Doctoral thesis

Extended Cuscuton Theory: Formulation and Dark Energy

拡張されたカスクートン理論

—理論の構築およびダークエネルギーモデルへの応用—

Aya Iyonaga

Department of Physics,
Graduate School of Science, Rikkyo University

Abstract

We construct a generic scalar-tensor theory with two degrees of freedom (DOFs) when a scalar field has a timelike gradient, which we dub the "extended cuscuton." This nature originates from the cuscuton theory being a subset of the k-essence model. The usual scalar-tensor theories have three dynamical DOFs, and the number of DOFs is generally independent of scalar field distributions. On the other hand, in the cuscuton, the number of DOFs depends on the gradient of the scalar field. This theory has three DOFs with a spacelike gradient and two DOFs with a timelike gradient. In the latter case, the scalar field obeys a constraint equation, and thus it acts as a nondynamical auxiliary field. Even so, the cuscuton indeed modifies gravity and appears many fascinating features.

In this thesis, we aim to construct a unifying framework of cuscuton-like theories. We particularly focus on the attractive two-DOFs situation with a timelike gradient, which allows us to choose the unitary gauge without loss of generality. We start from a generic scalar-tensor theory with three DOFs in general, and identify the specific form of the Lagrangian by requiring that the theory has only two DOFs in the unitary gauge. We first specify the cosmological prototype, that is, the class satisfying the cosmological features of the cuscuton. Next, we identify which of the theory among the cosmological prototype has two DOFs on an arbitrary background in the unitary gauge by means of the Hamiltonian analysis. We study a relation between the original and the extended cuscuton theories employing disformal transformations. We also compare the extended cuscuton with other related theories, and analyze the stabilities in the presence of a matter field. Finally, we investigate whether the extended cuscuton can account for the current cosmic acceleration. We present a simple example that admits analytic solutions for the cosmological background evolution that mimics ΛCDM cosmology. We argue that this example model can be constrained, like usual scalar-tensor theories, by the growth history of matter density perturbations and the time variation of Newton's constant.

Contents

Chapter 1

Introduction

General relativity (GR) is the most successful and standard gravitational theory. Unlike Newtonian gravity, GR interprets gravitation as the geometry of spacetime and is described in terms of a pseudo-Riemannian manifold. The action of GR is given by the Einstein-Hilbert action:

$$S = \int d^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}R, \tag{1.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ with Lorentzian signature $(-,+,+,+)$, and $M_{\rm Pl}^2 := c\hbar/(8\pi G)$ is the reduced Planck mass. We use the four-dimensional Riemann curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu} := R^{\rho}{}_{\mu\rho\nu}$, and the Ricci scalar $R := R^{\mu}{}_{\mu}$ defined in [1], and we work in natural units ($c = \hbar = 1$). The Einstein-Hilbert action has a very simple form, and this is one of the appealing features of GR.

Due to this geometrical paradigm of gravitation, GR can accurately describes astrophysical phenomena that Newtonian gravity fails to explain, such as the perihelion shift of Mercury. Also, GR predicts enormous never before seen phenomena, e.g., the Shapiro time delay, gravitational lensing, black holes, and gravitational waves (GWs). As of today, all of them have been ditected with high precision (see [2–4] for example). In the context of cosmology, the recent observations of Type Ia supernovae strongly support the current accelerated expansion of the universe [5, 6]. This cosmic expansion can also be described in the framework of GR once one adds the cosmological constant Λ and the matter terms $\mathcal{L}_{\rm m}$ including the cold dark matter (CDM) to the Einstein-Hilbert action,

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}(R - 2\Lambda) + \mathcal{L}_{\rm m}\right]. \tag{1.2}$$

This ΛCDM model passes all late-time observations so far. Thanks to its simplicity and consistency with observations, GR is widely accepted as a very successful gravitational theory.

On the other hand, we cannot verify the validity of GR out of our observable regime, and other gravitational theories are possibly more suitable than GR. Indeed, it is known that

GR will break down at high energy scales such as the Planck scale ($R \sim M_{\rm Pl}^2$), and therefore it is natural to interpret GR as a low-energy limit of more general frameworks of gravity. Even in the low-energy regime, it is no surprise that future measurements and observations will contradict predictions of GR. For this reason, one can consider novel gravitational theories called modified gravity (MG), which reproduce GR in the observable limit such as the low-energy or local scales but modify GR at the strong field or cosmological regimes (see [7] for a review).

Besides the above motivation, MG is worth studying for many viewpoints. First, it is widely believed that the early universe before the Big Bang expanded with increasing speed. This scenario is hard to explain by simple models within GR. Instead, we expect that MG can be a good candidate for the early universe models. Second, observations of the cosmic microwave background (CMB) tells us that the current universe is dominated by mysterious energy components; dark matter and dark energy [8]. These effects can be described by modifying the matter sector in GR, but it is also possible by modifying the gravitational sector, which is just the method of MG. Third, MG is helpful for testing and a better understanding of GR. There is no doubt that GR is feasible since it is consistent with any measurements and observations. However, we should also know the relative viability of GR: whether only GR is a viable theory, or other theories are also viable. Moreover, considering some deviations from GR, e.g., higher dimensions or breaking symmetries, tells us the uniqueness of GR. For these reasons, it is meaningful to compare GR and MG.

A guideline for modifying GR is given by the Lovelock theorem [9, 10]. This theorem implies that an action with only the metric tensor, the four-dimensional general covariance, and second-order Euler-Lagrange equations is just the Einstein-Hilbert action plus the cosmological constant. Given this, in order to construct MG, we must abandon some of the above assumptions, e.g., adding new degrees of freedom (DOFs) besides the metric tensor or considering higher dimensions. One obtains MG in various forms depending on the violating method of assumptions. However, many MG can be described by actions with additional fields on top of the metric tensor. Among additional-field actions, the scalar-tensor theory is the most fundamental model. This type of theory consists of scalar fields and the metric tensor, and is suitable for exploring intrinsic natures of MG. Also, the scalar-tensor theory is often used for studying the dynamics of the universe. For example, the most standard model for the early universe called inflation causes the accelerating cosmic expansion by using a scalar field.

Enormous models of the scalar-tensor theory have been constructed with various motivations, and now it is no longer easy to analyze each model individually. For this point, and for pure curiosity, some unifying frameworks of scalar-tensor theories have been proposed, e.g., the Horndeski theory [11–13], the Gleyzes-Langlois-Piazza-Vernizzi (GLPV, also known as beyond Horndeski) theory [14], and the degenerate higher-order scalar-tensor (DHOST, also known as extended scalar-tensor) theories [15–17]. At the stage of constructing generic theories, the theoretical restriction is evading the Ostrogradsky ghost [18–20]. To avoid the ghost, the general single-field scalar-tensor theories have three DOFs: two of that are tensor modes as in GR, and one of that is a scalar mode.

On the other hand, it is known that some classes of the Horndeski theory has only two dynamical DOFs. One example is the cuscuton theory [21], in which the number of DOFs depends on a gradient of the scalar field: three DOFs with a spacelike gradient, and two DOFs with a timelike gradient. In a timelike gradient case, the scalar field obeys a constraint equation and then acts as a nondynamical auxiliary field. Consequently, the scalar DOF does not propagate, and there remains only two tensor modes. The cuscuton has the same number of DOFs as GR, but actually modifies gravity, and moreover, has many appealing features. For example, the cuscuton is useful for viable early/late-time universe models [22–24], and the CMB and matter power spectra can be distinguished from those in GR [24]. The existence of the McVitte black hole solution has also been shown [25].

Given these fascinating natures, it would be intriguing to find more general theories sharing the same nature with the cuscuton model, i.e., general theories with only two physical DOFs when a gradient of the scalar field is timelike. This thesis aims to find such theories, which we dub the "extended cuscuton," and explore its theoretical/cosmological features. To this end, we start from the GLPV theories having three DOFs in general, and identify the specific form of the Lagrangian by requiring that the theory has only two DOFs when the scalar field has a timelike gradient. The timelike nature allows us to choose the unitary gauge without loss of generality, and thus we discuss in this gauge for simplicity. We first specify the cosmological prototype, that is, the subclass satisfying the cosmological features of the cuscuton. Next, we obtain the proper extended cuscuton by identifying which of the theory among the cosmological prototype has two DOFs on an arbitrary background in the unitary gauge. We study some relations between the original and the extended cuscuton theories by using disformal transformations (i.e., a redefinition of the metric which depends on a scalar field and its first derivative). We also analyze comparison with other related theories and cosmological perturbations in this theory in the presence of a matter field. Finally, we investigate whether the extended cuscuton can account for the current accelerated expansion of the universe.

The rest of the thesis is organized as follows. In Chap. 2, we first overview GR and the underlying principles. We then introduce MG, particularly the scalar-tensor theories and theories consisting of only the metric tensor (we call these "metric theories"), after explaining the Ostrogradsky ghost. We also review the Lorentz-violating and two-DOFs theories as groups sharing the same features with the cuscuton. Next, in Chap. 3, we formulate the cuscuton theory. We then study the relations between its DOFs and scalar field distributions, and explore the cuscuton cosmology and other various aspects. After that, we construct the extended cuscuton theories and analyze its stability in the presence of matter in Chap. 4. Some relations between the extended cuscuton and other theories, e.g., the original cuscuton and some two-DOFs/Lorentz-violating theories, are discussed. This chapter is based on our published paper [26]. In Chap. 5, we apply the extended cuscuton to the late-time cosmology. This chapter is based on our published paper [27]. Finally, we draw our conclusions in Chap. 6.

Chapter 2

Modified Gravity

In this chapter, we outline the modification flow of gravitational theories from Newtonian gravity to GR to MG. Newtonian gravity describes many gravitational phenomena observed on the Earth and in the Solar system well, however, it is incomplete especially at cosmological scales. GR overcame that defect by extending Newtonian gravity and reproducing it as an effective theory in the weak field limit. This extension unveils hidden principles and phenomena which do not appear in Newtonian gravity. The same things might be true of MG; that is, MG will possibly unveil hidden principles or phenomena which do not appear in GR. From this viewpoint, it is important to review the extension flow from Newtonian gravity to GR.

According to the Lovelock theorem, GR is the unique theory satisfying some theoretical requirements as mentioned below. Therefore, one must violate either of these requirements in order to modify GR. Resultant MG is categorized into several types depending on how to break the requirements, or on the form of the action. Among these, we mainly introduce the scalar-tensor theory and some related theories.

The rest of this chapter is organized as follows. First, we review GR as an extension of Newtonian gravity and motivations of further modifications of GR in §2.1. Next, in §2.2, we explain the Ostrogradsky theorem, which is the most crucial theorem to build generic theories, and how we avoid the Ostrogradsky ghost. Then, we overview the scalar-tensor theories in §2.3, and its related theories in §2.4. The extended cuscuton theory, which is the main theme of this thesis, belongs to two particular classes of the scalar-tensor theories: the Lorentz-violating theories and the two-DOFs theories. In §2.5 and §2.6, we introduce these classes and some specific theories.

2.1 General Relativity and Modified Gravity

GR was proposed by A. Einstein in 1915-1916 and is now the most standard gravitational theory. GR was constructed based on *the general principle of relativity* and *Einstein's equivalence principle*. The underlying ideas of these principles had been introduced in the Newtonian mechanics, such as

- *The principle of Galilean relativity*
  The laws of physics are the same in all inertial frames on the four-dimensional absolute spacetime.

- *The weak equivalence principle*
  The inertial mass and the gravitational mass are equivalent.

Newtonian mechanics is constructed on the four-dimensional absolute spacetime, including the three-dimensional Euclid space and absolute time which every inertial frame shares. In this spacetime, inertial frames are transformed by the Galilean transformation. This transformation does not change three-dimensional distance in the three-dimensional Euclid space and the equations of motion. Therefore, the Galilean principle of relativity means "*the laws of physics are invariant under the Galilean transformation*."

On the other hand, Michelson and Morley demonstrated that the propagating speed of light is constant in vacuum, which is called *the principle of invariant light speed*. This result is in conflict with the Galilean covariance of the Newtonian mechanics. To resolve this contradiction, the following extended principle was proposed:

- *The special principle of relativity*
  The laws of physics are the same in all inertial frames on the four-dimensional Minkowski spacetime.

To make the speed of light invariant, one adopts the Minkowski spacetime with relativistic time as well as space. The transformations between inertial frames on the Minkowski spacetime is the Lorentz transformation; thus, the special principle of relativity means "*the laws of physics are invariant under the Lorentz transformation*." The special relativity was constructed based on this principle, and it explains relations between space, time, and speed of objects especially in electrodynamics.

The special principle of relativity is valid only for physics other than the gravitational effects, and is incompatible with Newtonian gravity describing the law of universal gravitation. Because of the weak equivalence principle, the gravitational effects vanish locally. Lets consider free-falling observers to the Earth. An observer on the Earth should think the free-falling observers are in an accelerating frame. However, free-falling observers to the Earth think they are in an inertial frame because the inertial force cancels the gravitational force for every observer regardless of their mass. The gravitational force is canceled locally, although, it remains in a finite space. Indeed, two events located at a spatial distance must experience the tidal force as a gravitational effect. From these points, it is natural to think that the gravitational force arises from the spatial spread, and any local frame is an inertial frame represented by the Minkowski spacetime like the special relativity. Given that the Minkowski spacetime expresses a flat plane, the nonlocal spacetime with the nonvanishing gravitational force can be interpreted as a curved surface which is locally flat where the gravitational force is ignored. Therefore, spacetimes can be illustrated by the pseudo-Riemannian geometry, which locally has the Minkowski metric, and curvatures can be represented as gravitation. Considering the special principle of relativity is applied to local inertial reference frames, gravitational theories should satisfy the following principles:

- *The general principle of relativity*
  The laws of physics are the same in all reference frames.

- *Einstein's equivalence principle*
  Free falling observers are in local inertial reference frames.

GR was constructed based on these principles. [1]

GR has the Newtonian and post-Newtonian limit consistent with experiments [2], and predicts the existence of GWs and black holes, which were directly confirmed in recent years [3, 4]. Moreover, GR with the cosmological constant Λ can explain the current accelerated expansion of the universe strongly supported by recent observations of Type Ia supernovae [5, 6], and this ΛCDM model passes every cosmological test so far. Given these facts, GR is one of the most successful gravitational theories.

On the other hand, as mentioned in the previous chapter, it is meaningful to consider broader frameworks of gravitational theory: MG. For example, one can use these theories to explore undeveloped areas in cosmology, e.g., the early universe and the dark sector of the late-time universe. MG is also helpful to test the validity of GR by means of future measurements and observations. Not just concerning observational tests, MG gives us a deeper understanding of GR because one of the best ways to understand something is to break and reconstruct it. The action of MG should contain both the GR and an additional sector. *The Lovelock theorem* [9, 10] plays an essential role in considering such extensions, which states that *the possible second-order Euler-Lagrange equations derived from the four-dimensional general covariant action which consists of solely the metric tensor are only the Einstein equations with the cosmological constant.* Therefore, to modify the gravitational field equations from the Einstein equations, we must violate the assumptions of the Lovelock theorem: adding extra dynamical fields/extra dimensions/higher-derivative terms/non-locality, renouncing the four-dimensional general covariance, or extending into more general geometry.

2.2 Ostrogradsky Theorem

Constructing actions of MG, one must take the Ostrogradsky ghost into account. This ghost arises from a particular class of higher-derivative Lagrangians, which is stated by *the Ostrogradsky theorem: the Hamiltonians associated with nondegenerate Lagrangians involving more than first-order derivatives of time are unbounded below and above* [18]. Here, "nondegenerate" means that the kinetic matrix of a system is not regular. Naively, if a system with an unbounded Hamiltonian interacts with another system, energy of the system would diverge into $-\infty$. This instability is one aspect of the Ostrogradsky ghost. The Ostrogradsky theorem was widely known by [19, 20] with discussions of various natures of the ghost.

[1] Einstein's equivalence principle suggests the invariance of physics laws besides gravitational effects in local inertial reference frames. As an extended principle, including the gravitational laws, there is the strong equivalence principle. Various observations and experiments are still testing whether these three equivalence principles (weak, Einstein, strong) are correct or not.

We note that the Ostrogradsky ghost may show up once we define the Lagrangian, and the existence of the ghost is independent on every other factor, such as specific forms of spacetimes or perturbations. By contrast, other instabilities like the gradient and tachyonic instabilities appear after assuming specific perturbations or spacetimes, and hence we do not need to mind these instabilities at the stage of constructing actions.

The Ostrogradsky theorem is a powerful tool to avoid the ghost, but is not enough to build gerenal actions of MG. First, this theorem discusses the Lagrangians depending only on one variable which is a function of time in the context of classical mechanics: $L(x^{(N)}(t), \cdots, \dot{x}(t), x(t))$. In order to apply the Ostrogradsky theorem to gravitational theories, we need to extend it into field theories. Second, the Hamiltonians in the gauge systems including GR and its extensions always vanish regardless of the ghost. Therefore, we demand other analyses to check whether a gauge theory has the ghost or not. Third, the Ostrogradsky theorem only tells a sufficient condition to find the instability, and the no-ghost conditions are still an open question in this theorem.

To extend the Ostrogradsky theorem and achieve no-ghost conditions, many studies have been done. The authors of [28] unveiled the no-ghost conditions for multiple dynamical variables: $L(x^{a(N)}, \cdots, \dot{x}^a, x^a)$, where a represents the number of variables and x^a are functions of time. Also, the authors of [29] introduced the safe variables q^i into higher-derivative Lagrangians: $L(\ddot{x}^a, \dot{x}^a, x^a; \dot{q}^i, q^i)$. After that, [30, 31] showed the ghost-free conditions for arbitrary higher-derivatives and arbitrary number of variables: $L(x^{a_d(d+1)}, \ldots, x^{a_d}; x^{a_{d-1}(d)}, \ldots, x^{a_{d-1}}; \ldots; \dot{x}^{a_0}, x^{a_0})$. The above studies are extensions in the context of classical mechanics. An extension to multi-scalar field theories in a flat spacetime was explored in [32], and to scalar-tensor theories in [15], which leads the DHOST theories. The quantum Ostrogradsky theorem was analyzed in [19,20], and more recently in [33]. The authors of [34] investigated a relation between the ghost and constraint equations. One can see [35] for a recent review about the ghost.

In this section, we explain the essence of the Ostrogradsky ghost in §2.2.1 based on [20], and how to evade this ghost in §2.2.2.

2.2.1 Ostrogradsky ghost

We start with the usual classical mechanics with $L = L(\dot{x}, x)$. Let us consider the harmonic oscillator, whose Lagrangian is given by

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2, \tag{2.1}$$

with the mass m and the frequency ω. The general solution for its Euler-Lagrange equation is

$$x = x_0 \cos(\omega t) + \frac{\dot{x}_0}{\omega} \sin(\omega t) = \frac{1}{2}\left(x_0 + \frac{i}{\omega}\dot{x}_0\right) e^{-i\omega t} + \frac{1}{2}\left(x_0 - \frac{i}{\omega}\dot{x}_0\right) e^{i\omega t}, \tag{2.2}$$

with x_0 and $\dot{x}_0$ being the initial conditions for the position and the velocity. The canonical variables are given by

$$X := x, \qquad P := \frac{\partial L}{\partial \dot{X}} = m\dot{x}, \tag{2.3}$$

and the Hamiltonian is

$$H = \frac{P^2}{2m} + \frac{m}{2}\omega^2 X^2, \tag{2.4}$$

which is clearly bounded below.

Next, we consider a general form of nondegenerate Lagrangians $L = L(\ddot{x}, \dot{x}, x)$. This system has the 1×1 kinetic matrix $\partial^2 L/\partial \ddot{x}^2$, and we define the nondegeneracy as

$$\frac{\partial^2 L}{\partial \ddot{x}^2} \neq 0. \tag{2.5}$$

The Euler-Lagrange equation is given by

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0. \tag{2.6}$$

Note that this is a fourth-order differential equation, and hence it needs four initial conditions to be solved. If we add an auxiliary variable y associated with $\dot{x}$, the Lagrangian reads

$$L(\ddot{x}, \dot{x}, x) = \tilde{L}(\dot{y}, y, x) + \lambda(y - \dot{x}), \tag{2.7}$$

with λ being a Lagrange multiplier. In this case, we can determine the canonical variables in the usual way,

$$\begin{gathered} X_1 := x, \quad X_2 := y, \quad X_3 := \lambda, \\ P_1 := \frac{\partial L}{\partial \dot{X}_1} = -X_3, \quad P_2 := \frac{\partial L}{\partial \dot{X}_2} = \frac{\partial \tilde{L}}{\partial \dot{X}_2}, \quad P_3 := \frac{\partial L}{\partial \dot{X}_3} = 0. \end{gathered} \tag{2.8}$$

The nondegeneracy condition (2.5) is now

$$\frac{\partial^2 L}{\partial \dot{X}_2^2} = \frac{\partial P_2(\dot{X}_2, X_2, X_1)}{\partial \dot{X}_2} \neq 0, \tag{2.9}$$

and we find $\dot{X}_2 = \dot{X}_2(X_1, X_2, P_2)$. Using this, we define the Hamiltonian by the Legendre transformation as

$$\begin{aligned} H :=& P_1\dot{X}_1 + P_2\dot{X}_2 + P_3\dot{X}_3 - \tilde{L}(\dot{X}_2, X_2, X_1) + P_1(X_2 - \dot{X}_1) \\ =& P_1 X_2 + P_2\dot{X}_2(X_1, X_2, P_2) - \tilde{L}(\dot{X}_2(X_1, X_2, P_2), X_2, X_1). \end{aligned} \tag{2.10}$$

It is clear that the Hamiltonian linearly depends on the momentum P_1. Even in this case, the Hamiltonian is constant in isolated systems. However, once a system interacts with another one, the Hamiltonian diverges as P_1 diverge. [2] If the Hamiltonian goes to $-\infty$, there would be no stable state; namely, the system would be unstable. This instability is the ghost which appears in the Ostrogradsky theorem.

[2] Of course, when the range of P_1 is limited, the Hamiltonian can be bounded below.

Note that if one would not add any auxiliary variables, one may adopt "Ostrogradsky's choice" to determine the canonical variables:

$$X_1 := x, \qquad X_2 := \dot{x}, \qquad P_1 := \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}, \qquad P_2 := \frac{\partial L}{\partial \ddot{x}}. \tag{2.11}$$

Using these variables and the nondegeneracy condition, we define the Hamiltonian as

$$\begin{aligned} H :=& P_1\dot{X}_1 + P_2\dot{X}_2 - L(X_1, X_2, P_1, P_2) \\ =& P_1 X_2 + P_2\dot{X}_2(X_1, X_2, P_2) - L(X_1, X_2, P_1, P_2). \end{aligned} \tag{2.12}$$

This definition seems to be different from the usual Legendre transformation, but this ensures the equivalence between the Lagrangian and Hamiltonian formalism with respect to the time evolution of the system. This Hamiltonian also linearly depends on the momenta as well as (2.10), and one can see the ghost instability.

An unbounded Hamiltonian due to linear dependences on momenta is a critical nature of the Ostrogradsky instability. However, forms of Hamiltonian are unreliable as an indicator of that instability in some cases. Let us consider for example the previous harmonic oscillator (2.1), whose Hamiltonian is given by (2.4). After performing the following canonical transformation:

$$X = \sqrt{\frac{2p}{m\omega}}\sin x, \quad P = \sqrt{2pm\omega}\cos x, \tag{2.13}$$

one obtains the new Hamiltonian:

$$H = \omega p. \tag{2.14}$$

This H depends on the new momentum p linearly as if there was the Ostrogradsky ghost. Nevertheless, of course, this system has no ghost because the new position x is a cyclic coordinate, and hence its conjugate momentum p is a positive constant. Even if this system interacts another system, H is bounded below since p is larger than zero as one can see from (2.13).

Another example is the Hamiltonians of covariant systems. Due to the time-reparametrization invariance, the Hamiltonian vanishes on shell (see [35] for a review). To see this, we construct a time-reparametrization invariant action from a general Lagrangian $L_0(\dot{x}, x)$ for the dynamical variable $x(t)$ by introducing a new dynamical variable $\tau(t)$ as follows:

$$L(\dot{x}, x, \dot{\tau}) := \dot{\tau} L_0(\dot{x}/\dot{\tau}, x). \tag{2.15}$$

If we change time coodinate $t \to y(t)$, the action reads

$$S = \int dt L(\dot{x}, x, \dot{\tau}) = \int dt \dot{\tau} L_0(\dot{x}/\dot{\tau}, x) = \int dy \tau' L_0(x'/\tau', x) \tag{2.16}$$

with $'$:= d/dy. This action is obviously invariant under time reparametrization. To move to the Hamiltonian formalism, we introduce an auxiliary variable into L as follows:

$$L = NL_0(\dot{x}/N, x) + \pi(\dot{\tau} - N), \tag{2.17}$$

here we have added π as the conjugate momentum of τ in advance. The momentum conjugate to x is

$$p := \frac{\partial L}{\partial \dot{x}} = N\frac{\partial L_0}{\partial \dot{x}}. \tag{2.18}$$

Then, the canonical Hamiltonian is given by

$$H := \pi\dot{\tau} + p\dot{x} - L = N\left(\frac{\partial L_0}{\partial \dot{x}}\dot{x} - L_0 + \pi\right) =: N(H_0 + \pi), \tag{2.19}$$

where H_0 is the canonical Hamiltonian associated with L_0. $H_0 + \pi$ is a first-class constraint, and therefore H vanishes weakly, i.e., vanishes on the constraint surface. This occurs irrespective of higher-derivative terms, and we can not find out the ghost by looking the form of Hamiltonian.

A more rigorous method to check the instability is a mode decomposition. We consider a higher-derivatives deviation from the harmonic oscillator,

$$L = -\frac{\varepsilon m}{2\omega^2}\ddot{x}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2, \tag{2.20}$$

here ε quantifies a dimensionless parameter. Since this system satisfies the nondegeneracy condition (2.5), there exists the Ostrogradsky ghost. The Euler-Lagrange equation is

$$\frac{\varepsilon}{\omega^2}\ddot{\ddot{x}} + \ddot{x} + \omega^2 x = 0. \tag{2.21}$$

We assume a general solution as

$$x = C\cos(kt) + S\sin(kt). \tag{2.22}$$

Then, k^2 follows the dispersion relation such as

$$\frac{\varepsilon}{\omega^2}k^4 + k^2 + \omega^2 = 0, \tag{2.23}$$

whose solution is

$$k^2 = \omega^2\frac{1 \pm \sqrt{1 - 4\varepsilon}}{2\varepsilon}. \tag{2.24}$$

This solution of k^2 has two branches due to the fourth-order differential equation (2.21), and we obtain the general solution of x,

$$x = C_+\cos(k_+t) + S_+\sin(k_+t) + C_-\cos(k_-t) + S_-\sin(k_-t). \tag{2.25}$$

with the frequencies $k_\pm$, the coefficients $C_\pm$ and $S_\pm$ being

$$k_\pm = \omega\sqrt{\frac{1 \mp \sqrt{1-4\varepsilon}}{2\varepsilon}}, \qquad C_\pm = \frac{k_\mp^2 x_0 + \ddot{x}_0}{k_\mp^2 - k_\pm^2}, \qquad S_\pm = \frac{k_\mp^2 \dot{x}_0 + \dddot{x}_0}{k_\pm\left(k_\mp^2 - k_\pm^2\right)}. \tag{2.26}$$

The Hamiltonian (2.12) is expressed by the canonical variables (2.11),

$$H = P_1 X_2 - \frac{\omega^2}{2\varepsilon m} P_2^2 - \frac{m}{2} X_2^2 + \frac{m\omega^2}{2} X_1^2. \tag{2.27}$$

The Hamiltonian can diverge into $-\infty$ since its first term depends linearly on the momentum P_1. Substituting the solution (2.25) for the Hamiltonian, we obtain the energy of the isolated system as below:

$$H = \frac{m}{2}\sqrt{1-4\varepsilon}\,k_+^2\left(C_+^2 + S_+^2\right) - \frac{m}{2}\sqrt{1-4\varepsilon}\,k_-^2\left(C_-^2 + S_-^2\right). \tag{2.28}$$

We can see that the + modes carry positive energy and the − modes carry negative energy. The existence of this negative energy modes leads the ghost instability.

2.2.2 No-ghost Conditions

The Ostrogradsky theorem tells us only sufficient conditions to appear the Ostrogradsky ghost. The necessary and sufficient conditions, namely the no-ghost conditions, were revealed by subsequent studies in classical mechanics and covariant field theories. In this subsection, we illustrate the degeneracy of Lagrangian plays a key role in the no-ghost conditions using classical mechanics framework.

We consider the following Lagrangian:

$$\begin{aligned}\tilde{L}(\ddot{q}, \dot{q}, q; \dot{\xi}, \xi) &= \frac{A}{2}\ddot{q}^2 + \frac{B}{2}\ddot{q}\dot{\xi} + C\dot{\xi}^2 + D\dot{q}\dot{\xi} - V(q) \\ &= \frac{A}{2}\dot{Q}^2 + \frac{B}{2}\dot{Q}\dot{\xi} + C\dot{\xi}^2 + DQ\dot{\xi} - V(q) + \lambda(Q - \dot{q}),\end{aligned} \tag{2.29}$$

here A, B, C, D are constant, Q is an auxiliary variable associated with $\dot{q}$, and λ is a Lagrange multiplier. Variations with respect to Q, ξ, q, and λ lead the Euler-Lagrange equations as follows:

$$A\ddot{Q} + B\ddot{\xi} - D\dot{\xi} - \lambda = 0, \tag{2.30}$$

$$B\ddot{Q} + C\ddot{\xi} + D\dot{Q} = 0, \tag{2.31}$$

$$\dot{\lambda} - V'(q) = 0, \tag{2.32}$$

$$Q - \dot{q} = 0. \tag{2.33}$$

Eqs. (2.32) and (2.33) are the constraints which eliminate two dependent variables: for instance λ and Q. The equations of motion (EOMs) (2.30) and (2.31) are rewritten into

$$K\begin{pmatrix}\ddot{Q} \\ \ddot{\xi}\end{pmatrix} := \begin{pmatrix}A & B \\ B & C\end{pmatrix}\begin{pmatrix}\ddot{Q} \\ \ddot{\xi}\end{pmatrix} = \begin{pmatrix}D\dot{\xi} - \lambda \\ -D\dot{Q}\end{pmatrix}. \tag{2.34}$$

Here K represents the kinetic matrix,

$$K := \begin{pmatrix} L_{\dot{Q}\dot{Q}} & L_{\dot{Q}\dot{\xi}} \\ L_{\dot{\xi}\dot{Q}} & L_{\dot{\xi}\dot{\xi}} \end{pmatrix}, \qquad L_{\alpha\beta} := \frac{\partial^2 L}{\partial\alpha\partial\beta} \quad (\alpha, \beta = \dot{Q}, \dot{\xi}). \tag{2.35}$$

The degenerate Lagrangians have a singular kinetic matrix:

$$\det K = AC - B^2 = 0. \tag{2.36}$$

In a degenerate case, the variable redefinition $x := BQ + C\xi$ simplifies the equations (2.30)-(2.33) into

$$D\ddot{x} + CV'(q) = 0, \quad \dot{x} + D\ddot{q} = 0. \tag{2.37}$$

These equations obviously contain up to second-derivatives for x and q. This is the critical feature of degenerate systems: the Euler-Lagrange equations originally contain the higher-derivatives, but after variable redefinitions (equivalently linear combinations), the equations are reduced to at most second-derivatives. One can also find that the Hamiltonians of degenerate systems are bounded below, and there is no extra mode propagating negative energy.

A general Lagrangian $\tilde{L}(\ddot{q}, \dot{q}, q; \dot{\xi}, \xi)$ was discussed in [29], which found the ghost is absent if and only if the kinetic matrix is degenerate: $\det K = 0$. We can extend this result to single-scalar-tensor theories by associating q with a scalar field, and ξ with the metric tensor $g_{\mu\nu}$.

2.3 Scalar-Tensor Theories

In the previous section, we have explained the Ostrogradsky theorem as an important theorem to construct the action of MG. Then, we proceed to discussions about individual theories in the class of scalar-tensor theory. Regardless of modification methods, many MG can be (effectively) described by the metric tensor and additional fields. For celebrated example, $f(R)$ gravity [36, 37] can be recasted into GR with a canonical scalar field (see §2.4 for details). Effective theories of higher-dimensional gravity also naturally arise scalar-vector-tensor theories like Kaluza-Klein gravity [38]. In particular, the scalar-tensor theory consists of scalar fields and the metric tensor. Since a scalar field is the simplest and most fundamental field, the scalar-tensor theory is suitable for exploring intrinsic natures associated with additional fields. We note that the single-field scalar-tensor theories presented in this section generally have three dynamical DOFs: two tensor modes and one scalar mode. These theories can be interpreted as violating the Lovelock theorem by adding an extra scalar DOF.

The most simple scalar-tensor theory is Einstein's gravity with a canonical scalar field ϕ, which action is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + X - U(\phi)\right], \tag{2.38}$$

with the canonical kinetic term for the scalar field $X := -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$, and $U(\phi)$ being the potential for ϕ. This type can produce dynamical geometrical modifications, e.g., changing the time evolution of the Hubble parameter, thanks to the dynamical scalar DOF.

The Brans-Dicke theory [39, 40] introduces a nonminimal coupling term:

$$S = \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\partial_\mu\phi\partial^\mu\phi\right], \tag{2.39}$$

where ω is constant. In the first term of the action, ϕ nonminimally couples to $g_{\mu\nu}$ other than the kinetic term and $\sqrt{-g}$. Since the coefficient of R naively plays a role of the gravitational constant, one defines the effective gravitatiovnal constant G_{eff} as

$$\frac{1}{16\pi G_{\text{eff}}} := \phi. \tag{2.40}$$

This implies G_{eff} is no longer constant but a time-dependent quantity. The nonminimal coupling term ϕR can be extended into $f(\phi)R$, with $f(\phi)$ being an arbitrary function. We note that this type of action is mapped into minimally coupled action by the conformal transformation given by

$$\tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}. \tag{2.41}$$

This transformation is a field redefinition for the metric tensor, which changes only a measure mapped on spacetimes and does not change causality. After the conformal transformation, the scalar (and matter) sector becomes a non-trivial form instead of the coefficient of the Ricci scalar becoming constant. This frame is so-called the Einstein frame because of the existence of the Einstein-Hilbert term, and the original frame like (2.39) is so-called the Jordan frame. [3]

The k-essence model [41, 42] and k-inflation [43] have the following generalized kinetic term for ϕ:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\text{Pl}}^2}{2}R + P(\phi, X)\right], \tag{2.42}$$

with $P(\phi, X)$ being an arbitrary function, which generally has also nontrivial terms (even non-polynomial functions like $\ln X$ are allowed). A further modification is performed in the kinetic gravity braiding [44] or equivalently G-inflation [45], whose actions are given by

$$S = \int d^4x\sqrt{-g}\left[R + K(\phi, X) + G(\phi, X)\Box\phi\right], \tag{2.43}$$

with $K(\phi, X)$ and $G(\phi, X)$ being arbitrary functions. This action has a second-order derivatives term of ϕ, but the EOMs remain at most second-derivatives and the Ostrogradsky ghost is absent.

[3] The idea of nonminimal couplings has been originally proposed by P. Jordan [39].

The Galileon theory [4] [47] was constructed by adding higher-order terms of $\partial_\mu\partial_\nu\phi$ in the n-dimensional Minkowski spacetime, whose four-dimensional action is

$$S = \int d^4x \left[c_1\phi + c_2 X - c_3 X\Box\phi + c_4 X\left[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right] - \frac{c_5}{3}X\left[(\Box\phi)^3 - 3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi\right]\right], \tag{2.44}$$

with c_i's being constant. This is the most general action under the conditions that (i) the Galilean shift symmetry: $\phi \longrightarrow \phi + b_\mu x^\mu + c$ (b_μ, c: constant), (ii) up to second-derivatives in the EOMs so as to avoid the Ostrogradsky ghost, and (iii) the n-dimensional Minkowski spacetime. The Galilean shift symmetry is a generalization of the Galilean symmetry $\dot{x}_i(t) \longrightarrow \dot{x}_i(t) + v_i$ of non-relativistic mechanics. Note that a generic Galilean shift symmetric Lagrangian forms a polynomial of $\left(\partial^2\phi\right)^{n-1}\partial\phi\partial\phi$, and the corresponding EOM has exactly second-order derivatives of ϕ: $(\partial\partial\phi)^n = 0$.

The covariant Galileon [48] is an extension of the Galileon theory into curved spacetimes. One covariantizes (2.44) by replacing $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$ and $\partial \longrightarrow \nabla$, and then adds the counter terms in order to eliminate the higher derivative terms in the field equations. Then, one obtain the following action:

$$S = \int d^4x\sqrt{-g}\left[c_1\phi + c_2 X - c_3 X\Box\phi + \frac{c_4}{2}X^2 R + c_4 X\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right] + c_5 X^2 G^{\mu\nu}\phi_{\mu\nu} - \frac{c_5}{3}X\left[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^\mu_\lambda\right]\right], \tag{2.45}$$

with $\phi_\mu := \nabla_\mu\phi$, $\phi_{\mu\nu} := \nabla_\mu\nabla_\nu\phi$, and $G^{\mu\nu}$ being the Einstein tensor.

The generalized Galileons [12] was constructed in the same way as the covariant Galileon. One first derives the most general action which has up to second derivatives of ϕ and the second-order field equations in the Minkowski spacetime. After that, one covariantizes that action and adds the counterterms, and then one gets the generalized Galileons. This method can be performed in arbitrary dimensions.

On the other hand, in 1974, the author of [11] had proposed the Horndeski theory, which is the most general four-dimensional covariant action consisting of up to second-order derivatives of the metric and the scalar field whose field equations are at most second derivatives. The generalized Galileons and the Horndeski theory were motivated by different reasons and written in the different forms. Nonetheless, the authors of [13] showed that these are comlpletely equivalent in four dimensions whose action is given by

$$S_{\rm H} = \int d^4x\sqrt{-g}\left(\mathcal{L}_2^{\rm H} + \mathcal{L}_3^{\rm H} + \mathcal{L}_4^{\rm H} + \mathcal{L}_5^{\rm H}\right), \tag{2.46}$$

[4] [46] is a good review of theories from the Galileon to DHOST.

where

$$\begin{aligned}
\mathcal{L}_2^{\mathrm{H}} &= G_2(\phi, X), \\
\mathcal{L}_3^{\mathrm{H}} &= G_3(\phi, X)\Box\phi, \\
\mathcal{L}_4^{\mathrm{H}} &= G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - \phi_\mu^\nu\phi_\nu^\mu\right], \\
\mathcal{L}_5^{\mathrm{H}} &= G_5(\phi, X)G^{\mu\nu}\phi_{\mu\nu} - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)\phi_\mu^\nu\phi_\nu^\mu + 2\phi_\mu^\nu\phi_\nu^\lambda\phi_\lambda^\mu\right],
\end{aligned} \tag{2.47}$$

with $G_i(\phi, X)$ $(i = 2, \cdots, 5)$ being arbitrary functions. [5] It should be noted that the generalized Galileons is determined in arbitrary dimensions, while the Horndeski theory is still not generalized into higher dimensions. We also note that this theory (2.46) (we call this the Horndeski theory throughout the thesis) can reproduce a number of models; for example, $G_2 = \omega X/\phi, G_4 = \phi, G_3 = G_5 = 0$ corresponds to the Brans-Dicke theory (2.39), and $G_4 = M_{\mathrm{Pl}}^2/2, G_2 = g_3 = G_5 = 0$ corresponds to GR.

The Horndeski theory requires the field equations contains at most second derivatives of $g_{\mu\nu}$ and ϕ, which allows to circumvent the Ostrogradsky ghost trivially. However, it is known that the Horndeski theory can be more generalized into what is called the Gleyzes-Langlois-Piazza-Vernizzi (GLPV, also known as beyond Horndeski) theory [14]. The action of the GLPV theory is given by

$$S_{\mathrm{GLPV}} = S_{\mathrm{H}} + \int d^4x\sqrt{-g}(\mathcal{L}_4^{\mathrm{bH}} + \mathcal{L}_5^{\mathrm{bH}}), \tag{2.48}$$

where

$$\begin{aligned}
\mathcal{L}_4^{\mathrm{bH}} &= F_4(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'} \\
&= F_4(\phi, X)\delta^{\mu\nu\rho}_{\mu'\nu'\rho'}\phi_\mu\phi^{\mu'}\phi_\nu^{\nu'}\phi_\rho^{\rho'} \\
&= F_4(\phi, X)\left[-2X\left[(\Box\phi)^2 - \phi_\mu^\nu\phi_\nu^\mu\right] - 2\phi_\mu\phi_\nu^\mu(\phi^\nu\Box\phi - \phi_\lambda^\nu\phi^\lambda)\right], \\
\mathcal{L}_5^{\mathrm{bH}} &= F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'} \\
&= F_5(\phi, X)\delta^{\mu\nu\rho\sigma}_{\mu'\nu'\rho'\sigma'}\phi_\mu\phi^{\mu'}\phi_\nu^{\nu'}\phi_\rho^{\rho'}\phi_\sigma^{\sigma'} \\
&= F_5(\phi, X)\left[-2X\left[(\Box\phi)^3 - 3(\Box\phi)\phi_\mu^\nu\phi_\nu^\mu + 2\phi_\mu^\nu\phi_\nu^\lambda\phi_\lambda^\mu\right]\right. \\
&\qquad \left. - 3\phi_\lambda\phi_\sigma^\lambda\phi^\sigma\left[(\Box\phi)^2 - \phi_\mu^\nu\phi_\nu^\mu\right] + 6\phi_\mu\phi_\nu^\mu\phi^\sigma(\phi_\sigma^\nu\Box\phi - \phi_\lambda^\nu\phi_\sigma^\lambda)\right].
\end{aligned} \tag{2.49}$$

with $F_4(\phi, X)$ and $F_5(\phi, X)$ being arbitrary functions, $\epsilon_{\mu\nu\rho\sigma}$ being the totally antisymmetric Levi-Civita tensor, and $\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}$ being the generalized Kronecker delta,

$$\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n} := n!\delta^{[\mu_1\cdots\mu_n]}_{\nu_1\dots\nu_n}. \tag{2.50}$$

The Horndeski theory is the subset satisfying

$$F_4(\phi, X) = 0, \quad F_5(\phi, X) = 0. \tag{2.51}$$

[5] Some papers adopt the following styles: $\mathcal{L}_3^{\mathrm{H}} = -G_3(\phi, X)\Box\phi$ or $X = \phi_\mu\phi^\mu$. We indeed introduce $\bar{X} := \phi_\mu\phi^\mu$ in the DHOST theories. This difference does not affect any results, although, one must check which kind of style is used in each study.

To illustrate the method of constructing the GLPV theory, we consider the Arnowitt-Deser-Misner (ADM) decomposition of the Horndeski theory. In the unitary gauge $\phi = \phi(t)$, the ADM Horndeski action reads

$$S = \int dt d^3x N\sqrt{\gamma}\Big[A_2 + A_3 K + A_4(K^2 - K^i_j K^j_i) + B_4^{(3)} R + A_5(K^3 - 3KK^i_j K^j_i + 2K^i_j K^j_k K^k_i) + B_5 K^{ij}\left({}^{(3)}R_{ij} - \frac{1}{2}\gamma_{ij}{}^{(3)}R\right)\Big], \quad (2.52)$$

where N is the lupse function, γ_{ij} is the three-dimensional spatial metric, K_{ij} and ${}^{(3)}R_{ij}$ are the extrinsic and intrinsic curvature tensors of the constant time hypersurfaces, $\gamma := \det \gamma_{ij}$, $K := K^i_i$, ${}^{(3)}R :={}^{(3)} R^i_i$, and the coefficients A_2, A_3, A_4, A_5, B_4, and B_5 are functions of (t, N). The functions A_i, B_i are defined by the functions G_i, F_i in the covariant action (2.48) as follows:

$$\begin{aligned} A_2 &= G_2 - \sqrt{2X}\int^X \frac{G_{3\phi}}{\sqrt{2\bar{X}}}\mathrm{d}\bar{X}, & A_3 &= \int^X \sqrt{2\bar{X}}G_{3\bar{X}}\mathrm{d}\bar{X} - 2\sqrt{2X}G_{4\phi}, \\ A_4 &= -G_4 + 2XG_{4X} - XG_{5\phi}, & A_5 &= -\frac{1}{6}(2X)^{3/2}G_{5X} \\ B_4 &= G_4 - \sqrt{2X}\int^X \frac{G_{5\phi}}{2\sqrt{2\bar{X}}}\mathrm{d}\bar{X}, & B_5 &= -\int^X \frac{G_{5\bar{X}}}{\sqrt{2\bar{X}}}\mathrm{d}\bar{X}, \end{aligned} \quad (2.53)$$

here $G_{i\phi} := \partial G_i/\partial\phi, G_{iX} := \partial G_i/\partial X$. We note that, in the unitary gauge, functions of (ϕ, X) are mapped to functions of (t, N) due to $\phi = \phi(t)$ and $X = \dot{\phi}^2(t)/(2N^2)$. From (2.53), one finds the following relations:

$$A_4 = -B_4 - NB_{4N}, \quad A_5 = \frac{N}{6}B_{5N}, \quad (2.54)$$

here $B_{iN} := \partial B_i/\partial N$. These conditions correspond to the Horndeski conditions (2.51). Thanks to these conditions, only four of six functions A_i, B_i are independent in the case of the Horndeski theory. However, the Hamiltonian analysis shows that the number of DOFs of the action (2.52) is always three irrespective of the existence of the Horndeski conditions. Given that the Ostrogradsky ghost is absent if the number of physical DOFs is at most three, we find that the Horndeski conditions is not necessary to evade the Ostrogradsky ghost. In this way, the GLPV action in the ADM form was defined by (2.52) without the Horndeski conditions (2.51). Since the GLPV theory has more general action than the Horndeski, its EOMs contain higher-order derivatives. However, the EOMs are degenerate, and hence this theory has no ghost.

The GLPV theory removes the Horndeski restrictions and consequently becomes a degenerate system, but degenerate and ghost-free scalar-tensor theories are not only the GLPV. From this point of view, the authors of [15–17] constructed the degenerate higher-order scalar-tensor (DHOST) theories (see [49] for a review). These are the most general four-dimensional covariant theories consisting of at most second derivative terms of the scalar field which satisfying the no-ghost conditions (we call this degeneracy conditions), as we mentioned in §2.2.2. The specific action is determined for the quadratic and cubic terms of $\phi_{\mu\nu}$ as follows:

$$S_{\rm DHOST} = \int d^4x\sqrt{-g}\left[f_0(\bar{X},\phi) + f_1(\bar{X},\phi)\Box\phi + f_2(\bar{X},\phi)R + C_{(2)}^{\mu\nu\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma}\right. \\ \left. + f_3(\bar{X},\phi)G_{\mu\nu}\phi^{\mu\nu} + C_{(3)}^{\mu\nu\rho\sigma\alpha\beta}\phi_{\mu\nu}\phi_{\rho\sigma}\phi_{\alpha\beta}\right], \tag{2.55}$$

where $\bar{X} := \phi_\mu\phi^\mu$, $C_{(2)}^{\mu\nu\rho\sigma}$ and $C_{(3)}^{\mu\nu\rho\sigma\alpha\beta}$ are the most general tensors depending on $g_{\mu\nu}$ and ϕ_μ. One can find the quadratic terms of $\phi_{\mu\nu}$ reads

$$C_{(2)}^{\mu\nu\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma} = \sum_{A=1}^{5} a_A(\bar{X},\phi)L_A^{(2)}, \tag{2.56}$$

with

$$\begin{aligned} &L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}, \quad L_2^{(2)} = (\Box\phi)^2, \quad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu \\ &L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu, \quad L_5^{(2)} = \left(\phi^\mu\phi_{\mu\nu}\phi^\nu\right)^2, \end{aligned} \tag{2.57}$$

and a_A being functions of ϕ and $\bar{X}$. The cubic terms of $\phi_{\mu\nu}$ also reads

$$C_{(3)}^{\mu\nu\rho\sigma\alpha\beta}\phi_{\mu\nu}\phi_{\rho\sigma}\phi_{\alpha\beta} = \sum_{A=1}^{10} b_A(\bar{X},\phi)L_A^{(3)}, \tag{2.58}$$

with

$$\begin{aligned} &L_1^{(3)} = (\Box\phi)^3, \quad L_2^{(3)} = (\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu}, \quad L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho, \quad L_4^{(3)} = (\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu, \\ &L_5^{(3)} = \Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho, \quad L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma, \quad L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma \\ &L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda, \quad L_9^{(3)} = \Box\phi\left(\phi_\mu\phi^{\mu\nu}\phi_\nu\right)^2, \quad L_{10}^{(3)} = \left(\phi_\mu\phi^{\mu\nu}\phi_\nu\right)^3, \end{aligned} \tag{2.59}$$

and b_A being also functions of ϕ and $\bar{X}$. If a_A and b_A are entirely arbitrary, the above models obviously have the ghost. Therefore, these functions are restricted by the degeneracy conditions. As the Horndeski and GLPV theories, the DHOST theories can reproduce numerous models by specifying a_A and b_A. For instance, $a_1 = -a_2 = 2G_{4,\bar{X}}, a_3 = a_4 = a_5 = 0$ corresponds to $\mathcal{L}_4^H$, and

$$\frac{\beta_1}{\bar{X}} = -\frac{\beta_2}{3\bar{X}} = \frac{\beta_3}{2\bar{X}} = -\frac{\beta_4}{3} = \frac{\beta_5}{6} = \frac{\beta_6}{3} = -\frac{\beta_7}{6} = F_5 \tag{2.60}$$

corresponds to $\mathcal{L}_5^{\rm bH}$.

As §2.2.2, one obtains the ghost-free conditions using the kinetic matrix K so as to $\det K = 0$. Its specific forms for the quadratic DHOST are given by

$$D_0 = 0, \quad D_1 = 0, \quad D_2 = 0, \tag{2.61}$$

with

$$
\begin{aligned}
D_0 :=& -4\left(\alpha_1+\alpha_2\right)\left[\bar{X}G_4\left(2\alpha_1+\bar{X}\alpha_4+4G_{4,\bar{X}}\right)-2G_4^2-8\bar{X}^2G_{4,\bar{X}}^2\right]\\
D_1 :=& 4\left[\bar{X}^2\alpha_1\left(\alpha_1+3\alpha_2\right)-2G_4-4\bar{X}G_4\alpha_2\right]\alpha_4+4\bar{X}^2G_4\left(\alpha_1+\alpha_2\right)\alpha_5+8\bar{X}\alpha_1^3\\
&-4\left(G_4+4\bar{X}G_{4,\bar{X}}-6\bar{X}\alpha_2\right)\alpha_1^2-16\left(G_4+5\bar{X}G_{4,\bar{X}}\right)\alpha_1\alpha_2\\
&+4\bar{X}\left(3G_4-4\bar{X}G_{4,\bar{X}}\right)\alpha_1\alpha_3-\bar{X}^2G_4\alpha_3^2+32G_{4,\bar{X}}\left(G_4+2\bar{X}G_{4,\bar{X}}\right)\alpha_2\\
&-16G_4G_{4,\bar{X}}\alpha_1-8G_4\left(G_4-\bar{X}G_{4,\bar{X}}\right)\alpha_3+48G_4G_{4,\bar{X}}^2,\\
D_2 :=& 4\left[2G_4^2+4\bar{X}G_4\alpha_2-\bar{X}^2\alpha_1\left(\alpha_1+3\alpha_2\right)\right]\alpha_5+4\alpha_1^3+4\left(2\alpha_2-\bar{X}\alpha_3-4G_{4,\bar{X}}\right)\alpha_1^2\\
&+3\bar{X}^2\alpha_1\alpha_3^2-4\bar{X}G_4\alpha_3^2+8\left(G_4+\bar{X}G_{4,\bar{X}}\right)\alpha_1\alpha_3-32G_{4,\bar{X}}\alpha_1\alpha_2+16G_{4,\bar{X}}^2\alpha_1\\
&+32G_{4,\bar{X}}^2\alpha_2-16G_4G_{4,\bar{X}}\alpha_3.
\end{aligned}
\tag{2.62}
$$

We should note that the subclass having the following relation:

$$
\alpha_1+\alpha_2=0,\quad \alpha_3+\alpha_4=0,\quad \alpha_5=0, \tag{2.63}
$$

is completely equivalent to the theory described by $\mathcal{L}_4^{\mathrm{H}}+\mathcal{L}_4^{\mathrm{bH}}$. The ghost-free conditions for the cubic DHOST can be found in [50]. Moreover, some particular classes of the DHOST is mapped into the Horndeki or the GLPV theories by invertible disformal transformations. A disformal transformation is the following generalization of conformal transformations:

$$
\tilde{g}_{\mu\nu}=\Omega(\phi,X)g_{\mu\nu}+\Gamma(\phi,X)\phi_\mu\phi_\nu. \tag{2.64}
$$

This is invertible if

$$
\Omega(\Omega-X\Omega_X+2X^2\Gamma_X)\neq 0. \tag{2.65}
$$

Since the invertible disformal transformations have the corresponding inverse transformation, the above subclasses of the DHOST is identical to the Horndeki or the GLPV theories. [6] On the other hand, the DHOST theories which are not connected to the Horndeski theory by invertible disformal transformations are not equivalent to the Horndeski. Such DHOST class, which includes the cubic DHOST, suffers from the gradient instabilities of tensor or scalar modes on a cosmological background [54, 55].

DHOST theories have been explored in the range of cubic powers of $\nabla_\mu\nabla_\nu\phi$, and one can extend it to arbitrary powers in principle. Moreover, some studies have attempted further generalizations. For example, a series of papers [56–58] has been constructing ghost-free four-dimensional covariant theories which contain R^2 and $\nabla_\mu\nabla_\nu\nabla_\rho\phi$ by using spatially covariant theories and gauge recovery by the same method as XG3 theories (see §2.5). In any case, however, further generalizations require much time and effort.

In the scalar-tensor theories, the scalar field effects must approximately vanish at local scales since any measurements and observations support predictions of GR with

[6] More rigorously, the invertible field redefinitions do not change the number of DOFs in the theories [51–53]

high precision. Thus, one demands that theories have some additional mechanisms to remove such local effects, so-caled screening mechanisms. There are some types of these mechanisms. In the chameleon mechanism [59, 60], the scalar field is effectively massive in the vicinity of a source and screened. The symmetron screening mechanism [61] effectively suppress coupling to matter. In the Vainshtein mechanism [62], the scalar field is effectively weakly coupled to the source because of nonlinear perturbations. It is known that the Horndeski theory, particularly G_3, G_4, G_5, is equipped with the Vainshtein mechanism [63–66]. On the other hand, the DHOST theory partially breaks the Vainshtein screening inside stars [67–69].

2.4 Metric Theories

The Einstein equations, which are the field equations of GR, are derived only from the Einstein-Hilbert action (1.1) plus the total derivative terms. Conversely, other actions will lead different gravitational equations from the Einstein equations, even if the actions consist only of the metric. We consider such metric theories by introducing higher curvature terms like $(R_{....})^n$, or higher-derivatives of the curvature like $\Box R$. Generally, such terms lead higher-order field equations since the Riemann tensor contains $\partial\partial g_{\mu\nu}$. However, if its Lagrangian is degenerate, the equations will be recasted to at most second-order equations.

The authors of [70] hunted such degenerate metric theories in a general way. One starts from the four-dimensional general covariant and degenerate actions which depends at most on second derivatives of the metric. For the general covariance, the action consists of $R_{\mu\nu\rho\sigma}, g_{\mu\nu}, \varepsilon^{\mu\nu\rho\sigma}$. Also, for generacy, the rank of the kinetic matrix

$$\mathcal{A}^{ij,lm}(x,y) := \frac{\partial^2 L}{\partial \dot{K}_{ij}(x)\partial \dot{K}_{lm}(y)} \tag{2.66}$$

is supposed to be not full. All of the fully degenerate ($\mathrm{rank}(\mathcal{A}^{ij,lm}) = 0$) Lagrangian densities is derived in [71], as follows:

$$\begin{gathered} R, \quad GB := (\star R^{\mu\nu}{}_{\alpha\beta})(\star R^{\alpha\beta}{}_{\mu\nu}), \\ P := (\star R^{\mu\nu}{}_{\alpha\beta}) R^{\alpha\beta}{}_{\mu\nu}, \quad C := (\star R^{\mu\nu}{}_{\rho\sigma})(\star R^{\rho\sigma}{}_{\alpha\beta})(\star R^{\alpha\beta}{}_{\mu\nu}), \end{gathered} \tag{2.67}$$

with $\star$ holds for the Hodge dual $\star R^{\mu\nu}{}_{\rho\sigma} := \varepsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma}$. The Ricci scalar is nothing but the Einstein-Hilbert term and leads second-order field equations. GB and P, which are called the Gauss-Bonnet and the Pontryagin term respectively, are topological invariants in the four-dimensional spacetimes, and hence their variation yields no term to the field equations. The last term C is the only one whose EOMs are of third derivatives. Unfortunately, the theory $S = \int d^4x\sqrt{-g}C$ has five DOFs and three of them are ghosts, while this has spacetime diffeomorphisms.

R, GB, P, C are all the fully degenerate Lagrangian densities. Then, we consider partially degenerate Lagrangian densities, which means the rank of $\mathcal{A}^{ij,lm}$ is not zero but not full. One simple term of that is $f(Y)$ $(Y = R, GB, P, C)$. These theories can be rewritten into scalar-tensor forms by adding new scalar field ψ (see e.g., [37] for $f(R)$ term), whose action is given by

$$S = \int d^4x\sqrt{-g}[\phi Y - U(\phi)], \tag{2.68}$$

where $U(\phi)$ is a potential defined by

$$U(\phi) = \psi(\phi)\phi - f(\psi(\phi)), \quad \phi := f'(\psi). \tag{2.69}$$

Among the partially degenerate theories, $f(R)$ gravity has three DOFs but is ghost-free. Indeed, $f(R)$ gravity in the scalar-tensor form,

$$S = \int d^4x\sqrt{-g}[\phi R - U(\phi)], \tag{2.70}$$

can be mapped to GR with a canonical scalar field by a conformal transformation. $f(GB)$ gravity has also three DOFs, and is ghost-free if one adds a kinetic term of a scalar field in the scalar-tensor form. $f(P)$ gravity is related to Chern-Simons gravity [72]. In fact, the action of non-dynamical Chern-Simons gravity is

$$S_{CS} = \int d^4x\sqrt{-g}\phi P, \tag{2.71}$$

which is exactly the action (2.68) without the potential term $U(\phi)$. This theory has four DOFs but is ghost-free in the unitary gauge. On the other hand, generic $f(P)$ theory has five DOFs, and three of them are ghosts.

2.5 Lorentz-Violating Gravity

In general, scalar-tensor theories are Lorentz invariant following GR. Nevertheless, some theories break that invariance and consequently obtain unique natures. One famous example is the Einstein-æther theory [73]. This theory consists of the metric tensor and a unit timelike vector field named "æther." Due to its timelike nature, the æther never vanishes, and it always singles out preferred timelike trajectories in spacetimes. These trajectories will define a preferred time direction even in a local coordinate frame, which can choose the flat metric in a neighborhood of a certain event, and this implies the violation of local Lorentz invariance. We note that the Lorentz-breaking can occur in four-dimensional, generally covariant actions. Indeed, the action of the Einstein-æther theory is given by

$$S = \int d^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}[4R - M^{\mu\nu}{}_{\alpha\beta}\nabla_\mu u^\alpha \nabla_\nu u^\beta - \lambda(g_{\mu\nu}u^\mu u^\nu + 1)], \tag{2.72}$$

where u^μ is the æther, λ is a Lagrange multiplier constraining the æther to $g_{\mu\nu}u^\mu u^\nu = -1$, and $M^{\mu\nu}{}_{\alpha\beta}$ is defined by

$$M^{\mu\nu}{}_{\alpha\beta} := c_1 g^{\mu\nu}g_{\alpha\beta} + c_2\delta^\mu_\alpha\delta^\nu_\beta + c_3\delta^\mu_\beta\delta^\nu_\alpha - c_4 u^\mu u^\nu g_{\alpha\beta}, \tag{2.73}$$

with c_i $(i = 1, \cdots, 4)$ being constant.

Another Lorentz-violating example is Hořava-Lifshitz gravity (or simply Hořava gravity) [74]. This theory is a candidate for UV-completions of Einstein's gravity in which full spacetime diffeomorphism is abandoned and recovered only at low energies. In Hořava gravity, one introduces higher spatial derivatives, i.e., higher terms of the three-dimensional Riemann tensor ${}^{(3)}R_{....}$ (because schematically ${}^{(3)}R_{....} \sim D_i D_j \gamma_{kl}$). The generalized version of this theory is in [75], for example, whose action contains up to six spatial derivatives. The residual symmetries of Hořava gravity are space-independent time reparametrizations and spatial diffeomorphisms: $t \to \tilde{t}(t)$, $x^i \to \tilde{x}^i(t, x^i)$. Note in passing that a naive non-projectable extension of Hořava gravity causes a strange feature: the phase space is described by five fields, and consequently the canonical structure is lost [76]. This pathology is associated with an extra half DOF, and was discussed in [76, 77], for example.

Also, the effective field theory (EFT) of inflation [78, 79] describing the fluctuations around a background with time evolution, and more generally, ghost condensate [80] breaks the Lorentz invariance. The authors of [81] studied a general class of scalar-tensor models where at most three DOFs propagate in the unitary gauge but the ghost DOF seemingly revives in other gauges. This Theory is called the U-degenerate (or U-DHOST) theory. As described later, minimally modified gravity, which propagates only two tensorial degrees, is also a Lorentz-breaking theory.

We note that the Lorentz-violation on local scales is constrained by some observations, e.g., the constraints on the parameters of the parameterized post-Newtonian (PPN) formalism. Specifically, the PPN parameters $\alpha_1^{\rm PPN}, \alpha_2^{\rm PPN}$ describe preferred frame effects related to breaking of Lorentz symmetry (see [82] for example). If the Lorentz invariance does not violate on local scales, the Lorentz-breaking theories can be considered EFTs on cosmological scales (e.g., the EFT of inflation), or theories on higher scales of energy (e.g., Hořava gravity).

These Lorentz-breaking theories can often restore the full Lorentz invariance by introducing a scalar field with timelike gradient which is constant on each spacelike hypersurface in the ADM formalism. [7] For instance, Hořava gravity and the EFT of inflation can be reformulated in a fully covariant manner, as describing spacelike hypersurfaces specified by a scalar field coupled to a general background [75, 77, 78, 83]. In other words, Hořava gravity can be viewed as the gauge-fixed version of some Lorentz invariant theories having the full diffeomorphism. [84] briefly summarized some Lorentz-breaking theories and these covariantizations. The author of [84] also applied this method to generate more general scalar-tensor theories (so-called the XG3 theories) than the Horndeski theories. One starts from a general action covariant on the foliation of spacelike hypersurfaces, encoded into a scalar field ϕ with a timelike gradient. The normal vector to the foliation is $n_i := -N\nabla_i \phi$. The action is constructed by three-dimensional geometrical quantities used in the ADM formalism: γ_{ij}, K_{ij}, ${}^{(3)}R_{ij}$, the spatial covariant derivative D_i, and the acceleration

[7] This is one of the Stückelberg trick.

$a_a = n^b \nabla_b n_a \coloneqq D_a \ln N$. The explicit form is given by

$$\mathcal{L} = \sum_{n=1}^{3} \mathcal{K}_n + \mathcal{V}, \tag{2.74}$$

with

$$\mathcal{K}_n \coloneqq \mathcal{G}_{(n)}^{i_1 j_1, \dots, i_n j_n} K_{i_1 j_1} \dots K_{i_n j_n}, \tag{2.75}$$

here $\mathcal{G}_{(n)}$'s and $\mathcal{V}$ are general functions of $(\phi, N, \gamma_{ij}, {}^{(3)}R_{ij}, a_i, D_i)$. This action contains higher-order spatial derivatives as Hořava gravity, which extends scalar-tensor theories in the fully covariant formalism. After imposing that unwanted DOFs do not propagate to avoid the Ostrogradsky ghost, one obtains the XG3 theories. The Horndeski, the EFT of inflation, and Hořava gravity are different theories in the literature, but indeed all of these are special cases of the XG3 theories. A series of papers [56–58] mentioned in §2.3 were discussed in a similar way to the XG3 theories.

2.6 Two-DOFs Scalar-Tensor Theories

The single-field scalar-tensor theories generally have three dynamical DOFs: two tensor modes plus one scalar mode. On the other hand, given that GR is highly consistent with many observations, it is natural to expect nature to not allow considerable modifications besides at cosmological scales. Since additional DOFs can change gravitational effects at various scales, MG without any extra DOFs is preferred. In fact, the Lovelock theorem implies that four-dimensional generally covariant gravity containing only the metric tensor is just Einstein's gravity. Therefore, if we consider MG without additional physical DOFs in a four-dimensional spacetime range, we should break the symmetries. The cuscuton theory has this feature as well as the Lorentz-violation, as explained later.

Another example is "minimally modified gravity (MMG)" [85–92]. The authors of [85] aimed to construct two-DOFs theories by making all constraints first-class in the situation that only the spatial diffeomorphism invariance is imposed. To this end, we consider the ADM action linear in the lupse function N, such as

$$S = \int dt d^3x N \sqrt{\gamma} F(K_{ij}, {}^{(3)}R_{ij}, D_i, \gamma^{ij}, t). \tag{2.76}$$

This linearity ensures the Hamiltonian constraint does not contain N, and hence this constraint is expected to eliminate an extra DOF rather than fixing the lupse function. We define MMG as this action with the conditions to make the number of physical DOFs two, and find that all constraints of such theories are indeed first-class. If one add an independent term of N to the above action,

$$S = \int dt d^3x \sqrt{h} \left[N F(K_{ij}, {}^{(3)}R_{ij}, D_i, \gamma^{ij}, t) + G({}^{(3)}R_{ij}, D_i, \gamma^{ij}, t) \right], \tag{2.77}$$

it derives two second-class constraints instead of the first-class Hamiltonian constraint, and thus the number of DOF remains two. MMG is constructed as a spatially covariant theory, nevertheless, one can recover full diffeomorphism by introducing a scalar field ϕ with timelike gradient as other Lorentz-violating theories. After recovering the spacetime diffeomorphism, MMG in the ADM formalism can be viewed as gauge-fixed scalar-tensor theories in the unitary gauge $\phi = \phi(t)$. This point particularly reminds us of the structure of the U-degenerate theory. Moreover, the authors of [90] have shown that the cuscuton theory is a subset of MMG. [8]

MMG actions are classified into type-I and type-II [88]. The main difference between these two types is the existence of the Einstein frame. The type-I MMG theories can be recasted into the Einstein frame with a nontrivial coupling to matter by a field transformation. From this from, one finds gravity is modified due to a novel matter coupling. Furthermore, one obtains this type if one imposes a gauge-fixing condition after a canonical transformation to GR, and then adds matter fields. The gauge-fixing condition in this method violates the general covariance, and adding matter fields after a canonical transformation leads a novel matter coupling. Indeed, if one performs an appropriate field transformation to the whole action, one can get the Einstein frame with a nontrivial matter coupling. The type-II MMG theories do not have the Einstein frame, but it has an interesting subclass called "minimal theory of massive gravity" [93,94] in which the graviton is massive. There are more theories with only two tensor DOFs, and we will compare these theories and the extended cuscuton in § 4.4.2.

Recently, the author of [95] studied "scalarless scalar theories" in which scalar fluctuations do not propagate by using symmetry principles. Remind that the unitary gauge $\phi = \phi(t)$ singles out particular timelike trajectories, which suggests fixing the time coordinate. As naively pointed out in [85], if a scalar-tensor theory in the unitary gauge has a scalar DOF, this DOF is a Nambu-Goldstone boson associated with the broken temporal diffeomorphism. The author of [95] clarified this point and described the same mechanisms in the cases that scalar fields interacting with themselves, with vectors, or with tensors.

[8] As explained in §3.2, the cuscuton theory has intrinsically the same structure as the U-degenerate theory in the context of propagating DOFs.

Chapter 3

Cuscuton Theory

In the present thesis, we focus on one of the two-DOFs scalar-tensor theories: the cuscuton theory [21]. This theory is the unique subclass of the k-essence in which the scalar mode becomes nondynamical if and only if the gradient of the scalar field ϕ is timelike. The cuscuton shows intriguing characteristics within timelike $\partial_\mu \phi$, and hence this is suitable for exploring homogeneous and isotropic cosmology. The unfamiliar theory name, "cuscuton" (pronounced käs-kü-tän), is derived from a parasitic plant "cuscuta" (see Fig. 3.1 for its appearance). The authors of [21] made the scalar field look like this plant, because the scalar mode itself is nondynamical and merely follows the dynamics of the metric. The

Figure 3.1: "Cuscuta europaea in flower" by Michael Becker CC BY-SA 3.0, cited from [96]. The magenta and yellow string plant, that is a cuscuta, have coiled around the other green plant.

natures of the cuscuton DOF has been discussed from several aspects; the symplectic structure [21, 97], a Hamiltonian analysis [97], and relations among the physical DOFs, homogeneity and direction of the cuscuton [26, 97]. Many theoretically and cosmologically fascinating features of the cuscuton have been also unveiled by [22–24, 98–116].

In this chapter, we first formulate the cuscuton theory based on [21] in § 3.1. Then, we show relations between the scalar DOFs and the cuscuton field distributions in § 3.2. In the next section, we investigate cosmological background and scalar parturbations based on [98] in § 3.3.1, the viable power-low inflation model based on [22] and some other early universe models in § 3.3.2, applications to the dark energy cosmology based on [24] in § 3.3.3, and other aspects of the cuscuton in § 3.3.4.

3.1 Formulation

Originally, the cuscuton was constructed as a subclass of the k-essence characterized by the at most first-order field equation for the scalar field in a homogeneous limit. Note that a first-order differential equation is not an EOM but a constraint equation. Indeed, a first-order differential equation in the phase space represents a constraint between canonical variables, and one of these will be a dependent variable. We start with the k-essence action (2.42), particularly

$$P(\phi, X) = \frac{1}{2}F(X,\phi) - V(\phi), \tag{3.1}$$

with $F(X,\phi), V(\phi)$ being arbitrary functions [41, 117]. We identify the form of $F(X,\phi)$ which becomes a total derivative on a local Minkowski spacetime $ds^2 = -dt^2 + dx_i dx^i$ to drop itself out of the field equation, and on the homogeneous scalar field $\phi = \phi(t)$ satisfying $\dot{\phi} \neq 0$. In this limit, $F(X,\phi)$ reads

$$F(X,\phi(x)) \quad \longrightarrow \quad F\left(\dot{\phi}^2, \phi(t)\right) = \frac{d}{dt}J(\phi,\dot{\phi},\dots) = \dot{\phi}\frac{\partial J}{\partial \phi} + \ddot{\phi}\frac{\partial J}{\partial \dot{\phi}} + \dots. \tag{3.2}$$

Since F contains at most first derivative of ϕ,

$$F\left(\dot{\phi}^2, \phi\right) = \frac{d}{dt}J(\phi) = \sqrt{\dot{\phi}^2}\frac{\partial J(\phi)}{\partial \phi}. \tag{3.3}$$

Here the sign of $\dot{\phi}$ can be absorbed into $J(\phi)/\partial\phi$. Using this, the homogeneous limit of the scalar sector of the action (3.1) will be

$$S_\phi^{\text{homog}} = -\int d^4x V(\phi). \tag{3.4}$$

Variation with respect to ϕ leads the algebraic equation:

$$\frac{\partial V}{\partial \phi} = 0. \tag{3.5}$$

This equation shows ϕ has solely the constraint equation, not an EOM, and thus ϕ has no dynamical DOF. When ϕ couples to another field χ, the total action is

$$S_{\phi-\chi} = \int d^4x \left[\mathcal{L}_\chi(\chi,\phi) - V(\phi)\right], \tag{3.6}$$

and the field equation with respect to ϕ is

$$-\frac{\partial V}{\partial \phi} + \frac{\partial \mathcal{L}_\chi(\chi, \phi)}{\partial \phi} = 0. \tag{3.7}$$

Because of the interaction terms, the dynamics of χ can be modified by $V(\phi)$. Moreover, as long as χ is nondynamical, ϕ behaves like an auxiliary field.

We have derived the action in which ϕ becomes nondynamical in the homogeneous and Minkowski limit. Now let us define the most general four-dimensional covariant action in the form of (3.1) which is equivalent to the action (3.3) and (3.4) in the homogeneous limit. In order to that, all you need to do is replacement of the kinetic term $\sqrt{\dot{\phi}^2} \to \sqrt{2|X|}$, and then the generalized action is

$$S_\phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}\frac{\partial J(\phi)}{\partial \phi}\sqrt{2\,|X|} - V(\phi)\right]. \tag{3.8}$$

It is true that $\partial J/\partial\phi$ might be a function of ϕ, but as long as this term is positive, we can redefine ϕ so that [9]

$$\frac{\partial J(\phi)}{\partial \phi} = 2\mu^2 = \text{const.} \tag{3.9}$$

The action (3.8) is well defined even in $X > 0$ (spacelike $\partial_\mu\phi$). However, in this case, we could not say the cuscuton field is nondynamical because any dynamics of that in inhomogeneous situations is not assumed. Indeed, the inhomogeneous field equation in the Minkowski spacetime reads

$$2\mu^2 X\Box\phi - \mu^2\partial^\mu\phi\partial^\nu\phi\partial_\mu\partial_\nu\phi - (2X)^{\frac{3}{2}}V'(\phi) = 0, \tag{3.10}$$

which is the second-order differential equation, that is the EOM; therefore, ϕ following the above equation is dynamical. [10] Note in passing that the variation of this action is ill-defined at $X = 0$, and hence we should define the EOM without the variation. To summarize, the action of the cuscuton theory is written as

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + \mu^2\sqrt{2|X|} - V(\phi)\right], \tag{3.11}$$

where μ are nonvanishing constants.

3.2 DOF and Distributions of Cuscuton

The most critical feature of the cuscuton is a nondynamical scalar field. The variation of the number of DOFs depending on a cuscuton field distribution is also a unique nature

[9] One can also take a negative $\partial J/\partial\phi$ and $\partial J/\partial\phi = -2\mu^2$. We will deal with the scalar perturbations and the inflation model for both cases in § 3.3.

[10] However, as shown in the next section, an inhomogeneous cuscuton with timelike gradient does not have dynamical DOF.

not existing in usual scalar-tensor theories. For the original cuscuton with timelike $\partial_\mu\phi$, the authors of [97] claimed that the cuscuton field in general carries a scalar DOF and it vanishes only in the homogeneous limit. However, this result is counterintuitive as one can always make the scalar field homogeneous, $\phi = \phi(t)$, when $\partial_\mu\phi$ is timelike by choosing the coordinate system appropriately, namely taking the unitary gauge.

We resolved such a contradiction in our paper [26] by showing that the potentially existing scalar DOF does not propagate if an appropriate boundary condition is imposed. Thus, provided that $\partial_\mu\phi$ is timelike, choosing the unitary gauge does not change the number of physical DOFs. We note that the same idea had already been proposed in the context of U-degenerate theory which was mentioned in §2.5. They have claimed that the ghost DOF does not propagate once a physically reasonable boundary condition is imposed at spatial infinity. Thus, the U-degenerate theory is free of the Ostrogradsky ghost as long as one can take the unitary gauge.

In what follows, we see the specific analysis of the above idea in the cuscuton. Assuming $V(\phi) = 0,\ \mu^2 = 1$ and the Minkowski spacetime for simplicity, the field equation for ϕ with a spacelike gradient (3.10) reads

$$2X\Box\phi + \partial^\mu\phi\partial^\nu\phi\partial_\mu\partial_\nu\phi = 0, \tag{3.12}$$

which is satisfied even for a timelike gradient. (3.12) is a second-order differential equation and propagates the scalar DOF. Hereafter, we suppose ϕ depends only on t and x.

At first, let us consider a timelike $\partial_\mu\phi$. The field equation (3.12) in this case is

$$(\phi')^2\ddot{\phi} - 2\dot{\phi}\phi'\dot{\phi}' + \dot{\phi}^2\phi'' = 0, \tag{3.13}$$

where $\dot{\phi} := \partial_t\phi$ and $\phi' := \partial_x\phi$. Now we can introduce the following non-unitary gauge background:

$$\overline{\phi} = t + \alpha x, \quad -1 < \alpha < 1, \quad \alpha \neq 0. \tag{3.14}$$

We assume a small fluctuation on this background: $\phi = \overline{\phi} + \pi(t, \vec{x})$, and then the quadratic Lagrangian for π is given by

$$\mathcal{L}^{(2)} = -\frac{1}{2(1-\alpha^2)^{3/2}}\left(\alpha\dot{\pi} - \partial_x\pi\right)^2 - \frac{1}{2(1-\alpha^2)^{1/2}}\left[(\partial_y\pi)^2 + (\partial_z\pi)^2\right]. \tag{3.15}$$

The kinetic term seems to have a wrong sign for $\alpha \neq 0$,

$$-\frac{\alpha^2}{2(1-\alpha^2)^{3/2}}\dot{\pi}^2, \tag{3.16}$$

which implies a ghost. The EOM for π is given by

$$\alpha^2\ddot{\pi} - 2\alpha\partial_x\dot{\pi} + \partial_x^2\pi + (1-\alpha^2)(\partial_y^2\pi + \partial_z^2\pi) = 0. \tag{3.17}$$

Substituting $\pi = e^{-i\omega t + i\vec{k}\cdot\vec{x}}$, we obtain the dispersion relation,

$$(\alpha\omega - k_x)^2 + (1-\alpha^2)(k_y^2 + k_z^2) = 0. \tag{3.18}$$

There exist two complex solutions,

$$\omega = \frac{k_x}{\alpha} \pm i\frac{(1-\alpha^2)^{1/2}}{\alpha}\sqrt{k_y^2 + k_z^2}, \tag{3.19}$$

one of which leads a blowing up mode. However, as is discussed in Ref. [81], we expect that the regularity at spatial infinity could remove this dangerous ghost mode. To see this, we would perform the following boost transformation:

$$\tilde{t} = \frac{t + \alpha x}{\sqrt{1-\alpha^2}}, \quad \tilde{x} = \frac{\alpha t + x}{\sqrt{1-\alpha^2}}, \quad \tilde{y} = y, \quad \tilde{z} = z. \tag{3.20}$$

Then, Eq. (3.17) becomes the Laplace equation,

$$\left(\partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 + \partial_{\tilde{z}}^2\right)\pi = 0. \tag{3.21}$$

The solutions of this equation are at most linear in spatial coordinates, and once one require regularity at spatial infinity, the allowed solution is solely $\pi = \text{const}$. Therefore, the ghost mode does not propagate if such an appropriate boundary condition is imposed.

The other case is spacelike $\partial_\mu\phi$, and in this case, $|\alpha|$ is larger than unity. We then obtain

$$\mathcal{L}^{(2)} = -\frac{1}{2(\alpha^2-1)^{3/2}}\left(\alpha\dot{\pi} - \partial_x\pi\right)^2 + \frac{1}{2(\alpha^2-1)^{1/2}}\left[(\partial_y\pi)^2 + (\partial_z\pi)^2\right], \tag{3.22}$$

and the equaiton for π is again given by Eq. (3.17). Given $\alpha^2 > 1$, we define the following coordinate system:

$$\tilde{t} = \frac{\alpha t + x}{\sqrt{\alpha^2-1}}, \quad \tilde{x} = \frac{t + \alpha x}{\sqrt{\alpha^2-1}}, \quad \tilde{y} = y, \quad \tilde{z} = z, \tag{3.23}$$

by which Eq. (3.17) is transformed into

$$\left(-\partial_{\tilde{t}}^2 + \partial_{\tilde{y}}^2 + \partial_{\tilde{z}}^2\right)\pi = 0. \tag{3.24}$$

This is clearly a hyperbolic equation, and thus the dangerous mode π propagates.

To summerize, the key point of above discussions is the magnitude of $|\alpha|$. The EOM for π is (3.17) irrespective of the value of $|\alpha|$, but the sign of $(1-\alpha^2)(\partial_y^2\pi + \partial_z^2\pi)$ in this equation flips depending on $|\alpha| \gtrless 1$, namely $\partial_\mu\phi$ being spacelike/timelike. This difference consequently leads the different types of differential equations like (3.21) and (3.24).

3.3 Various Aspects of the Cuscuton

In this section, we review various aspects of the cuscuton theory, mainly in the context of cosmology: applications to the inflation model, nonsingular bouncing cosmology, and the late-time universe.

3.3.1 Cosmological Backgrounds and Scalar Perturbations

The cuscuton field is nondynamical and causes a non-local modification to Einstein gravity. Details are discussed later, but at the beginning of this subsection, let us think about what would happen if we consider a homogeneous and isotropic cosmology in the model (3.11) in vacuum without other fields. A homogeneous and isotropic spacetime is given by

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad \phi = \phi(t), \tag{3.25}$$

with the lapse function $N(t)$ and the scale factor $a(t)$. The field equations are obtained by substituting this metric into the action (3.11) and varying it with respect to N, a, and ϕ. Thereafter, we may set $N = 1$. [11] The resultant equations are

$$3M_{\rm Pl}^2 H^2 - V(\phi) = 0, \tag{3.26}$$

$$2M_{\rm Pl}^2 \dot{H} + \mu^2|\dot{\phi}| = 0, \tag{3.27}$$

$$\mathrm{sign}(\dot{\phi})3\mu^2 H + V'(\phi) = 0. \tag{3.28}$$

with $H \coloneqq \dot{a}/a$ being the Hubble parameter. (3.27) is clearly the second-order differential equation for a, although, the other two equations are at most first-order differential equation for both a and ϕ. From (3.26) and (3.28), once one determine the form of $V(\phi)$ one get

$$3M_{\rm Pl}^2 H^2 - V\left(V^{-1}\left(\mp\,\mathrm{sign}(\dot{\phi})3\mu^2 H\right)\right) = 0, \tag{3.29}$$

where V'^{-1} is the inverse function of $V'(\phi)$. This equation implies that the cuscuton drops out from the Friedmann equations, but the form of its solutions H is controlled by the cuscuton potential term $V(\phi)$ (see § 3.3.3 for details). It is also manifest that the cuscuton only modifies gravity on large scales since the cuscuton itself has no dynamics. Therefore, to produce dynamical geometrical modifications in a cuscuton scenario, other sources with propagating DOFs are needed (see § 3.3.2).

If one considers a quadratic action for the perturbations around the background, the scalar mode does not have a kinetic term. Hence, its field equation is not an EOM but a constraint by which one can integrate the scalar mode out from the perturbed action, while the tensor mode propagates as in GR. Hence, to obtain a viable cosmological scenario, we must add some extra fields that fluctuate the scale factor like the inflaton.

According to [98], we add the following scalar field χ with a canonical kinetic term and a potential term,

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - U(\chi) \pm \mu^2\sqrt{-2X} - V(\phi)\right]. \tag{3.30}$$

In §3.1, we have formulated the cuscuton action viable in both signs of X. On the other hand, now we want to consider a cosmological setup, so that we have ignored the $X > 0$ case and

[11] If one sets $N = 1$ at the action level, one will obtain only (4.1) and (4.2) besides the Euler-Lagrange equation for N [118].

replaced $|X| \to -X$. Furthermore, we treat the both signs of $\mu^2\sqrt{-2X}$ in this discussion. The field equations from (3.30) are

$$3M_{\rm Pl}^2H^2 - V(\phi) - U(\chi) - \frac{1}{2}\dot{\chi}^2 = 0, \tag{3.31}$$

$$2M_{\rm Pl}^2\dot{H} \pm \mu^2|\dot{\phi}| + \dot{\chi}^2 = 0, \tag{3.32}$$

$$\pm{\rm sign}(\dot{\phi})3\mu^2H + V'(\phi) = 0, \tag{3.33}$$

$$\ddot{\chi} + 3H\dot{\chi} + U'(\chi) = 0. \tag{3.34}$$

Eq. (3.34) is the same as the EOM for the inflaton in the standard inflaton model. Eq. (3.33), which is almost same as (3.27), determines ϕ as a function of H by considering the inverse function of $V'(\phi)$: V'^{-1}. Substituting it into (3.31), we obtain

$$3M_{\rm Pl}^2H^2 - V\left(V'^{-1}\left(\mp{\rm sign}(\dot{\phi})3\mu^2H\right)\right) - U(\chi) - \frac{1}{2}\dot{\chi}^2 = 0. \tag{3.35}$$

Furthermore, unlike in Einstein's gravity with additional scalar DOF, no scalar contributions are in the equation. Hence, one can derive the dynamics of H without initial or boundary conditions for ϕ. Even so, this algebraic equation can be controlled by the choice of function $V(\phi)$, which clearly represents the modification of gravity.

After we use (3.32), the first slow-roll parameter is given by

$$\epsilon := -\frac{\dot{H}}{H^2} = \frac{\pm\mu^2|\dot{\phi}| + \dot{\chi}^2}{2M_{\rm Pl}^2H^2}, \tag{3.36}$$

which indicates that the cuscuton affects the slow-roll dynamics. To represent the cuscuton contribution to deviation from the conventional single field inflation, we define the following quantities:

$$\sigma := \epsilon - \alpha = \pm\frac{\mu^2|\dot{\phi}|}{2M_{\rm Pl}^2H^2}, \qquad \alpha := \frac{\dot{\chi}^2}{2M_{\rm Pl}^2H^2}. \tag{3.37}$$

Here $\sigma = 0$ represents $\mu = 0$, namely the absence of the cuscuton.

Now we will derive the quadratic action for the curvature perturbation ζ to investigate the stability. We choose the uniform field gauge for χ: $\delta\chi = 0$, and we write the perturbed metric and the cuscuton field as

$$\begin{aligned} ds^2 &= -(1+2\delta N)dt^2 + 2\partial_i\psi dt dx^i + a^2(1+2\zeta)\delta_{ij}dx^idx^j, \\ \phi &= \phi(t) + \delta\phi(t,\vec{x}), \end{aligned} \tag{3.38}$$

where $\delta N, \psi, \zeta$ and $\delta\phi$ are the scalar fluctuations. [12] The equations for δN and ψ are constraints, and one can remove them by substituting these constraints into the action. The equations for $\delta\phi$ is also a constraint, though, this contains the spatial derivative for $\delta\phi$ like the equation for inhomogeneous cuscuton field (3.10). Consequently, the solution of that

[12] Of course, the manner of gauge fixing does not affect the number of the propagating scalar fluctuations.

constraint involves non-local operators, namely inverses of derivative operators. To avoid such terms in the action, we recast the action into Fourier-space one before substituting $\delta\phi$. After we take $\delta N, \psi$ and $\delta\phi$ away, the quadratic action for scalar perturbations is

$$S^{(2)} = \int dtd^3xz^2 \left[\dot{\zeta}^2(t,\mathbf{k}) - \frac{c_s^2}{a^2}\mathbf{k}^2\zeta^2(t,\mathbf{k})\right]. \tag{3.39}$$

Here $z(t,\mathbf{k})$ and $c_s(t,\mathbf{k})$ are given by

$$z^2 := a^3\alpha\left(\frac{(\mathbf{k}/a)^2 + 3\alpha H^2}{(\mathbf{k}/a)^2 + \alpha H^2(3-\sigma)}\right), \qquad c_s^2 := \frac{(\mathbf{k}/a)^4 + (\mathbf{k}/a)^2H^2\mathcal{B}_1 + H^4\mathcal{B}_2}{(\mathbf{k}/a)^4 + (\mathbf{k}/a)^2H^2\mathcal{A}_1 + H^4\mathcal{A}_2}, \tag{3.40}$$

and the other quantities are

$$\begin{aligned}
\eta &:= \frac{\dot{\epsilon}}{H\epsilon}, \qquad \beta := \frac{\dot{\alpha}}{H\alpha}, \\
\mathcal{A}_1 &:= 6\alpha - \alpha\sigma, \qquad \mathcal{A}_2 := 9\alpha^2 - 3\alpha^2\sigma, \\
\mathcal{B}_1 &:= \mathcal{A}_1 + \sigma(6+\eta+\beta-2\epsilon) + \alpha(\eta-\beta), \\
\mathcal{B}_2 &:= \mathcal{A}_2 + \sigma\alpha(12-4\sigma+3\eta) + 3\alpha^2(\eta-\beta),
\end{aligned} \tag{3.41}$$

where η and β are the second slow-roll parameters.

At this stage, we must check two instabilities. One is the ghost instability. This arises from the negative sign of the coefficient of the kinetic term: $z^2 < 0$. If this is the case, the enegy of this system will be unbounded below in the same way as the Ostrogradsky ghost. The other one is the gradient instability, which appears with the wrong sign of the spatial derivatives. Now we explain this instability briefly using the simple model such as

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\partial\phi)^2, \tag{3.42}$$

with ∂ being the spatial derivatives. The solutions in the Fourier space are given by $\phi_k(t) \sim e^{\pm kt}$. Among them, e^{kt} represents the growing mode that time scale is $t \sim k^{-1}$. Therefore, the high energy mode leads to an instability.

In (3.39), taking the $\sigma \to 0$ limit or the UV limit $\mathbf{k} \to \infty$, we get the standard single scalar field result of $c_s^2 \sim 1$ and $z^2 \sim a^2\alpha$. In the UV limit, particularly, all the cuscuton contributions vanish and there are neither ghost nor gradient instabilities. Furthermore, when $-\mu^2$ branch leads a negative σ and hence z^2, which is the coefficient of the kinetic term for the curvature perturbation, is always positive regardless of scale. We note that, in the infrared (IR) regime, both ghost/gradient instabilities are not necessarily problematic: Even if the kinetic term has a wrong sign, it is legitimate to ignore the ghost instability if its energy scale is much lower than the cutoff scale. The gradient instability is also irrelevant when the timescale of interest is much shorter than the instability.

3.3.2 Early Universe

As we mentioned above, the cuscuton field can affect the dynamics of other dynamical fields in the absence of an additional DOF. As the next step, we would like to see if the cuscuton contributions might circumvent some pathologies in present cosmological models or not. One successful example of this has been presented by [22], in which the cuscuton can reconcile the inflation model so-called "power-law inflation" [119] with observations. [13] This model is known as a simple and exact solution, though, has been ruled out by recent observations [120]. Ref. [22] has shown that the cuscuton can yield consistent values of the scalar spectral index n_s and the tensor-to-scalar ratio r with CMB data.

We consider the action (3.11) and regard χ as the inflaton field. We assume the potential term of χ the same as the standard power-law inflation model, such as

$$U(\chi) = U_0 e^{u\frac{\chi}{M_{\rm Pl}}}, \tag{3.43}$$

here U_0 and u are constant. We would like to obtain an exact solution of the background field equations (3.31)-(3.34) satisfying $H = p/t$ with the following ansatz:

$$V(\phi) = \frac{1}{2}m^2\phi^2, \quad \frac{\chi}{M_{\rm Pl}} = s\ln M_{\rm Pl}t, \quad \phi = \frac{q}{t}, \tag{3.44}$$

here m, p, s, and q are parameters which are related with each other by the field equations. One of nontrivial branches of the remaining parameters satisfying (3.31)-(3.34) is

$$\begin{aligned} u &= -\frac{2}{s}, \quad q = -\frac{3\mu^2 s^2}{m^2}\left(2 - \frac{3\mu^4}{M_{\rm Pl}^2 m^2}\right)^{-1}, \\ p &= s^2\left(2 - \frac{3\mu^4}{M_{\rm Pl}^2 m^2}\right)^{-1}, \quad \frac{U_0}{M_{\rm Pl}^4} = \frac{s^2}{2}\left(\frac{3s^2}{2 - \frac{3\mu^4}{M_p^2 m^2}} - 1\right), \end{aligned} \tag{3.45}$$

where $2 - \frac{3\mu^4}{M_{\rm Pl}^2 m^2} > 0$ should be satisfied to keep p positive. This branch has two out of six free parameters, and the first and second slow-roll parameter ϵ and η reads

$$\epsilon = \frac{1}{p}, \quad \eta = 0, \tag{3.46}$$

which is of cource the same result of the standard power-law inflation.

To calculate n_s and r, let us consider the following scalar and tensor perturbations of the metric:

$$ds^2 = -(1 + 2\delta N)dt^2 + 2\partial_i\psi\, dt\, dx^i + a^2(1 + 2\zeta)(\delta_{ij} + h_{ij})dx^i dx^j. \tag{3.47}$$

The quadratic action of the scalar perturbations is (3.39), and α and σ in (3.37) become

$$\alpha = \frac{1}{2s^2}\left(2 - \frac{3\mu^4}{M_{\rm Pl}^2 m^2}\right)^2, \qquad \sigma = \frac{3\mu^4}{2M_{\rm Pl}^2 m^2 s^2}\left(2 - \frac{3\mu^4}{M_{\rm Pl}^2 m^2}\right). \tag{3.48}$$

[13] The power-law inflation represents an accelerating universe satisfying $a(t) \propto t^p$, $p = $ const. > 1. Note in passing that $p = $ const. $\gg 1$ brings $\epsilon = $ const. $\ll 1$, which implies this model needs an additional mechanism of ending the inflationary era.

The power spectrum of the curvature perturbations, which is associated with two point correlation function of the curvature perturbations, is given by

$$P_\zeta = \left. \frac{H^2}{8\pi^2 M_{\rm Pl}^2 \alpha} \right|_{aH=k} . \tag{3.49}$$

One the other hand, the tensor perturbations is not mixed with the scalar ones at the linear order, and hence the power spectrum of tensor perturbations is the same as that of conventional single field inflation,

$$P_h = \left. \frac{2H^2}{\pi^2 M_{\rm Pl}^2} \right|_{aH=k} . \tag{3.50}$$

Using these quantities, we get n_s and r as follows:

$$n_s - 1 = \frac{d \ln P_\zeta}{d \ln k} \simeq -2\epsilon, \quad r = \frac{P_h}{P_\zeta} = 16\alpha, \tag{3.51}$$

which shows that we can control n_s and r independently as long as σ is nonvanishing. More explicitly, these reads

$$n_s - 1 = -\frac{2}{s^2}\left(2 - \frac{3\mu^4}{M_{\rm Pl}^2 m^2}\right), \qquad r = \frac{8}{s^2}\left(2 - \frac{3\mu^4}{M_{\rm Pl}^2 m^2}\right)^2 . \tag{3.52}$$

From these, we find

$$r = 2s^2 (n_s - 1)^2. \tag{3.53}$$

Since s^2 is a model parameter, r can take any values even if one fixes n_s to the observational value. Therefore, by tuning model parameters, we can obtain consistent n_s and r values with observations.

Finally, we comment that (3.51) is the general result in an inflationary scenario with the (original) cuscuton. We thus believe the cuscuton can bypass observational constraints at least the linear level.

The inflationary scenario resolves the horizon and flatness problems in the standard Big Bang cosmology, and moreover provides the seeds for large-scale structure of the universe by quantum fluctuations of the inflaton field. Even though inflationary models are successful, such universe has the curvature singularity at the beginning of the universe, classical gravitational theories become useless. To evade this problem, one may consider alternative models, e.g., nonsingular bouncing cosmology (see [121] for a review). This model is never singular so that the universe initially contracts and then "bounces" and starts expansion. In the context of the cuscuton theory, the bounce scenarios seem naively to be viable. As said in §3.3.1, the negative sign of μ^2 in (3.30) automatically circumvents the ghost instabilities regardless of scale. Moreover, this sign can make the first slow-roll parameter

ϵ in (3.36) negative. A negative ϵ, namely $\dot{H} > 0$, means the dynamics of the universe can turn from contraction ($H < 0$) to expansion ($H > 0$) as time goes by, which is exactly a bounce scenario. The authors of [23] have found a healthy cuscuton bounce solution for the action (3.30) with $U(\chi) = 0$, which has no pathologies associated with the violation of null energy condition (NEC) (see also [99]). In this model, NEC is effectively broken in FRW backgrounds while the actual matter sources satisfy NEC. More intrinsic interpretation of this point has been studied in [100, 101], associating with the limiting (extrinsic) curvature principle [102–104].

Utilizing the cosmic expansion, one can compactify extra space dimensions into a relatively smaller size than our three-dimensional space. Specifically, all spatial dimensions were initially compact, and subsequently, along with the cosmological evolution, the only four-dimensional universe has expanded up to the present scale. [14] The authors of [105] have studied this scenario in the context of the (4+1)-dimensional cuscuton theory in which the cuscuton field couples with a vector field. They have shown that a solution of this model describes accelerating expansion of four-dimensional spacetime with a completely static extra dimension. This seems to be the first concrete model of accelerating universes besides the de Sitter spacetime with an extra static dimension. This discussion can be extended into $(n + 1)$-dimensional spacetime.

3.3.3 Dark Energy

As previously mentioned, the present cosmic accelerated expansion is strongly supported by recent observations. All matters attract each other by gravitational force, and hence an unknown repulsive long-range force should cause this accelerated expansion. The source of the repulsive force is called "dark energy," and it is the dominant component of the energy density in the current universe [8] after the matter dominant era. The simplest candidate for the dark energy is the cosmological constant Λ. Let us consider Einstein gravity only with dark energy, that is, the Einstein-Hilbert action plus $\Lambda > 0$ whose action is given by (1.2), to study the dark energy dominant era. The Friedmann equation reads $3H^2 = \Lambda$, namely $a = e^{\sqrt{\Lambda/3}t}$, and this implies an accelerating cosmic expansion. On the other hand, another unknown non-relativistic component called "dark matter" is also supported. The ΛCDM model containing Λ as dark energy and cold dark matter (CDM) passes all of the precise cosmological observations so far.

Other than the ΛCDM model, one can apply MG to the dark energy models. The authors of [24] have studied the cuscuton as dark energy. The present universe is the dark energy dominant era. Still, it is important to consider the non-relativistic matter because measurements for matter, e.g., the gravitational lensing and the large-scale structures, are powerful tools to test the dark energy models. Hence, we consider the cuscuton (3.11) with a minimally coupled matter component. If this is the case, the Einstein equations are similar

[14] The original idea was in [122], but details of this scenario is different from the above (see [105]).

to (3.26) and (3.27),

$$3M_{\rm Pl}^2H^2 - V(\phi) - \rho_{\rm m} = 0, \tag{3.54}$$

$$2M_{\rm Pl}^2\dot{H} + \mu^2|\dot{\phi}| + p_{\rm m} = 0, \tag{3.55}$$

with $\rho_{\rm m}$ and $p_{\rm m}$ is the energy density and pressure of matter respectively. The variation with respect to ϕ is the same as (3.28). For simplicity, hereafter, we assume $\dot{\phi} > 0$. From these equations, we obtain an algebraic equation for H,

$$3M_{\rm Pl}^2H^2 - V\left(V'^{-1}\left(\mp(\dot{\phi})3\mu^2H\right)\right) - \rho_{\rm m} = 0. \tag{3.56}$$

In addition to the features of (3.35), the dependence between H and $\rho_{\rm m}$ is also changed by the potential term $V(\phi)$.

For example, let us consider the quadratic potential for ϕ: $V(\phi) = V_0 + m^2\phi^2/2$ with $V_0 = \text{const}$. In this case, (3.56) reads

$$3\left(M_{\rm Pl}^2 - \frac{3\mu^4}{2m^2}\right)H^2 - \rho_{\rm m} = 0. \tag{3.57}$$

This is equal to the conventional Friedmann equation with the following renormalized Planck mass,

$$\tilde{M}_{\rm Pl}^2 = M_{\rm Pl}^2 - \frac{3\mu^4}{2m^2}. \tag{3.58}$$

Indeed, this type of potential derives the exactly equivalent expansion history to that of a ΛCDM cosmology. To see this, we define the following quantity:

$$\Omega_{\rm Q} := -\frac{\tilde{M}_{\rm Pl}^2 - M_{\rm Pl}^2}{M_{\rm Pl}^2} = \frac{3\mu^4}{2M_{\rm Pl}^2m^2} = \text{const}. \tag{3.59}$$

This represents actually a part of the density parameter of dark energy contributed by the quadratic term of $V(\phi)$. We definie the energy density of dark energy as follows:

$$\rho_{\rm DE} := 3M_{\rm Pl}^2H^2 - \rho_{\rm m} = V_0 + \frac{1}{2}m^2\phi^2, \tag{3.60}$$

and then the density parameter of dark energy is given by

$$\Omega_{\rm DE} := \frac{\rho_{\rm DE}}{3M_{\rm Pl}^2H^2} = \frac{V_0}{3M_{\rm Pl}^2H^2} + \frac{\frac{1}{2}m^2\phi^2}{3M_{\rm Pl}^2H^2} =: \Omega_{V_0} + \frac{3\mu^4}{2M_p^2m^2} = \Omega_{V_0} + \Omega_{\rm Q}, \tag{3.61}$$

where we have used Eq. (3.28). Considering $V_0 = \text{const.}$, Ω_{V_0} is identical to the density parameter of the cosmological constant and causes the same time evolution of H as that in the ΛCDM model. On the other hand, $\Omega_{\rm Q} = \text{const.}$ maintains a constant fraction of the total energy density of the universe by transforming each density parameter besides $\Omega_{\rm Q}$: $\Omega_i \to \left(1 - \Omega_{\rm Q}\right)\Omega_i$. Thus, the quadratic term does not affect the cosmic expansion history, and consequently, that history is identical to that in the ΛCDM cosmology as long as $V_0 \neq 0$. On the contrary, the CMB and matter power spectra can be distinguished from those in ΛCDM (see [24] for details). Therefore, geometrical tests such as supernovae Ia, or the angular scale of baryonic acoustic oscillations, are blind to a quadratic term in the Cuscuton potential, while the integrated Sachs-Wolfe effect in the CMB might detect its influence.

3.3.4 Others

Besides the DOFs and cosmology, other various aspects of the cuscuton have been still actively investigated. The Cuscuton theory is a subclass of minimally modified gravity [85]. Also, the cuscuton appears as the extreme relativistic limit of a five-dimensional brane theory [106] and as the UV limit of an anti-Dirac-Born-Infeld theory [107]. Furthermore, relations between the cuscuton and other Lorentz-violationg theories, especially the Einstein-æther and the Hořava-Lifshitz theory (see § 2.5 for brief overview) [108, 109]. The authors of [110] have pointed out the absence of caustic singularities in cuscuton-like scalar field theories, and cuscuton kinks and the braneworld scenario have been explored in [111]. The McVitte solution [25], which is a solution of GR describing time-dependent black holes, is an exact solution for cuscuton gravity [112, 113]. More formally, the cuscuton admits extra symmetries other than the Poincaré symmetry [114–116].

Chapter 4

Extended Cuscuton: Formulation

The cuscuton theory (3.11) is the unique k-essence subclass that the scalar mode is nondynamical if and only if the scalar field has a timelike gradient. Given that the k-essence is also a subclass of more general scalar-tensor theories, e.g., the Horndeski or the DHOST theories, it is natural to expect that the cuscuton is not the unique subclass of these general frameworks as two-DOFs theories, and there are some comprehensive classes sharing the same nature as the cuscuton. In this chapter, we dub such extended theories the "extended cuscuton" and find its specific form. If we identify the extended cuscuton, we can explore the following questions. The first one is whether the intriguing features of the original cuscuton are unique to itself or shared with the whole extended framework. The second one is how relevant are the extended cuscuton to other two-DOFs theories with timelike $\phi_\mu := \partial_\mu \phi$ developed with different motivations from ours (for example, [85, 87, 108, 110, 123]).

To this end, we start from the GLPV theory as a general scalar-tensor theory with three DOFs in general, and identify the forms of the free functions in the Lagrangian by requiring that the theory has only two DOFs. Here we suppose that ϕ_μ is timelike, and use the unitary gauge. As the first step, we consider a relatively easy situation: we specify the Lagrangian having some cosmological properties. As mentioned in §3.3, the cuscuton has mainly two properties in a homogeneous and isotropic cosmology:

[a] *The field equation of the scalar field is at most of first order in the case of homogeneous and isotropic cosmology.* In this cosmological setup, one may safely choose the unitary gauge $\phi = \phi(t)$. Then, the second term in Eq. (3.11) takes the form $\mu^2|\dot{\phi}|$ with a dot denoting $\partial/\partial t$, so the Euler-Lagrange equation for ϕ does not contain second or higher derivatives of ϕ. Thus, the scalar field becomes nondynamical and its evolution is determined by the dynamics of the metric through the constraint equation.

[b] *The kinetic term of scalar cosmological perturbations vanishes.* If the action (3.11) is expanded to second order in scalar perturbations around a cosmological background, one ends up with the quadratic action for a single variable ζ (the curvature perturbation), where it turns out that the coefficient of the kinetic term $\dot{\zeta}^2$ vanishes. This is due to the nondynamical nature of ϕ in the cuscuton theory.

As is anticipated, the two properties [a] and [b] are closely related to each other (see §4.1). These conditions ensure the two-DOFs nature only in a cosmological background, and hence these are just necessary conditions for the theory we aim to construct. Next, we identify which of the theory among this cosmological prototype of the extended cuscuton has two DOFs on an arbitrary background in the unitary gauge. Note that if one starts from the k-essence model, the above procedure leads to the original cuscuton theory. We believe that the same procedure can, in principle, be applied to even broader classes such as DHOST theories as a starting point, but this topic is out of scope of this work.

The rest of this chapter is organized as follows. In §4.1, we construct a prototype model for the extended cuscuton theory as a subclass of the GLPV theory, which has two DOFs at least in a cosmological background. Then, in §4.2, we perform a nonlinear Hamiltonian analysis of the prototypes in an arbitrary background in which one can choose the unitary gauge, and identify the theory with only two propagating DOFs which is just our desired extended cuscuton theory. The above discussions are performed in the ADM formalism, and we transform (a part of) the extended cuscuton in the covariantized form in §4.3. In §4.4, we study relations between the extended cuscuton and other theories. The relation between the original and the extended cuscuton theories is studied employing disformal transformations in §4.4.1, and we discuss the other two-DOFs models in §4.4.2. In §4.5, we also analyze cosmological perturbations in the presence of a matter field and study the stability conditions for the tensor and scalar modes.

4.1 Cosmological Prototype for Extended Cuscuton

4.1.1 Prototype in Flat Cosmology

We start with the GLPV theory, whose action is given by (2.48), and a homogeneous and isotropic universe (3.25). In this spacetime, variations of (2.48) with respect to a and ϕ lead the following dynamical equations (see, e.g., Ref. [67]):

$$\mathcal{E}_a = 2\mathcal{G}_T\dot{H} - 2\mathcal{M}\ddot{\phi} + \mathcal{U} = 0, \tag{4.1}$$

$$\mathcal{E}_\phi = 6\mathcal{M}\dot{H} + \mathcal{K}\ddot{\phi} + \mathcal{V} = 0, \tag{4.2}$$

where

$$
\begin{aligned}
\mathcal{G}_T &:= 2\left(G_4 - 2XG_{4X} + XG_{5\phi} - H\dot{\phi}XG_{5X} + 4X^2F_4 - 12H\dot{\phi}X^2F_5\right),\\
\mathcal{M} &:= -XG_{3X} + G_{4\phi} - 2XG_{4\phi X} + H^2X\left(3G_{5X} + 2XG_{5XX} + 60XF_5 + 24X^2F_{5X}\right)\\
&\quad + 2H\dot{\phi}\left(G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X} - 8XF_4 - 4X^2F_{4X}\right),\\
\mathcal{K} &:= G_{2X} + 2XG_{2XX} + 2\left(G_{3\phi} + XG_{3\phi X}\right)\\
&\quad - 6H\dot{\phi}\left(G_{3X} + XG_{3XX} + 3G_{4\phi X} + 2XG_{4\phi XX}\right)\\
&\quad + 6H^2\left(G_{4X} + 8XG_{4XX} + 4X^2G_{4XXX} - G_{5\phi} - 5XG_{5\phi X}\right.\\
&\qquad\left. - 2X^2G_{5\phi XX} - 24XF_4 - 36X^2F_{4X} - 8X^3F_{4XX}\right)\\
&\quad + 2H^3\dot{\phi}\left(3G_{5X} + 7XG_{5XX} + 2X^2G_{5XXX} + 120XF_5 + 132X^2F_{5X} + 24X^3F_{5XX}\right),\\
\mathcal{U} &:= G_2 + 2XG_{3\phi} + 4XG_{4\phi\phi} + 4H\dot{\phi}\left(G_{4\phi} - 2XG_{4\phi X} + XG_{5\phi\phi} + 4X^2F_{4\phi}\right)\\
&\quad + 2H^2\left(3G_4 - 6XG_{4X} + 3XG_{5\phi} - 2X^2G_{5\phi X} + 12X^2F_4 - 24X^3F_{5\phi}\right)\\
&\quad - 4H^3\dot{\phi}\left(XG_{5X} + 12X^2F_5\right),\\
\mathcal{V} &:= -G_{2\phi} + 2XG_{2\phi X} + 2XG_{3\phi\phi} + 3H\dot{\phi}\left(G_{2X} + 2G_{3\phi} - 2XG_{3\phi X} - 4XG_{4\phi\phi X}\right)\\
&\quad - 6H^2\left(3XG_{3X} + 2G_{4\phi} + 6XG_{4\phi X} - 4X^2G_{4\phi XX}\right.\\
&\qquad + XG_{5\phi\phi} - 3H^2XG_{5X} - 2H^2X^2G_{5XX} + 2X^2G_{5\phi\phi X}\\
&\qquad\left. + 12X^2F_{4\phi} + 8X^3F_{4\phi X} - 60H^2X^2F_5 - 24H^2X^3F_{5X}\right)\\
&\quad + 2H^3\dot{\phi}\left(9G_{4X} + 18XG_{4XX} - 9G_{5\phi} - 7XG_{5\phi X} + 2X^2G_{5\phi XX}\right.\\
&\qquad\left. - 72XF_4 - 36X^2F_{4X} + 48X^2F_{5\phi} + 24X^3F_{5\phi X}\right).
\end{aligned}
\tag{4.3}
$$

These quantities (4.3) contain at most first derivatives of the metric and the scalar field.

In the case of the k-essence (2.42), we have $G_3 = G_5 = 0$, $G_4 =$ const, and hence $\mathcal{M}$ vanishes. Then, the property [a] reads

$$
\mathcal{K} = G_{2X} + 2XG_{2XX} = 0 \quad \Rightarrow \quad G_2 = c_1(\phi)\sqrt{|X|} + c_2(\phi). \tag{4.4}
$$

The original cuscuton theory (3.11) is thus recovered. However, we have $\mathcal{M} \neq 0$ in general, which signals a kinetic mixing of gravity and the scalar field. In this case, the statement of [a] is not necessarily correct, and instead it is more appropriate to require the following extension of [a]:

[a′] *The system composed of the two dynamical equations* (4.1) *and* (4.2) *is degenerate*:

$$
\det\begin{pmatrix} 2\mathcal{G}_T & -2\mathcal{M} \\ 6\mathcal{M} & \mathcal{K} \end{pmatrix} = 2\left(\mathcal{G}_T\mathcal{K} + 6\mathcal{M}^2\right) = 0. \tag{4.5}
$$

This condition can be rewritten into the polynomial in H,

$$
\mathcal{G}_T\mathcal{K} + 6\mathcal{M}^2 = \sum_{n=0}^{4} a_n(\phi, \dot{\phi})H^n = 0, \tag{4.6}
$$

where a_n's are functions of ϕ and $\dot{\phi}$. The property [a′] is satisfied if

$$a_n = 0 \quad (n = 0, 1, 2, 3, 4), \tag{4.7}$$

which is a set of differential equations satisfied by $G_2, G_3, \cdots$ of the extended cuscuton.

To provide conditions for the property [b], we consider the quadratic action for the curvature perturbation ζ in the GLPV theory. Following the standard procedure (see, e.g., Ref. [14] or §3.3.1 in this thesis), we have

$$S_S^{(2)} = \int dt d^3x N a^3 \left[\mathcal{G}_S \dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2}(\partial_k \zeta)^2\right], \tag{4.8}$$

where it is found that

$$\mathcal{G}_S \propto \mathcal{G}_T \mathcal{K} + 6\mathcal{M}^2. \tag{4.9}$$

Therefore, the two requirements [a′] and [b] are consequently equivalent.

Although Eq. (4.7) provides some restrictions on the functions in the GLPV action, and one can specify the subclass satisfying this in principle, the actual manipulation is tedious. To bypass this nonessential issue, we move to the ADM formalism rather than sticking to the covariant formulation. It turns out that the ADM formalism dramatically simplifies the analysis. The GLPV action (2.48) is translated to the ADM language in the unitary gauge $\phi = \phi(t)$, (2.52). In terms of (A_i, B_j) instead of (G_i, F_j), a_n can be expressed as

$$a_0 \propto 3\left(A_3'\right)^2 - 4\left(A_2' + A_2''\right)A_4, \tag{4.10}$$

$$a_1 \propto \left(A_3' + A_3''\right)A_4 - 2A_3'A_4' + \left(A_2' + A_2''\right)A_5, \tag{4.11}$$

$$a_2 \propto 2\left(A_4' + A_4''\right)A_4 - 4\left(A_4'\right)^2 + 3\left(A_3' + A_3''\right)A_5 - 3A_3'A_5', \tag{4.12}$$

$$a_3 \propto 3\left(A_4' + A_4''\right)A_5 - 6A_4'A_5' + A_4\left(A_5' + A_5''\right), \tag{4.13}$$

$$a_4 \propto 3\left(A_5'\right)^2 - 2\left(A_5' + A_5''\right)A_5, \tag{4.14}$$

where $' := \partial/\partial \ln N$.

In the following, we solve the system of differential equations $a_n = 0$ to obtain the prototype of the extended cuscuton. Since the structure of the system is different for $A_5 = 0$ and $A_5 \neq 0$, we treat these two cases separately. It is worth noting that the coefficients a_n are independent of B_4 and B_5. This in particular means that no restrictions on B_4 and B_5 can be imposed from the analysis of the cosmological setup. It should also be noted that the condition $a_n = 0$ is a sufficient but not a necessary condition for $\mathcal{G}_S = 0$: There is still a possibility that $\mathcal{G}_S$ vanishes after imposing the Hamiltonian constraint for the background. This is indeed the case in theories generated from the original cuscuton theory via generic disformal transformation (see §4.4.1).

We first focus on the case $A_5 = 0$ (and $A_4 \neq 0$). In this case, $a_4 = 0$ and $a_3 = 0$ are automatically satisfied. From $a_2 = 0$, we obtain

$$A_4 = -\frac{v_4 N}{N + u_4}, \tag{4.15}$$

with u_4 and v_4 being arbitrary functions of t. Hereafter, we assume $v_4 \neq 0$ so that $A_4 \neq 0$. Then, $a_1 = 0$ yields

$$A_3 = u_3 + \frac{v_3}{N + u_4}, \tag{4.16}$$

and $a_0 = 0$ can be solved to give

$$A_2 = u_2 + \frac{v_2}{N} - \frac{3v_3^2}{8v_4 N(N + u_4)}, \tag{4.17}$$

where u_2, u_3, v_2, and v_3 are arbitrary functions of t. Since u_3 in Eq. (4.16) can be absorbed into v_2 through integration by parts (see the form of the Lagrangian (2.52)), we take $u_3 = 0$ from the beginning. Thus, we have obtained for the $A_5 = 0$ case,

$$A_5 = 0, \quad A_4 = -\frac{v_4 N}{N + u_4}, \quad A_3 = \frac{v_3}{N + u_4}, \quad A_2 = u_2 + \frac{v_2}{N} - \frac{3v_3^2}{8v_4 N(N + u_4)}. \tag{4.18}$$

Next, in $A_5 \neq 0$ case, $a_4 = 0$ leads to the following solution for A_5:

$$A_5 = \frac{\pm N^2}{(\mu_5 N + \nu_5)^2}, \tag{4.19}$$

with μ_5 and ν_5 being arbitrary functions of t that do not vanish simultaneously. Throughout this section, double signs are in the same order. One can then successively solve $a_3 = 0$, $a_2 = 0$, and $a_1 = 0$ to obtain

$$\begin{aligned} A_4 &= \frac{N(\mu_4 N + \nu_4)}{(\mu_5 N + \nu_5)^2}, \\ A_3 &= \mu_3 + \frac{\nu_3}{\mu_5 N + \nu_5} \pm \frac{2(\mu_4 N + \nu_4)^2}{3(\mu_5 N + \nu_5)^2}, \\ A_2 &= \mu_2 + \frac{\nu_2}{N} \pm \frac{\nu_3(\mu_4 N + \nu_4)}{N(\mu_5 N + \nu_5)} + \frac{2(\mu_4 N + \nu_4)^3}{9N(\mu_5 N + \nu_5)^2}, \end{aligned} \tag{4.20}$$

where $\mu_2, \mu_3, \mu_4, \nu_2, \nu_3$, and ν_4 are arbitrary functions of t. Finally, $\nu_3 = 0$ is imposed from $a_0 = 0$, so that we now have

$$\begin{aligned} A_5 &= \frac{\pm N^2}{(\mu_5 N + \nu_5)^2}, \quad A_4 = \frac{N(\mu_4 N + \nu_4)}{(\mu_5 N + \nu_5)^2}, \\ A_3 &= \mu_3 \pm \frac{2(\mu_4 N + \nu_4)^2}{3(\mu_5 N + \nu_5)^2}, \quad A_2 = \mu_2 + \frac{\nu_2}{N} + \frac{2(\mu_4 N + \nu_4)^3}{9N(\mu_5 N + \nu_5)^2}. \end{aligned} \tag{4.21}$$

Here, μ_3 can be absorbed into ν_2, but we avoid doing so for later convenience. Note that one can take a smooth limit $\mu_5 \to 0$ or $\nu_5 \to 0$ in Eq. (4.21). It should also be noted that the result of the case with $A_5 = 0$ can be reproduced by choosing the integration functions as

$$\begin{aligned} &\nu_5 = u_4 \mu_5, \quad \mu_4 = -v_4 \mu_5^2, \quad \nu_4 = \mp \frac{3v_3}{4v_4} - u_4 v_4 \mu_5^2, \quad \mu_3 = \mp \frac{2}{3} v_4^2 \mu_5^2, \\ &\mu_2 = u_2 + \frac{2}{9} v_4^3 \mu_5^4, \quad nu_2 = v_2 \pm \frac{1}{2} v_3 v_4 \mu_5^2 + \frac{2}{9} u_4 v_4^3 \mu_5^4, \end{aligned} \tag{4.22}$$

and then taking the limit $\mu_5 \to \infty$.

4.1.2 Prototype in Non-Flat Cosmology

We have determined a prototype for the extended cuscuton in a flat cosmological background, and we have found that one cannot determine the form of B_4 and B_5 by this approach. In what follows, we show that one can fix the form of B_4 and B_5 by considering a non-flat cosmological background.

For a non-flat cosmological background with

$$ds^2 = -N^2 dt^2 + a^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right], \tag{4.23}$$

with K being the spatial curvature. The two dynamical equations take the same form as Eqs. (4.1) and (4.2),

$$\mathcal{E}_a = 2\mathcal{G}_T \dot{H} - 2\mathcal{M}\ddot{\phi} + \cdots = 0, \tag{4.24}$$
$$\mathcal{E}_\phi = 6\mathcal{M}\dot{H} + \mathcal{K}\ddot{\phi} + \cdots = 0, \tag{4.25}$$

but now with

$$\begin{gathered} \mathcal{G}_T = \mathcal{G}_{T\,\mathrm{flat}}, \quad \mathcal{M} = \mathcal{M}_{\mathrm{flat}} + XG_{5X}\frac{K}{a^2}, \\ \mathcal{K} = \mathcal{K}_{\mathrm{flat}} + 6\left[G_{4X} + 2XG_{4X} - G_{5\phi} - XG_{5\phi X} + H\dot{\phi}(G_{5X} + XG_{5XX})\right]\frac{K}{a^2}, \end{gathered} \tag{4.26}$$

where the quantities labeled by “flat” represent the corresponding ones in the flat case found in Eq. (4.3). This leads to

$$\mathcal{G}_T\mathcal{K} + 6\mathcal{M}^2 = \sum_{n=0}^{4} a_n H^n + a_5\frac{K}{a^2} + a_6\frac{K^2}{a^4} + a_7 H\frac{K}{a^2} + a_8 H^2\frac{K}{a^2}, \tag{4.27}$$

where the coefficients of the four additional terms must vanish.

Switching from the (G_i, F_j) representation to the (A_i, B_j) representation, first we see that

$$a_6 = 6(XG_{5X})^2 \propto (B_{5N})^2 = 0. \tag{4.28}$$

Substituting this to the other coefficients, we obtain

$$a_5 \propto A_4(NB_4)_{NN}, \quad a_7 \propto A_5(NB_4)_{NN}, \quad a_8 = 0. \tag{4.29}$$

We thus arrive at

$$(NB_4)_{NN} = 0, \quad B_{5N} = 0 \qquad \Rightarrow \qquad B_4 = b_0(t) + \frac{b_1(t)}{N}, \quad B_5 = 0. \tag{4.30}$$

It is not until one consider a non-flat cosmological background that one obtain these new conditions for B_4 and B_5.

4.2 Extended Cuscuton from Hamiltonian Analysis

Having constructed the cosmological prototype of the extended cuscuton, we now perform its Hamiltonian analysis to identify the theories having two DOFs in the unitary gauge without any assumption on the underlying spacetime. The method of the Hamiltonian analysis in the constrained systems are in [124].

4.2.1 General Discussion

Before proceeding to the Hamiltonian analysis of the cosmological prototype of the extended cuscuton, we derive a (sufficient) condition for a theory written in the ADM language to have DOFs less than three. We start from a general ADM action of the form

$$S = \int dtd^3x\, N\sqrt{\gamma}\left[L(t, N, \gamma_{ij}, R_{ij}, Q_{ij}) + v^{ij}(Q_{ij} - K_{ij})\right], \tag{4.31}$$

respecting the three-dimensional spatial diffeomorphism invariance, and explore the condition for L to yield two DOFs. Since K_{ij} contains $\dot{\gamma}_{ij}$, N, and N^i, the original Lagrangian density $L(t, N, \gamma_{ij}, R_{ij}, K_{ij})$ will make the following analysis complicated. To bypass this trouble, here we have introduced Lagrange multipliers v^{ij} to replace K_{ij} in L by auxiliary variables Q_{ij}. This is thought of as the ADM expression of general scalar-tensor theories in the unitary gauge. Note that some DHOST theories yield the velocity of the lapse function $\dot{N}$ [125], which is beyond the scope of this paper. We shall revisit later in §4.2.3 the Hamiltonian structure when L is at most quadratic in K_{ij}, which is the case for the extended cuscuton theory with $A_5 = 0$.

Switched to the Hamiltonian formalism, this theory has 44 canonical variables:

$$\begin{pmatrix} N, & N^i, & \gamma_{ij}, & Q_{ij}, & v^{ij} \\ \pi_N, & \pi_i, & \pi^{ij}, & P^{ij}, & U_{ij} \end{pmatrix}, \tag{4.32}$$

where the lower variables are the conjugate momenta as follows:

$$\pi_N := \frac{\partial L}{\partial \dot{N}}, \quad \pi_i := \frac{\partial L}{\partial \dot{N}^i}, \quad \pi^{ij} := \frac{\partial L}{\partial \dot{\gamma}_{ij}}, \quad P^{ij} := \frac{\partial L}{\partial \dot{Q}_{ij}}, \quad U_{ij} := \frac{\partial L}{\partial \dot{v}^{ij}}. \tag{4.33}$$

From the action (4.31), we obtain the primary constraints as

$$\pi_N \approx 0, \quad \pi_i \approx 0, \quad P^{ij} \approx 0, \quad U_{ij} \approx 0, \quad \Psi^{ij} := \pi^{ij} + \frac{\sqrt{\gamma}}{2}v^{ij} \approx 0, \tag{4.34}$$

here the symbol $\approx$ represents weak equalities, i.e., equalities that hold on the surface of the constraints. We will use the following notations for derivatives of L with respect to Q_{ij}:

$$L_Q^{ij} := \frac{\partial L}{\partial Q_{ij}}, \quad L_{QQ}^{ij,kl} := \frac{\partial^2 L}{\partial Q_{ij}\partial Q_{kl}}. \tag{4.35}$$

The canonical Hamiltonian can be obtained by the Legendre transformation as

$$H = \int d^3x \left(N\mathcal{H}_0 + N^i\gamma_i\right), \tag{4.36}$$

with

$$\mathcal{H}_0 := -\sqrt{\gamma}L(t, N, \gamma_{ij}, R_{ij}, Q_{ij}) + 2\pi^{ij}Q_{ij}, \quad \mathcal{H}_i := -2\sqrt{\gamma}D^j\left(\frac{\pi_{ij}}{\sqrt{\gamma}}\right), \tag{4.37}$$

The total Hamiltonian is written as

$$H_T = H + \int d^3x(\lambda_N\pi_N + \lambda^i\pi_i + \chi_{ij}P^{ij} + \varphi^{ij}U_{ij} + \lambda_{ij}\Psi^{ij}), \tag{4.38}$$

with the coefficients of the primary constraints being spatial functions. Time evolutions in the Hamiltonian formalism is generated by the total Hamiltonian.

Some of the consistency relations for the primary constraints produce the following secondary constraints:

$$\begin{aligned} \dot{\pi}_N &\approx \sqrt{\gamma}(NL)_N - 2\pi^{ij}Q_{ij} =: C \approx 0, \\ \dot{\pi}_i &\approx -\mathcal{H}_i \approx 0, \\ \dot{P}^{ij} &\approx N(\sqrt{\gamma}L_Q^{ij} - 2\pi^{ij}) =: N\Pi^{ij} \approx 0, \end{aligned} \tag{4.39}$$

while $\dot{U}_{ij} \approx 0$ and $\dot{\Psi}^{ij} \approx 0$ just fix the multipliers λ_{ij} and φ^{ij}, respectively. The consistency relation from the time evolution of the secondary constraint $\mathcal{H}_i \approx 0$, i.e., $\dot{\mathcal{H}}_i \approx 0$, is automatically satisfied on the constraint surface. Among the constraints derived so far, $\pi_i \approx 0$ is first class, which reflects the fact that one can freely specify the shift vector. The momentum constraint $\mathcal{H}_i \approx 0$ can be promoted to a first-class constraint by adding appropriate terms that vanish weakly, i.e.,

$$\mathcal{H}_i \quad \to \quad \bar{\mathcal{H}}_i := \mathcal{H}_i + \pi_N D_i N + P^{jk}D_iQ_{jk} - 2\sqrt{\gamma}D_j\left(\frac{P^{jk}}{\sqrt{\gamma}}Q_{ik}\right), \tag{4.40}$$

so that $\bar{\mathcal{H}}_i$ defines the generator of spatial diffeomorphisms for γ_{ij}, N, and Q_{ij}.

Now we proceed to the consistency relations for $C \approx 0$ and $\Pi^{ij} \approx 0$. One finds

$$\dot{C} \approx \{C, H\}_{\mathrm{P}} + \sqrt{\gamma}\left[\lambda_N(NL)_{NN} + \chi_{kl}NL_{QN}^{kl}\right] \approx 0, \tag{4.41}$$

$$\dot{\Pi}^{ij} \approx \left\{\Pi^{ij}, H\right\}_{\mathrm{P}} + \sqrt{\gamma}\left[\lambda_N L_{QN}^{ij} + \chi_{kl}L_{QQ}^{ij,kl}\right] \approx 0, \tag{4.42}$$

with $\{\}_{\mathrm{P}}$ being the Poisson bracket. Therefore, if the matrix

$$M := \begin{pmatrix} (NL)_{NN} & NL_{QN}^{kl} \\ L_{QN}^{ij} & L_{QQ}^{ij,kl} \end{pmatrix} \tag{4.43}$$

has a nonvanishing determinant, the above consistency relations fix λ_N and χ_{kl}, and the Poisson algebra closes here. If this is the case, we would have 6 first-class and 26 second-class constraints, resulting in three DOFs. Hence, we require

$$\det M = \left(\det L_{QQ}^{ij,kl}\right)\left[(NL)_{NN} - NL_{QN}^{ij}(L_{QQ}^{-1})_{ij,kl}L_{QN}^{kl}\right] = 0 \tag{4.44}$$

$$\Rightarrow \quad \Delta := (NL)_{NN} - NL_{QN}^{ij}(L_{QQ}^{-1})_{ij,kl}L_{QN}^{kl} = 0, \tag{4.45}$$

so that the theory (4.31) has DOFs less than three. Here, we have assumed $\det L_{QQ}^{ij,kl} \neq 0$ to guarantee the existence of two propagating tensor DOFs. Note that this requirement might be too strong for the absence of the third DOF, because it should be sufficient that $\det M$ vanishes only weakly, $\det M \approx 0$ (see §4.4.1). Nevertheless, in this thesis, we demand the presumably stronger condition (4.44) for simplicity. Then, combining Eqs. (4.41) and (4.42) we obtain the tertiary constraint

$$\Xi := \{C, H\}_{\mathrm{P}} - N\left\{\Pi^{ij}, H\right\}_{\mathrm{P}}(L_{QQ}^{-1})_{ij,kl}L_{QN}^{kl} \approx 0. \tag{4.46}$$

The manipulations required hereafter are too involved to carry out, but we can estimate the upper limit for the number of dynamical DOFs by a naive analysis. The time evolution of the tertiary constraint will produce the quaternary constraint: $\dot{\Xi} \approx 0 \Rightarrow \Phi \approx 0$, because otherwise the number of phase-space dimensions would be odd and the theory would be inconsistent. Finally, the consistency relation $\dot{\Phi} \approx 0$ will fix the multiplier λ_N. As we have two more second-class constraints than what we would have in the $\Delta \neq 0$ case, the system has only two physical DOFs. [15] We note that the above estimation is not confirmed by rigorous calculations, and we leave it a future work.

4.2.2 The form of A_i and B_j

In the previous section, we have obtained the extended cuscuton theory candidate from the cosmological considerations. We now check whether or not the candidate can satisfy the condition (4.45). For theories whose action can be written in the form (2.52), we have

$$\begin{aligned}
(NL)_{NN} &= (NA_2)_{NN} + (NA_3)_{NN}Q + (NA_4)_{NN}\mathcal{Q}_2 + (NA_5)_{NN}\mathcal{Q}_3 + (NB_4)_{NN}{}^{(3)}R \\
&\quad + (NB_5)_{NN}\left({}^{(3)}R^{ij}Q_{ij} - \frac{{}^{(3)}R}{2}Q\right), \\
L_{QN}^{ij} &= (A_{3N} + 2A_{4N}Q + 3A_{5N}\mathcal{Q}_2)\gamma^{ij} - (2A_{4N} + 6A_{5N}Q)Q^{ij} + A_{5N}Q^i_kQ^{kj} \\
&\quad + B_{5N}\left({}^{(3)}R^{ij} - \frac{{}^{(3)}R}{2}\gamma^{ij}\right), \\
L_{QQ}^{ij,kl} &= -(2A_4 + 6A_5Q)\mathcal{G}^{ij,kl} + 6A_5\left(Q^{k(i}\gamma^{j)l} + Q^{l(i}\gamma^{j)k} - Q^{ij}\gamma^{kl} - \gamma^{ij}Q^{kl}\right),
\end{aligned} \tag{4.47}$$

[15] There may be another possibility for the system to have two physical DOFs: If $\Xi \approx 0$ is automatically satisfied by the existing primary/secondary constraints, then $\pi_N \approx 0$ and $C \approx 0$ should be first-class constraints, and thus the number of DOFs is again two. In any case, $\Delta = 0$ is a sufficient condition for the theory to have DOFs less than three.

where $\mathcal{G}^{ij,kl} := \gamma^{k(i}\gamma^{j)l} - \gamma^{ij}\gamma^{kl}$, and

$$\mathcal{Q} := Q^i_i, \quad \mathcal{Q}_2 := \mathcal{Q}^2 - Q^i_j Q^j_i, \quad \mathcal{Q}_3 := \mathcal{Q}^3 - 3\mathcal{Q}Q^i_j Q^j_i + 2Q^i_j Q^j_k Q^k_i. \tag{4.48}$$

The inverse of $L^{ij,kl}_{QQ}$ can be written as

$$\begin{aligned}(L^{-1}_{QQ})_{ij,kl} =& -\frac{1}{2A_4}\left(\gamma_{k(i}\gamma_{j)l} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right) \\ &+ \frac{3A_5}{4A_4^2}\left[\left(2\gamma_{k(i}\gamma_{j)l} - \gamma_{ij}\gamma_{kl}\right)\mathcal{Q} + \gamma_{ij}Q_{kl} + \gamma_{kl}Q_{ij} - 2\gamma_{k(i}Q_{j)l} - 2\gamma_{l(i}Q_{j)k}\right] \\ &+ \cdots,\end{aligned} \tag{4.49}$$

where the ellipsis denotes the terms quadratic and higher in Q_{ij}. Thus, we obtain the equation of the form

$$\begin{aligned}\Delta =& \tilde{c}_0(t,N) + \tilde{c}_1(t,N)\mathcal{Q} + \cdots + \tilde{d}_1(t,N)^{(3)}R + \tilde{d}_2(t,N)\left({}^{(3)}R_{ij}{}^{(3)}R^{ij} - \frac{3}{8}{}^{(3)}R^2\right) \\ &+ \tilde{d}_3(t,N)\mathcal{Q}^{(3)}R + \cdots \\ =& 0,\end{aligned} \tag{4.50}$$

and all the coefficients must vanish. Here, the $\tilde{d}_i$ coefficients contain B_4 and B_5. We see that $\tilde{d}_2 \propto (B_{5N})^2 = 0 \implies B_5 = b_2(t)$. Then, $\tilde{d}_1 \propto (NB_4)_{NN} = 0 \implies B_4 = b_0(t) + b_1(t)/N$. However, b_2 can be absorbed into the redefinition of b_1. We thus get

$$B_4 = b_0(t) + \frac{b_1(t)}{N}, \quad B_5 = 0, \tag{4.51}$$

with b_0 and b_1 being free functions of t, which is identical to the conditions arising from a non-flat cosmological background (4.30). Now B_4 and B_5 are found to be eliminated from Eq. (4.47) and Δ, and hence all the $\tilde{d}_i$ coefficients vanish.

Let us then check that the form of A_i we have found in the previous section is consistent with $\Delta = 0$. First, we take a look at the case with $A_5 = 0$, for which simple explicit expressions of the equations can be obtained. In this case, the inverse of the matrix $L^{ij,kl}_{QQ} = -2A_4\mathcal{G}^{ij,kl}$ is given explicitly by

$$(L^{-1}_{QQ})_{ij,kl} = -\frac{1}{2A_4}\left(\gamma_{k(i}\gamma_{j)l} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right), \tag{4.52}$$

and hence we have

$$\Delta = \frac{4(A_2' + A_2'')A_4 - 3(A_3')^2}{4NA_4} + \frac{(A_3' + A_3'')A_4 - 2A_3'A_4'}{NA_4}\mathcal{Q} + \frac{(A_4' + A_4'')A_4 - 2(A_4')^2}{NA_4}\mathcal{Q}_2, \tag{4.53}$$

where recall that the prime denotes $\partial/\partial \ln N$. As is clear from Eqs. (4.10), (4.11), and (4.12), the three coefficients vanish if and only if $a_0 = a_1 = a_2 = 0$ (with $A_5 = 0$), and therefore $\Delta = 0$ is satisfied for the functions (4.18).

In the $A_5 \neq 0$ case, one cannot express L^{-1}_{QQ} in a closed form, but rather one has an infinite sum of the form (4.49). Then, we obtain Δ as

$$
\begin{aligned}
\Delta = \; & \frac{4(A_2' + A_2'')A_4 - 3(A_3')^2}{4NA_4} + \frac{4(A_3' + A_3'')A_4^2 - 8A_3'A_4A_4' + 3A_3'^2A_5}{4NA_4^2}Q \\
& + \frac{8(A_4' + A_4'')A_4^3 - 16A_4^2(A_4')^2 + 12A_3'A_4(2A_4'A_5 - A_4A_5') - 9A_3'^2A_5^2}{8NA_4^3}Q_2 \\
& + \frac{8(A_5' + A_5'')A_4^4 + 3(3A_3'A_5 - 4A_4A_4')(4A_4^2A_5' - 4A_4A_4'A_5 + 3A_3'A_5^2)}{8NA_4^4}Q_3 \\
& + \left(2A_4^2A_5' - 4A_4A_4'A_5 + 3A_3'A_5^2\right)^2 \tilde{\Delta}_{\geq 4},
\end{aligned}
\tag{4.54}
$$

where $\tilde{\Delta}_{\geq 4}$ denotes higher-order terms of Q_{ij}. It should be noted that this reduces to Eq. (4.53) in the limit $A_5 \to 0$. Although Eq. (4.54) has infinitely many terms for generic choices of the A_i functions, one can check directly that $\Delta = 0$ is satisfied if and only if the A_i functions are given by (4.21).

Thus, we have established that the cosmological prototype constructed in §4.1 can be promoted to a theory with two DOFs in arbitrary spacetime in the unitary gauge, i.e., the extended cuscuton. It turns out that we do not need to impose further constraints on the form of the A_i and B_j functions obtained from the non-flat cosmological analysis. In general, the extended cuscuton theory contains a nonminimal derivative coupling to the curvature. This is the reason why we have worked in the GLPV framework. The Horndeski conditions (2.54) are satisfied if and only if $A_5 = 0$, $u_4 = 0$, and $v_4 = b_0(t)$. Only in this case, the extended cuscuton theory can be described as a particular case of the Horndeski theory.

4.2.3 More on the Hamiltonian analysis in the $A_5 = 0$ case

At the last subsection, we examine the Hamiltonian structure of the extended cuscuton theory with $A_5 = 0$ in more detail to show that (i) the Hamiltonian can be recasted into the form in which the lapse function appears only linearly, as in the theories studied in Ref. [85], via a canonical transformation; and that (ii) this linearity directly appears without any canonical transformations if we do not introduce the auxiliary variables Q_{ij} from the beginning. For the extended cuscuton model with $A_5 = 0$, the explicit form of the Lagrangian is given by

$$
L = u_2 + \frac{v_2}{N} - \frac{3v_3^2}{8v_4N(N+u_4)} + \frac{v_3}{N+u_4}Q + \frac{v_4N}{N+u_4}(Q^i_jQ^j_i - Q^2) + \left(b_0 + \frac{b_1}{N}\right)R, \tag{4.55}
$$

plus the Lagrange multiplier term enforcing $Q_{ij} = K_{ij}$. The subsequent analysis can be done in the same way as in §4.2. Using the notation, the total Hamiltonian is given by

$$
\begin{aligned}
H_T &= H + \int d^3x(\lambda_N\pi_N + \lambda^i\pi_i + \chi_{ij}P^{ij} + \varphi^{ij}U_{ij} + \lambda_{ij}\Psi^{ij}), \\
H &= \int d^3x\left(-N\sqrt{\gamma}L + 2N\pi^{ij}Q_{ij} + 2\pi^{ij}D_iN_j\right).
\end{aligned}
\tag{4.56}
$$

This Hamiltonian depends nontrivially on N. Now we perform the following canonical transformation:

$$\begin{aligned} Q_{ij} &\to \frac{N+u_4}{N}Q_{ij} + \frac{v_3}{4v_4N}\gamma_{ij}, \qquad \gamma_{ij} \to \gamma_{ij}, \\ P^{ij} &\to \frac{N}{N+u_4}P^{ij}, \qquad \pi^{ij} \to \pi^{ij} - \frac{v_3}{4v_4(N+u_4)}P^{ij}. \end{aligned} \tag{4.57}$$

Then, H is transformed to

$$\begin{aligned} H \to \int d^3x \Big\{ &-\sqrt{\gamma}\left[Nu_2 + v_2 + (N+u_4)(Q^i_jQ^j_i - Q^2) + (Nb_0+b_1)R\right] \\ &+ 2\pi^{ij}\left[(N+u_4)Q_{ij} + \frac{v_3}{4v_4}\gamma_{ij}\right] + 2\pi^{ij}D_iN_j\Big\}, \end{aligned} \tag{4.58}$$

where the terms proportional to P_{ij} were absorbed into the redefinition of χ_{ij}. Now we see that the new Hamiltonian depends on N at most linearly. If B_4 satisfies the condition (4.51), this form of the total Hamiltonian form belongs to the model studied in [85].

We have employed auxiliary fields Q_{ij} for convenience of calculations so far. However, if one does not introduce Q_{ij} from the beginning, one can see the linear dependence of the Hamiltonian on N without canonical transformations. Indeed, after straightforward calculations, the total Hamiltonian is obtained as

$$\begin{aligned} H_T &= H + \int d^3x(\lambda_N\pi_N + \lambda^i\pi_i), \\ H &= \int d^3x \left\{\sqrt{\gamma}\left[-Nu_2 - v_2 - (Nb_0+b_1)R + \frac{N+u_4}{2v_4}\frac{2\pi^i_j\pi^j_i - \pi^2}{\gamma} + \frac{v_3}{2v_4}\frac{\pi}{\sqrt{\gamma}}\right]\right. \\ &\qquad \left. + 2\pi^{ij}D_iN_j\right\}, \end{aligned} \tag{4.59}$$

where $\pi := \pi^i_i$, and thus it is found without invoking the canonical transformation that the dependence of H on N is at most linear.

$$H = \int d^3x\left[N\tilde{C}(Q_{ij}, R_{ij}, \gamma_{ij}, \pi^{ij}, t, N) + N^i\mathcal{H}_i + \tilde{G}\right], \tag{4.60}$$

where

$$\tilde{C} := -\sqrt{\gamma}\left[-\frac{1}{v_4\gamma}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} + u_2 + B_4R\right], \tag{4.61}$$

$$\tilde{G} := -\sqrt{\gamma} - \left[\frac{u_2}{v_4\gamma}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} - \frac{v_3}{2v_4\sqrt{\gamma}}\pi^{ij}\gamma_{ij} + v_2\right]. \tag{4.62}$$

Thus, again the Hamiltonian is linear in the lapse function if B_4 satisfies the condition (4.51). This is intrinsically the same result as the case using Q_{ij}, namely if B_4 satisfies the condition (4.51) this theory will be contained in the minimally modified gravity theories.

4.3 Covariantized Form of the Extended Cuscuton

Now we have obtained the action of the extended cuscuton in the ADM form. Then, it is straightforward to recast the theory to a covariant form via Stückelberg trick, though the resultant expression is messy. We present, therefore, the covariantized form of the extended cuscuton model, particularly with $A_5 = 0$. To restore general covariance, we introduce a Stückelberg field ϕ so that its gradient is proportional to the unit normal vector to a constant-time hypersurface: $n_\mu = -\phi_\mu/\sqrt{2X}$ [77,82]. Then, the ingredients of the ADM action can be rewritten in the following way:

$$\begin{aligned} N &\to \frac{1}{\sqrt{2X}}, \quad \gamma_{ij} \to h_{\mu\nu} := g_{\mu\nu} + \frac{1}{2X}\phi_\mu\phi_\nu, \\ K_{ij} &\to \mathcal{K}_{\mu\nu} := h^\lambda_\mu \nabla_\lambda n_\nu, \quad R_{ij} \to h^\alpha_\mu h^\gamma_\nu h^{\beta\delta}\mathcal{R}_{\alpha\beta\gamma\delta} - \mathcal{K}^\alpha_\alpha \mathcal{K}_{\mu\nu} + \mathcal{K}^\alpha_\mu \mathcal{K}_{\alpha\nu}, \end{aligned} \tag{4.63}$$

while the functions of t are replaced with those of ϕ: $u_i(t) \to \tilde{u}_i(\phi)$, $v_i(t) \to \tilde{v}_i(\phi)$, and $b_i(t) \to \tilde{b}_i(\phi)$. The result is given by

$$\begin{aligned} G_2 &= \tilde{u}_2 + \tilde{v}_2\sqrt{2X} - 4\tilde{b}_0''X + 2\tilde{b}_1''(2X)^{3/2} - \frac{\tilde{v}_3 X}{1+\tilde{u}_4\sqrt{2X}}\left(\frac{3\tilde{v}_3}{4\tilde{v}_4} + 2\tilde{u}_4'\sqrt{2X}\right) \\ &\quad + 2\tilde{v}_3' X \log\frac{\sqrt{2X}}{1+\tilde{u}_4\sqrt{2X}} + 2\tilde{b}_0'' X\log X, \\ G_3 &= -4\tilde{b}_1'\sqrt{2X} - \tilde{v}_3\left(\frac{1}{1+\tilde{u}_4\sqrt{2X}} + \log\frac{\sqrt{2X}}{1+\tilde{u}_4\sqrt{2X}}\right) - \tilde{b}_0'\log X, \\ G_4 &= \tilde{b}_0 + \tilde{b}_1\sqrt{2X}, \\ F_4 &= \frac{1}{4X^2}\left(-\tilde{b}_0 + \frac{\tilde{v}_4}{1+\tilde{u}_4\sqrt{2X}}\right), \\ G_5 &= 0, \quad F_5 = 0, \end{aligned} \tag{4.64}$$

where a prime here denotes $\partial/\partial\phi$. One may further add to this any terms that vanish when the unitary gauge is chosen. Note that, in the above expressions, we have assumed that ϕ_μ is timelike because our extended cuscuton was obtained under the unitary gauge $\phi = \phi(t)$. If one makes a replacement $X \to |X|$, one could incorporate a case where ϕ_μ is spacelike, but this is beyond the scope of the present paper.

The case with $A_5 \neq 0$ can be divided into three subtypes: (i) $\mu_5 = 0$ and $\nu_5 \neq 0$; (ii) $\mu_5 \neq 0$ and $\nu_5 = 0$; and (iii) $\mu_5 \neq 0$ and $\nu_5 \neq 0$. One can straightforwardly obtain the full expressions for G_i and F_j in each case, but we do not present them here because the result is too complicated to be illuminating.

4.4 Relations between Other Theories

It is known that a certain class of the GLPV theories can be reproduced from the Horndeski theory by invertible disformal transformations [126, 127]. Given that the original cuscuton is the subclass of the Horndeski, one may expect that the original cuscuton can generate a part or all of the extended cuscuton by disformal transformations. In §4.4.1, we study the behavior of the extended cuscuton theory under disformal transformation and show that a particular subclass with $A_5 = 0$ can be generated from the original cuscuton theory. We also compare our extended cuscuton theory with some other related theories in the literature in §4.4.2.

4.4.1 Disformal Transformations

The original cuscuton model (3.11) can be represented in the language of the GLPV action (2.48) as

$$A_2 = -V(\phi(t)) + \frac{\sigma(t)}{N}, \quad A_4 = -B_4 = -\frac{M_{\rm Pl}^2}{2}, \quad A_3 = A_5 = B_5 = 0, \tag{4.65}$$

with $\sigma(t) := \mu^2|\dot{\phi}(t)|$. Let us consider a (invertible) disformal transformation $g_{\mu\nu} \to \Omega(t)g_{\mu\nu} + \Gamma(t,N)\phi_\mu\phi_\nu$ of the original cuscuton model, with

$$\Omega = \frac{2}{M_{\rm Pl}^2}v_4, \quad \Gamma = -\frac{\Omega u_4}{\dot{\phi}^2}(2N + u_4)\,. \tag{4.66}$$

The above transformation contains two arbitrary functions, u_4 and v_4, of t. Then, the original theory with the coefficients (4.65) is mapped to another GLPV theory with the following coefficients:

$$\begin{gathered} A_5 = 0, \quad A_4 = -\frac{v_4 N}{N + u_4}, \quad A_3 = \frac{v_3}{N + u_4}, \quad A_2 = u_2 + \frac{v_2}{N} - \frac{3v_3^2}{8v_4 N(N + u_4)}, \\ B_5 = 0, \quad B_4 = v_4\left(1 + \frac{u_4}{N}\right), \end{gathered} \tag{4.67}$$

where v_3, u_2, and v_2 are given by

$$v_3 = -2\dot{v}_4, \quad u_2 = -\Omega^2 V, \quad v_2 = \Omega^{3/2}\sigma - \Omega^2 u_4 V. \tag{4.68}$$

These A_i and B_j functions are of the form of (4.18) and (4.30), but the t-dependent functions are subject to (4.68). Therefore, the theory generated from the original cuscuton via the disformal transformation (4.66) resides in a particular subclass of the extended cuscuton theory. The generated theory has two DOFs on any spacetime, which is compatible with the unitary gauge. This result is reasonable as an invertible disformal transformation does not change the number of physical DOFs.

One could perform more general disformal transformations, but then the resultant theories generically lie beyond the current framework in the sense that the condition $\Delta = 0$ for the absence of the third DOF (see §4.2.1) is satisfied only *weakly*. Although it may offer a possible generalization of the present formulation of cuscuton theories retaining two DOFs, we leave it for future study.

4.4.2 Comparison with Other Related Theories

The authors of [108] extended the cuscuton theory to include $G_3(\phi, X)\Box\phi$ to obtain consistently a generalization of the McVittie solution. Their theory is included as a special case in our extended cuscuton, but seemingly they have not addressed the kinetic mixing of gravity and the scalar field or the importance of the property [a′]. Another extension is the "cuscuta-Galileon" proposed in [110]. This model is a subclass of the generalized Galileons in arbitrary dimensions that can avoid caustic singularities. The cuscuta-Galileon is defined only in flat spacetime, so a direct comparison with our extended cuscuton would not be meaningful. Nevertheless, another model was developed in [123] as an extension of the Hořava-Lifshitz theory respecting the power-counting renormalizability. This theory was shown to have two DOFs in the unitary gauge, and it contains terms quadratic or higher in the curvature tensor, which are not incorporated in our extended cuscuton. However, at the same time, many extended cuscuton models do not fall into the theory studied in [123].

Besides the above concrete models, some general classes of two-DOF theories are constructed in different ways than ours. The authors of [85] studied a class of theories depending on the lapse function at most linearly, and derived a condition on the Lagrangian to yield two DOFs (see §2.6). Although this theory generically lies outside our theory, it does not cover the whole of the extended cuscuton since our Lagrangian depends on N nonlinearly. In [87], another general class of scalar-tensor theories with two DOFs was invented by performing a canonical transformation on GR. There should be some relation between this theory and ours, but the comparison would be far from trivial, and thus we leave it for future work.

4.5 Stability in the Presence of Matter

In this section, we discuss the stability of cosmological solutions in the extended cuscuton theory in the presence of a matter field, generalizing the result of [98]. We add a scalar field χ minimally coupled to gravity, whose Lagrangian has the form

$$\mathcal{L}_{\rm m} = P(Y), \quad Y := -\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi. \tag{4.69}$$

For simplicity, we assume that P is a function of Y and does not depend on χ explicitly. Such a scalar field can mimic a barotropic perfect fluid [43]. The energy density, pressure, and sound speed of χ are respectively written as

$$\rho_{\rm m} = 2YP_Y - P, \quad p_{\rm m} = P, \quad c_s^2 = \frac{dp_{\rm m}}{d\rho_{\rm m}} = \frac{P_Y}{P_Y + 2YP_{YY}}. \tag{4.70}$$

Now we consider scalar perturbations around a cosmological background. We choose the unitary gauge for the cuscuton field, $\phi = \phi(t)$, and write each constituent of the metric as

$$N = 1 + \delta N, \quad N_i = \partial_i\psi, \quad \gamma_{ij} = a^2 e^{2\zeta}\left(e^h\right)_{ij} = a^2 e^{2\zeta}\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{kj} + \cdots\right), \tag{4.71}$$

where δN, ψ, and ζ are scalar perturbations and h_{ij} denotes transverse-traceless tensor perturbations. The matter scalar field also fluctuates as $\chi = \chi(t) + \delta\chi(t, \vec{x})$.

The quadratic action for the tensor perturbations h_{ij} is independent of the matter sector, which takes the form

$$S_T^{(2)} = \frac{1}{8}\int dt d^3 x a^3 \left[\mathcal{G}_T \dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2}(\partial_k h_{ij})^2\right], \tag{4.72}$$

where

$$\mathcal{G}_T \coloneqq -2(A_4 + 3HA_5), \quad \mathcal{F}_T \coloneqq 2B_4 + \dot{B}_5. \tag{4.73}$$

Thus, the ghost and gradient instability for the tensor perturbation are absent if $\mathcal{G}_T > 0$ and $\mathcal{F}_T > 0$. The equations are entirely the same as in the GLPV theory and we do not see any cuscuton nature at this point.

The quadratic Lagrangian for the scalar perturbations is $L^{(2)} = a^3\left(\mathcal{L}_H^{(2)} + \mathcal{L}_\chi^{(2)}\right)$ with

$$\mathcal{L}_H^{(2)} = -3\mathcal{G}_T\dot{\zeta}^2 + \frac{\mathcal{F}_T}{a^2}(\partial_k\zeta)^2 + \Sigma\delta N^2 - 2\Theta\delta N\frac{\partial^2\psi}{a^2} + 2\mathcal{G}_T\dot{\zeta}\frac{\partial^2\psi}{a^2} + 6\Theta\delta N\dot{\zeta} - 2\bar{\mathcal{G}}_T\delta N\frac{\partial^2\zeta}{a^2}, \tag{4.74}$$

$$\mathcal{L}_\chi^{(2)} = \frac{P_Y}{c_s^2}\left[-\frac{c_s^2}{2a^2}(\partial_k\delta\chi)^2 + c_s^2\dot{\chi}\frac{\partial^2\psi}{a^2}\delta\chi + Y\delta N^2 - \dot{\chi}\left(\delta N - 3c_s^2\zeta\right)\delta\dot{\chi} + \frac{1}{2}\delta\dot{\chi}^2\right], \tag{4.75}$$

where

$$\begin{aligned}
\bar{\mathcal{G}}_T \coloneqq& 2(B_4 + B_{4N}) - HB_{5N}, \\
\Sigma \coloneqq& A_{2N} + \frac{1}{2}A_{2NN} + \frac{3}{2}HA_{3NN} + 3H^2\left(2A_4 - 2A_{4N} + A_{4NN}\right) \\
&+ 3H^3\left(6A_5 - 4A_{5N} + A_{5NN}\right), \\
\Theta \coloneqq& \frac{1}{2}A_{3N} - 2H\left(A_4 - A_{4N}\right) - 3H^2\left(2A_5 - A_{5N}\right).
\end{aligned} \tag{4.76}$$

Note that $\mathcal{G}_S$ in Eq. (4.8) can be written as $\mathcal{G}_S = (\mathcal{G}_T/\Theta^2)(\Sigma\mathcal{G}_T + 3\Theta^2)$, so the condition $\mathcal{G}_S = 0$, which any cuscuton theory must satisfy (see §4.1), implies

$$\Sigma\mathcal{G}_T + 3\Theta^2 = 0. \tag{4.77}$$

Variations of $L^{(2)}$ with respect to the auxiliary variables δN and ψ yield

$$\left(\Sigma + \frac{YP_Y}{c_s^2}\right)\delta N - \Theta\frac{\partial^2\psi}{a^2} + 3\Theta\dot{\zeta} - \mathcal{G}_T\frac{\partial^2\zeta}{a^2} - \frac{\dot{\chi}P_Y}{c_s^2}\delta\dot{\chi} = 0, \tag{4.78}$$

$$\Theta\delta N - \mathcal{G}_T\dot{\zeta} - \frac{1}{2}\dot{\chi}P_Y\delta\chi = 0, \tag{4.79}$$

by which we can eliminate δN and ψ from $L^{(2)}$:

$$L^{(2)} = a^3 \left[\frac{\mathcal{G}_T^2 Y P_Y}{c_s^2 \Theta^2} \left(\dot{\zeta} - \frac{\Theta}{\mathcal{G}_T} \frac{\dot{\delta\chi}}{\dot{\chi}} \right)^2 - \frac{2(YP_Y)^2}{c_s^2\Theta} \frac{\dot{\delta\chi}\delta\chi}{\dot{\chi}^2} \right. \\ + \left(\Sigma + \frac{YP_Y}{c_s^2} \right) \frac{YP_Y}{\Theta^2} \left(2\mathcal{G}_T \dot{\zeta} \frac{\delta\chi}{\dot{\chi}} + YP_Y \frac{\delta\chi^2}{\dot{\chi}^2} \right) \\ \left. - \frac{\mathcal{F}_S}{a^2} (\partial_k \zeta)^2 + 2\bar{\mathcal{G}}_T \frac{YP_Y}{\Theta} \frac{\partial_k \zeta \partial_k \delta\chi}{a^2 \dot{\chi}} - \frac{YP_Y}{a^2} \frac{(\partial_k \delta\chi)^2}{\dot{\chi}^2} \right], \tag{4.80}$$

where we have defined

$$\mathcal{F}_S := \frac{1}{a}\frac{d}{dt}\left(\frac{a}{\Theta} \mathcal{G}_T \bar{\mathcal{G}}_T \right) - \mathcal{F}_T, \tag{4.81}$$

and used the background EOM for χ, $\ddot{\chi} + 3c_s^2 H \dot{\chi} = 0$. One can remove the kinetic term for $\delta\chi$ by making the field redefinition

$$\tilde{\zeta} := \zeta - \frac{\Theta}{\mathcal{G}_T} \frac{\delta\chi}{\dot{\chi}}. \tag{4.82}$$

Then, $\delta\chi$ becomes an auxiliary variable and thus can be eliminated by using its EOM. After tedious but straightforward manipulations, we finally arrive at

$$L^{(2)} = a^3 \left[\mathcal{A}(t, \partial^2) \dot{\tilde{\zeta}}^2 - \mathcal{B}(t, \partial^2) \frac{(\partial_k \tilde{\zeta})^2}{a^2} \right], \tag{4.83}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given respectively by

$$\mathcal{A} = \frac{\mathcal{G}_T^2 Y P_Y}{c_s^2 \Theta^2} \frac{\partial^2/a^2 - \alpha_1}{\partial^2/a^2 - \alpha_2}, \quad \mathcal{B} = \Upsilon \frac{\mathcal{G}_T^2 Y P_Y}{\Theta^2} \frac{\partial^4/a^4 - \beta_1 \partial^2/a^2 + \beta_2}{(\partial^2/a^2 - \alpha_2)^2}. \tag{4.84}$$

Here, we have defined

$$\begin{aligned} \alpha_1 &:= \frac{3}{f}\alpha_2, \quad \alpha_2 := -\frac{\bar{\Upsilon}^2 c_s^2 \Theta^2 Y P_Y}{\mathcal{F}_S \Theta^2 - \Upsilon \mathcal{G}_T^2 Y P_Y} f(f-3), \\ \beta_1 &:= \alpha_2 \left(1 + \frac{\mathcal{F}_S \Theta^2}{\Upsilon \mathcal{G}_T^2 Y P_Y} \right) - \frac{\Theta^2}{a^3 \Upsilon \mathcal{G}_T^2 Y P_Y} \frac{d}{dt} \left[\frac{a^3 \bar{\Upsilon} \mathcal{G}_T Y P_Y}{\Theta} (f-3) \right], \\ \beta_2 &:= \frac{\Theta^2 \alpha_2^2}{\Upsilon \mathcal{G}_T^2 Y P_Y} \left\{ \mathcal{F}_S - \frac{1}{a}\frac{d}{dt} \left[\frac{a \bar{\Upsilon} \mathcal{G}_T Y P_Y}{\Theta \alpha_2} (f-3) \right] \right\}, \end{aligned} \tag{4.85}$$

with

$$\begin{aligned} \Upsilon &:= \frac{\mathcal{F}_S \Theta^2 - \bar{\mathcal{G}}_T^2 Y P_Y}{\mathcal{F}_S \Theta^2 - \mathcal{G}_T (2\bar{\mathcal{G}}_T - \mathcal{G}_T) Y P_Y}, \\ \bar{\Upsilon} &:= \frac{\mathcal{F}_S \Theta^2 - \mathcal{G}_T \bar{\mathcal{G}}_T Y P_Y}{\mathcal{F}_S \Theta^2 - \mathcal{G}_T (2\bar{\mathcal{G}}_T - \mathcal{G}_T) Y P_Y}, \\ f &:= \frac{\mathcal{G}_T Y P_Y}{c_s^2 \Theta^2} - \frac{1}{c_s^2}\frac{d}{dt}\left(\frac{\mathcal{G}_T}{\Theta} \right) + \frac{3\mathcal{G}_T H}{\Theta}. \end{aligned} \tag{4.86}$$

Thus, we have a single scalar DOF associated with the matter field. Interestingly, the quadratic action is of a nonlocal form and as a result the dispersion relation is nonstandard. This means that the nature of scalar cosmological perturbations is different from that in GR in the presence of a perfect fluid. In other words, gravity is indeed modified in the cuscuton theory. Note in passing that under the Horndeski tuning (2.54), $\mathcal{G}_T$ and $\bar{\mathcal{G}}_T$ coincide, and hence $\Upsilon = \bar{\Upsilon} = 1$.

It follows that as long as

$$\rho_{\mathrm{m}} + p_{\mathrm{m}} = 2YP_Y > 0, \quad c_s^2 > 0, \quad \Upsilon > 0, \tag{4.87}$$

are satisfied, scalar perturbations are stable in the UV regime. Previously said in §3.3.1, both ghost/gradient instabilities are not necessarily problematic in the IR regime given the magnitudes of their energy/time scale. Note that the first two conditions are related only to the matter field, stating that χ must be "usual" matter in the sense that it satisfies the null energy condition and has a positive sound speed squared. However, the last condition, $\Upsilon > 0$, depends on the concrete form of the cuscuton Lagrangian as well as the matter field, and hence is nontrivial.

Chapter 5

Extended Cuscuton: Dark Energy

For models of the late-time cosmic acceleration, GR with the cosmological constant has been an appealing candidate due to its simplicity and consistency with any cosmological observations so far. In order to test this paradigm, it is helpful to compare the ΛCDM model with alternative ones, i.e., dark energy/MG models. The two-DOFs scalar-tensor theories can be regarded as minimal modifications of GR, providing the second most economical explanation of the accelerated expansion next to the cosmological constant. Indeed, the authors of [88] have shown that the model proposed in [87] can explain dark energy. This model has been obtained by performing a canonical transformation on GR, utilizing the idea that a canonical transformation preserves the number of physical DOFs [52, 53]. Along this line, we in this chapter aim to investigate its cosmological aspects as to whether the extended cuscuton can account for the current accelerated expansion of the universe. If a solution of the extended cuscuton mimics the cosmological background evolution in the ΛCDM model, one can consider this theory as a candidate for the viable dark energy models. One should also need the observational consistency for the matter density fluctuation, but once our model fulfills these constraints, deviations from the ΛCDM model will be useful to test these models' validity.

The rest of this chapter is organized as follows. In §5.1, we present the models on which we focus. Then, in §5.2, we study cosmology in these models with a matter field. We derive the background field equations and the quadratic action for scalar perturbations, particularly the density fluctuation of matter, to investigate the effective gravitational coupling and the Newton's constant measured in the local scale. Also, we propose some requirements for the extended cuscutons to be a viable dark energy model. In §5.3, we focus on an analytically solvable case and obtain the criteria for the model to satisfy the above requirements. We find that this model can mimic the cosmological background evolution in the ΛCDM model, though the density fluctuations evolution deviates from the one in the ΛCDM model.

5.1 The Model

From the simultaneous detection of the gravitational waves and the γ-ray burst from a binary neutron star merger, GW180817/GRB170817A [128–131], the deviation of $c_{\rm GW}$ from the speed of light ($c_{\rm light} \coloneqq 1$) is strongly constrained at the low-redshift universe:

$$|c_{\rm GW} - 1| \lesssim 10^{-15} \quad \text{for} \quad z \lesssim 0.01. \tag{5.1}$$

Now we emphasize that $c_{\rm GW}$ can deviate from unity in the early universe. Furthermore, according to Ref. [132], the energy scale observed by LIGO lies close to the cutoff scale of many dark energy models. In other words, dark energy models whose cutoff scale is lower than the observed energy scale do not have to fulfill the above constraint. From this point of view, the constraint (5.1) might be too intense for our model, but partly for simplicity, we focus on the GLPV subclass having $c_{\rm GW} = 1$.

In the GLPV theory satisfying this condition irrespective of the background spacetime, the functions G_4, G_5, F_4, and F_5 in Eq. (2.48) must obey [133–135]

$$F_4 = -8\frac{G_{4X}}{X}, \quad G_5 = F_5 = 0. \tag{5.2}$$

Let us apply this requirement to the extended cuscuton Lagrangian. Since $G_5 = F_5 = 0$ implies $A_5 = B_5 = 0$, we employ the $A_5 = 0$ case with Eq. (4.51). Then, imposing the condition $F_4 = -8G_{4X}/X$ we obtain

$$A_2 = u_2 + \frac{v_2}{N} - \frac{3v_3^2}{8v_4N^2}, \quad A_3 = \frac{v_3}{N}, \quad A_4 = -B_4 = -v_4, \tag{5.3}$$

in the ADM representation, which is translated to the covariant form (2.48) with

$$\begin{aligned} G_2 &= \tilde{u}_2 + \tilde{v}_2\sqrt{2X} - \left(2\tilde{v}_3' + 4\tilde{v}_4'' + \frac{3\tilde{v}_3^2}{4\tilde{v}_4}\right) X + \left(\tilde{v}_3' + 2\tilde{v}_4''\right) X\log X, \\ G_3 &= -\left(\frac{\tilde{v}_3}{2} + \tilde{v}_4'\right)\log X, \quad G_4 = \tilde{v}_4, \quad G_5 = F_4 = F_5 = 0, \end{aligned} \tag{5.4}$$

where $\tilde{u}_2$, $\tilde{v}_2$, $\tilde{v}_3$, and $\tilde{v}_4$ are arbitrary functions of ϕ. Hereafter, we omit tildes of $\tilde{u}_i, \tilde{v}_i$ since this will not confuse us. Interestingly, this model is conformally equivalent to the one studied in Ref. [113]. Note also that the original cuscuton model (3.11) is recovered by $v_3 = 0$ and $v_4 = M_{\rm Pl}^2/2$. And note in passing that the GLPV theory with $G_4 = G_4(\phi)$, $G_5 = F_4 = F_5 = 0$ satisfies $c_{\rm GW} = 1$ without "cuscuton tuning" of G_2, G_3 in (5.4), since it is conformally equivalent to the Einstein-Hilbert action with a scalar field in the form of kinetic gravity braiding [44] (see also [45]).

5.2 Cosmology

5.2.1 Background

We study a homogeneous and isotropic universe (3.25) in the presence of a matter field χ, and consider the following action with (5.4) and (4.69):

$$S = \int d^4x\sqrt{-g}\left[G_2(\phi, X) + G_3(\phi, X)\Box\phi + G_4(\phi)R + P(Y)\right]. \tag{5.5}$$

In this chapter, we consider the contributions of the cuscuton, instead of the cosmological constant Λ, as the dark energy which gives rise to the current cosmic acceleration. The matter Lagrangian $P(Y)$ mimics a barotropic perfect fluid, and its energy density, pressure, and squared sound speed are given by (4.70).

We substitute the ansatz (3.25) into the action (5.5), and derive the field equations for N, a, ϕ, and χ. Among these EOMs, only three of the four equations are independent, and the EOM for N cannot be reproduced from the other ones. Therefore, one may set $N = 1$ only after deriving the EOM for N [118] and then we focus on those for N, a, and ϕ. We also note that the dust limit $p_{\rm m} = 0$ and $c_{\rm s} = 0$, which is introduced in late-time cosmology where only the dust component is essential, is now well-defined. One may naively think that this dust limit is ill-defined in (4.69) since $p_{\rm m} \to 0$ implies that $P(Y)$ in the action goes to zero. Nevertheless, once we rewrite every P and its derivative in terms of $\rho_{\rm m}$, $p_{\rm m}$, and $c_{\rm s}$, we can safely take the dust limit [136]. In deriving the field equations, we assume $\dot\phi > 0$ to fix the sign of the terms originating from the $\sqrt{2X}$ term in the action. It is possible to assume $\dot\phi < 0$ instead, and in that case, one should replace $v_2 \to -v_2$ in the following analysis.

The equations for N, a, and ϕ read, respectively,

$$\mathcal{E}_N := 6v_4H^2 + u_2 - 3v_3H\dot\phi + \frac{3v_3^2}{8v_4}\dot\phi^2 - \rho_{\rm m} = 0, \tag{5.6}$$

$$\mathcal{E}_a := 2v_4(3H^2 + 2\dot H) + u_2 + v_2\dot\phi + 4v_{4\phi}H\dot\phi - \frac{3v_3^2}{8v_4}\dot\phi^2 - v_{3\phi}\dot\phi^2 - v_3\ddot\phi + p_{\rm m} = 0, \tag{5.7}$$

$$\begin{aligned}\mathcal{E}_\phi := &-\frac{3v_3^2}{4v_4}\ddot\phi + 3v_3\dot H - \frac{9v_3^2}{4v_4}H\dot\phi - \frac{3v_3(2v_{3\phi}v_4 - v_3v_{4\phi})}{8v_4^2}\dot\phi^2 - u_{2\phi} \\ &+ 3v_2H + 3H^2(3v_3 + 2v_{4\phi}) = 0.\end{aligned} \tag{5.8}$$

Taking a linear combination $4v_4\mathcal{E}_\phi - 3v_3\mathcal{E}_a$, one can simultaneously remove $\dot H$ and $\ddot\phi$ and obtain a constraint equation, which degeneracy is a essential property of the extended cuscuton theories as said in the previous section. Note that, when $v_3 = 0$, there is no $\dot H$ or $\ddot\phi$ in $\mathcal{E}_\phi$ from the beginning. Also, one can get the continuity equation $\dot\rho_{\rm m} + 3H(\rho_{\rm m} + p_{\rm m}) = 0$ by combining the EOMs (5.6), (5.7), and (5.8).

Here we comment on the equation of state (EOS) parameter $w_{\rm DE}$. We can define the dark energy component by regarding the equations $\mathcal{E}_N = 0$ and $\mathcal{E}_a = 0$ as

$$\mathcal{E}_N = 3G_4(\phi_0)H^2 - \rho_t = 0, \quad \mathcal{E}_a = G_4(\phi_0)(3H^2 + 2\dot H) + p_t = 0, \tag{5.9}$$

where ϕ_0 is the present value of ϕ, and ρ_t and p_t denotes the total energy density and pressure

$$\rho_t = \rho_{\rm m} + \rho_{\rm DE}, \quad p_t = p_{\rm m} + p_{\rm DE}. \tag{5.10}$$

In this case, we can obtain $w_{\rm DE}$ as follows:

$$w_{\rm DE} := \frac{p_{\rm DE}}{\rho_{\rm DE}} = \frac{\mathcal{E}_a - p_{\rm m} - G_4(\phi_0)(3H^2 + 2\dot{H})}{-\mathcal{E}_N - \rho_{\rm m} + G_4(\phi_0)H^2}. \tag{5.11}$$

However, $w_{\rm DE}$ is not an observable, and its restrictions from the CMB data depend on the underlying ansatz. Furthermore, the definition of $w_{\rm DE}$ is subtle in MG, and one may define it other manners. Therefore, we will focus on the evolutions of the Hubble parameter and the matter density fluctuation.

In what follows, let us discuss some viability requirements for the present framework to serve as a dark energy model. Later in §5.3, these requirements are used to constrain model parameters.

[A] *Asymptotic behavior of the Hubble parameter*
We require the following asymptotic behavior for the Hubble parameter:

$$\begin{cases} H \longrightarrow \text{const} \cdot a^{-3/2} & \text{for } t \longrightarrow t_{\rm i}, \\ H \longrightarrow \text{const} & \text{for } t \longrightarrow \infty, \end{cases} \tag{5.12}$$

so that it behaves as in the matter-dominated universe for $t \longrightarrow t_{\rm i}$ (with $t_{\rm i}$ being some early initial time) and the de Sitter universe for $t \longrightarrow \infty$.

[B] *Accelerating universe at the present time*
Whether the universe is experiencing an accelerated expansion can be judged by looking at the Hubble slow-roll parameter $\epsilon_H := -\dot{H}/H^2$. Since $\ddot{a} \propto 1 - \epsilon_H$, the accelerated (decelerated) expansion corresponds to $\epsilon_H < 1$ ($\epsilon_H > 1$). We require that the current value of ϵ_H should be less than unity.

[C] *Positive $\dot{\phi}$*
Since we assumed $\dot{\phi} > 0$ as mentioned above, we require that $\dot{\phi}$ must remain positive throughout its time evolution.

[D] *Positive nonminimal coupling function*
A negative coupling to the Ricci scalar leads to unstable tensor perturbations. Moreover, it also results in negative Newton's constant, as we shall see in the next section. Therefore, we require that $G_4 = 2v_4(\phi) > 0$.

5.2.2 Scalar Perturbations

We now consider the cosmological scalar perturbations. We have derived the quadratic action for the curvature perturbation ζ to the stability analysis in §4.5. On the other hand, in this subsection, we would instead focus on the matter density fluctuations to examine the evolution equation for these. In any case, one can remove all fluctuations from the action other than only one scalar mode, since the extended cuscuton has just one scalar propagating DOF.

As in §4.5, we consider scalar perturbations around the cosmological background (3.25). One can take the dust limit by $p_{\rm m} \to 0, c_{\rm s} \to 0$, but we keep $p_{\rm m}$ and $c_{\rm s}$ for the moment and take this limit in the final step. We use the perturbed metric (3.38) and perturbed matter field$\chi = \chi(t) + \delta\chi(t, \vec{x})$, and take the unitary gauge for the cuscuton: $\phi = \phi(t)$. The following quantity represents the gauge-invariant density fluctuation of χ:

$$\delta = \frac{\rho_{\rm m} + p_{\rm m}}{\rho_{\rm m} c_{\rm s}^2}\left(\frac{\dot{\delta\chi}}{\dot{\chi}} - \delta N\right) + 3\frac{\rho_{\rm m} + p_{\rm m}}{\rho_{\rm m}}\zeta. \tag{5.13}$$

Below, we organize the Lagrangian. We first recast the real-space Lagrangian into the Fourier-space one. To this end, we perform integration by parts so that each variable has an even number of spatial derivatives and replace $\partial^2 \to -k^2$. We then proceed to reexpress the Lagrangian in terms of δ instead of $\delta\chi$. The Lagrangian contains the following terms associated with $\delta\chi$:

$$\mathcal{L} \supset a^3\left(\frac{\rho_{\rm m} + p_{\rm m}}{4c_{\rm s}^2 Y}\dot{\delta\chi}^2 - \frac{\rho_{\rm m} + p_{\rm m}}{4Y}\frac{k^2}{a^2}\delta\chi^2 + \delta\chi \cdot f(\delta N, \psi, \zeta)\right), \tag{5.14}$$

where $f(\delta N, \psi, \zeta)$ denotes the linear terms in δN, ψ, and ζ. One can add the following term to $\mathcal{L}$ without changing the dynamics:

$$\mathcal{L}_{\delta\chi\to\delta} = -a^3\frac{\rho_{\rm m} + p_{\rm m}}{4c_{\rm s}^2 Y}\left\{\dot{\delta\chi} - \dot{\chi}\left[c_{\rm s}^2\left(\frac{\rho_{\rm m}}{\rho_{\rm m} + p_{\rm m}}\delta - 3\zeta\right) + \delta N\right]\right\}^2, \tag{5.15}$$

because after substituting the solution to the Euler-Lagrange equation for δ, namely (5.13), this Lagrangian vanishes. Note that we have chosen the overall normalization of (5.15) so that $\mathcal{L}' \coloneqq \mathcal{L} + \mathcal{L}_{\delta\chi\to\delta}$ becomes linear in $\dot{\delta\chi}$. Consequently, one can eliminate $\delta\chi$ by use of its Euler-Lagrange equation, and we are left with the quadratic action written in terms of $(\delta N, \psi, \zeta, \delta)$: [16]

$$\begin{aligned}
\mathcal{L}' = a^3\Bigg\{&-6v_4\dot{\zeta}^2 + \left[2v_4\frac{k^2}{a^2} - \frac{9}{2}c_{\rm s}^2(\rho_{\rm m} + p_{\rm m})\right]\zeta^2 - \frac{3\Theta^2}{2v_4}\delta N^2 + 2\Theta\frac{k^2}{a^2}\delta N\psi - 4v_4\frac{k^2}{a^2}\psi\dot{\zeta} \\
&+ 6\Theta\delta N\dot{\zeta} + \left[4v_4\frac{k^2}{a^2} + 3(\rho_{\rm m} + p_{\rm m})\right]\delta N\zeta + \frac{a^2\rho_{\rm m}^2}{2k^2(\rho_{\rm m} + p_{\rm m})}\left[\dot{\delta} + \frac{k^2(\rho_{\rm m} + p_{\rm m})}{a^2\rho_{\rm m}}\psi\right]^2 \\
&- \frac{\rho_{\rm m}}{2(\rho_{\rm m} + p_{\rm m})}\left(\rho_{\rm m}c_{\rm s}^2 + \frac{3a^2}{k^2}\left\{5H^2(\rho_{\rm m}c_{\rm s}^2 - p_{\rm m}) + \frac{d}{dt}\left[(\rho_{\rm m}c_{\rm s}^2 - p_{\rm m})H\right]\right\}\right)\delta^2 \\
&- \rho_{\rm m}\delta N\delta + 3H(\rho_{\rm m}c_{\rm s}^2 - p_{\rm m})\psi\delta + 3\rho_{\rm m}c_{\rm s}^2\zeta\delta\Bigg\},
\end{aligned} \tag{5.16}$$

[16]This Lagrangian modification $\mathcal{L} \to \mathcal{L}'$ might be not valid in general. However, in this case, we can verify that the Lagrangian $\mathcal{L}'$ can reproduce the EOMs derived from the original Lagrangian $\mathcal{L}$.

where

$$\Theta := 2v_4 H - \frac{1}{2} v_3 \dot{\phi}. \tag{5.17}$$

After we eliminate δN and ψ by using their Euler-Lagrange equations, the Lagrangian can be written in the form

$$\mathcal{L}'' = a^3 \left[a_1(t,k)\dot{\delta}^2 + a_2(t,k)\delta^2 + 2a_3(t)\zeta\delta + a_4(t,k)\zeta^2 \right]. \tag{5.18}$$

Finally, by integrating out ζ, we obtain the quadratic action for δ as

$$\mathcal{L}_\delta = a^3 \left(\mathcal{A}\dot{\delta}^2 + \mathcal{B}\delta^2 \right), \tag{5.19}$$

from which we obtain the evolution equation for δ as follows:

$$\ddot{\delta} + \left(3H + \frac{\dot{\mathcal{A}}}{\mathcal{A}} \right) \dot{\delta} - \frac{\mathcal{B}}{\mathcal{A}} \delta = 0. \tag{5.20}$$

In the case of generic scalar-tensor theories where the scalar field is dynamical, we still have an additional dynamical DOF other than δ at this stage. In order to extract the effective dynamics of the density fluctuations on subhorizon scales, one usually makes the quasi-static approximation. In the present case of the extended cuscutons, however, the quadratic action is written solely in terms of the density fluctuations even before taking the subhorizon limit. This is one of the distinct properties of cuscuton-like theories.

In what follows, we consider a dust fluid by taking the limits $p_{\rm m} \to 0$ and $c_{\rm s} \to 0$. [17] Then, the coefficients $\mathcal{A}$ and $\mathcal{B}$ are respectively written as

$$\mathcal{A} = \frac{2v_4 \rho_{\rm m}}{4v_4 + 3(a^2/k^2)\rho_{\rm m}} \frac{a^2}{k^2}, \quad \mathcal{B} = \frac{2\lambda v_4 \rho_{\rm m}^2}{[4v_4 + 3(a^2/k^2)\rho_{\rm m}]^2} \frac{a^2}{k^2}, \tag{5.21}$$

with

$$\lambda := \frac{4v_4 \left[2\left(\dot{v}_4 + v_4 H\right)^2 - v_4 \left(\rho_{\rm m} + 2\dot{\Theta} + 2H\Theta \right) \right] - 3(a^2/k^2)\rho_{\rm m} \left[v_4 \left(\rho_{\rm m} + 2\dot{\Theta} \right) - 2\dot{v}_4 \Theta \right]}{4v_4 \left[\Theta \left(4\dot{v}_4 - \Theta \right) - v_4 \left(\rho_{\rm m} + 2\dot{\Theta} - 2H\Theta \right) \right] - 3(a^2/k^2)\rho_{\rm m} \left[v_4 \left(\rho_{\rm m} + 2\dot{\Theta} \right) - 2\dot{v}_4 \Theta \right]}. \tag{5.22}$$

In the subhorizon limit $k \to \infty$, (5.20) reduces to

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\rm eff} \rho_{\rm m} \delta = 0, \tag{5.23}$$

where we have defined the effective gravitational coupling $G_{\rm eff}$ for the density fluctuations as

$$4\pi G_{\rm eff} := \lim_{k\to\infty} \frac{\mathcal{B}}{\rho_{\rm m}\mathcal{A}} = \frac{1}{4v_4} \left[1 + \frac{\left(2\dot{v}_4 + 2v_4 H - \Theta\right)^2}{\Theta\left(4\dot{v}_4 - \Theta\right) - v_4 \left(\rho_{\rm m} + 2\dot{\Theta} - 2H\Theta \right)} \right]. \tag{5.24}$$

[17] This limiting procedure is justified in [94, 137]. Considering the action for irrotational dust [138] is a simple way as our procedure. However, we have already calculated the quadratic Lagrangian for the scalar perturbations on the same setup as this paper in [26], so we take over this setup. Instead, one may consider the action for a dust fluid from the beginning [138].

The Poisson equations for two gauge-invariant gravitational potentials, $\Psi := \delta N + \dot{\psi}$ and $\Phi := -\zeta - H\psi$, are given by

$$-\frac{k^2}{a^2}\Psi = 4\pi G_{\rm eff}\rho_{\rm m}\delta, \quad -\frac{k^2}{a^2}\Phi = 4\pi \bar{G}_{\rm eff}\rho_{\rm m}\delta, \tag{5.25}$$

with

$$4\pi \bar{G}_{\rm eff} := \frac{1}{4v_4}\left[1 + \frac{(2v_4 H - \Theta)(2\dot{v}_4 + 2v_4 H - \Theta)}{\Theta(4\dot{v}_4 - \Theta) - v_4\left(\rho_{\rm m} + 2\dot{\Theta} - 2H\Theta\right)}\right]. \tag{5.26}$$

Note that, if and only if $\dot{v}_4(2\dot{v}_4 + 2v_4 H - \Theta) = 0$, i.e., $v_{4\phi} = 0$ or $v_3 + 4v_{4\phi} = 0$, we have $G_{\rm eff} = \bar{G}_{\rm eff}$ and then the so-called gravitational slip parameter $\eta := \Psi/\Phi$ is equal to unity as in GR. [18]

It is important to see the difference between the above effective gravitational coupling for linear density fluctuations and the locally measured value of Newton's constant, G_N. The deviations of $G_{\rm eff}$ from G_N at present can change the dynamics of matter on the cosmological scales, i.e., the large scale structure from that in GR. Such effects might be observed by the integrated Sachs-Wolfe effect or weak gravitational lensing. Also, the time derivative of G_N is strongly restricted by local observations, such as the lunar laser ranging [140]. Therefore, $\dot{G}_N$ is one of the essential quantities to judge the validity of the dark energy models.

To evaluate G_N in the extended cuscuton theory, one can follow the discussion for the Vainshtein solution of [141]. Although ϕ is nondynamical in the present setup due to the particular choice of the functions in the action (5.5), this "cuscuton tuning" does not change the procedure to derive a static and spherically symmetric solution in the weak gravity regime. Thus, regardless of whether ϕ is dynamical or not, its nonlinearities play an essential role below a specific scale to reproduce Newtonian gravity, provided that $G_{3X} \neq 0$. [19] It then follows that G_N is given by [141] [20]

$$4\pi G_N = \frac{1}{4v_4}, \tag{5.27}$$

which is different from $G_{\rm eff}$ as long as $v_3 + 4v_{4\phi} \neq 0$. Note that G_N depends on time and is not a constant since $v_4(\phi)$ varies in time.

To sum up, although ϕ is nondynamical in the extended cuscuton theory, the evolution of density fluctuations is modified in the same way as in usual scalar-tensor theories.

---

[18] As was shown in [139], the deviation of the slip parameter from unity is characterized by the functions called $\alpha_{\rm M}$ and $\alpha_{\rm T}$, which are fixed once the arbitrary functions in the action (5.5) are fixed. Specifically, the slip parameter becomes unity if and only if $\alpha_{\rm M} = \alpha_{\rm T} = 0$. On the other hand, for our model satisfying (5.4), we have $\alpha_{\rm T} = 0$ and $\alpha_{\rm M} \propto \dot{G}_4 = v_{4\phi}\dot{\phi} \neq 0$ in general, and thus the slip parameter deviates from unity. Therefore, our result is consistent with the one in [139].

[19] Although it is not directly related to the subject, we mention that the authors of [142] have recently derived the subclass of the Horndeski theories, which does not require the Vainshtein screening mechanism because the scalar DOF is effectively not sourced by matter. Such action is given by $G_3 = f'(\phi)\log X, G_4 = f(\phi), G_5 = 0$, where $f(\phi)$ has no term linear in ϕ. This model is quite similar to our model (5.28) introduced in §5.3 when $f(\phi) = M_*^2/2 + \mu\phi^2$, but the sign of G_3 is different and hence these are not related to each other.

[20] Some assumptions on the size of various coefficients are made in [141]. All these assumptions are valid as well in the extended cuscuton theory if it accounts for the present accelerated expansion of the Universe.

5.3 Exactly Solvable Model

In the previous section, we obtained the background field equations, the effective gravitational coupling $G_{\rm eff}$, and the Newton's constant G_N for generic models described by (5.5). Now, we proceed to more specific discussions using a simple subclass that can be solved analytically.

5.3.1 The Lagrangian and Basic Equations

We consider the extended cuscuton theory with a quadratic nonminimal coupling,

$$S_{\rm EC} = \int d^4x\sqrt{-g}\left[\left(\frac{M_*^2}{2}+\mu\phi^2\right)R-\frac{1}{2}m^2\phi^2 \right. \\ \left. +(\alpha+\beta\phi)\sqrt{2X}+4\mu X(-2+\log X)-2\mu\phi\log X\Box\phi\right], \quad (5.28)$$

which corresponds to the following choice of the functions in (5.4):

$$u_2=-\frac{1}{2}m^2\phi^2, \quad v_2=\alpha+\beta\phi, \quad v_3=0, \quad v_4=\frac{M_*^2}{2}+\mu\phi^2. \quad (5.29)$$

Here, M_*, μ, α, β, and m are nonvanishing constant. Note that $\mu\to 0$ and $\beta\to 0$ limit reproduces the original cuscuton; hence the terms with μ and β characterize the difference from the original model. We also note that nonvanishing μ leads to $G_{3X}\neq 0$, meaning that Newtonian gravity is reproduced except for the time dependence of G_N. The field equations read

$$\mathcal{E}_N = 3(M_*^2+2\mu\phi^2)H^2-\frac{1}{2}m^2\phi^2-\rho_{\rm m}=0, \quad (5.30)$$

$$\mathcal{E}_a = (M_*^2+2\mu\phi^2)(3H^2+2\dot{H})-\frac{1}{2}m^2\phi^2+(\alpha+\beta\phi)\dot{\phi}+8\mu H\phi\dot{\phi}=0, \quad (5.31)$$

$$\mathcal{E}_\phi = 3(\alpha+\beta\phi)H+(m^2+12\mu H^2)\phi=0, \quad (5.32)$$

where we have set $p_{\rm m}=0$. Previouly said, $\mathcal{E}_\phi$ does not have $\dot{H}$ or $\ddot{\phi}$ from the beginning since $v_3=0$. We use the redshift $z:=a(t_0)/a(t)-1$ (with t_0 being the present time) as the time coordinate. Provided that the scale factor is monotonically increasing from zero to infinity in time, then $z=\infty$ corresponds to the initial time, and $z=-1$ corresponds to the infinite future. Let us define the following dimensionless variables:

$$M:=\frac{H_0^2}{m^2}\mu, \quad A:=\frac{\alpha}{mM_*}, \quad B:=\frac{H_0}{m^2}\beta, \quad \hat{\phi}(z):=\frac{m}{M_*H_0}\phi(z), \quad \hat{H}(z):=\frac{H(z)}{H_0}, \quad (5.33)$$

with $H_0:=H(z=0)$. In terms of the above variables, Eqs. (5.31) and (5.32) are rewritten as

$$\frac{\mathcal{E}_a}{M_*^2H_0^2} = (1+2M\hat{\phi}^2)\hat{H}\left[3\hat{H}-2(1+z)\hat{H}'\right]-\frac{1}{2}\hat{\phi}^2-(A+B\hat{\phi}+8M\hat{H}\hat{\phi})(1+z)\hat{H}\hat{\phi}' \\ =0, \quad (5.34)$$

$$\frac{\mathcal{E}_\phi}{mM_*H_0} = 3A\hat{H}+(1+3B\hat{H}+12M\hat{H}^2)\hat{\phi}=0, \quad (5.35)$$

where a prime denotes a derivative with respect to z. Removing $\hat{\phi}$ from (5.34) by using (5.35), we are left with the following first-order differential equation for $\hat{H}$:

$$(1+z)\hat{H}' = \frac{3\hat{H}}{2}\frac{(1+3B\hat{H}+12M\hat{H}^2)\left[2(1+3B\hat{H}+12M\hat{H}^2)^2-3A^2(1-12M\hat{H}^2)\right]}{2(1+3B\hat{H}+12M\hat{H}^2)^3-3A^2(1-36M\hat{H}^2-36MB\hat{H}^3)}. \quad (5.36)$$

Here we require $1+3B\hat{H}+12M\hat{H}^2 \neq 0$ for any z so that (5.35) can always be solved for $\hat{\phi}$. This equation is surely solvable, Note that, in the limit $\hat{H}\to\infty$, Eq. (5.36) takes the form

$$(1+z)\hat{H}' = \frac{3\hat{H}}{2}, \quad (5.37)$$

which yields the desired behavior of the Hubble parameter at early times, namely, $H \to \mathrm{const}\cdot a^{-3/2} \propto (1+z)^{3/2}$. We will see that (5.36) can be solved analytically in §5.3.3.

Equation (5.30) is used to determine the matter energy density ρ_{m}. In terms of the matter density parameter $\Omega_{\mathrm{m0}} \coloneqq 8\pi G_N\rho_{\mathrm{m}}/3H^2|_{z=0}$, Eq. (5.30) can be written as

$$\Omega_{\mathrm{m0}} = 1 - \frac{3A^2}{2[(1+3B+12M)^2+18MA^2]}, \quad (5.38)$$

showing that Ω_{m0} is fixed by the parameters M, A, and B.

5.3.2 Viable Parameter Region

Now we apply the requirements [A]–[D] to the present case and find the viable region in the three-dimensional parameter space (M, A, B) by studying the dynamics of $\hat{H}$ based on (5.36).

We first demand [A], namely, we require that $\hat{H}$ starts from a great value at some early initial time and approaches to a constant (denoted by $\hat{H}_{\mathrm{dS}}$) in the infinite future. Then, the asymptotic value $\hat{H}_{\mathrm{dS}}$ should correspond to the largest stable equilibrium point of (5.36). Here, an equilibrium point $\hat{H}=\hat{H}_*$ is said to be stable if and only if $\hat{H}'<0$ (i.e., $d\hat{H}/dt>0$) for $\hat{H}\in(\hat{H}_*-\epsilon,\hat{H}_*)$ and $\hat{H}'>0$ (i.e., $d\hat{H}/dt<0$) for $\hat{H}\in(\hat{H}_*,\hat{H}_*+\epsilon)$, with ϵ being an infinitesimal positive number. Given that $\hat{H}>0$ and $1+3B\hat{H}+12M\hat{H}^2\neq 0$, $\hat{H}_{\mathrm{dS}}$ is given by one of the positive solutions (if they exist) of the following quartic equation:

$$2(1+3B\hat{H}+12M\hat{H}^2)^2-3A^2(1-12M\hat{H}^2)=0. \quad (5.39)$$

Provided that this equation has positive solutions, the largest one is a candidate of $\hat{H}_{\mathrm{dS}}$.

Let us now demand [C]: $-\dot{\phi}\propto\hat{\phi}'<0$. By using (5.35), $\hat{\phi}'$ reads

$$\hat{\phi}' = -\frac{3A(1-12M\hat{H}^2)\hat{H}'}{(1+3B\hat{H}+12M\hat{H}^2)^2}. \quad (5.40)$$

When M is positive, the factor $1-12M\hat{H}^2$ should be negative definite as otherwise $\hat{\phi}'$ changes sign during its evolution. However, this contradicts the fact that $\hat{H}$ travels to $\hat{H}_{\mathrm{dS}}$ because

$$1-12M\hat{H}_{\mathrm{dS}}^2 = \frac{2(1+3B\hat{H}_{\mathrm{dS}}+12M\hat{H}_{\mathrm{dS}}^2)^2}{3A^2} > 0. \quad (5.41)$$

Hence, in what follows, we require $M < 0$. In this case, one can show that (5.39) has at least one positive solution and that the largest solution provides a stable equilibrium point of (5.36). Then, this largest solution can be identified as $\hat{H}_{\rm dS}$. One can also verify that $\hat{H}' > 0$ for $\hat{H} > \hat{H}_{\rm dS}$, and therefore one always has $\hat{\phi}' < 0$ as long as $A > 0$. Moreover, we require $\hat{H}_{\rm dS} < 1$ so that the evolution of $\hat{H}$ is consistent with the condition $\hat{H}(z=0) = 1$. Given that $M < 0$, the requirement $\hat{H}_{\rm dS} < 1$ is satisfied if

$$1 + 3B + 12M < 0, \qquad \frac{2(1+3B+12M)^2}{3A^2(1-12M)} > 1. \tag{5.42}$$

Regarding [D], it is trivially satisfied as

$$\frac{2v_4}{M_*^2} = 1 + \frac{18MA^2\hat{H}^2}{(1+3B\hat{H}+12M\hat{H}^2)^2} > 1 + \frac{18MA^2\hat{H}_{\rm dS}^2}{(1+3B\hat{H}_{\rm dS}+12M\hat{H}_{\rm dS}^2)^2} = \frac{1}{1-12M\hat{H}_{\rm dS}^2} > 0. \tag{5.43}$$

Thus, the requirement [D] does not narrows down the viable parameter region.

Finally, let us consider [B]. The present value of the Hubble slow-roll parameter is written as

$$\epsilon_H(z=0) = \hat{H}'(z=0) = \frac{3(1+3B+12M)\left[2(1+3B+12M)^2 - 3A^2(1-12M)\right]}{2\left[2(1+3B+12M)^3 - 3A^2(1-36M-36MB)\right]}. \tag{5.44}$$

Requiring $\epsilon_H(z=0) < 1$ to guarantee the accelerated expansion of the Universe at the present time, we have

$$\frac{3A^2\left[1+72M-432M^2+9B(1-4M)\right]}{2(1+3B+12M)^3} > 1. \tag{5.45}$$

In summary, the requirements [A]–[D] are satisfied if the following four conditions are fulfilled:

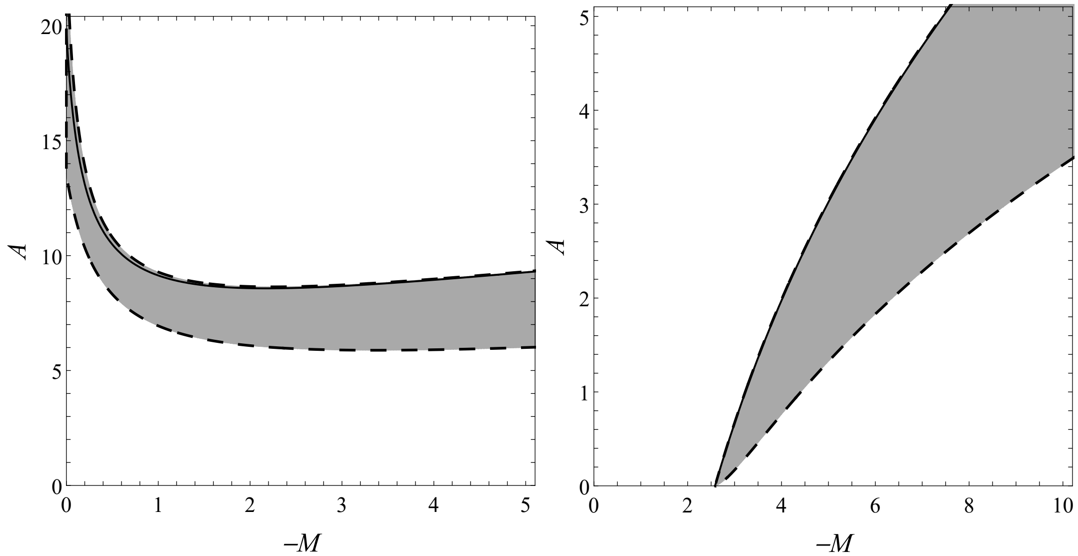
$$\begin{gathered} M < \min\left(0, -\frac{1+3B}{12}\right), \quad A > 0, \quad \frac{2(1+3B+12M)^2}{3A^2(1-12M)} > 1, \\ \frac{3A^2\left[1+72M-432M^2+9B(1-4M)\right]}{2(1+3B+12M)^3} > 1. \end{gathered} \tag{5.46}$$

We present two-dimensional sections of the viable parameter region (5.46) at some fixed values of B in Fig. 5.1. The matter density parameter $\Omega_{\rm m0}$ is given in terms of M, A, and B as (5.38). For a fiducial value $\Omega_{\rm m0} = 0.3$, Eq. (5.38) defines a two-dimensional surface in the parameter space (M, A, B), which appears as the solid curves in Fig. 5.1. For the parameters in the vicinity of these curves, one expects to have a background cosmological evolution similar to the one in the currently viable ΛCDM model.

5.3.3 The Solution

Having obtained the viable parameter region, we are now able to analyze the exact solution to (5.36). It is straightforward to integrate (5.36) to obtain the following algebraic equation for $\hat{H}$:

$$\hat{H}^2\left[2(1+3B\hat{H}+12M\hat{H}^2)^2 - 3A^2(1-12M\hat{H}^2)\right] + C(1+z)^3(1+3B\hat{H}+12M\hat{H}^2)^2 = 0, \tag{5.47}$$



(a) $B = -10$ (b) $B = 10$

Figure 5.1: Two-dimensional sections of the parameter space (M, A, B) satisfying (5.46) are colored gray (the boundary is indicated by dashed curves). The solid curves correspond to the parameters that yield $\Omega_{\mathrm{m}0} = 0.3$, which almost overlap with the upper dashed curves.

where the integration constant C is determined from $\hat{H}(z = 0) = 1$ as

$$C = -2 + \frac{3A^2(1 - 12M)}{(1 + 3B + 12M)^2}. \tag{5.48}$$

Note that (5.39) is recovered in the limit $z \to -1$.

The Newton's constant (5.27) and the effective gravitational coupling (5.24) are given, respectively, by

$$8\pi G_N M_*^2 = 1 - \frac{18MA^2\hat{H}^2}{(1 + 3B\hat{H} + 12M\hat{H}^2)^2 + 18MA^2\hat{H}^2}, \tag{5.49}$$

$$8\pi G_{\mathrm{eff}} M_*^2 = 8\pi G_N M_*^2 + \frac{864M^2A^2(1 - 12M\hat{H}^2)(1 + 3B\hat{H} + 12M\hat{H}^2)\hat{H}^3}{(1 - 36M\hat{H}^2)\left[(1 + 3B\hat{H} + 12M\hat{H}^2)^2 + 18MA^2\hat{H}^2\right]^2}(1 + z)\hat{H}', \tag{5.50}$$

One can draw some information on the asymptotic behavior of these quantities from (5.49) and (5.50). In the infinite future, we have $\hat{H}' \to 0$, and thus $G_{\mathrm{eff}}/G_N \to 1$, while for large z where $\hat{H} \propto (1 + z)^{3/2}$, we have

$$8\pi G_{\mathrm{eff}} M_*^2 \to 1 + \frac{A^2}{8M\hat{H}^2}, \quad 8\pi G_N M_*^2 \to 1 - \frac{A^2}{8M\hat{H}^2}. \tag{5.51}$$

As an illustrative example, we plot the evolution of $\hat{H}$, ϵ_H, and the gravitational couplings for $(M, A, B) = (-0.02, 17.5, -10)$ in Fig. 5.2. Note that this parameter choice fulfills the viability conditions (5.46) (see Fig. 5.1a). From these examples, we see that the background evolution is similar to the conventional ΛCDM model. In contrast, the evolution of the density fluctuations can be used to test the extended cuscuton as dark energy by observations associated with the density fluctuations, e.g., the integrated Sachs-Wolfe effect or weak gravitational lensing. The time variation of Newton's constant can also be used to constrain the model, which, in the present case, is given by

$$\left.\frac{\dot{G}_N}{HG_N}\right|_{z=0} = \frac{54MA^2(1-12M)\left[2(1+3B+12M)^2-3A^2(1-12M)\right]}{\left[(1+3B+12M)^2+18MA^2\right]\left[2(1+3B+12M)^3-3A^2(1-36M-36MB)\right]}, \tag{5.52}$$

while the observational bound reads $|\dot{G}_N/G_N| \lesssim 10^{-3}H_0$. One can check that the parameter choice $(M, A, B) = (-0.02, 17.5, -10)$ satisfies this bound. We note that the parameter region fulfills this bound is only a neighborhood of the $\Omega_{\rm m0} = 0.3$ line in 5.1, and hence (5.52) imposes a substantial restriction on the viable parameter region.

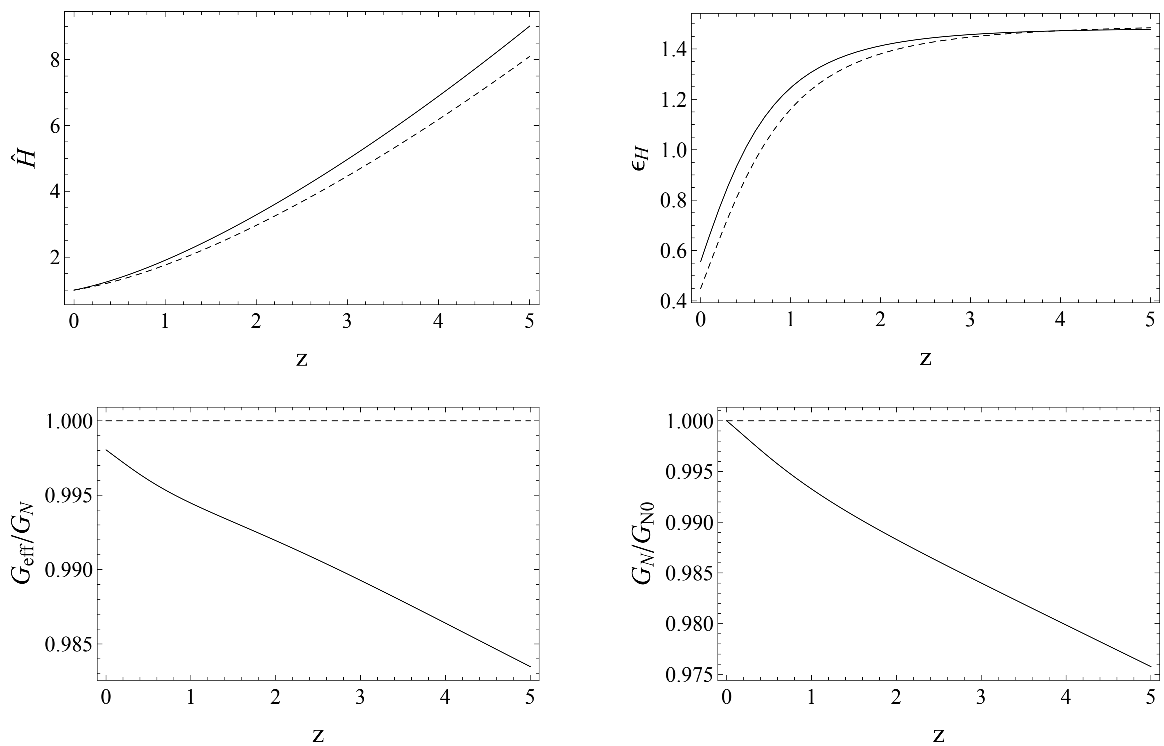


Figure 5.2: Time evolution of $\hat{H}$, ϵ_H, $G_{\rm eff}/G_N$, and G_N/G_{N0}, with $G_{N0} := G_N(z=0)$. The solid lines correspond to $(M, A, B) = (-0.02, 17.5, -10)$ and the dashed lines represent the result of the ΛCDM model with $\Omega_{\rm m0} = 0.3$.

Before closing this subsection, let us mention some limiting cases where one of the model parameters in (5.28) is vanishing. When $\alpha = 0$ (i.e., $A = 0$), we obtain $\phi = 0$ from (5.32), which contradicts the assumption that $\partial_\mu\phi$ is timelike (see §5.1). On the other hand, when $\mu = 0$ (i.e., $M = 0$), we obtain $G_N = G_{\rm eff} = (8\pi M_*^2)^{-1}$ from (5.49) and (5.50), while the spacetime and the cuscuton field can evolve in a nontrivial manner.

Chapter 6

Conclusions

The cuscuton theory is a special case of single-field scalar-tensor theories having only two DOFs, i.e., no propagating scalar DOF, with timelike $\partial_\mu\phi$. Given that GR is two-DOFs gravity, the cuscuton minimally modifies GR in terms of dynamical DOFs. In this thesis, we have extended the framework of the cuscuton into the GLPV theories, which we call the "extended cuscuton," and have investigated its theoretical and cosmological features. In particular, we have found a viable dark energy solution mimicing background dynamics of the ΛCDM model with different evolutions for the matter fluctuations from that in ΛCDM model.

At the beginning of this thesis, in Chap. 2, we overviewed the modification flow of gravitational theories from Newtonian gravity to GR. Then, we introduced MG and explained the Ostrogradsky ghost and the ghost-free conditions, which are the most important issues to construct new frameworks of gravity. Among several types of MG, we mainly mentioned the scalar-tensor theories and the metric theories. Also, many MG satisfies the Lorentz invariance, but some special classes violate this invariance. Furthermore, there are only two-DOFs scalar-tensor theories other than the (extended) cuscuton. We referred these two types of MG.

Next, we formulated the cuscuton theory and reviewed its various fascinating features in Chap. 3. The cuscuton (3.11) was constructed as a subset of the k-essence model characterized by the at most first-order EOM for ϕ in a homogeneous limit. A homogeneous limit of ϕ is identical to the unitary gauge $\phi = \phi(t)$, but indeed this limit is not necessary to kill the scalar propagating mode. Actually, the scalar mode with timelike $\partial_\mu\phi$ does not propagate if an appropriate boundary condition at spatial infinity is imposed. This implies the number of propagating DOFs of the timelike cuscuton is always two irrespective to homogeneity of the scalar distributions. Then, we reviewed the cuscuton cosmology with and without an extra field χ, and explored its applications to the inflation, bounce, and dark energy models. Various aspects of the cuscuton found so far was also referred.

In Chap. 4, we explored a possible extension of the cuscuton theory in the context of the GLPV theory. At first, we constructed the cosmological prototype of the extended cuscuton theory by requiring the conditions [a′] and [b] on the GLPV action. At this stage, the B_i functions remain arbitrary in a flat cosmological background, while both the A_i and B_j functions are fixed in a non-flat cosmological background. Thereafter, to obtain the complete form of the extended cuscuton theory, i.e., the theory having two physical DOFs on any background spacetime under the unitary gauge, we performed a Hamiltonian analysis of the precursory models. Then, the obtained model is identical to the non-flat cosmological prototype. So that this class violates the Lorentz symmetry and only has the three-dimensional diffeomorphism, the Hamiltonian constraint is not the first-class constraint. And we found the condition to less-than-three DOFs as that to appear some tertiary constraints; namely the condition not to determine the either of the multipliers λ_N or χ_{ij} from the consistency conditions for the secondary constraints Θ and Π^{ij}, which is given by (4.45), whose explicit form for GLPV theories is (4.51). Furthermore, we showed that the theory that are mapped from the original cuscuton model by the disformal transformation (4.66) belong to the $A_5 = 0$ case of our extended cuscuton theory. We also studied scalar and tensor cosmological perturbations in the presence of another scalar field as matter. The scalar modes acquire nonlocal interaction as in (4.83) and the stability conditions read (4.87).

In Chap. 5, we applied the extended cuscuton to the late-time cosmology. We studied homogeneous and isotropic cosmology in the extended cuscutons satisfying $c_{\rm GW} = 1$ in the presence of a matter field. First, we derived the background field equations and proposed the requirements [A]–[D] for these theories to serve as a viable dark energy model. Also, we investigated scalar perturbations to derive the evolution equation for the density fluctuations and the gravitational Poisson equations. Then, we turned to more specific discussions using a simple model (5.28) that can be solved analytically. The model parameter α appears as a coefficient of $\sqrt{2X}$, which is typical in the original cuscuton model. On the other hand, the parameters μ and β characterize the difference from the original model. In order to avoid technical complexity, we defined dimensionless parameters M, A, and B, corresponding to μ, α, and β, respectively. We obtained the viable region in the parameter space (M, A, B) which satisfies the requirements [A]–[D]. We also plotted the evolution of the dimensionless Hubble parameter $\hat{H}$, the Hubble slow-roll parameter ϵ_H, the ratio of the effective gravitational coupling $G_{\rm eff}$ to the Newton's constant G_N, and G_N normalized by its present value for the parameter choice $(M, A, B) = (-0.02, 17.5, -10)$, which lies in the viable parameter region. We found that the background evolution in this model can mimic the conventional ΛCDM model while the evolution of the density fluctuation deviates from the one in the ΛCDM case. Moreover, this set of parameters satisfies the observational constraint on the time variation of the Newton's constant, $|\dot{G}_N/G_N| < \mathcal{O}(10^{-3})H_0$. Hence, one can test the extended cuscuton as dark energy by observations associated with the density fluctuations, e.g., the integrated Sachs-Wolfe effect or weak gravitational lensing, which we leave for future study.

Having formulated the extended cuscuton theory, it would be intriguing to study its phenomenological aspects such as the early universe or large scale structure, which about the original cuscuton had been considered in [24]. In parallel to phenomenology, we expect that further extension of the cuscuton framework is still possible. In §4.4.1, we found that the original cuscuton can produce a certain class of the extended cuscuton by a disformal transformation. Then, we expect that more general disformal transformation than (4.66) may map the original cuscuton into a class of DHOST theories. This prediction may offer further generalization of the extended cuscuton theory. Also, in §4.2, we performed the

Hamiltonian analysis to identify the theories truly having two DOFs in the unitary gauge. However, our analysis after obtaining the tertiary constraint (4.46) was just a brief analysis. On the other hand, the authors of [143] found the conditions to only two-DOFs in a specific theories. Refering to this result, we might perform more precise analysis. We should also study more details on its cosmology: the history of cosmic acceleration, inflation models or large scale structure, which about the original cuscuton had been considered in [24]. We hope to discuss this point in the near future.

Acknowledgments

First of all, I owe my deepest gratitude to my supervisor Tsutomu Kobayashi since I was an undergraduate student. I have learned an attitude towards study from his insightful suggestions. I am grateful to my collaborators Kazufumi Takahashi, Asuka Ito, Suro Kim, and Jiro Soda. Discussions with them have given me a much better understanding of physics. Also, I would like to thank all the members of the group of theoretical physics at Rikkyo University, especially Shingo Akama, Yuji Akita, Hiromu Ogawa, Tomohiro Harada, Takashi Hiramatsu, Shin'ichi Hirano, Tact Ikeda, Yosuke Mishima, Daiki Miyata, Yuko Mori, Masataka Nagashima, and Keitaro Tomikawa. Their various comments have made me aware of something new, and daily discussions with them have provided me an extensive knowledge not only of physics but of mathematics. Kumiko Inagawa has been helpful to dealing with cumbersome office procedures and with going overseas. I am grateful to my friends outside my department, Katsuki Aoki, Kohei Fujikura, Tomohiro Fujita, Satsuki Matsuno, Taisaku Mori, Hayato Motohashi, Shintaro Nakamura, Sirachak Panpanich, Tomo Tanaka, and Mai Yashiki. I have been always able to have a really fantastic time as a researcher and a person. Finally, my heartfelt thanks to my family, Toshiko and Takuichiro. I could not have live out the doctral cource without their dedicated support. This work was supported by the Japan Society for the Promotion of Science (JSPS) research fellowships for Young Scientists.

Bibliography

[1] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984, 10.7208/chicago/9780226870373.001.0001.

[2] C. M. Will, *Theory and Experiment in Gravitational Physics*. Cambridge University Press, 2 ed., 2018, 10.1017/9781316338612.

[3] LIGO Scientific, Virgo collaboration, *Binary Black Hole Mergers in the first Advanced LIGO Observing Run*, Phys. Rev. **X6** (2016) 041015 [1606.04856].

[4] Event Horizon Telescope collaboration, *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, Astrophys. J. **875** (2019) L1 [1906.11238].

[5] Supernova Search Team collaboration, *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, Astron. J. **116** (1998) 1009 [astro-ph/9805201].

[6] Supernova Cosmology Project collaboration, *Measurements of Omega and Lambda from 42 high redshift supernovae*, Astrophys. J. **517** (1999) 565 [astro-ph/9812133].

[7] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, *Modified Gravity and Cosmology*, Phys. Rept. **513** (2012) 1 [1106.2476].

[8] Planck collaboration, *Planck 2018 results. VI. Cosmological parameters*, 1807.06209.

[9] D. Lovelock, *The Einstein tensor and its generalizations*, J. Math. Phys. **12** (1971) 498.

[10] D. Lovelock, *The four-dimensionality of space and the einstein tensor*, J. Math. Phys. **13** (1972) 874.

[11] G. W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, Int. J. Theor. Phys. **10** (1974) 363.

[12] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, *From k-essence to generalised Galileons*, Phys. Rev. D **84** (2011) 064039 [1103.3260].

[13] T. Kobayashi, M. Yamaguchi and J. Yokoyama, *Generalized G-inflation: Inflation with the most general second-order field equations*, *Prog. Theor. Phys.* **126** (2011) 511 [1105.5723].

[14] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, *Healthy theories beyond Horndeski*, *Phys. Rev. Lett.* **114** (2015) 211101 [1404.6495].

[15] D. Langlois and K. Noui, *Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability*, *JCAP* **1602** (2016) 034 [1510.06930].

[16] M. Crisostomi, K. Koyama and G. Tasinato, *Extended Scalar-Tensor Theories of Gravity*, *JCAP* **1604** (2016) 044 [1602.03119].

[17] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui and G. Tasinato, *Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order*, *JHEP* **12** (2016) 100 [1608.08135].

[18] M. Ostrogradsky, *Mémoires sur les équations différentielles, relatives au problème des isopérimètres*, *Mem. Acad. St. Petersbourg* **6** (1850) 385.

[19] R. P. Woodard, *Avoiding dark energy with 1/r modifications of gravity*, *Lect. Notes Phys.* **720** (2007) 403 [astro-ph/0601672].

[20] R. P. Woodard, *Ostrogradsky's theorem on Hamiltonian instability*, *Scholarpedia* **10** (2015) 32243 [1506.02210].

[21] N. Afshordi, D. J. H. Chung and G. Geshnizjani, *Cuscuton: A Causal Field Theory with an Infinite Speed of Sound*, *Phys. Rev. D* **75** (2007) 083513 [hep-th/0609150].

[22] A. Ito, A. Iyonaga, S. Kim and J. Soda, *Dressed power-law inflation with a cuscuton*, *Phys. Rev.* **D99** (2019) 083502 [1902.08663].

[23] S. S. Boruah, H. J. Kim, M. Rouben and G. Geshnizjani, *Cuscuton bounce*, *JCAP* **1808** (2018) 031 [1802.06818].

[24] N. Afshordi, D. J. H. Chung, M. Doran and G. Geshnizjani, *Cuscuton Cosmology: Dark Energy meets Modified Gravity*, *Phys. Rev. D* **75** (2007) 123509 [astro-ph/0702002].

[25] G. McVittie, *The mass-particle in an expanding universe*, *Mon. Not. Roy. Astron. Soc.* **93** (1933) 325.

[26] A. Iyonaga, K. Takahashi and T. Kobayashi, *Extended Cuscuton: Formulation*, *JCAP* **1812** (2018) 002 [1809.10935].

[27] A. Iyonaga, K. Takahashi and T. Kobayashi, *Extended Cuscuton as Dark Energy*, *JCAP* **07** (2020) 004 [2003.01934].

[28] H. Motohashi and T. Suyama, *Third order equations of motion and the Ostrogradsky instability*, *Phys. Rev. D* **91** (2015) 085009 [1411.3721].

[29] H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi and D. Langlois, *Healthy degenerate theories with higher derivatives*, *JCAP* **1607** (2016) 033 [1603.09355].

[30] H. Motohashi, T. Suyama and M. Yamaguchi, *Ghost-free theory with third-order time derivatives*, 1711.08125.

[31] H. Motohashi, T. Suyama and M. Yamaguchi, *Ghost-free theories with arbitrary higher-order time derivatives*, *JHEP* **06** (2018) 133 [1804.07990].

[32] M. Crisostomi, R. Klein and D. Roest, *Higher Derivative Field Theories: Degeneracy Conditions and Classes*, *JHEP* **06** (2017) 124 [1703.01623].

[33] H. Motohashi and T. Suyama, *Quantum Ostrogradsky theorem*, *JHEP* **20** (2020) 032 [2001.02483].

[34] K. Aoki and H. Motohashi, *Ghost from constraints: a generalization of Ostrogradsky theorem*, *JCAP* **08** (2020) 026 [2001.06756].

[35] A. Ganz and K. Noui, *Reconsidering the Ostrogradsky theorem: Higher-derivatives Lagrangians, Ghosts and Degeneracy*, 2007.01063.

[36] T. P. Sotiriou and V. Faraoni, *$f(R)$ Theories Of Gravity*, *Rev. Mod. Phys.* **82** (2010) 451 [0805.1726].

[37] A. De Felice and S. Tsujikawa, *$f(R)$ theories*, *Living Rev. Rel.* **13** (2010) 3 [1002.4928].

[38] J. Overduin and P. Wesson, *Kaluza-Klein gravity*, *Phys. Rept.* **283** (1997) 303 [gr-qc/9805018].

[39] P. Jordan, *The present state of Dirac's cosmological hypothesis*, *Z. Phys.* **157** (1959) 112.

[40] C. Brans and R. H. Dicke, *Mach's principle and a relativistic theory of gravitation*, *Phys. Rev.* **124** (1961) 925.

[41] C. Armendáriz-Picón, V. F. Mukhanov and P. J. Steinhardt, *A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration*, *Phys. Rev. Lett.* **85** (2000) 4438 [astro-ph/0004134].

[42] T. Chiba, T. Okabe and M. Yamaguchi, *Kinetically driven quintessence*, *Phys. Rev. D* **62** (2000) 023511 [astro-ph/9912463].

[43] C. Armendáriz-Picón, T. Damour and V. F. Mukhanov, *k - inflation*, *Phys. Lett. B* **458** (1999) 209 [hep-th/9904075].

[44] C. Deffayet, O. Pujolas, I. Sawicki and A. Vikman, *Imperfect Dark Energy from Kinetic Gravity Braiding*, *JCAP* **10** (2010) 026 [1008.0048].

[45] T. Kobayashi, M. Yamaguchi and J. Yokoyama, *G-inflation: Inflation driven by the Galileon field*, *Phys. Rev. Lett.* **105** (2010) 231302 [1008.0603].

[46] T. Kobayashi, *Horndeski theory and beyond: a review*, *Rept. Prog. Phys.* **82** (2019) 086901 [1901.07183].

[47] A. Nicolis, R. Rattazzi and E. Trincherini, *The Galileon as a local modification of gravity*, *Phys. Rev. D* **79** (2009) 064036 [0811.2197].

[48] C. Deffayet, G. Esposito-Farese and A. Vikman, *Covariant Galileon*, *Phys. Rev. D* **79** (2009) 084003 [0901.1314].

[49] D. Langlois, *Dark energy and modified gravity in degenerate higher-order scalar–tensor (DHOST) theories: A review*, *Int. J. Mod. Phys.* **D28** (2019) 1942006 [1811.06271].

[50] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui and G. Tasinato, *Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order*, *JHEP* **12** (2016) 100 [1608.08135].

[51] F. Arroja, N. Bartolo, P. Karmakar and S. Matarrese, *The two faces of mimetic Horndeski gravity: disformal transformations and Lagrange multiplier*, *JCAP* **1509** (2015) 051 [1506.08575].

[52] G. Domènech, S. Mukohyama, R. Namba, A. Naruko, R. Saitou and Y. Watanabe, *Derivative-dependent metric transformation and physical degrees of freedom*, *Phys. Rev. D* **92** (2015) 084027 [1507.05390].

[53] K. Takahashi, H. Motohashi, T. Suyama and T. Kobayashi, *General invertible transformation and physical degrees of freedom*, *Phys. Rev. D* **95** (2017) 084053 [1702.01849].

[54] K. Takahashi and T. Kobayashi, *Extended mimetic gravity: Hamiltonian analysis and gradient instabilities*, *JCAP* **1711** (2017) 038 [1708.02951].

[55] D. Langlois, M. Mancarella, K. Noui and F. Vernizzi, *Mimetic gravity as DHOST theories*, *JCAP* **1902** (2019) 036 [1802.03394].

[56] X. Gao, *Higher derivative scalar-tensor monomials and their classification*, 2003.11978.

[57] X. Gao and Y.-M. Hu, *Higher derivative scalar-tensor theory and spatially covariant gravity: the correspondence*, 2004.07752.

[58] X. Gao, *Higher derivative scalar-tensor theory from the spatially covariant gravity: a linear algebraic analysis*, 2006.15633.

[59] J. Khoury and A. Weltman, *Chameleon fields: Awaiting surprises for tests of gravity in space*, *Phys. Rev. Lett.* **93** (2004) 171104 [astro-ph/0309300].

[60] J. Khoury and A. Weltman, *Chameleon cosmology*, *Phys. Rev. D* **69** (2004) 044026.

[61] K. Hinterbichler and J. Khoury, *Screening long-range forces through local symmetry restoration*, *Physical Review Letters* **104** (2010) .

[62] A. I. Vainshtein, *To the problem of nonvanishing gravitation mass*, *Phys. Lett.* **39B** (1972) 393.

[63] T. Narikawa, T. Kobayashi, D. Yamauchi and R. Saito, *Testing general scalar-tensor gravity and massive gravity with cluster lensing*, *Phys. Rev. D* **87** (2013) 124006 [1302.2311].

[64] K. Koyama, G. Niz and G. Tasinato, *Effective theory for the Vainshtein mechanism from the Horndeski action*, *Phys. Rev. D* **88** (2013) 021502 [1305.0279].

[65] A. De Felice, R. Kase and S. Tsujikawa, *Vainshtein mechanism in second-order scalar-tensor theories*, *Phys. Rev. D* **85** (2012) 044059 [1111.5090].

[66] R. Kase and S. Tsujikawa, *Screening the fifth force in the Horndeski's most general scalar-tensor theories*, *JCAP* **08** (2013) 054 [1306.6401].

[67] T. Kobayashi, Y. Watanabe and D. Yamauchi, *Breaking of Vainshtein screening in scalar-tensor theories beyond Horndeski*, *Phys. Rev. D* **91** (2015) 064013 [1411.4130].

[68] A. De Felice, R. Kase and S. Tsujikawa, *Existence and disappearance of conical singularities in Gleyzes-Langlois-Piazza-Vernizzi theories*, *Phys. Rev. D* **92** (2015) 124060 [1508.06364].

[69] R. Kase, S. Tsujikawa and A. De Felice, *Conical singularities and the Vainshtein screening in full GLPV theories*, *JCAP* **03** (2016) 003 [1512.06497].

[70] M. Crisostomi, K. Noui, C. Charmousis and D. Langlois, *Beyond Lovelock: on higher derivative metric theories*, 1710.04531.

[71] D. Lovelock, *The uniqueness of the einstein field equations in a four-dimensional space*, *Archive for Rational Mechanics and Analysis* **33** (1969) 54.

[72] R. Jackiw and S. Y. Pi, *Chern-Simons modification of general relativity*, *Phys. Rev. D* **68** (2003) 104012 [gr-qc/0308071].

[73] T. Jacobson and D. Mattingly, *Gravity with a dynamical preferred frame*, *Phys. Rev. D* **64** (2001) 024028 [gr-qc/0007031].

[74] P. Hořava, *Quantum Gravity at a Lifshitz Point*, *Phys. Rev. D* **79** (2009) 084008 [0901.3775].

[75] D. Blas, O. Pujolàs and S. Sibiryakov, *Consistent extension of hořava gravity*, *Phys. Rev. Lett.* **104** (2010) 181302.

[76] M. Li and Y. Pang, *A Trouble with Horava-Lifshitz Gravity*, *JHEP* **08** (2009) 015 [0905.2751].

[77] D. Blas, O. Pujolàs and S. Sibiryakov, *On the Extra Mode and Inconsistency of Hořava Gravity*, *JHEP* **10** (2009) 029 [0906.3046].

[78] P. Creminelli, M. A. Luty, A. Nicolis and L. Senatore, *Starting the Universe: Stable Violation of the Null Energy Condition and Non-standard Cosmologies*, *JHEP* **12** (2006) 080 [hep-th/0606090].

[79] C. Cheung, P. Creminelli, A. Fitzpatrick, J. Kaplan and L. Senatore, *The Effective Field Theory of Inflation*, *JHEP* **03** (2008) 014 [0709.0293].

[80] N. Arkani-Hamed, H.-C. Cheng, M. A. Luty and S. Mukohyama, *Ghost condensation and a consistent infrared modification of gravity*, *JHEP* **05** (2004) 074 [hep-th/0312099].

[81] A. De Felice, D. Langlois, S. Mukohyama, K. Noui and A. Wang, *Generalized instantaneous modes in higher-order scalar-tensor theories*, *Phys. Rev. D* **98** (2018) 084024 [1803.06241].

[82] D. Blas, O. Pujolàs and S. Sibiryakov, *Models of non-relativistic quantum gravity: The Good, the bad and the healthy*, *JHEP* **04** (2011) 018 [1007.3503].

[83] C. Germani, A. Kehagias and K. Sfetsos, *Relativistic quantum gravity at a lifshitz point*, *Journal of High Energy Physics* **2009** (2009) 060.

[84] X. Gao, *Unifying framework for scalar-tensor theories of gravity*, *Phys. Rev. D* **90** (2014) 081501 [1406.0822].

[85] C. Lin and S. Mukohyama, *A Class of Minimally Modified Gravity Theories*, *JCAP* **1710** (2017) 033 [1708.03757].

[86] R. Carballo-Rubio, F. Di Filippo and S. Liberati, *Minimally modified theories of gravity: a playground for testing the uniqueness of general relativity*, *JCAP* **06** (2018) 026 [1802.02537].

[87] K. Aoki, C. Lin and S. Mukohyama, *Novel matter coupling in general relativity via canonical transformation*, *Phys. Rev. D* **98** (2018) 044022 [1804.03902].

[88] K. Aoki, A. De Felice, C. Lin, S. Mukohyama and M. Oliosi, *Phenomenology in type-I minimally modified gravity*, *JCAP* **1901** (2019) 017 [1810.01047].

[89] C. Lin, *The Self-consistent Matter Coupling of a Class of Minimally Modified Gravity Theories*, *JCAP* **05** (2019) 037 [1811.02467].

[90] S. Mukohyama and K. Noui, *Minimally Modified Gravity: a Hamiltonian Construction*, *JCAP* **07** (2019) 049 [1905.02000].

[91] A. De Felice, A. Doll and S. Mukohyama, *A theory of type-II minimally modified gravity*, *JCAP* **09** (2020) 034 [2004.12549].

[92] K. Aoki, A. De Felice, S. Mukohyama, K. Noui, M. Oliosi and M. C. Pookkillath, *Minimally modified gravity fitting Planck data better than ΛCDM*, *Eur. Phys. J. C* **80** (2020) 708 [2005.13972].

[93] A. De Felice and S. Mukohyama, *Minimal theory of massive gravity*, *Phys. Lett. B* **752** (2016) 302 [1506.01594].

[94] A. De Felice and S. Mukohyama, *Phenomenology in minimal theory of massive gravity*, *JCAP* **04** (2016) 028 [1512.04008].

[95] G. Tasinato, *Symmetries for scalarless scalar theories*, 2009.02157.

[96] Wikipedia contributors, "Cuscuta — Wikipedia, the free encyclopedia." https://en.wikipedia.org/w/index.php?title=Cuscuta&oldid=979839263, 2020.

[97] H. Gomes and D. C. Guariento, *Hamiltonian analysis of the cuscuton*, *Phys. Rev. D* **95** (2017) 104049 [1703.08226].

[98] S. S. Boruah, H. J. Kim and G. Geshnizjani, *Theory of Cosmological Perturbations with Cuscuton*, *JCAP* **1707** (2017) 022 [1704.01131].

[99] A. E. Romano, *General background conditions for K-bounce and adiabaticity*, *Eur. Phys. J. C* **77** (2017) 147 [1607.08533].

[100] J. Quintin and D. Yoshida, *Cuscuton gravity as a classically stable limiting curvature theory*, 1911.06040.

[101] Y. Sakakihara, D. Yoshida, K. Takahashi and J. Quintin, *Limiting extrinsic curvature theory and stable non-singular anisotropic universe*, 2005.10844.

[102] M. A. Markov, *Limiting density of matter as a universal law of nature*, *JETP Lett* **36** (1982) 265.

[103] M. A. Markov, *Possible state of matter just before the collapse stage*, *JETP Lett* **46** (1987) 431.

[104] V. Ginsburg, V. F. Mukhanov and V. P. Frolov, *Cosmology of the Superearly Universe and the 'Fundamental Length'*, *Sov. Phys. JETP* **67** (1988) 649.

[105] A. Ito, Y. Sakakihara and J. Soda, *Accelerating Universe with a stable extra dimension in cuscuton gravity*, *Phys. Rev.* **D100** (2019) 063531 [1906.10363].

[106] J. Chagoya and G. Tasinato, *A geometrical approach to degenerate scalar-tensor theories*, *JHEP* **02** (2017) 113 [1610.07980].

[107] N. Afshordi and J. Magueijo, *The critical geometry of a thermal big bang*, *Phys. Rev. D* **94** (2016) 101301 [1603.03312].

[108] N. Afshordi, *Cuscuton and low energy limit of Horava-Lifshitz gravity*, *Phys. Rev. D* **80** (2009) 081502 [0907.5201].

[109] J. Bhattacharyya, A. Coates, M. Colombo, A. E. Gümrükçüoğlu and T. P. Sotiriou, *Revisiting the cuscuton as a Lorentz-violating gravity theory*, *Phys. Rev. D* **97** (2018) 064020 [1612.01824].

[110] C. de Rham and H. Motohashi, *Caustics for Spherical Waves*, *Phys. Rev. D* **95** (2017) 064008 [1611.05038].

[111] I. Andrade, M. Marques and R. Menezes, *Cuscuton kinks and branes*, *Nucl. Phys. B* **942** (2019) 188 [1806.01923].

[112] E. Abdalla, N. Afshordi, M. Fontanini, D. C. Guariento and E. Papantonopoulos, *Cosmological black holes from self-gravitating fields*, *Phys. Rev. D* **89** (2014) 104018.

[113] N. Afshordi, M. Fontanini and D. C. Guariento, *Horndeski meets McVittie: A scalar field theory for accretion onto cosmological black holes*, *Phys. Rev. D* **90** (2014) 084012 [1408.5538].

[114] E. Pajer and D. Stefanyszyn, *Symmetric Superfluids*, *JHEP* **06** (2019) 008 [1812.05133].

[115] T. Grall, S. Jazayeri and E. Pajer, *Symmetric Scalars*, 1909.04622.

[116] W. Barker, A. Lasenby, M. Hobson and W. Handley, *Mapping Poincar\'{e} gauge cosmology to Horndeski theory for emergent dark energy*, 6, 2020, 2006.03581.

[117] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, *Essentials of k essence*, *Phys. Rev. D* **63** (2001) 103510 [astro-ph/0006373].

[118] H. Motohashi, T. Suyama and K. Takahashi, *Fundamental theorem on gauge fixing at the action level*, *Phys. Rev. D* **94** (2016) 124021 [1608.00071].

[119] F. Lucchin and S. Matarrese, *Power Law Inflation*, *Phys. Rev. D* **32** (1985) 1316.

[120] Planck collaboration, *Planck 2018 results. X. Constraints on inflation*, 1807.06211.

[121] D. Battefeld and P. Peter, *A Critical Review of Classical Bouncing Cosmologies*, *Phys. Rept.* **571** (2015) 1 [1406.2790].

[122] A. Chodos and S. L. Detweiler, *Where Has the Fifth-Dimension Gone?*, *Phys. Rev. D* **21** (1980) 2167.

[123] J. Chagoya and G. Tasinato, *A new scalar− tensor realization of Hořava− Lifshitz gravity*, *Class. Quant. Grav.* **36** (2019) 075014 [1805.12010].

[124] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*. 1992.

[125] D. Langlois, M. Mancarella, K. Noui and F. Vernizzi, *Effective Description of Higher-Order Scalar-Tensor Theories*, *JCAP* **1705** (2017) 033 [1703.03797].

[126] J. D. Bekenstein, *The Relation between physical and gravitational geometry*, *Phys. Rev. D* **48** (1993) 3641 [gr-qc/9211017].

[127] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, *Exploring gravitational theories beyond Horndeski*, *JCAP* **1502** (2015) 018 [1408.1952].

[128] B. P. Abbott et al., *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*, *Phys. Rev. Lett.* **119** (2017) 161101 [1710.05832].

[129] B. P. Abbott et al., *Multi-messenger Observations of a Binary Neutron Star Merger*, *Astrophys. J.* **848** (2017) L12 [1710.05833].

[130] B. P. Abbott et al., *Gravitational Waves and Gamma-rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A*, *Astrophys. J.* **848** (2017) L13 [1710.05834].

[131] J. Sakstein and B. Jain, *Implications of the Neutron Star Merger GW170817 for Cosmological Scalar-Tensor Theories*, *Phys. Rev. Lett.* **119** (2017) 251303 [1710.05893].

[132] C. de Rham and S. Melville, *Gravitational Rainbows: LIGO and Dark Energy at its Cutoff*, 1806.09417.

[133] J. M. Ezquiaga and M. Zumalacárregui, *Dark Energy After GW170817: Dead Ends and the Road Ahead*, *Phys. Rev. Lett.* **119** (2017) 251304 [1710.05901].

[134] P. Creminelli and F. Vernizzi, *Dark Energy after GW170817 and GRB170817A*, *Phys. Rev. Lett.* **119** (2017) 251302 [1710.05877].

[135] D. Langlois, R. Saito, D. Yamauchi and K. Noui, *Scalar-tensor theories and modified gravity in the wake of GW170817*, *Phys. Rev. D* **97** (2018) 061501 [1711.07403].

[136] L. Boubekeur, P. Creminelli, J. Norena and F. Vernizzi, *Action approach to cosmological perturbations: the 2nd order metric in matter dominance*, *JCAP* **08** (2008) 028 [0806.1016].

[137] E. Babichev, S. Ramazanov and A. Vikman, *Recovering $P(X)$ from a canonical complex field*, *JCAP* **11** (2018) 023 [1807.10281].

[138] J. Brown and K. V. Kuchar, *Dust as a standard of space and time in canonical quantum gravity*, *Phys. Rev. D* **51** (1995) 5600 [gr-qc/9409001].

[139] I. D. Saltas, I. Sawicki, L. Amendola and M. Kunz, *Anisotropic Stress as a Signature of Nonstandard Propagation of Gravitational Waves*, *Phys. Rev. Lett.* **113** (2014) 191101 [1406.7139].

[140] J. G. Williams, S. G. Turyshev and D. H. Boggs, *Progress in lunar laser ranging tests of relativistic gravity*, *Phys. Rev. Lett.* **93** (2004) 261101 [gr-qc/0411113].

[141] R. Kimura, T. Kobayashi and K. Yamamoto, *Vainshtein screening in a cosmological background in the most general second-order scalar-tensor theory*, *Phys. Rev.* **D85** (2012) 024023 [1111.6749].

[142] J. Noller, L. Santoni, E. Trincherini and L. G. Trombetta, *Scalar-tensor cosmologies without screening*, 2008.08649.

[143] X. Gao and Z.-B. Yao, *Spatially covariant gravity theories with two tensorial degrees of freedom: the formalism*, 1910.13995.