guarantee stability. Based on this strategy, some healthy non-singular cosmological solutions have been found so far. (See, e.g., Refs [43, 49].)

Cuscuton theory

When we consider a scalar field ϕ in GR, ϕ contributes to the dynamical degrees of freedom in general. However, some specific Lagrangian of ϕ in which the coefficient of $\ddot{\phi}$ in the equation of motion for ϕ vanishes can keep the degrees of freedom unchanged. In Ref. [50]. such a theory, the so-called Cuscuton theory, has been proposed, and the Lagrangian with a matter field σ is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + X - \mu^2\sqrt{2|Y|} - V(\sigma)\right], \tag{3.127}$$

where $Y := -(1/2)g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma$. In this theory, the scalar degree of freedom regarding σ does not propagate, and thus the dynamical degrees of freedom are one scalar (ϕ) and two tensor modes. Then, the quadratic action of the curvature perturbation under the unitary gauge, $\sigma = \sigma(t)$, can be obtained as [51]

$$S_\zeta^{(2)} = \int \mathrm{d}t\mathrm{d}^3ka^3\left[\mathcal{G}_\zeta\dot{\zeta}^2 - \frac{\mathcal{F}_\zeta k^2}{a^2}\zeta^2\right], \tag{3.128}$$

where $\mathcal{F}_\zeta$ for the high frequency modes is given by

$$\mathcal{F}_\zeta \simeq \frac{X}{H^2} = M_{\rm Pl}^2\left(-\frac{\dot{H}}{H^2}\right) + \frac{\mu^2\sqrt{2Y}}{H^2}, \tag{3.129}$$

which can be positive even for the case $\dot{H} > 0$, and the stability condition does not lead to the geodesic incompleteness for graviton. In Ref. [51], a concrete bouncing solution without any pathologies for the perturbations has been constructed.

3.6.3 Inconsistency with observational evidences

In order for the early universe models to be viable, those should be consistent with CMB observations. In this section, we introduce a problem regarding the large tensor-to-scalar ratio and/or a large scalar non-Gaussianity based on [52, 53]

First, we consider a matter-dominated contracting model in GR with a canonical scalar field. The Friedmann and evolution equations are given, respectively, by

$$\mathcal{E} = -3M_{\rm Pl}^2H^2 + \rho_\phi \sim t^{-2}, \tag{3.130}$$
$$\mathcal{P} = M_{\rm Pl}^2(3H^2 + 2\dot{H}) + P_\phi \sim t^{-2}, \tag{3.131}$$

where $\rho_\phi := X + V$ and $P_\phi := X - V$. The phase which we are focusing on is a matter-dominated contracting one, and thus $a \propto \eta^2$ with η being the conformal time. Within this theory, both the canonically normalized scalar and tensor perturbations obey the same equation as,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\eta^2} + \left(k^2 - \frac{2}{\eta^2}\right)u = 0. \tag{3.132}$$

By solving this equations, we obtain the curvature and tensor perturbations in Fourier space as

$$\zeta(\eta, k) = -\frac{i}{2M_{\rm Pl}\sqrt{\epsilon}}\frac{k^{-3/2}}{a\eta}(1+ik\eta)e^{-ik\eta}, \tag{3.133}$$

$$h_{ij}(\eta, k) = -\frac{\sqrt{2}i}{M_{\rm Pl}}\frac{k^{-3/2}}{a\eta}(1+ik\eta)e^{-ik\eta}e_{ij}. \tag{3.134}$$

It can be seen from the above that both curvature and tensor perturbations scale as $\sim \eta^{-3}$ at the superhorizon scales, $|k\eta| \ll 1$. This is in contrast with the case of the standard inflation model in which the amplitudes of the perturbations freeze out at the superhorizon scales.[*5]

Then, the power spectra of ζ and h_{ij} evaluated at the end of a contracting phase ($\eta = \eta_*$) are obtained as

$$\mathcal{P}_\zeta = \frac{k^3}{2\pi^2}|\zeta|^2 \simeq \frac{1}{8\pi^2 M_{\rm Pl}^2\epsilon}\frac{1}{a^2(\eta_*)\eta_*^2}, \tag{3.136}$$

$$\mathcal{P}_h = \frac{k^3}{2\pi^2}|h_{ij}|^2 \simeq \frac{2}{\pi^2 M_{\rm Pl}^2}\frac{1}{a^2(\eta_*)\eta_*^2}, \tag{3.137}$$

giving us the tensor-to-scalar ratio

$$r := \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} \simeq 16\epsilon = 24, \tag{3.138}$$

where I used $H = 2/(3t)$. We have $r = \mathcal{O}(10)$ since the usual slow-roll parameter is $\epsilon = \mathcal{O}(1)$ as opposed to inflation, $\epsilon \ll 1$. Therefore, the model presented here is excluded. However, several ways to obtain $r \ll 1$ in matter-dominated contracting models have been explored so far. We introduce two examples based on a single-field model in the next section.

Before moving to the next section, we also refer to the primordial non-Gaussianity. The successful point of the above example is to be able to generate a small scalar non-Gaussianity. In Ref. [54], it has been found that the non-linearity parameter is

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[*5] One may think that the observed value of the amplitude of ζ, i.e., $\mathcal{P}_\zeta = \mathcal{O}(10^{-9})$ requires some fine-tuning for the amplitude at the initial time. However, the amplitude in the sub-horizon scales behaves as

$$k^{3/2}|\zeta| = \mathcal{O}\left(\frac{k}{aM_{\rm Pl}}\right). \tag{3.135}$$

In a contracting universe, the physical wave number $k_{\rm phys} = k/a$ is sufficiently below the Planck scale, i.e., $k_{\rm phys} \ll M_{\rm Pl}$, and thus the very small amplitude of ζ in the past infinity can be naturally obtained.

$f_{\rm NL} = \mathcal{O}(\epsilon) = \mathcal{O}(1)$ at the local and equilateral configurations while the standard inflation models predict $f_{\rm NL} \ll 1$ due to the slow-roll suppression. Therefore, we need to discuss whether or not r can be $\ll 1$ while $f_{\rm NL}$ remains $\mathcal{O}(1)$. In light of this, we introduce the predictions from some bounce models below.

Growth of the curvature perturbation

Only the contracting phase has been focused on so far. In Ref. [52], it has been investigated whether or not the growth of the curvature perturbation during a bouncing phase can reduce the tensor-to-scalar ratio. Here, let us suppose that the curvature perturbation grows during the bouncing phase as

$$\zeta(\eta_b) = \zeta(\eta_*) + \Delta\zeta, \tag{3.139}$$

where η_b is the time of the end of a bouncing phase. By assuming that the amplitudes of the tensor modes do not change during the bouncing phase, the tensor-to-scalar ratio is modified as

$$r(\eta_b) = r(\eta_*)|1 + \mathcal{O}(\Delta\zeta/\zeta(\eta_*))|^{-2}. \tag{3.140}$$

Therefore, the growth such that $\Delta\zeta/\zeta(\eta_*) \gg 1$ can realize $r(\eta_b) \ll 1$ even though one has $r(\eta_*) = \mathcal{O}(10)$. However, the growth of ζ can enhance the amplitude of the scalar non-Gaussianity. Actually, in Ref. [52], it has been found that $f_{\rm NL}$ has a lower bound which is particularly proportional to $(\Delta\zeta/\zeta(\eta_*))^{5/2} \gg 1$, and the resultant $f_{\rm NL}$ satisfies $f_{\rm NL} \geq \mathcal{O}(10^2)$. We thus face the another problem, a large scalar non-Gaussianity.

Small propagation speed of the curvature perturbation

The different approach to obtain a small r has been explored by invoking the k-essence field as the matter field driving the matter-dominated contraction of the universe [53]. Let us begin with the k-essence theory. In this theory, the Friedmann and evolution equations are given, respectively, by

$$\mathcal{E} = 2XK_X - K - 3M_{\rm Pl}^2 H^2 = 0, \tag{3.141}$$

$$\mathcal{P} = K + M_{\rm Pl}^2(3H^2 + 2\dot{H}) = 0. \tag{3.142}$$

The arbitrary function $K(\phi, X)$ modifies the coefficients in the quadratic action of the curvature perturbation as

$$\mathcal{G}_S = \frac{XK_X + 2X^2K_{XX}}{H^2}, \quad \mathcal{F}_S = M_{\rm Pl}^2\left(-\frac{\dot{H}}{H^2}\right) = \frac{XK_X}{H^2}, \tag{3.143}$$

where we used the background equations. For simplicity, we consider the case in which the propagation speed of the curvature perturbation, $c_s^2 := \mathcal{F}_S/\mathcal{G}_S$, is a constant. In this case, the equation of motion for the canonically normalized ζ is modified as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\eta^2} + \left(c_s^2 k^2 - \frac{2}{\eta^2}\right) u = 0. \tag{3.144}$$

Here, the equation for the tensor mode does not change. After similar calculations with the case of the canonical scalar field model, we can obtain

$$\mathcal{P}_\zeta = \frac{1}{8\pi^2 M_{\mathrm{Pl}}^2 \epsilon c_s} \frac{1}{a_*^2 \eta_*^2}, \tag{3.145}$$

and Eq. (3.137). Finally, the tensor-to-scalar ratio can change as

$$r = 24 c_s, \tag{3.146}$$

which gives us the possibility to have $r \ll 1$ by the small propagation speed, $c_s \ll 1$ (, or equivalently $|K_X| \ll |XK_{XX}|$).

As the next step, we consider the scalar non-Gaussianity. In Ref. [53], the nonlinearity parameter f_{NL} for the local and equilateral configurations has been evaluated as

$$f_{\mathrm{NL}} \sim \mathcal{O}(c_s^{-2}), \tag{3.147}$$

indicating that $f_{\mathrm{NL}} \gg 1$ for $c_s \ll 1$. Thus, similarly to the first example, a small r can be obtained while a small f_{NL} cannot.

In light of these results, it is important to investigate whether one can realize a small r and a small f_{NL} or not. We investigate how general the inconsistency with CMB observations is in more general theories in the next Chapter.

Chapter 4

Primordial Non-Gaussianities in General Bounce Cosmology

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The matter-dominated contracting (or bounce) universe can be mimicked by a canonical scalar field and this model can generate a scale-invariant curvature perturbations [56, 57, 52]. However, as explained so far, this model yields a too large tensor-to-scalar ratio and thus is excluded [52] (see, however, Refs. [58, 59]). One may use a k-essence field to reduce the tensor-to-scalar ratio by taking a small sound speed, but then this in turn enhances the production of non-Gaussianity, making the model inconsistent with observations [53]. At this stage, it is not evident whether or not this "no-go theorem" holds in more general scalar-tensor theories. (The scalar-tensor theory is a generic term for theories of single or multiple scalar fields and the gravitational field)

The purpose of the present chapter is clarifying to what extent the previous no-go theorem (which was formulated in the context of a k-essence field minimally coupled to gravity as an extension of Ref. [52]) holds in more general setups. To do so, we consider a general power-law contracting universe in the Horndeski theory [21], the most general second-order scalar-tensor theory, and evaluate the power spectra and the bispectra of scalar and tensor perturbations generated during the contracting phase. Throughout the paper we assume that the statistical nature of these primordial perturbations does not change during the subsequent bouncing and expanding phases. (In some cases in matter bounce cosmology, this has been justified. See, e.g., Ref. [60].) In calculating tensor non-Gaussianity we explore peculiar signatures of a contracting phase as compared to inflation, and show that the two scenarios can potentially be distinguishable due to the non-Gaussian amplitudes and shapes.

This chapter is organized as follows. In the next section, we introduce our setup of the general contracting cosmological background. In Sec. 4.3, we evaluate the power spectra for curvature and tensor perturbations, and derive the conditions under which they are scale-invariant. In Sec. 4.4, we calculate primordial non-Gaussianities of curvature and tensor perturbations, and investigate whether a small tensor-to scalar ratio and small scalar non-Gaussianity are compatible or not in the Horndeski theory. We also discuss how one can distinguish bounce cosmology with inflation based on tensor non-Gaussianity. The conclusion of this paper is drawn in Sec. 4.5.

4.1 Setup

We begin with a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{4.1}$$

where the scale factor describes a contracting phase,

$$a = \left(\frac{-t}{-t_b}\right)^n = \left(\frac{-\eta}{-\eta_b}\right)^{n/(1-n)} \qquad (0 < n < 1)\,, \tag{4.2}$$

with $\mathrm{d}\eta = \mathrm{d}t/a$. Here, we denoted the time at the end of the contracting phase as $t_b(<0)$ and $\eta_b(<0)$, and we normalized the scale factor so that $a(t_b) = 1 = a(\eta_b)$. The two time coordinates are related with

$$-\eta = \frac{(-t_b)^n}{1-n}(-t)^{1-n}, \tag{4.3}$$

where t and η coordinates run from $-\infty$ to t_b and η_b, respectively. In this paper, we do not assume n to take any particular value, so that our setup includes models other than the familiar matter bounce scenario [22]. Note, however, that it will turn out that models with different n are related to each other via conformal transformation (see Sec. 4.2.3).

We work with the Horndeski action which gives the most general second-order scalar-tensor theory, and hence a vast class of contracting scenarios reside within this theory. Therefore, the Horndeski theory is adequate for studying generic properties of cosmological perturbations from contracting models. Note, however, that nonsingular cosmological solutions suffer from gradient instabilities if the entire history of the universe were described by the Horndeski theory [42, 61, 46, 45, 47] as seen in Chapter 3. We circumvent this issue by assuming that beyond-Horndeski operators come into play at some moment, but at least the contracting phase we are focusing on is assumed to be described by the Horndeski theory.

The Friedmann and evolution equations are written, respectively, in the form

$$\mathcal{E} := \sum_{i=2}^{5} \mathcal{E}_i = 0, \quad \mathcal{P} := \sum_{i=2}^{5} \mathcal{P}_i = 0, \tag{4.4}$$

where $\mathcal{E}_i = \mathcal{E}_i(H, \phi, \dot{\phi})$ and $\mathcal{P}_i = \mathcal{P}_i(H, \dot{H}, \phi, \dot{\phi}, \ddot{\phi})$ come from the variation of the action involving G_i, whose explicit expressions are given in Appendix A. Here a dot stands for differentiation with respect to t and $H := \dot{a}/a$. In this paper, we do not consider any concrete background models, but just assume that each term in the background equations scales as

$$\mathcal{E}_i,\ \mathcal{P}_i \propto (-t)^{2\alpha}, \tag{4.5}$$

where α is a constant to be specified below. This scaling is reasonable as far as one focuses on one phase, e.g., only the phase of the matter-dominated contraction, ekpyrotic contraction, etc. The impact of spatial curvature and anisotropies is discussed in Appendix B.

4.2 Scale-invariant power spectra

Some functions appeared in the coefficients of the quadratic actions can be written in terms of the partial derivatives with respect to X and H as

$$\Sigma = X\frac{\partial \mathcal{E}}{\partial X} + \frac{H}{2}\frac{\partial \mathcal{E}}{\partial H}, \tag{4.6}$$

$$\Theta = -\frac{1}{6}\frac{\partial \mathcal{E}}{\partial H}. \tag{4.7}$$

(The explicit expressions for Θ and Σ are given in Appendix C.) As inferred from Eqs. (4.5), (4.6), and (4.7), it is natural to assume that $\Sigma \propto (-t)^{2\alpha}$ and $\Theta \propto$

$(-t)^{2\alpha+1}$. In addition, the time dependences of $\mathcal{G}_T$ and $\mathcal{F}_T$ can be approximately written as $\mathcal{G}_T, \mathcal{F}_T \sim \mathcal{E}_4/H^2, \mathcal{E}_5/H^2, \mathcal{P}_4/H^2, \mathcal{P}_5/H^2$. These imply

$$\mathcal{G}_T,\ \mathcal{F}_T,\ \mathcal{G}_S,\ \mathcal{F}_S \propto (-t)^{2(\alpha+1)} \propto (-\eta)^{2(\alpha+1)/(1-n)}. \tag{4.8}$$

Under these assumptions, the propagation speed of the curvature perturbation, $c_s^2 = \mathcal{F}_S/\mathcal{G}_S$, and that of the tensor perturbations, $c_t^2 = \mathcal{F}_T/\mathcal{G}_T$, are constant. Note that only $\alpha = -1$ is possible if ϕ is minimally coupled to gravity.

Let us move to derive a relation between α and n by imposing that the primordial curvature and tensor perturbations have scale-invariant power spectra.

4.2.1 Curvature Perturbation

The mode function $u_{\mathbf{k}}(\eta)$ of the canonically normalized perturbation, $u_{\mathbf{k}} = \sqrt{2}a(\mathcal{F}_S\mathcal{G}_S)^{1/4}\zeta_{\mathbf{k}}$, obeys

$$u_{\mathbf{k}}'' + \left[c_s^2k^2 - \frac{1}{\eta^2}\left(\nu_s^2 - \frac{1}{4}\right)\right]u_{\mathbf{k}} = 0, \tag{4.9}$$

where a prime denotes differentiation with respect to η and

$$\nu_s := \frac{-1-3n-2\alpha}{2(1-n)}. \tag{4.10}$$

The positive frequency solution is then given by

$$\zeta_{\mathbf{k}} = \frac{1}{\sqrt{2}a(\mathcal{F}_S\mathcal{G}_S)^{1/4}} \cdot \frac{\sqrt{\pi}}{2}\sqrt{-c_s\eta}H_{\nu_s}^{(1)}(-c_sk\eta). \tag{4.11}$$

Here we chose the initial condition as

$$\lim_{\eta\to-\infty} u_{\mathbf{k}} = \frac{1}{\sqrt{2k}}e^{-ic_sk\eta}. \tag{4.12}$$

similar to the case of inflation. The power spectrum of the curvature perturbation and its spectral index are parametrized as

$$\mathcal{P}_\zeta \propto k^{3-2|\nu_s|}. \tag{4.13}$$
$$n_s - 1 = 3 - 2|\nu_s|. \tag{4.14}$$

Let us focus on the exactly scale-invariant spectrum, which corresponds to

$$\nu_s = \frac{3}{2} \quad \Rightarrow \quad \alpha = -2, \tag{4.15}$$
$$\nu_s = -\frac{3}{2} \quad \Rightarrow \quad \alpha = 1-3n. \tag{4.16}$$

On superhorizon scales, $c_s k|\eta| \ll 1$, we have $\zeta_{\mathbf{k}} \propto |\eta|^{\nu_s - |\nu_s|}$. Therefore, the perturbations freeze out on superhorizon scales in the former case (as in the inflationary universe), while they grow as $\zeta_{\mathbf{k}} \propto |\eta|^{-3}$ in the latter case (as in the contracting universe). In this paper, we consider the growing superhorizon perturbations having a scale-invariant spectrum, which is a characteristic feature of contracting models. Note that the Planck results [4] require a slightly red tilted spectrum, $n_s \simeq 0.96$. This can be obtained by slightly detuning the relation (4.16) between n and α, though for simplicity in this paper we only consider the exactly scale-invariant case.

Taking $\alpha = 1 - 3n$, the scale-invariant power spectrum can now be derived as

$$\mathcal{P}_\zeta = \frac{1}{8\pi^2}\frac{1}{\mathcal{F}_S c_s}\frac{1}{\eta^2}\bigg|_{t=t_b} = \frac{1}{8\pi^2}\left(1-\frac{1}{n}\right)^2 \frac{H^2}{\mathcal{F}_S c_s}\bigg|_{t=t_b}, \tag{4.17}$$

where the time-dependent quantities are evaluated at the end of the contracting phase.

4.2.2 Tensor Perturbations

The mode function $v_{\mathbf{k}}^{(s)}(\eta)$ of the canonically normalized perturbations, $v_{\mathbf{k}}^{(s)} = a(\mathcal{F}_T\mathcal{G}_T)^{1/4} h_{\mathbf{k}}^{(s)}/2$, obeys

$$v_{\mathbf{k}}^{(s)\prime\prime} + \left[c_t^2 k^2 - \frac{1}{\eta^2}\left(\nu_t^2 - \frac{1}{4}\right)\right] v_{\mathbf{k}}^{(s)} = 0, \tag{4.18}$$

where $\nu_t = \nu_s$. The positive frequency solution is then given by

$$h_{\mathbf{k}}^{(s)} = \frac{2}{a(\mathcal{F}_T\mathcal{G}_T)^{1/4}} \cdot \frac{\sqrt{\pi}}{2}\sqrt{-c_t\eta}\, H_{\nu_t}^{(1)}(-c_t k\eta), \tag{4.19}$$

with the initial condition being

$$\lim_{\eta\to-\infty} v_{\mathbf{k}}^{(s)} = \frac{1}{\sqrt{2k}} e^{-ic_t k\eta}. \tag{4.20}$$

The behavior of the tensor perturbations is essentially the same as that of $\zeta_{\mathbf{k}}$. For $\alpha = 1 - 3n$ ($\nu_t = \nu_s = -3/2$), $h_{\mathbf{k}}$ grows on superhorizon scales as $h_{\mathbf{k}} \propto |\eta|^{-3}$ and the tensor power spectrum is scale invariant.

For $\alpha = 1 - 3n$, we have the scale-invariant power spectrum

$$\mathcal{P}_h = \frac{2}{\pi^2}\frac{1}{\mathcal{F}_T c_t}\frac{1}{\eta^2}\bigg|_{t=t_b} = \frac{2}{\pi^2}\left(1-\frac{1}{n}\right)^2 \frac{H^2}{\mathcal{F}_T c_t}\bigg|_{t=t_b}, \tag{4.21}$$

where time-dependent quantities are evaluated at $t = t_b$.

The tensor-to-scalar ratio is given by

$$r = \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} = 16\,\frac{\mathcal{F}_S}{\mathcal{F}_T}\frac{c_s}{c_t}\bigg|_{t=t_b}, \tag{4.22}$$

which is constrained as Eq. (3.63).

For example, in the case of matter contracting models within the k-essence theory, we have $n = 2/3$, $\alpha = -1$, $c_t = 1$, and $\mathcal{F}_S = (3/2)\mathcal{F}_T =$ const. Therefore, the tensor-to-scalar ratio is

$$r = 24c_s, \tag{4.23}$$

which can satisfy the upper bound on r only for $c_s \ll 1$. However, as argued in Ref. [53] and introduced in Chapter 3, $c_s \ll 1$ implies large scalar non-Gaussianity, and hence bounce models within the k-essence theory are ruled out. In the next section, we revisit this issue and study whether or not upper bounds on the tensor-to-scalar ratio and non-Gaussianity can be satisfied at the same time in a wider class of theories.

4.2.3 Conformal Frames

At this stage it is instructive to perform a conformal transformation and clarify the relation among models with different n.

Let us consider a conformally related metric

$$\widetilde{\mathrm{d}s}^2 = \Omega^2(t)\left(-\mathrm{d}t^2 + a^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right), \quad \Omega \propto (-t)^{\alpha+1}. \tag{4.24}$$

In this tilde frame, the time coordinate and the scale factor are given respectively by

$$\alpha = -2 \quad \Rightarrow \quad -\tilde{t} \propto \ln(-t), \quad \tilde{a} \propto e^{\tilde{H}\tilde{t}}, \tag{4.25}$$

$$\alpha \neq -2 \quad \Rightarrow \quad -\tilde{t} \propto (-t)^{\alpha+2}, \quad \tilde{a} \propto (-\tilde{t})^{(n+\alpha+1)/(\alpha+2)}. \tag{4.26}$$

where $\mathrm{d}\tilde{t} = \Omega\mathrm{d}t$ and $\tilde{a} = \Omega a$. By inspecting the quadratic action for scalar and tensor perturbations we see that in the tilde frame all the four coefficients reduce to constants.

We find that the case of $\nu_s = \nu_t = 3/2$ $(\alpha = -2)$ can be regarded as de Sitter inflation in the tilde frame (see, e.g., Ref. [62]).

In the case of $\nu_s = \nu_t = -3/2$ $(\alpha = 1 - 3n)$, we have

$$\tilde{a} \propto (-\tilde{t})^{2/3}, \tag{4.27}$$

which describes a matter-dominated contracting universe. Therefore, the dynamics of cosmological perturbations in our contracting models (with general n) is equivalent to that in the more familiar matter-dominated contracting model. However, it should be emphasized that the magnitudes of the coefficients in the perturbation action are still arbitrary even in the tilde frame.

4.3 Primordial non-Gaussianities

4.3.1 Scalar Perturbations

The cubic Lagrangian of the curvature perturbation can be written in the form [63, 64, 65]

$$\mathcal{L}_\zeta^{(3)} = a^3\mathcal{G}_S\left[\frac{\Lambda_1}{H}\dot{\zeta}^3 + \Lambda_2\zeta\dot{\zeta}^2 + \Lambda_3\zeta\frac{(\partial_i\zeta)^2}{a^2} + \frac{\Lambda_4}{H^2}\dot{\zeta}^2\frac{\partial^2\zeta}{a^2} + \Lambda_5\dot{\zeta}\partial_i\zeta\partial_i\psi + \Lambda_6\partial^2\zeta\,(\partial_i\psi)^2 \right.$$
$$\left. + \frac{\Lambda_7}{H^2}\frac{1}{a^4}\left[\partial^2\zeta\,(\partial_i\zeta)^2 - \zeta\partial_i\partial_j\,(\partial_i\zeta\partial_j\zeta)\right] + \frac{\Lambda_8}{H}\frac{1}{a^2}\left[\partial^2\zeta\partial_i\zeta\partial_i\psi - \zeta\partial_i\partial_j\,(\partial_i\zeta\partial_j\psi)\right]\right]$$
$$+ F(\zeta)E_S, \tag{4.28}$$

where $\psi := \partial^{-2}\dot{\zeta}$ and Λ_i are dimensionless coefficients. The complete form of the cubic Lagrangian is summarized in Appendix C. Based on the scaling argument similar to that in the previous section, it can be seen that the coefficients Λ_i are constant.

The last term in Eq. (4.28) can be eliminated by means of a field redefinition[*1]

$$\zeta \to \zeta - F(\zeta). \tag{4.30}$$

In Fourier space, this redefinition is equivalent to

$$\zeta(\mathbf{k}) \to \zeta(\mathbf{k}) - \frac{3(1-n)}{n}\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\left[B + \frac{A}{2}\left(\frac{\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')}{k'^2} - \frac{(\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}'))}{k^2k'^2}\right)\right]$$
$$\times\,\zeta(\mathbf{k}')\zeta(\mathbf{k}-\mathbf{k}') + \cdots, \tag{4.31}$$

[*1] The field redefinition induces the additional contribution from the quadratic action to the cubic one as

$$\delta S_\zeta^{(2)} \supset -\int \mathrm{d}t\mathrm{d}^3x a^3 E_S F(\zeta), \tag{4.29}$$

which has an inverse sign compared to the last one in Eq. (4.28).

where

$$A := \frac{H\mathcal{G}_S}{\Theta\mathcal{G}_T}\frac{\partial\Theta}{\partial H} - \frac{H\mathcal{G}_S}{\mathcal{G}_T^2}\frac{\partial\mathcal{G}_T}{\partial H} = \text{const}, \tag{4.32}$$

$$B := \frac{H\mathcal{G}_T\mathcal{G}_S}{\Theta\mathcal{F}_S} = \text{const}. \tag{4.33}$$

Here we approximated the time derivative of the curvature perturbation on superhorizon scales as

$$\dot{\zeta} \simeq -\frac{3(1-n)}{n}H\zeta \tag{4.34}$$

which can be derived from the form of the superhorizon mode, $\zeta \propto (-t)^{-3(1-n)}$, and ignored sub-leading contributions denoted by the ellipsis $(\cdots)$.

The bispectrum B_ζ is defined by

$$\langle\hat{\zeta}(\mathbf{k}_1)\hat{\zeta}(\mathbf{k}_2)\hat{\zeta}(\mathbf{k}_3)\rangle = (2\pi)^3\delta\left(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\right)B_\zeta, \tag{4.35}$$

where we write

$$B_\zeta := (2\pi)^4\frac{\mathcal{P}_\zeta^2}{k_1^3k_2^3k_3^3}\mathcal{A}_{\text{total}}, \tag{4.36}$$

and evaluate the amplitude $\mathcal{A}_{\text{total}}$. In our setup, $\mathcal{A}_{\text{total}}$ reads

$$\mathcal{A}_{\text{total}} = \mathcal{A}_{\text{original}} + \mathcal{A}_{\text{redefine}}, \tag{4.37}$$

where $\mathcal{A}_{\text{original}}$ and $\mathcal{A}_{\text{redefine}}$ are the contributions respectively from the interaction Hamiltonian and from the field redefinition (4.31):

$$\begin{aligned}\mathcal{A}_{\text{original}} = \frac{1}{8}\Bigg[&\left(\frac{9(1-n)}{n}\Lambda_1 - \Lambda_2 + \frac{\Lambda_5}{2}\right)\sum_i k_i^3 + \frac{\Lambda_6}{2}\sum_{i\neq j}k_i^2k_j \\ &+ \frac{1}{2k_1^2k_2^2k_3^2}\left(\Lambda_6\sum_i k_i^9 - (\Lambda_5+\Lambda_6)\sum_{i\neq j}k_i^7k_j^2 - \Lambda_6\sum_{i\neq j}k_i^6k_j^3\right. \\ &\left.+ (\Lambda_5+\Lambda_6)\sum_{i\neq j}k_i^5k_j^4\right)\Bigg],\end{aligned} \tag{4.38}$$

$$\begin{aligned}\mathcal{A}_{\text{redefine}} = \frac{3}{8}\frac{(1-n)}{n}\Bigg[&(A-4B)\sum_i k_i^3 + \frac{A}{4}\sum_{i\neq j}k_i^2k_j \\ &- \frac{A}{4}\frac{1}{k_1^2k_2^2k_3^2}\left(\sum_{i\neq j}k_i^7k_j^2 + \sum_{i\neq j}k_i^6k_i^3 - 2\sum_{i\neq j}k_i^5k_j^4\right)\Bigg].\end{aligned} \tag{4.39}$$

One can check that the result of the calculation of the primordial bispectra involving the procedure of the field redefinition is identical to that involving boundary terms in the cubic action with the linear equation of motion $E_S = 0$ being imposed. (See Refs. [66, 67, 68].) The explicit form of the boundary terms is given in Appendix C.

Based on the above result we also evaluate the nonlinearity parameter. At the squeezed limit ($k_1 \ll k_2 = k_3$), the equilateral limit ($k_1 = k_2 = k_3$), and the folded limit ($k_1 = 2k,\ k_2 = k_3 = k$), the parameter is given respectively by

$$f_{\rm NL}^{\rm local} = \frac{5}{12}\left[\frac{9(1-n)}{n}\Lambda_1 - \Lambda_2 + 3(A-4B)\frac{1-n}{n}\right], \tag{4.40}$$

$$f_{\rm NL}^{\rm equil} = \frac{5}{12}\left[\frac{9(1-n)}{n}\Lambda_1 - \Lambda_2 + \frac{\Lambda_5}{2} + \frac{\Lambda_6}{2} + \left(\frac{9}{2}A - 12B\right)\frac{1-n}{n}\right], \tag{4.41}$$

$$f_{\rm NL}^{\rm folded} = \frac{5}{12}\left[\frac{9(1-n)}{n}\Lambda_1 - \Lambda_2 - \frac{8}{5}\Lambda_5 + \frac{16}{5}\Lambda_6 - 12B\frac{1-n}{n}\right]. \tag{4.42}$$

In the case of the matter contracting models within the k-essence theory, these are written as

$$f_{\rm NL}^{\rm local} = \frac{5}{12}\left[-6c_s^2\frac{\lambda}{M_{\rm Pl}^2H^2} - \frac{15}{2} + \frac{9}{4c_s^2}\right], \tag{4.43}$$

$$f_{\rm NL}^{\rm equil} = \frac{5}{12}\left[-6c_s^2\frac{\lambda}{M_{\rm Pl}^2H^2} - \frac{15}{2} + \frac{87}{32c_s^2}\right], \tag{4.44}$$

$$f_{\rm NL}^{\rm folded} = \frac{5}{12}\left[-6c_s^2\frac{\lambda}{M_{\rm Pl}^2H^2} - \frac{15}{2} + \frac{24}{5c_s^2}\right], \tag{4.45}$$

where $\lambda := X^2G_{2XX} + (2/3)X^3G_{2XXX}$. These results reproduce those in [53, 54]. In order for these nonlinearity parameters to be $\lesssim \mathcal{O}(1)$, one requires $c_s^2 = \mathcal{O}(1)$. In the context of k-essence, this leads to $r > \mathcal{O}(10)$, which is ruled out. Instead one may take $c_s^2 \ll 1$ to have $r < 0.064$, but then the nonlinearity parameters are too large to be consistent with observations:

$$f_{\rm NL}^{\rm local},\ f_{\rm NL}^{\rm equil},\ f_{\rm NL}^{\rm folded} \sim \frac{1}{c_s^2} = \left(\frac{24}{r}\right)^2 > \mathcal{O}(10^5), \tag{4.46}$$

indicating that any matter bounce models in the k-essence theory are excluded.

Although small r is incompatible with small scalar non-Gaussianity in the k-essence theory, this is not always the case in the Horndeski theory. Thanks to a sufficient number of independent functions, one can make r small while retaining A, B, and Λ_i less than $\mathcal{O}(1)$. We will discuss this point in more detail in the next subsection.

4.3.2 Example

Let us consider a concrete Lagrangian characterized by

$$\begin{aligned} G_2 &= \frac{M_{\rm Pl}^2}{\mu^2} e^{-2\phi/\mu} g_2(Y), \quad G_3 = \frac{M_{\rm Pl}^2}{\mu} g_3(Y), \\ G_4 &= \frac{M_{\rm Pl}^2}{2}, \quad G_5 = 0, \end{aligned} \tag{4.47}$$

where $Y := X e^{2\phi/\mu}$. We seek for a solution of the matter-dominated contracting universe, $H = 2/3t$, with a time-dependent scalar field,

$$\phi = \mu \ln(-Mt). \tag{4.48}$$

It then follows that $Y = \bar{Y} := M^2\mu^2/2 = $const. This indeed solves the background equations provided that the functions $g_2(Y)$ and $g_3(Y)$ satisfy

$$g_2(\bar{Y}) = 0, \tag{4.49}$$

$$g_2'(\bar{Y}) + 2\bar{Y} g_3'(\bar{Y}) = \frac{4}{3}, \tag{4.50}$$

where a prime in this subsection denotes differentiation with respect to Y.

Let us further impose that

$$\bar{Y} g_3'(\bar{Y}) = \delta_1 - 1, \tag{4.51}$$

$$\bar{Y}\left[g_2''(\bar{Y}) + 2\bar{Y} g_3''(\bar{Y})\right] = \frac{1}{3}\left(21\delta_1 + 5\delta_2 - 14\right), \tag{4.52}$$

where δ_1 and δ_2 are some small positive numbers, $\delta_1 \sim \delta_2 \ll 1$. We then have

$$\mathcal{F}_S \simeq \frac{3}{5}\delta_1 M_{\rm Pl}^2, \quad \mathcal{G}_S \simeq \frac{3}{5}\delta_2 M_{\rm Pl}^2, \tag{4.53}$$

and a small tensor-to-scalar ratio can be obtained, $r = 16\delta_1^{3/2}\delta_2^{-1/2} \ll 1$, while $c_s^2 = \delta_1/\delta_2 = \mathcal{O}(1)$, which cannot be achieved in the k-essence theory.

A would-be dangerous contribution to $f_{\rm NL}$ comes from Λ_1:

$$\Lambda_1 = -\frac{4}{25\delta_2}\left[8 + \bar{Y}^2\left(g_2''' - 12 g_3'' + 2\bar{Y} g_3'''\right)\right] + \mathcal{O}(1). \tag{4.54}$$

This can be made safe if one requires

$$\bar{Y}^2\left[g_2'''(\bar{Y}) - 12 g_3''(\bar{Y}) + 2\bar{Y} g_3'''(\bar{Y})\right] = \delta_3 - 8, \tag{4.55}$$

where $\delta_3 (\lesssim \delta_1)$ is another small number. All the other terms give at most $\mathcal{O}(1)$ contributions.

To sum up, by introducing the functions $g_2(Y)$ and $g_3(Y)$ satisfying the conditions (4.49), (4.50), (4.51), (4.52), and (4.55), one has $r \ll 1$ and $f_{\rm NL} \lesssim 1$ simultaneously. Clearly, this is indeed possible. One can thus circumvent the no-go theorem presented in [53] by appropriately choosing the functions in the Lagrangian which is more general than the k-essence theory. Here, one should note that in the present case $\mathcal{O}(0.01)$ fine-tuning is required for the parameters. However, our aim is to give a proof of concept. It would therefore be interesting to explore more natural models without fine-tuning based on some symmetry argument.

4.3.3 Tensor Perturbations

The interaction Hamiltonian, $H_{\rm int}$, is given by

$$H_{\rm int} = -\int \mathrm{d}^3x \mathcal{L}_h^{(3)}, \tag{4.56}$$

where [39]

$$\mathcal{L}_h^{(3)} = a^3\left[\frac{\mu}{12}\dot{h}_{ij}\dot{h}_{jk}\dot{h}_{ki} + \frac{\mathcal{F}_T}{4a^2}\left(h_{ik}h_{jl} - \frac{1}{2}h_{ij}h_{kl}\right)h_{ij,kl}\right], \tag{4.57}$$

with $\mu := -(1/2)\partial\mathcal{G}_T/\partial H$ which scales as $\mu \sim (-t)^{3+2\alpha}$, as seen from Eq. (4.8). The first term, $\dot{h}^3$, is the new contribution due to $G_{5X} \neq 0$, while the second one, which is of the form $h^2\partial^2 h$, is identical to the corresponding term in general relativity except for the overall normalization. We attach the label "new" (respectively, "GR") to the quantities associated with the former (respectively, latter) interaction.

Similarly to the case of the curvature perturbation, the bispectrum is defined by

$$\langle \hat{\xi}^{s_1}(\mathbf{k}_1)\hat{\xi}^{s_2}(\mathbf{k}_2)\hat{\xi}^{s_3}(\mathbf{k}_3)\rangle = (2\pi)^3\delta\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3\right)\left(\mathcal{B}^{s_1s_2s_3}_{\rm (new)} + \mathcal{B}^{s_1s_2s_3}_{\rm (GR)}\right), \tag{4.58}$$

where

$$\mathcal{B}^{s_1s_2s_3}_{\rm (new)} = (2\pi)^4\frac{\mathcal{P}_h^2}{k_1^3k_2^3k_3^3}\mathcal{A}^{s_1s_2s_3}_{\rm (new)}, \tag{4.59}$$

$$\mathcal{B}^{s_1s_2s_3}_{\rm (GR)} = (2\pi)^4\frac{\mathcal{P}_h^2}{k_1^3k_2^3k_3^3}\mathcal{A}^{s_1s_2s_3}_{\rm (GR)}, \tag{4.60}$$

and we evaluate the amplitudes $\mathcal{A}^{s_1s_2s_3}_{\rm (new)}$ and $\mathcal{A}^{s_1s_2s_3}_{\rm (new)}$. In our setup we obtain

$$\mathcal{A}^{s_1s_2s_3}_{\rm (new)} = \frac{3}{16}\frac{1-n}{n}\frac{H\mu}{\mathcal{G}_T}\bigg|_{t=t_b} F(s_1k_1, s_2k_2, s_3k_3)\sum_i k_i^3, \tag{4.61}$$

$$\mathcal{A}^{s_1s_2s_3}_{\rm (GR)} = -\frac{1}{128}c_t^2\eta_b^2(s_1k_1 + s_2k_2 + s_3k_3)^2F(s_1k_1, s_2k_2, s_3k_3)\sum_i k_i^3, \tag{4.62}$$

Figures 4.1 and 4.2 show that both $\mathcal{A}^{+++}_{(\text{new})}$ and $\mathcal{A}^{+++}_{(\text{GR})}$ have peaks at the squeezed limit[*2]. Note that $\mathcal{A}^{s_1s_2s_3}_{(\text{GR})}$ has a specific scale-dependence $c_t^2k_i^2\eta_b^2$. This has been obtained in the context of matter bounce cosmology driven by a scalar field minimally coupled to gravity [69]. However, this factor makes the detection more challenging [70].

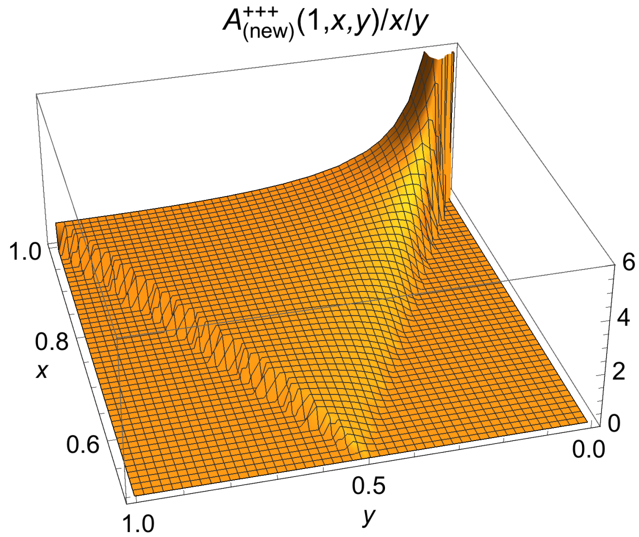

Fig.4.1 $\mathcal{A}^{+++}_{(\text{new})}(1, k_2/k_1, k_3/k_1)(k_1/k_2)(k_1/k_3)$ as a function of $x = k_2/k_1$ and $y = k_3/k_1$. We take $n = 2/3$ and $H\mu/\mathcal{G}_T|_{t_b} = 1$. The plot is normalized to 1 for the equilateral configuration, $x = 1 = y$.

Now let us compare the above results with the prediction from generalized G-inflation [20]. The amplitudes of non-Gaussianities of tensor perturbations in (quasi-de Sitter) inflation are given by [39]

$$\mathcal{A}^{s_1s_2s_3}_{(\text{new})} = \frac{H\mu}{4\mathcal{G}_T}\frac{k_1^2k_2^2k_3^2}{K^3}F(s_1k_1, s_2k_2, s_3k_3), \tag{4.63}$$

$$\mathcal{A}^{s_1s_2s_3}_{(\text{GR})} = \frac{\mathcal{A}}{2}(s_1k_2 + s_2k_2 + s_3k_3)^2F(s_1k_1, s_2k_2, s_3k_3). \tag{4.64}$$

Let us first look at their shapes. As shown in [39], $\mathcal{A}^{+++}_{(\text{new})}$ of inflation models has a peak at the equilateral limit. This is in contrast with the case of contracting models. On the other hand, $\mathcal{A}^{+++}_{(\text{GR})}$ has a peak at the squeezed limit both in inflation and contracting models. Therefore, the detection of the equilateral-type tensor non-Gaussianities would rule out our contracting models.

[*2] The shape function, $S(k_1, k_2, k_3)$ introduced in Chapter 2, is corresponding to $\mathcal{A}^{s_1s_2s_3}_{(\text{GR,new})}/(k_1k_2k_3)$.

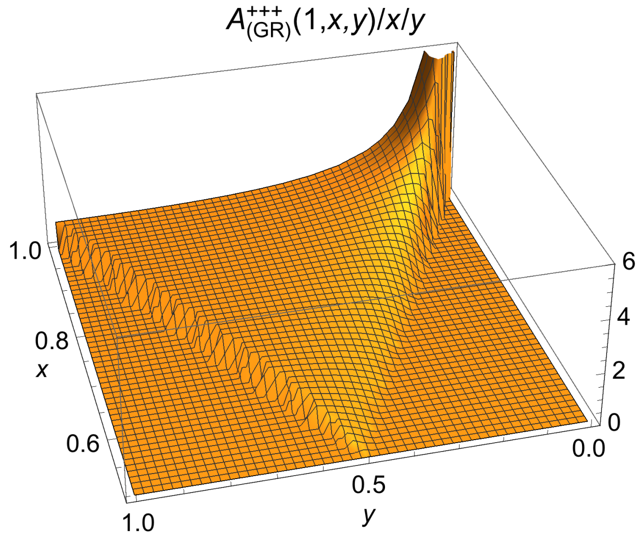

Fig.4.2 $\mathcal{A}^{+++}_{(\mathrm{GR})}(1, k_2/k_1, k_3/k_1)\,(k_1/k_2)(k_1/k_3)$ as a function of $x = k_2/k_1$ and $y = k_3/k_1$. We take $c_t^2\eta_b^2(k_1+k_2+k_3)^2/128 = 10^{-6}$. The plot is normalized to 1 for the equilateral configuration, $x = 1 = y$.

Next, let us compare the amplitudes. Squeezed tensor non-Gaussianity from inflation has the fixed amplitude, as Eq. (4.64) is independent of the functions in the Horndeski action. This is not the case for squeezed non-Gaussianity from contracting models, as is clear from Eqs. (4.61) and (4.62), whichever is dominant.

Finally, notice that the non-Gaussian amplitudes (4.61) and (4.62) agree with those obtained in a kind of non-attractor inflation models, where tensor perturbations grow on superhorizon scales during inflation due to non-attractor dynamics of the non-minimally coupled inflaton [71]. This is because both our contracting models and the non-attractor phase of inflation are conformally equivalent to the matter-dominated contracting scenario.

Chapter 5

Non-Bunch-Davies Effects on Tensor Non-Gaussianities from Inflation

This chapter is cited from, S. Akama, S. Hirano and T. Kobayashi, "Primordial tensor non-Gaussianities from general single-field inflation with non-Bunch-Davies initial states," Phys. Rev. D **102**, no.2, 023513 (2020) doi:10.1103/PhysRevD.102.023513 [arXiv:2003.10686 [gr-qc]] [72].

The inflationary perturbations have been studied mainly by assuming the Bunch-Davies initial state [30]. However, in principle, the initial state is not necessarily given by the Bunch-Davies one, and the validity of the assumptions on the initial state must be tested against observations in the end. Deviations from the Bunch-Davies state mean that the initial state is excited, i.e., there exist particles initially. In this case, the particles present initially can interact with each other at early times, leading possibly to the generation of non-Gaussianities on subhorizon scales. Therefore, assuming non-Bunch-Davies initial states would result in novel non-Gaussian signatures compared to the standard case of the Bunch-Davies initial state.

The nature of primordial perturbations from non-Bunch-Davies initial states has been explored so far in the literature [37, 8, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 9, 91, 92, 93, 94, 95, 10]. In particular, it has been found that the non-Gaussianity of the curvature perturbation at the squeezed and flattened configurations can be enhanced compared with those in the case of the Bunch-Davies state [8, 73, 74, 75, 78, 81, 82, 85, 90, 95]. It is therefore natural to ask whether or not the non-Gaussianities associated with the tensor modes can be enhanced as well. There have already been several studies regarding tensor non-Gaussianities from non-Bunch-Davies initial states [9, 10]. To address this question in more detail, in this paper we investigate the auto-bispectra of tensor modes and the cross-bispectra involving one tensor and two scalar modes in more general gravity theory than in the previous literature.

One naively expects that higher-derivative interactions have more impacts on non-Gaussianities due to non-Bunch-Davies initial states. Generalizing the underlying gravity theory yields such higher-derivative interactions. As a framework including higher derivative interactions, we use an effective description of scalar-tensor gravity, writing down the operators composed of the geometrical quantities such as extrinsic and intrinsic curvature tensors [96, 97]. Based on this effective description, in the present paper we will estimate the size of tensor non-Gaussianities from non-Bunch-Davies initial states in general single-field inflation models.

This chapter is outlined as follows. In the next section, we consider general quadratic and cubic actions for tensor modes and introduce non-Bunch-Davies initial states from a Bogoliubov transform of the usual Bunch-Davies modes. In Sec. 5.2, we calculate the auto-bispectrum of the tensor modes, and investigate whether the enhanced non-Gaussian amplitudes can be obtained or not. We then study in Sec. 5.3 the cross-bispectrum involving one tensor and two scalar modes, and discuss how it can be enhanced compared with the case of the Bunch-Davies initial state. A summary of the present paper is given in Sec. 5.4.

5.1 Tensor modes

5.1.1 General quadratic and cubic interactions

In the present paper, we investigate the properties of the tensor modes with non-Bunch-Davies initial states in order to see whether the tensor non-Gaussianities could be enhanced or not. Although we also study the cross-bispectrum with the scalar modes briefly, here we only summarize the quadratic and cubic interactions of the tensor modes.

To derive the generic action for the tensor modes during inflation, it is convenient to employ the ADM decomposition with uniform inflaton hypersurfaces as constant time hypersurfaces and write down the possible operators composed of the extrinsic curvature tensor K_{ij} and the intrinsic curvature tensor $R^{(3)}_{ij}$ of the constant time hypersurfaces. First, the operators having the dimension of mass squared are

$$\mathcal{L}_{\rm GR} \supset K_{ij}^2,\ K^2,\ R^{(3)}, \tag{5.1}$$

where K is the trace of K_{ij}. All these terms are present in general relativity. Then, one can consider the leading-order corrections to Eq. (5.1):

$$\mathcal{L}_{\rm cor} \supset K_{ij}^3,\ KK_{ij}^2,\ K^3,\ K^{ij}R^{(3)}_{ij},\ KR^{(3)}. \tag{5.2}$$

One may anticipate that these corrections play a crucial role in the generation of non-Gaussianities. Therefore, in the present study, we consider the Lagrangian up to this order, and evaluate the contributions on the non-Gaussian amplitudes from these correction terms.

More specifically, we consider the following wide class of the ADM action,

$$S = \int {\rm d}t{\rm d}^3x\sqrt{\gamma}N\mathcal{L}, \tag{5.3}$$

where γ is the determinant of the spacial metric γ_{ij}, N is the lapse function, and

$$\begin{aligned}\mathcal{L} = &M_0^4(t,N) + M_1^3(t,N)K + M_2^2(t,N)\left(K^2 - K_{ij}^2\right) + M_3^2(t,N)R^{(3)}\\ &+ M_4(t,N)\left(K^3 - 3KK_{ij}^2 + 2K_{ij}^3\right) + M_5(t,N)\left(K^{ij}R^{(3)}_{ij} - \frac{1}{2}KR^{(3)}\right),\end{aligned} \tag{5.4}$$

with $M_i(t,N)$ being a function having the dimension of mass. Here we have included the lower-order terms M_0^4 and M_1^3K, though they do not contribute to the action for the tensor modes. Equation (5.4) is nothing but the Lagrangian of the so-called Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theory [96], and it includes the Horndeski theory as a subclass. By introducing a Stückelberg field ϕ, one can restore the full 4D covariance.

At the level of the background, we may always reparameterize the time coordinate so that we hereafter take $N = 1$ and write $M_i(t,N(t)) = M_i(t)$. Since $\sqrt{\gamma}$ and the trace part K do not involve h_{ij}, the terms such as M_0^4, M_1^3K, and $M_2^2K^2$ in the Lagrangian do not contribute to the dynamics of the tensor perturbations.

Substituting the above metric into Eq. (5.4), the action for the tensor perturbations up to cubic order in h_{ij} can be obtained as [20, 39]

$$S_h = S_h^{(2)} + S_h^{(3)}, \tag{5.5}$$

where

$$S_h^{(2)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \frac{M_T^2}{c_h^2}\left[\dot{h}_{ij}^2 - \frac{c_h^2}{a^2}(\partial_k h_{ij})^2\right], \tag{5.6}$$

$$\begin{aligned} S_h^{(3)} &= \int \mathrm{d}t\mathrm{d}^3x a^3 \left[\frac{M_T^2}{4a^2}\left(h_{ik}h_{jl} - \frac{1}{2}h_{ij}h_{kl}\right)\partial_k\partial_l h_{ij} + \frac{M_4}{4}\dot{h}_{ij}\dot{h}_{jk}\dot{h}_{ki}\right] \\ &=: -\int \mathrm{d}t H_{\mathrm{int}}, \end{aligned} \tag{5.7}$$

with

$$M_T^2 := 2M_3^2 + \dot{M}_5, \tag{5.8}$$

$$c_h^2 := -2(M_2^2 + 3HM_4)/M_T^2. \tag{5.9}$$

The interaction Hamiltonian H_{int} is introduced for later convenience. We assume that $M_T \sim M_{\mathrm{Pl}}$. The terms in the first line in Eq. (5.7) are present in general relativity, while the one in the second line is a new operator introduced as a result of the extension of general relativity with $M_4(t) \neq 0$. For example, this operator is obtained from the so-called G_5 term in the Horndeski theory [39]. One might think that the third line in Eq. (5.4) could also lead to a new cubic operator, but it turns out that this can be integrated by parts to yield the same terms as in the first line in Eq. (5.7).

5.1.2 Non-Bunch-Davies initial states

We now move to the Fourier domain,

$$h_{ij}(t,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\tilde{h}_{ij}(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.10}$$

In the standard setup, one expands the quantized tensor modes as

$$\tilde{h}_{ij}(t,\mathbf{k}) = \sum_s \left[u_k(t)e_{ij}^{(s)}(\mathbf{k})a_{\mathbf{k}}^{(s)} + u_k^*(t)e_{ij}^{(s)*}(-\mathbf{k})a_{-\mathbf{k}}^{(s)\dagger}\right], \tag{5.11}$$

The equation of motion for the mode function u_k is derived from Eq. (5.6) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3M_T^2}{c_h^2}\dot{u}_k\right) + aM_T^2k^2u_k = 0. \tag{5.12}$$

We solve Eq. (5.12) under the assumption that M_T^2, c_h^2 = const in the de Sitter background, H = const. Then, the mode function with the Bunch-Davies initial state is obtained as

$$u_k = \frac{\sqrt{\pi}}{a}\frac{c_h}{M_T}\sqrt{-\eta}H_{3/2}^{(1)}(-c_hk\eta), \tag{5.13}$$

We write the state annihilated by $\hat{a}_{\mathbf{k}}^{(s)}$ as $|0_a\rangle$: $\hat{a}_{\mathbf{k}}^{(s)}|0_a\rangle = 0$.

In this thesis, we instead expand $\tilde{h}_{ij}$ as

$$\tilde{h}_{ij} = \sum_s \left[\psi_k^{(s)} e_{ij}^{(s)}(\mathbf{k}) b_{\mathbf{k}}^{(s)} + \psi_k^{(s)*} e_{ij}^{(s)*}(-\mathbf{k}) b_{-\mathbf{k}}^{(s)\dagger}\right], \tag{5.14}$$

where $\psi_k^{(s)}$ is a Bogoliubov transform of the Bunch-Davies modes,

$$\psi_k^{(s)} = \alpha_k^{(s)} u_k + \beta_k^{(s)} u_k^*. \tag{5.15}$$

The Bogoliubov coefficients are normalized as $|\alpha_k^{(s)}|^2 - |\beta_k^{(s)}|^2 = 1$ and the creation and annihilation operators satisfy

$$a_{\mathbf{k}}^{(s)} = \alpha_k^{(s)} b_{\mathbf{k}}^{(s)} + \beta_k^{(s)*} b_{-\mathbf{k}}^{(s)\dagger}, \tag{5.16}$$
$$b_{\mathbf{k}}^{(s)} = \alpha_k^{(s)*} a_{\mathbf{k}}^{(s)} - \beta_k^{(s)*} a_{-\mathbf{k}}^{(s)\dagger}. \tag{5.17}$$

We write the state annihilated by $b_{\mathbf{k}}^{(s)}$ as $|0_b\rangle$:

$$b_{\mathbf{k}}^{(s)}|0_b\rangle = 0. \tag{5.18}$$

Nonvanishing $\beta_k^{(s)}$ coefficients indicate that the tensor modes get excited from the Bunch-Davies vacuum, $a_{\mathbf{k}}^{(s)}|0_b\rangle \neq 0$.

Let us assume that the deviations from the Bunch-Davies initial states are characterized by some small, real parameters as

$$\beta_k^{(s)} = \delta_1^{(s)}(k) + i\delta_2^{(s)}(k), \tag{5.19}$$
$$\alpha_k^{(s)} = 1 + i\delta_3^{(s)}(k), \tag{5.20}$$

where $\delta_1^{(s)} \sim \delta_2^{(s)} \sim \delta_3^{(s)} \ll 1$. This is a reasonable assumption because the magnitude of $\beta_k^{(s)}$ has an upper bound in order for the inflationary background not to be spoiled by the excited tensor modes, which is typically given by $|\beta_k^{(s)}| \lesssim 10^{-6}$ as argued in Appendix D. The assumption on the form of $\alpha_k^{(s)}$ [Eq. (5.20)] follows from $|\alpha_k^{(s)}|^2 - |\beta_k^{(s)}|^2 = 1$.

5.1.3 Primordial power spectrum

The two-point correlation function is defined by

$$\langle 0_b|\tilde{h}_{ij}(\mathbf{k})\tilde{h}_{kl}(\mathbf{k}')|0_b\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\mathcal{P}_{ij,kl}, \tag{5.21}$$

where

$$\mathcal{P}_{ij,kl} := \sum_{s,s'} \left[\psi_k^{(s)} \psi_k^{(s')*} e_{ij}^{(s)}(\mathbf{k}) e_{kl}^{(s')*}(\mathbf{k}) \right]. \tag{5.22}$$

Using Eqs. (5.13)–(5.15), we obtain the power spectrum $\mathcal{P}_h$ as

$$\mathcal{P}_h := \frac{k^3}{2\pi^2} \mathcal{P}_{ij,ij} = \frac{1}{\pi^2} \frac{H^2}{M_T^2 c_h} \sum_s \left| \alpha_k^{(s)} - \beta_k^{(s)} \right|^2, \tag{5.23}$$

evaluated at the time of horizon crossing, $c_h k = aH$. Its tilt is then derived as

$$\begin{aligned} n_t &:= \frac{\mathrm{d}\ln \mathcal{P}_h}{\mathrm{d}\ln k} \\ &\simeq -2\epsilon - s_h - 2m_T + \frac{\mathrm{d}}{\mathrm{d}\ln k} \sum_s \left| \alpha_k^{(s)} - \beta_k^{(s)} \right|^2, \end{aligned} \tag{5.24}$$

where

$$\epsilon := -\frac{\dot{H}}{H^2}, \quad s_h := \frac{\dot{c}_h}{H c_h}, \quad m_h := \frac{\dot{M}_T}{H M_T}, \tag{5.25}$$

are assumed to be small. To leading order in $\beta_k^{(s)}$ we have $|\alpha_k^{(s)} - \beta_k^{(s)}|^2 \simeq 1 - 2\mathrm{Re}[\beta_k^{(s)}]$, and so

$$n_t \simeq -2\epsilon - s_h - 2m_T - 2\sum_s \frac{\mathrm{d}\,\mathrm{Re}[\beta_k^{(s)}]}{\mathrm{d}\ln k}. \tag{5.26}$$

This is a rather straightforward generalization of previous results, simultaneously taking into account the different effects on the spectral tilt: the time variation of the inflationary Hubble parameter, the speed of gravitational waves, and the effective Planck mass, as well as the k-dependence of the Bogoliubov coefficients. Note that in principle the sign of each term in Eq. (5.26) is not constrained. In particular, a blue tensor spectrum can be obtained as a consequence of a time-dependent speed of gravitational waves [98, 99, 100, 101, 102] and/or k-dependent $\beta_k^{(s)}$ [103] even if the null energy condition is preserved, $\epsilon > 0$.

5.2 Auto-bispectrum

Let us now calculate the tensor three-point correlation functions with non-Bunch-Davies initial states. Since the cubic interaction (5.7) is composed of the two different contributions, i.e., the one present in general relativity and the new one beyond general relativity, we write the bispectrum as $\mathcal{B}^{s_1s_2s_3}_{(\mathrm{GR})} + \mathcal{B}^{s_1s_2s_3}_{(\mathrm{new})}$, where $\mathcal{B}^{s_1s_2s_3}_{(\mathrm{GR})}$

and $\mathcal{B}^{s_1s_2s_3}_{(\mathrm{new})}$ are originated from the former and the latter, respectively. The auto-bispectrum (evaluated at $t = t_f$) is defined by

$$\langle \xi^{s_1}(t_f,\mathbf{k}_1)\xi^{s_2}(t_f,\mathbf{k}_2)\xi^{s_3}(t_f,\mathbf{k}_3)\rangle = \langle \Omega_b|\xi^{s_1}(t_f,\mathbf{k}_1)\xi^{s_2}(t_f,\mathbf{k}_2)\xi^{s_3}(t_f,\mathbf{k}_3)|\Omega_b\rangle \quad (5.27)$$
$$= (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\left(\mathcal{B}^{s_1s_2s_3}_{(\mathrm{GR})}+\mathcal{B}^{s_1s_2s_3}_{(\mathrm{new})}\right), \quad (5.28)$$

with $\xi^s(t,\mathbf{k}) = \tilde{h}_{ij}(t,\mathbf{k})e^{(s)*}_{ij}(\mathbf{k})$ and

$$|\Omega_b\rangle = T\exp\left(-i\int H_{\mathrm{int}}(t')\mathrm{d}t'\right)|0_b\rangle, \quad (5.29)$$

and the three-point correlation function can be calculated as

$$\langle \xi^{s_1}(t_f,\mathbf{k}_1)\xi^{s_2}(t_f,\mathbf{k}_2)\xi^{s_3}(t_f,\mathbf{k}_3)\rangle$$
$$= -i\int_{t_i}^{t_f}\mathrm{d}t'\langle 0_b|[\xi^{s_1}(t_f,\mathbf{k}_1)\xi^{s_2}(t_f,\mathbf{k}_2)\xi^{s_3}(t_f,\mathbf{k}_3),H_{\mathrm{int}}(t')]|0_b\rangle. \quad (5.30)$$

In terms of the conformal time defined by $\mathrm{d}\eta := \mathrm{d}t/a$, we take $\eta_f = 0$. As for the initial time, we do not simply take $\eta_0 \to -\infty$, but we keep it finite, $\eta_i = \eta_0\,(<0)$, where η_0 is associated with the cutoff scale M_* as $M_* = k/a(\eta_0) \simeq (-k\eta_0)H_{\mathrm{inf}}$, because the physical momentum k/a is larger than M_* for $\eta < \eta_0$.

Before moving to an explicit calculation of the bispectrum (5.30), we comment on the crucial difference between the calculation with the Bunch-Davies state and that with non-Bunch-Davies initial states. This difference explains the reason why we keep η_0 finite. Formally, Eq. (5.30) includes an integral of the form:

$$S(\tilde{k}) := \int_{\eta_0}^{0}\mathrm{d}\eta(-\eta)^n e^{ic_h\tilde{k}\eta}, \quad (5.31)$$

where $n = 1$ for the standard cubic term with two spatial derivatives and $n = 2$ for the $\dot{h}_{ij}\dot{h}_{jk}\dot{h}_{ki}$ term.

In the case of the Bunch-Davies initial state in which there are only the positive-frequency modes participating in this integral, we have $\tilde{k} = k_t$ with

$$k_t := k_1 + k_2 + k_3 > 0, \quad (5.32)$$

and so

$$S(\tilde{k}) \propto \frac{1}{(ic_h\tilde{k})^{n+1}}, \quad (5.33)$$

because the exponential function rapidly oscillates for $|c_h\tilde{k}\eta| \gg 1$. In contrast to this standard case, in the case of non-Bunch Davies states, we have both positive and negative frequency modes in the integral, leading to $\tilde{k} = -k_m + k_{m+1} + k_{m+2}$ with m being defined modulo 3. Note that $\tilde{k}$ exactly vanishes at the flattened configuration, $k_m = k_{m+1} + k_{m+2}$. For this configuration, the exponential function no longer oscillates even for $\eta \sim \eta_0$, and thus the integral reads

$$S(\tilde{k}) \simeq \frac{(-\eta_0)^{n+1}}{n+1}, \tag{5.34}$$

which depends explicitly on η_0. For the other configurations, the results of the integral are identical to Eq. (5.33). In this section, we therefore need to calculate the primordial bispectra in the two different cases separately, the non-flattened and flattened configurations.

Throughout this paper, we specify the exact flattened configuration as

$$k_0 := k_1 - k_2 - k_3 = 0. \tag{5.35}$$

5.2.1 Non-flattened configurations ($k_0 \neq 0$)

We first focus on the non-flattened configurations, i.e., $k_0 \neq 0$. Assuming the de Sitter background and M_T, c_h, $M_4 = $ const, the two contributions in the bispectrum (5.28) respectively read

$$\mathcal{B}^{s_1 s_2 s_3}_{(\mathrm{GR})} = \mathrm{Re}\left[\tilde{\mathcal{B}}^{s_1 s_2 s_3}_{(\mathrm{GR})}\right] (s_1 k_1 + s_2 k_2 + s_3 k_3)^2 F(s_i, k_i), \tag{5.36}$$

$$\mathcal{B}^{s_1 s_2 s_3}_{(\mathrm{new})} = \mathrm{Re}\left[\tilde{\mathcal{B}}^{s_1 s_2 s_3}_{(\mathrm{new})}\right] F(s_i, k_i), \tag{5.37}$$

where

$$\begin{aligned}\tilde{\mathcal{B}}^{s_1s_2s_3}_{(\mathrm{GR})} =&\frac{2H^4}{c_h^2M_T^4}\frac{1}{k_1^3k_2^3k_3^3}\left[\Pi_i\left(\alpha_{k_i}^{(s_i)*}-\beta_{k_i}^{(s_i)*}\right)\right]\\&\times\left\{\left(\alpha_{k_1}^{(s_1)}\alpha_{k_2}^{(s_2)}\alpha_{k_3}^{(s_3)}+\beta_{k_1}^{(s_1)}\beta_{k_2}^{(s_2)}\beta_{k_3}^{(s_3)}\right)\mathcal{I}_0(k_1,k_2,k_3)\right.\\&+\left[\left(\alpha_{k_1}^{(s_1)}\alpha_{k_2}^{(s_2)}\beta_{k_3}^{(s_3)}+\beta_{k_1}^{(s_1)}\beta_{k_2}^{(s_2)}\alpha_{k_3}^{(s_3)}\right)\mathcal{I}_1(k_1,k_2,k_3)\right.\\&\left.\left.+(k_1,s_1\leftrightarrow k_2,s_2)+(k_1,s_1\leftrightarrow k_3,s_3)\right]\right\},\end{aligned}\tag{5.38}$$

$$\begin{aligned}\tilde{\mathcal{B}}^{s_1s_2s_3}_{(\mathrm{new})} =&\frac{192M_4H^5}{M_T^6}\frac{1}{k_1k_2k_3}\left[\Pi_i\left(\alpha_{k_i}^{(s_i)*}-\beta_{k_i}^{(s_i)*}\right)\right]\\&\left\{\left(\alpha_{k_1}^{(s_1)}\alpha_{k_2}^{(s_2)}\alpha_{k_3}^{(s_3)}+\beta_{k_1}^{(s_1)}\beta_{k_2}^{(s_2)}\beta_{k_3}^{(s_3)}\right)\frac{1}{k_t^3}\right.\\&-\left[\left(\alpha_{k_1}^{(s_1)}\beta_{k_2}^{(s_2)}\beta_{k_3}^{(s_3)}+\beta_{k_1}^{(s_1)}\alpha_{k_2}^{(s_2)}\alpha_{k_3}^{(s_3)}\right)\frac{1}{(-k_1+k_2+k_3)^3}\right.\\&\left.\left.+(k_1,s_1\leftrightarrow k_2,s_2)+(k_1,s_1\leftrightarrow k_3,s_3)\right]\right\},\end{aligned}\tag{5.39}$$

and

$$\mathcal{I}_0(k_1,k_2,k_3):=-k_t+\frac{k_1k_2k_3}{k_t^2}+\frac{k_1k_2+k_2k_3+k_3k_1}{k_t},\tag{5.40}$$

$$\mathcal{I}_1(k_1,k_2,k_3):=k_1+k_2-k_3+\frac{k_1k_2k_3}{(k_1+k_2-k_3)^2}+\frac{-k_1k_2+k_2k_3+k_1k_3}{k_1+k_2-k_3}.\tag{5.41}$$

These expressions are a generalization of Ref. [39], and reproduce the previous results by taking the Bunch-Davies states ($\alpha_k^{(s)}=1$ and $\beta_k^{(s)}=0$). Note that we have derived the auto-bispectrum from the $\dot{h}_{ij}\dot{h}_{jk}\dot{h}_{ki}$ term for the first time in the context of the non-Bunch-Davies states.

Taking into account the smallness of $\beta_k^{(s)}$ [Eqs. (5.19) and (5.20)], the resultant bispectra to first order in $\beta_k^{(s)}$ are given by

$$\begin{aligned}\mathcal{B}^{s_1s_2s_3}_{(\mathrm{GR})} =&\frac{2H^4}{c_h^2M_T^4}\frac{1}{k_1^3k_2^3k_3^3}(s_1k_1+s_2k_2+s_3k_3)^2F(s_i,k_i)\\&\times\left\{(1-\sum_i\mathrm{Re}[\beta_{k_i}^{(s_i)}])\mathcal{I}_0(k_1,k_2,k_3)+\left[\mathrm{Re}[\beta_{k_3}^{(s_3)}]\mathcal{I}_1(k_1,k_2,k_3)+\cdots\right]\right\},\end{aligned}\tag{5.42}$$

$$\mathcal{B}^{s_1s_2s_3}_{(\mathrm{new})} =\frac{192M_4H^5}{M_T^6}\frac{F(s_i,k_i)}{k_1k_2k_3}\left\{\frac{1-\sum_i\mathrm{Re}[\beta_{k_i}^{(s_i)}]}{k_t^3}-\left[\frac{\mathrm{Re}[\beta_{k_1}^{(s_1)}]}{(-k_1+k_2+k_3)^3}+\cdots\right]\right\},\tag{5.43}$$

where the ellipsis denotes permutations.

Let us consider the squeezed configuration with $k_L := k_3 \ll k_S := k_1 = k_2$. In the squeezed limit, the expressions in the curly brackets in Eqs. (5.42) and (5.43) are written respectively as

$$\{\cdots\} \simeq -\frac{3}{2}k_S\left(1 - \frac{4}{3}\mathrm{Re}[\beta_{k_S}^{(s_1)} + \beta_{k_S}^{(s_2)}]\frac{k_S}{k_L}\right), \tag{5.44}$$

and

$$\{\cdots\} \simeq \frac{1}{8k_S^3}\left(1 - 8\mathrm{Re}[\beta_{k_S}^{(s_1)} + \beta_{k_S}^{(s_2)}]\frac{k_S^3}{k_L^3}\right). \tag{5.45}$$

These equations show that the effect of nonvanishing $\beta_k^{(s)}$ could be enhanced and seen in the squeezed configuration. In particular, the generation of squeezed non-Gaussianity from the $\dot{h}_{ij}\dot{h}_{jk}\dot{h}_{ki}$ term is in contrast with the standard case of the Bunch-Davies state in which the bispectrum has a peak at the equilateral configuration [39].*1

To see whether this enhancement effect is significant or not, let us take $k_S/k_L \sim 10^2$. The non-Bunch-Davies contributions in Eqs. (5.44) and (5.45) are then of $\mathcal{O}(10^2|\beta_{k_S}^{(s)}|)$ and $\mathcal{O}(10^6|\beta_{k_S}^{(s)}|)$, respectively. As in Appendix D, the upper bound on the Bogoliubov coefficients is obtained from the backreaction constraint, which depends on the ratio $M_*/M_T(\sim M_*/M_{\mathrm{Pl}})$. If one takes $M_* \sim M_T \sim M_{\mathrm{Pl}}$, one has $|\beta_{k_S}^{(s)}| \lesssim 10^{-6}$, so that the non-Bunch-Davies contribution in $\mathcal{B}_{(\mathrm{GR})}^{s_1s_2s_3}$ is small, $\sim 10^{-4}$, while that in $\mathcal{B}_{(\mathrm{new})}^{s_1s_2s_3}$ is of $\mathcal{O}(1)$. This can be larger if one assumes smaller M_*. For example, one gets $|\beta_{k_S}^{(s)}| \lesssim 10^{-2}$ if $M_* \sim 10^{-2}M_T \sim 10^{-2}M_{\mathrm{Pl}}$. In this case, the non-Bunch-Davies contribution in $\mathcal{B}_{(\mathrm{GR})}^{s_1s_2s_3}$ is of $\mathcal{O}(1)$ and that in $\mathcal{B}_{(\mathrm{new})}^{s_1s_2s_3}$ is as large as $\mathcal{O}(10^4)$. Therefore, tensor squeezed non-Gaussianity could be generated from the non-Bunch-Davies initial states, depending on the parameters.

5.2.2 Flattened Configuration ($k_0 \to 0$)

So far we have assumed that $k_0 = k_1 - k_2 - k_3 \neq 0$. Let us now investigate the flattened configuration, $k_0 \simeq 0$, using Eq. (5.34). In this case, $\tilde{\mathcal{B}}_{(\mathrm{GR})}^{s_1s_2s_3}$ and $\tilde{\mathcal{B}}_{(\mathrm{new})}^{s_1s_2s_3}$

*1 Squeezed tensor non-Gaussianities from the the $\dot{h}_{ij}\dot{h}_{jk}\dot{h}_{ki}$ operator has been found also in the non-attractor inflation models [71] and bouncing models [55].

in Eqs. (5.36) and (5.37) are given respectively by

$$\tilde{\mathcal{B}}^{s_1s_2s_3}_{(\mathrm{GR})} \simeq \frac{2H^4}{c_h^2 M_T^4}\frac{1}{k_1^3k_2^3k_3^3}\left[\Pi_i\left(\alpha^{(s_i)*}_{k_i}-\beta^{(s_i)*}_{k_i}\right)\right] \times\left[\left(\alpha^{(s_1)}_{k_1}\alpha^{(s_2)}_{k_2}\alpha^{(s_3)}_{k_3}+\beta^{(s_1)}_{k_1}\beta^{(s_2)}_{k_2}\beta^{(s_3)}_{k_3}\right)\mathcal{I}_0(k_1,k_2,k_3) - \frac{k_1k_2k_3}{2}c_h^2\eta_0^2\left(\beta^{(s_1)}_{k_1}\alpha^{(s_2)}_{k_2}\alpha^{(s_3)}_{k_3}+\alpha^{(s_1)}_{k_1}\beta^{(s_2)}_{k_2}\beta^{(s_3)}_{k_3}\right)\right], \tag{5.46}$$

$$\tilde{\mathcal{B}}^{s_1s_2s_3}_{(\mathrm{new})} \simeq \frac{192M_4H^5}{M_T^6}\frac{1}{k_1k_2k_3}\left[\Pi_i\left(\alpha^{(s_i)*}_{k_i}-\beta^{(s_i)*}_{k_i}\right)\right] \times\left[\left(\alpha^{(s_1)}_{k_1}\alpha^{(s_2)}_{k_2}\alpha^{(s_3)}_{k_3}+\beta^{(s_1)}_{k_1}\beta^{(s_2)}_{k_2}\beta^{(s_3)}_{k_3}\right)\frac{1}{k_t^3} + \frac{i}{6}c_h^3\eta_0^3\left(\beta^{(s_1)}_{k_1}\alpha^{(s_2)}_{k_2}\alpha^{(s_3)}_{k_3}-\alpha^{(s_1)}_{k_1}\beta^{(s_2)}_{k_2}\beta^{(s_3)}_{k_3}\right)\right], \tag{5.47}$$

where we used $k_0 \ll k_i$, $|c_h k_i \eta_0| \gg 1$, and $|c_h k_0 \eta_0| \ll 1$. In Ref. [10], the flattened tensor non-Gaussianity has already been studied, but the interactions among the different polarization modes have not been considered.

Similarly to the non-flattened configurations, we express the resultant bispectra to first order in $\mathcal{O}(\beta^{(s)}_k)$ as

$$\mathcal{B}^{s_1s_2s_3}_{(\mathrm{GR})} \simeq \frac{2H^4}{c_h^2 M_T^4}\frac{1}{k_1^3k_2^3k_3^3}(s_1k_1+s_2k_2+s_3k_3)^2F(s_i,k_i) \times\left\{(1-\sum_i \mathrm{Re}[\beta^{(s_i)}_{k_i}])\mathcal{I}_0(k_1,k_2,k_3) - \frac{k_1k_2k_3}{2}c_h^2\eta_0^2\,\mathrm{Re}[\beta^{(s_1)}_{k_1}]\right\}, \tag{5.48}$$

$$\mathcal{B}^{s_1s_2s_3}_{(\mathrm{new})} \simeq \frac{192M_4H^5}{M_T^6}\frac{F(s_i,k_i)}{k_1k_2k_3}\left\{\frac{1-\sum_i\mathrm{Re}[\beta^{(s_i)}_{k_i}]}{k_t^3} - \frac{c_h^3\eta_0^3}{6}\,\mathrm{Im}[\beta^{(s_1)}_{k_1}]\right\}. \tag{5.49}$$

Now we see that the primordial bispectra always vanish at the exact flattened configurations, because $F(s_i,k_i) = 0$ for $k_0 = 0$. This universal feature can be understood intuitively from the viewpoint of angular momentum conservation [104]. Although the expressions in the curly brackets could be enhanced by powers of $k_i\eta_0$, it would be difficult to obtain large flattened non-Gaussianities due to this universal factor.*2 This is in sharp contrast to the result of the similar analysis for the curvature perturbation. However, this is not the case for the cross-interaction, as shown in the next section.

*2 A different conclusion was obtained in [10] because the overall factor $F(s_i,k_i)$ was overlooked.

5.3 Cross-bispectrum

In this section, we consider a scalar-scalar-tensor bispectrum, rather than a tensor-tensor-tensor bispectrum, and explore the possibility of enhancing it with nontrivial initial states of the tensor modes. The cross-bispectrum we will consider is defined by

$$\langle\tilde{\zeta}(0,\mathbf{k}_1)\tilde{\zeta}(0,\mathbf{k}_2)\xi^{(s)}(0,\mathbf{k}_3)\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\mathcal{B}^s_{\zeta\zeta h}. \tag{5.50}$$

For the Lagrangian (5.4), the quadratic action for the curvature perturbation in the unitary gauge, ζ, takes the form [20]

$$S^{(2)}_\zeta = \int \mathrm{d}t\mathrm{d}^3x\, \frac{a^3M_S^2}{c_s^2}\left[\dot{\zeta}^2 - \frac{c_s^2}{a^2}(\partial_i\zeta)^2\right], \tag{5.51}$$

where we do not need the concrete expression for M_S and c_s in the present discussion. These are time-dependent functions in general, but in the inflationary universe we may assume that they are approximately constant. We assume that the Fourier component of the curvature perturbation, $\tilde{\zeta}(t,\mathbf{k})$, can be written as

$$\tilde{\zeta} = \psi_k a_{\mathbf{k}} + \psi_k^* a^\dagger_{-\mathbf{k}}, \tag{5.52}$$

where

$$\psi_k = \frac{\sqrt{\pi}}{2\sqrt{2}a}\frac{c_s}{M_S}\sqrt{-\eta}H^{(1)}_{3/2}(-c_sk\eta) \tag{5.53}$$

is the Bunch-Davies mode function and the initial state is in a vacuum state annihilated by $a_{\mathbf{k}}$. By assuming this we focus on the effect of the excited tensor modes.

It has been found that the generic action [Eqs. (5.3) and (5.4)] introduces various cubic operators that are not present in the simple case where the inflaton is minimally coupled to gravity [32]. Among such operators it is sufficient to consider one representative term that is expected to be a dominant source of the non-Gaussianities in order to see whether the bispectrum can be enhanced or not. Naively, operators with many derivatives are important for the generation of non-Gaussianities on subhorizon scales, and thus we focus on the following interaction Hamiltonian:

$$H^{\zeta\zeta h}_{\mathrm{int}} = -\int \mathrm{d}^3x\frac{M_S^2\Lambda_c}{ac_s^2H^2}\partial^2h_{ij}\partial_i\zeta\partial_j\zeta, \tag{5.54}$$

where we assume that Λ_c = const. This term is indeed present in the general Horndeski class of theories [65].

Similarly to the auto-correlation function, the cross-correlation function includes the integral

$$S_c(\tilde{k}_c) := \int_{\eta_0}^{0} \mathrm{d}\eta(-\eta)^3 e^{i\tilde{k}_c\eta}, \tag{5.55}$$

where

$$\tilde{k}_c := c_h k_3 - c_s(k_1 + k_2). \tag{5.56}$$

For the configuration satisfying $\tilde{k}_c = 0$, the cross-bispectrum depends on η_0 and is enhanced by powers of $k_i\eta_0$ due to the excited tensor modes. Note that this configuration depends on the propagation speeds. For given c_s/c_h (< 1), one has a one-parameter family of different shapes satisfying $\tilde{k}_c = 0$ away from the flattened configuration.

In the same way as the previous calculations, we derive the cross-bispectrum to first order in $\beta^{(s)}_{k_3}$:

$$\mathcal{B}^s_{\zeta\zeta h} = \mathcal{B}^s_{\zeta\zeta h,(\mathrm{BD})}\Big|_{\tilde{k}_c=0}\left\{1 - \mathrm{Re}[\beta^{(s)}_{k_3}] + \frac{(2/5)k_1k_2}{2k_1^2 + 5k_1k_2 + 2k_2^2}c_s^4(k_1+k_2)^4\eta_0^4\mathrm{Re}[\beta^{(s)}_{k_3}]\right\}, \tag{5.57}$$

where $\mathcal{B}^s_{\zeta\zeta h}$ is the cross-bispectrum in the case of the Bunch-Davies initial state. This quantity is obtained in [65] as

$$\mathcal{B}^s_{\zeta\zeta h,(\mathrm{BD})} = \frac{H^4\Lambda_c}{M_S^2 M_T^2 c_s^4 c_h} \cdot \frac{k_t}{16k_1^3k_2^3k_3^3} \cdot \frac{(k_1-k_2-k_3)(k_1+k_2-k_3)(k_1-k_2+k_3)}{[c_s(k_1+k_2)+c_hk_3]^4} \times \left\{c_s^2[c_s(k_1+k_2)+4c_hk_3](k_1^2+3k_1k_2+k_2^2) + c_h^2k_3^2[4c_s(k_1+k_2)+c_hk_3]\right\}. \tag{5.58}$$

From the above result we see that the non-Bunch-Davies contribution is of $\mathcal{O}(\beta^{(s)}_k c_s^4 k_i^4 \eta_0^4)$.

In the actual observables, we anticipate that this non-Bunch-Davies enhancement will be softened by (at least) one power of $|k\eta_0|$ due to the angular averaging [8]. Let us therefore estimate roughly how large $\beta^{(s)}_k(c_sk_i\eta_0)^n$ could be. As argued in Appendix D, the Bogoliubov coefficients have an upper bound from the backreaction constraint, which depends on the cutoff scale. We also have $|c_sk_i\eta_0| \lesssim c_sM_*/H_{\mathrm{inf}}$. Combining these, we find

$$\beta^{(s)}_k(c_sk_i\eta_0)^n \lesssim \frac{c_s^n}{c_h^{1/2}}\frac{M_{\mathrm{Pl}}M_*^{n-2}}{H_{\mathrm{inf}}^{n-1}}. \tag{5.59}$$

Even for $n = 2$ the upper bound is typically larger than $\mathcal{O}(1)$. We thus conclude that initially excited tensor modes can leave a potentially observable imprint in the cross-bispectrum[*3].

[*3] In the present chapter, we have considered the scalar-scalar-tensor bispectrum, but initially excited scalar modes would be able to enhance the scalar-tensor-tensor bispectrum as well.

Chapter 6

Conclusions

In Chapter 2, we have briefly reviewed the standard Big Bang cosmology and its problems, and then introduced the early universe models, inflation and bounce models.

In Chapter 3, we have briefly reviewed the cosmological perturbation theory, and applied to inflation and alternatives. In the case of inflation, the model can be easily consistent with the observations, and thus we have calculated the primordial power spectrum and the primordial non-Gaussianities. In particular, we have shown that the amplitudes and the shapes of non-Gaussianities are useful to discriminate the models. As for the alternatives to inflation, we have explained the generic instabilities of the non-singular cosmological solutions and also introduced how to evade those. As different problems, we have also introduced the inconsistency with the observations, i.e., a large tensor-to-scalar ratio or a large scalar non-Gaussianity.

In Chapter 4, we have studied the primordial power spectra and the bispectra of scalar and tensor perturbations generated during a general contracting phase in the Horndeski theory. It can be shown that under certain conditions the power spectra of scalar and tensor perturbations are scale invariant. We have found that the previous no-go theorem [53] prohibiting the simultaneous realization of small tensor-to-scalar ratio and small scalar non-Gaussianity in matter bounce cosmology driven by a k-essence field no longer holds in more general setups. A concrete example with small r and small f_{NL} has been presented. Then, we have found that the non-Gaussianities of tensor perturbations from the contracting universes have two specific features which are in contrast with the predictions from generalized G-inflation. First, our contracting models predict only squeezed-type non-Gaussianities, while inflation can in principle generate both squeezed- and equilateral-type ones. Second, the squeezed-type non-Gaussian amplitude from inflation is model-independently fixed, while that from the contracting scenario is model-dependent. We thus conclude that our general bounce model can be distinguished from generalized G-inflation by combining the information of the non-Gaussian amplitudes and shapes.

In Chapter 5, we have considered primordial tensor perturbations with non-Bunch-Davies initial states. Employing a general scalar-tensor theory, we have described non-minimal couplings between gravity and the inflaton. First, we evaluated the size of tensor three-point functions and showed that the squeezed non-Gaussianities in particular from the newly introduced operator in non-minimally coupled theories can potentially be enhanced. In contrast to the case of the scalar three-point functions [8], the tensor three-point function always vanishes at the flattened momentum triangles. This is as it should be, as can be seen from the momentum conservation argument [104]. Next, we have studied the cross-bispectrum involving one tensor and two scalar modes. We have found that the enhancement due to the non-Bunch-Davies effect can be large at non-trivial triangle shapes.

Throughout this thesis, we have calculated the primordial tensor non-Gaussianity. It would therefore be interesting to investigate how such non-Gaussian signature is imprinted e.g. on CMB bispectra [70, 105], which we leave for further studies. Also, it would be important to estimate the non-Gaussian amplitudes and shapes from different early universe models, e.g., Galilean Genesis [106].

Note that a part of the above is cited from, S. Akama, S. Hirano and T. Kobayashi, "Primordial non-Gaussianities of scalar and tensor perturbations in general bounce cosmology: Evading the no-go theorem," Phys. Rev. D **101**, no.4, 043529 (2020) doi:10.1103/PhysRevD.101.043529 [arXiv:1908.10663 [gr-qc]] [55], and S. Akama, S. Hirano and T. Kobayashi, "Primordial tensor non-Gaussianities from general single-field inflation with non-Bunch-Davies initial states," Phys. Rev. D **102**, no.2, 023513 (2020) doi:10.1103/PhysRevD.102.023513 [arXiv:2003.10686 [gr-qc]] [72].

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Appendix A

Background Equations

This chapter is based on S. Akama, S. Hirano and T. Kobayashi, “Primordial non-Gaussianities of scalar and tensor perturbations in general bounce cosmology: Evading the no-go theorem,” Phys. Rev. D **101**, no.4, 043529 (2020) doi:10.1103/PhysRevD.101.043529 [arXiv:1908.10663 [gr-qc]] [55]. Copyright (2020) by the American Physical Society.

For a flat FLRW universe the gravitational field equations read [20]

$$\mathcal{E} := \sum_{i=2}^{5} \mathcal{E}_i = 0, \quad \mathcal{P} := \sum_{i=2}^{5} \mathcal{P}_i = 0, \tag{A.1}$$

where

$$\mathcal{E}_2 = 2XG_{2X} - G_2, \tag{A.2}$$
$$\mathcal{E}_3 = 6X\dot{\phi}HG_{3X} - 2XG_{3\phi}, \tag{A.3}$$
$$\mathcal{E}_4 = -6H^2G_4 + 24H^2X(G_{4X} + XG_{4XX}) - 12HX\dot{\phi}G_{4\phi X} - 6H\dot{\phi}G_{4\phi}, \tag{A.4}$$
$$\mathcal{E}_5 = 2H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) - 6H^2X(3G_{5\phi} + 2XG_{5\phi X}), \tag{A.5}$$

and

$$\mathcal{P}_2 = G_2, \tag{A.6}$$
$$\mathcal{P}_3 = -2X(G_{3\phi} + \ddot{\phi}G_{3X}), \tag{A.7}$$
$$\begin{aligned}\mathcal{P}_4 = {} & 2(3H^2 + 2\dot{H})G_4 - 12H^2XG_{4X} - 4H\dot{X}G_{4X} - 8\dot{H}XG_{4X} \\ & - 8HX\dot{X}G_{4XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4\phi} + 4XG_{4\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4\phi X},\end{aligned} \tag{A.8}$$
$$\begin{aligned}\mathcal{P}_5 = {} & -2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5X} - 4H^2X^2\ddot{\phi}G_{5XX} \\ & + 4HX(\dot{X} - HX)G_{5\phi X} + 2\left[2(HX)^{\cdot} + 3H^2X\right]G_{5\phi} + 4HX\dot{\phi}G_{5\phi\phi}.\end{aligned} \tag{A.9}$$

The scalar-field equation follows from the above two equations.

Appendix B

Spatial Curvature and Anisotropies in a Contracting Background

This chapter is based on S. Akama, S. Hirano and T. Kobayashi, "Primordial non-Gaussianities of scalar and tensor perturbations in general bounce cosmology: Evading the no-go theorem," Phys. Rev. D **101**, no.4, 043529 (2020) doi:10.1103/PhysRevD.101.043529 [arXiv:1908.10663 [gr-qc]] [55]. Copyright (2020) by the American Physical Society.

In the simple, standard case of a scalar field minimally coupled to gravity, spatial curvature and anisotropies in the Friedmann and evolution equations evolve in proportion to a^{-2} and a^{-6}, respectively. As a result, it has been known that a contracting universe is plagued with the instability associated with large anisotropies [107]. Some resolutions of the problem have been proposed so far. See, e.g., Refs. [108, 23, 109, 24, 110]. However, the impact of spatial curvature and anisotropies has not been clear yet in more general cases where the scalar field is nonminimally coupled to gravity. Hence, we investigate the evolution of spatial curvature and anisotropies in a general contracting background in the Horndeski theory.

First, we investigate the impact of spatial curvature (denoted hereafter as $\mathcal{K}$). To do so, we consider open ($\mathcal{K} < 0$) and closed ($\mathcal{K} > 0$) universes in the Horndeski theory. In the presence of spatial curvature, the background equations reduce to [111, 48]

$$\mathcal{E} + \mathcal{E}_{\mathcal{K}} = 0, \quad \mathcal{P} + \mathcal{P}_{\mathcal{K}} = 0, \tag{B.1}$$

where

$$\mathcal{E}_{\mathcal{K}} = -3\mathcal{G}_T\frac{\mathcal{K}}{a^2}, \quad \mathcal{P}_{\mathcal{K}} = \mathcal{F}_T\frac{\mathcal{K}}{a^2}. \tag{B.2}$$

It can be seen from the scaling argument that $\mathcal{E}_{\mathcal{K}}/\mathcal{E}$, $\mathcal{P}_{\mathcal{K}}/\mathcal{P} \propto (-t)^{2(1-n)}$, which implies that the relative magnitudes of the curvature terms decrease with time so that the effect of the spatial curvature on the background equations can be neglected in our setups.

Next, let us consider the effect of anisotropies on the contracting background by investigating an anisotropic Kasner universe whose metric is written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\left[e^{2(\beta_+ + \sqrt{3}\beta_-)}\mathrm{d}x^2 + e^{2(\beta_+ - \sqrt{3}\beta_-)}\mathrm{d}y^2 + e^{-4\beta_+}\mathrm{d}z^2\right]. \tag{B.3}$$

The differences between the expansion rates in different directions, $\beta_\pm$, obey [111, 112]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{a^3\left[\mathcal{G}_T\dot{\beta}_+ - 2\mu\left(\dot{\beta}_+^2 - \dot{\beta}_-^2\right)\right]\right\} = 0, \tag{B.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{a^3\left[\mathcal{G}_T\dot{\beta}_- + 4\mu\dot{\beta}_+\dot{\beta}_-\right]\right\} = 0. \tag{B.5}$$

Since we have $\mathcal{O}(\mathcal{G}_T) \gtrsim \mathcal{O}(\mu H)$, the nonlinear terms can be ignored as long as initially small anisotropies are considered, $\dot{\beta}_\pm \ll H$. Then, these equations can be integrated to give $\dot{\beta}_\pm \propto (a^3\mathcal{G}_T)^{-1} \propto (-t)^{-(2+2\alpha+3n)}$. We thus see that $\dot{\beta}_\pm/H \propto (-t)^{-(1+2\alpha+3n)}$, which decreases with time if $1 + 2\alpha + 3n < 0$ and increases if $1 + 2\alpha + 3n > 0$. The case of $\alpha = -2$ ($\nu_s = \nu_t = 2/3$) corresponds to the former, while $\alpha = 1 - 3n$ ($\nu_s = \nu_t = -2/3$) to the latter. This result implies the contracting background we are considering requires some mechanism to evade the unwanted growth of anisotropies. In the present paper, we simply assume that the contracting universe enjoys a bounce before the anisotropies spoil its background evolution.

Appendix C

Cubic Action for Scalar Perturbations in the Horndeski Theory

This chapter is based on S. Akama, S. Hirano and T. Kobayashi, "Primordial non-Gaussianities of scalar and tensor perturbations in general bounce cosmology: Evading the no-go theorem," Phys. Rev. D **101**, no.4, 043529 (2020) doi:10.1103/PhysRevD.101.043529 [arXiv:1908.10663 [gr-qc]] [55]. Copyright (2020) by the American Physical Society.

Substituting the perturbed FLRW metric into the Horndeski action, expanding it to cubic order in perturbations and using the background equations, we obtain the cubic action for scalar perturbations [63, 64, 65]:

$$S_\zeta^{(3)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \mathcal{L}^{(3)}, \tag{C.1}$$

where

$$
\begin{aligned}
\mathcal{L}^{(3)} = {} & \mathcal{G}_T\Bigg(-9\zeta\dot{\zeta}^2+\frac{2\dot{\zeta}}{a^2}\left(\zeta\partial^2\chi+\partial_i\zeta\partial_i\chi\right)+\frac{1}{a^4}\left(\partial_i\chi\right)^2\partial^2\zeta+\frac{1}{2a^4}\zeta\left(\left(\partial^2\chi\right)^2-\left(\partial_i\partial_j\chi\right)^2\right)\Bigg)\\
& -\mathcal{G}_T\frac{\delta n}{a^2}\left(\left(\partial_i\zeta\right)^2+2\zeta\partial^2\zeta\right)+\frac{\mathcal{F}_T}{a^2}\zeta\left(\partial_i\zeta\right)^2+3\Sigma\zeta\delta n^2+2\Theta\delta n\left(9\zeta\dot{\zeta}-\zeta\partial^2\chi-\partial_i\zeta\partial_i\chi\right)\\
& +\mu\Bigg(2\dot{\zeta}^3-\frac{2}{a^2}\partial^2\chi\dot{\zeta}^2+\frac{\dot{\zeta}}{a^4}\left(\left(\partial^2\chi\right)^2-\left(\partial_i\partial_j\chi\right)^2\right)+4\delta n\dot{\zeta}\frac{\partial^2\zeta}{a^2}\\
& -\frac{2\delta n}{a^4}\left(\partial^2\zeta\partial^2\chi-\partial_i\partial_j\zeta\partial_i\partial_j\chi\right)\Bigg)+\Gamma\Bigg(3\delta n\dot{\zeta}^2-\frac{2}{a^2}\delta n\dot{\zeta}\partial^2\chi\\
& +\frac{1}{2a^4}\delta n\left(\left(\partial^2\chi\right)^2-\left(\partial_i\partial_j\chi\right)^2\right)\Bigg)+\Xi\delta n^2\left(\dot{\zeta}-\frac{\partial^2\chi}{3a^2}\right)+\left(\Gamma-\mathcal{G}_T\right)\frac{\delta n^2}{a^2}\partial^2\zeta\\
& -\frac{1}{3}\left(\Sigma+2X\Sigma_X+H\Xi\right)\delta n^3.
\end{aligned}
\tag{C.2}
$$

From the first-order constraint equations we have

$$
\delta n=\frac{\mathcal{G}_T}{\Theta}\dot{\zeta}, \tag{C.3}
$$

$$
\chi=\frac{1}{a\mathcal{G}_T}\left(a^3\mathcal{G}_S\psi-\frac{a\mathcal{G}_T^2}{\Theta}\zeta\right), \tag{C.4}
$$

where $\partial^2\psi=\dot{\zeta}$. Substituting these solutions into the cubic action, we obtain

$$
\begin{aligned}
S_\zeta^{(3)} = {} & \int \mathrm{d}t\mathrm{d}^3x a^3\mathcal{G}_S\Bigg\{\frac{\Lambda_1}{H}\dot{\zeta}^3+\Lambda_2\zeta\dot{\zeta}^2+\Lambda_3\zeta\frac{\left(\partial_i\zeta\right)^2}{a^2}+\frac{\Lambda_4}{H^2}\dot{\zeta}^2\frac{\partial^2\zeta}{a^2}+\Lambda_5\dot{\zeta}\partial_i\zeta\partial_i\psi\\
& +\Lambda_6\partial^2\zeta\left(\partial_i\psi\right)^2+\frac{\Lambda_7}{H^2}\frac{1}{a^4}\left[\partial^2\zeta\left(\partial_i\zeta\right)^2-\zeta\partial_i\partial_j\left(\partial_i\zeta\partial_j\zeta\right)\right]\\
& +\frac{\Lambda_8}{H}\frac{1}{a^2}\left[\partial^2\zeta\partial_i\zeta\partial_i\psi-\zeta\partial_i\partial_j\left(\partial_i\zeta\partial_j\psi\right)\right]\Bigg\}\\
& +\int \mathrm{d}t\mathrm{d}^3x F(\zeta)E_S,
\end{aligned}
\tag{C.5}
$$

where

$$\begin{aligned}
\Lambda_1 = H\Bigg[&\frac{\mathcal{G}_T}{\Theta}\left(\frac{\mathcal{G}_S}{\mathcal{F}_S}+3\frac{\mathcal{G}_T}{\mathcal{G}_S}-1\right)+\frac{\Xi\mathcal{G}_T}{3\Theta^2}\left(3\frac{\mathcal{G}_T}{\mathcal{G}_S}-1\right)+2\mu\left(\frac{1}{\mathcal{G}_S}-\frac{1}{\mathcal{G}_T}\right)\\
&+\frac{\Gamma}{\Theta}\left(3\frac{\mathcal{G}_T}{\mathcal{G}_S}-2\right)+\frac{2}{3}\frac{\mathcal{G}_T^3}{\Theta^3\mathcal{G}_S}\left(\Sigma-X\Sigma_X\right)-\frac{H}{3}\frac{\mathcal{G}_T^3\Xi}{\Theta^3\mathcal{G}_S}\Bigg],
\end{aligned} \tag{C.6}$$

$$\Lambda_2 = 3-\frac{H\mathcal{G}_T\mathcal{G}_S}{\mathcal{F}_S\Theta}\left(3-g_T+f_S+f_\Theta\right), \tag{C.7}$$

$$\Lambda_3 = \frac{\mathcal{F}_T}{\mathcal{G}_S}+\frac{H\mathcal{G}_T}{\Theta}\left(1+g_T+g_S-f_\Theta\right)-\frac{H\mathcal{G}_T^2}{\mathcal{G}_S\Theta}\left(1+2g_T-f_\Theta\right), \tag{C.8}$$

$$\Lambda_4 = H^2\left[\frac{\Xi}{3}\frac{\mathcal{G}_T^3}{\mathcal{G}_S\Theta^3}+6\mu\frac{\mathcal{G}_T}{\mathcal{G}_S\Theta}+\left(3\Gamma-\mathcal{G}_T\right)\frac{\mathcal{G}_T^2}{\mathcal{G}_S\Theta^2}\right], \tag{C.9}$$

$$\Lambda_5 = -\frac{1}{2}\frac{\mathcal{G}_S}{\mathcal{G}_T}-\frac{H}{2}\frac{\Gamma\mathcal{G}_S}{\mathcal{G}_T\Theta}\left(3+g_T-f_\Gamma+f_\Theta\right)-\mu H\frac{\mathcal{G}_S}{\mathcal{G}_T^2}\left(3+2g_T-f_\mu\right), \tag{C.10}$$

$$\Lambda_6 = \frac{3}{4}\frac{\mathcal{G}_S}{\mathcal{G}_T}-\frac{\mathcal{G}_S}{4\mathcal{G}_T}\frac{\Gamma H}{\Theta}\left(3+g_T-f_\Gamma+f_\Theta\right)-\mu H\frac{\mathcal{G}_S}{\mathcal{G}_T^2}\left(\frac{3}{2}+g_T-\frac{1}{2}f_\mu\right), \tag{C.11}$$

$$\begin{aligned}
\Lambda_7 = \frac{H^2}{6}\Bigg[&\frac{\mathcal{G}_T^3}{\mathcal{G}_S\Theta^2}-\frac{H\Gamma\mathcal{G}_T^3}{\mathcal{G}_S\Theta^3}\left(1-3g_T+3f_\Theta-f_\Gamma+3\frac{\Theta\mathcal{F}_S}{H\mathcal{G}_T^2}\right)\\
&-6\mu H\frac{\mathcal{G}_T^2}{\mathcal{G}_S\Theta^2}\left(1-2g_T-f_\mu+2f_\Theta+2\frac{\Theta\mathcal{F}_S}{H\mathcal{G}_T^2}\right)\Bigg],
\end{aligned} \tag{C.12}$$

$$\begin{aligned}
\Lambda_8 = H\Bigg[&-\frac{\mathcal{G}_T}{\Theta}+\frac{\mu H}{\Theta}\left(4+2f_\Theta-2f_\mu+2\frac{\Theta\mathcal{F}_S}{H\mathcal{G}_T^2}\right)\\
&+H\frac{\Gamma\mathcal{G}_T}{\Theta^2}\left(1-\frac{1}{2}g_T-\frac{1}{2}f_\Gamma+f_\Theta+\frac{\Theta\mathcal{F}_S}{H\mathcal{G}_T^2}\right)\Bigg],
\end{aligned} \tag{C.13}$$

$$\begin{aligned}
F(\zeta) = &-\frac{\mathcal{G}_T\mathcal{G}_S}{\Theta\mathcal{F}_S}\zeta\dot{\zeta}-\frac{1}{2}\left(\frac{\Gamma\mathcal{G}_S}{\Theta\mathcal{G}_T}+2\mu\frac{\mathcal{G}_S}{\mathcal{G}_T^2}\right)\left(\partial_i\zeta\partial_i\psi-\partial^{-2}\partial_i\partial_j\left(\partial_i\zeta\partial_j\psi\right)\right)\\
&+\frac{1}{4a^2}\left(\frac{\Gamma\mathcal{G}_T}{\Theta^2}+\frac{4\mu}{\Theta}\right)\left(\left(\partial_i\zeta\right)^2-\partial^{-2}\partial_i\partial_j\left(\partial_i\zeta\partial_j\zeta\right)\right),
\end{aligned} \tag{C.14}$$

$$E_S = -2\left[\partial_t\left(a^3\mathcal{G}_S\dot{\zeta}\right)-a\mathcal{F}_S\partial^2\zeta\right]. \tag{C.15}$$

Here we defined

$$\Theta := -\dot{\phi}XG_{3X} + 2HG_4 - 8HXG_{4X} - 8HX^2G_{4XX} + \dot{\phi}G_{4\phi} + 2X\dot{\phi}G_{4\phi X} - H^2\dot{\phi}(5XG_{5X} + 2X^2G_{5XX}) + 2HX(3G_{5\phi} + 2XG_{5\phi X}), \tag{C.16}$$

$$\begin{aligned}\Sigma := {} & XG_{2X} + 2X^2G_{2XX} + 12H\dot{\phi}XG_{3X} + 6H\dot{\phi}X^2G_{3XX} - 2XG_{3\phi} - 2X^2G_{3\phi X} - 6H^2G_4 \\ & + 6\left[H^2(7XG_{4X} + 16X^2G_{4XX} + 4X^3G_{4XXX}) - H\dot{\phi}(G_{4\phi} + 5XG_{4\phi X} + 2X^2G_{4\phi XX})\right] \\ & + 2H^3\dot{\phi}\left(15XG_{5X} + 13X^2G_{5XX} + 2X^3G_{5XXX}\right) \\ & - 6H^2X(6G_{5\phi} + 9XG_{5\phi X} + 2X^2G_{5\phi XX}),\end{aligned} \tag{C.17}$$

$$\Gamma := 2G_4 - 8XG_{4X} - 8X^2G_{4XX} - 2H\dot{\phi}(5XG_{5X} + 2X^2G_{5XX}) + 2X(3G_{5\phi} + 2XG_{5\phi X}), \tag{C.18}$$

$$\begin{aligned}\Xi := {} & 12\dot{\phi}XG_{3X} + 6\dot{\phi}X^2G_{3XX} - 12HG_4 + 6\Big[2H(7XG_{4X} + 16X^2G_{4XX} + 4X^3G_{4XXX}) \\ & - \dot{\phi}(G_{4\phi} + 5XG_{4\phi X} + 2X^2G_{4\phi XX})\Big] + 90H^2\dot{\phi}XG_{5X} + 78H^2\dot{\phi}X^2G_{5XX} \\ & + 12H^2\dot{\phi}X^3G_{5XXX} - 12HX(6G_{5\phi} + 9XG_{5\phi X} + 2X^2G_{5\phi XX}),\end{aligned} \tag{C.19}$$

and

$$g_T = \frac{\dot{\mathcal{G}}_T}{H\mathcal{G}_T}, \quad g_S = \frac{\dot{\mathcal{G}}_S}{H\mathcal{G}_S}, \quad f_S = \frac{\dot{\mathcal{F}}_S}{H\mathcal{F}_S}, \quad f_\Theta = \frac{\dot{\Theta}}{H\Theta}, \quad f_\Gamma = \frac{\dot{\Gamma}}{H\Gamma}, \quad f_\mu = \frac{\dot{\mu}}{H\mu}. \tag{C.20}$$

Note that we can write the Eqs. (C.18), (C.19) as

$$\Gamma = \frac{\partial\Theta}{\partial H}, \quad \Xi = \frac{\partial\Sigma}{\partial H}. \tag{C.21}$$

It is therefore natural to assume that these quantities scale as

$$\Gamma \propto (-t)^{2+2\alpha}, \quad \Xi \propto (-t)^{1+2\alpha}. \tag{C.22}$$

In Eq. (C.5), we neglected some boundary terms having the form of a total time derivative. They are given by

$$\begin{aligned}S_B = \int \mathrm{d}t\mathrm{d}^3x \frac{\mathrm{d}}{\mathrm{d}t}\Bigg[& -a^3\frac{\mathcal{G}_T\mathcal{G}_S^2}{\Theta\mathcal{F}_S}\zeta\dot{\zeta}^2 + a^3\frac{\mathcal{G}_S^2}{2\mathcal{G}_T^2}\left(2\mu + \frac{\Gamma\mathcal{G}_T}{\Theta}\right)\left(\zeta\dot{\zeta}^2 - \zeta(\partial_i\partial_j\psi)^2\right) \\ & - \frac{a\mathcal{G}_S}{2\Theta}\left(4\mu + \frac{\Gamma\mathcal{G}_T}{\Theta}\right)\left(\zeta\dot{\zeta}\partial^2\zeta - \zeta\partial_i\partial_j\psi\partial_i\partial_j\zeta\right) \\ & + \frac{9a^3}{2}(A_3 - 2H\mathcal{G}_T - 2\mu H^2)\zeta^3 + a\left(\frac{\mathcal{G}_T^2}{\Theta} - B_5\right)\zeta(\partial_i\zeta)^2 \\ & - \frac{\mathcal{G}_T^2}{6a\Theta^2}\left(6\mu + \frac{\Gamma\mathcal{G}_T}{\Theta}\right)\left(\zeta(\partial_i\partial_j\zeta)^2 - \zeta(\partial^2\zeta)^2\right)\Bigg],\end{aligned} \tag{C.23}$$

where

$$A_3 = -\int^X G_{3X'}\sqrt{2X'}\mathrm{d}X' - 2\sqrt{2X}G_{4\phi}, \tag{C.24}$$

$$B_5 = -\int^X G_{5X'}\sqrt{2X'}\mathrm{d}X'. \tag{C.25}$$

Appendix D

Backreaction constraint on $\beta_k^{(s)}$

This chapter is cited from, S. Akama, S. Hirano and T. Kobayashi, "Primordial tensor non-Gaussianities from general single-field inflation with non-Bunch-Davies initial states," Phys. Rev. D **102**, no.2, 023513 (2020) doi:10.1103/PhysRevD.102.023513 [arXiv:2003.10686 [gr-qc]] [72]. Copyright (2020) by the American Physical Society.

If a scalar field is minimally coupled to gravity, the energy-momentum tensor of tensor perturbations is derived by expanding the Einstein tensor to second order in h_{ij}. Even if the scalar field is non-minimally coupled to gravity, one may proceed essentially in the same way and expand the field equations to second order in h_{ij} to estimate the energy density of tensor perturbations. This is how one can evaluate the backreaction of excited tensor modes to the homogeneous background. The effective energy density of subhorizon tensor perturbations is thus given by

$$\rho_h \sim \frac{M_T^2}{a^2 c_h^2} {h'_{ij}}^2 \sim M_T^2 \frac{(\partial_i h_{jk})^2}{a^2}, \tag{D.1}$$

where a dash stands for differentiation with respect to η. The backreaction can safely be ignored if

$$\langle 0_b | \hat{\rho}_h | 0_b \rangle \lesssim \bar{\mathcal{E}}, \tag{D.2}$$

where $\bar{\mathcal{E}}$ is the homogeneous part of the field equation, which can be estimated naively as

$$\bar{\mathcal{E}} \sim M_{\rm Pl}^2 H_{\rm inf}^2, \tag{D.3}$$

where $H_{\rm inf}$ is the inflationary Hubble parameter and $M_{\rm Pl} \sim M_T$.

The backreaction from the excited modes of tensor perturbations can be estimated at $\eta = \eta_0$ from

$$\langle 0_b|\hat{\rho}_h|0_b\rangle \sim \frac{M_T^2}{a^2c_h^2}\langle 0_b|\hat{h}_{ij}'^2|0_b\rangle \sim \frac{c_h}{a^4(\eta_0)}\int_0^{M_*a(\eta_0)} |\beta_k^{(s)}|^2k^3\mathrm{d}k, \tag{D.4}$$

where we discarded the vacuum energy. Then, by requiring that

$$\frac{c_h}{a^4(\eta_0)}\int_0^{M_*a(\eta_0)} |\beta_k^{(s)}|^2k^3\mathrm{d}k \lesssim M_{\rm Pl}^2H_{\rm inf}^2, \tag{D.5}$$

one can save the inflationary background from being spoiled by the backreaction.

To derive a more explicit constraint, we need to assume the momentum dependence of the Bogoliubov coefficients. Here, let us suppose that $\beta_k^{(s)}$ is of the form

$$\beta_k^{(s)} \sim \beta\exp\left[-\frac{k^2}{M_*^2a^2(\eta_0)}\right] \tag{D.6}$$

as a simple model, where β is a constant parameter. Substituting this into Eq. (D.5), we obtain

$$|\beta|^2 \lesssim \frac{1}{c_h}\left(\frac{M_{\rm Pl}}{M_*}\right)^2\left(\frac{H_{\rm inf}}{M_*}\right)^2. \tag{D.7}$$

As is explained in the main text, the deviation of the tensor power spectrum from the standard Bunch-Davies result is at most of $\mathcal{O}(|\beta_k^{(s)}|) \ll 1$, and thus we may use $\mathcal{P}_h \sim H_{\rm inf}^2/(c_hM_T^2)$. Then, the constraint (D.7) can be rewritten as

$$\begin{aligned}|\beta|^2 &\lesssim \mathcal{P}_h\frac{M_{\rm Pl}^2}{M_*^2}\frac{M_T^2}{M_*^2} \sim r\mathcal{P}_\zeta\frac{M_{\rm Pl}^2M_T^2}{M_*^4} \\ &\lesssim 10^{-11}\frac{M_{\rm Pl}^2M_T^2}{M_*^4}.\end{aligned} \tag{D.8}$$

For example, if we take $M_* \sim M_{\rm Pl} \sim M_T$, then we have $|\beta| \lesssim 10^{-6}$, while if we assume that the cutoff scale is much smaller, say, $M_* \sim 10^{-2}M_{\rm Pl} \sim 10^{-2}M_T$, the bound is looser, $|\beta| \lesssim 10^{-2}$.

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