

On Values of Zeta Functions of Arakawa-Kaneko Type

by

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Abstract. For these two decades, the Arakawa-Kaneko zeta function has been studied actively. Recently Kaneko and Tsumura constructed its variants from the viewpoint of poly-Bernoulli numbers. In this paper, we generalize their zeta functions of Arakawa-Kaneko type to those with indices in which positive and nonpositive integers are mixed. We show that values of these functions at positive integers can be expressed in terms of the multiple Hurwitz zeta star values.

1. Introduction

The Arakawa-Kaneko zeta function is defined by

$$\xi(k_1, k_2, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{k_1, k_2, \dots, k_r}(1 - e^{-t})}{e^t - 1} dt \quad (\Re(s) > 0) \quad (1)$$

for $k_1, k_2, \dots, k_r \in \mathbb{Z}_{\geq 1}$ in [1]. This zeta function has been studied and generalized by a lot of authors (see, for example, [3, 4, 7, 8, 9, 10, 12, 14, 15, 16, 17]). It should be noted that $\xi(-k_1, -k_2, \dots, -k_r; s)$ ($k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$) has not been defined yet.

As variants of (1), Kaneko and Tsumura [13] defined the following η and $\tilde{\xi}$ functions by

$$\eta(k_1, k_2, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{k_1, k_2, \dots, k_r}(1 - e^t)}{1 - e^t} dt \quad (\Re(s) > 0) \quad (2)$$

for $k_1, k_2, \dots, k_r \in \mathbb{Z}_{\geq 1}$,

$$\eta(-k_1, -k_2, \dots, -k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{-k_1, -k_2, \dots, -k_r}(1 - e^t)}{1 - e^t} dt \quad (\Re(s) > 0) \quad (3)$$

for $k_1, k_2, \dots, k_r \in \mathbb{Z}_{\geq 0}$, and

$$\tilde{\xi}(-k_1, -k_2, \dots, -k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{-k_1, -k_2, \dots, -k_r}(1 - e^t)}{e^{-t} - 1} dt \quad (\Re(s) > 0) \quad (4)$$

for $k_1, k_2, \dots, k_r \in \mathbb{Z}_{\geq 0}$ with $(k_1, k_2, \dots, k_r) \neq (0, \dots, 0)$.

It is also noted that we cannot define $\tilde{\xi}(k_1, k_2, \dots, k_r; s)$ by replacing $\{-k_j\}$ by $\{k_j\}$ in (4) for $(k_j) \in \mathbb{Z}_{\geq 1}^r$.

We emphasize that indices of these zeta functions consist of all positive integers or all nonpositive integers.

In this paper, we aim to consider the cases of these zeta functions with indices in which positive and nonpositive integers are mixed. To explain our results in some detail, we give two overviews of necessary background. First, we recall the multiple zeta values (MZVs, for short) given by

$$\zeta(p_1, p_2, \dots, p_n) = \sum_{0 < m_1 < m_2 < \dots < m_n} \frac{1}{m_1^{p_1} m_2^{p_2} \cdots m_n^{p_n}}, \quad (5)$$

and the multiple zeta star values (MZSVs, for short) are given by

$$\zeta^*(p_1, p_2, \dots, p_n) = \sum_{0 < m_1 \leq m_2 \leq \dots \leq m_n} \frac{1}{m_1^{p_1} m_2^{p_2} \cdots m_n^{p_n}} \quad (6)$$

for $p_1, p_2, \dots, p_n \in \mathbb{Z}_{\geq 1}$ with $p_n \geq 2$. Further, the multiple Hurwitz zeta star values are given by

$$\begin{aligned} & \zeta^*(p_1, p_2, \dots, p_n; \alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \sum_{0 \leq m_1 \leq m_2 \leq \dots \leq m_n} \frac{1}{(m_1 + \alpha_1)^{p_1} (m_2 + \alpha_2)^{p_2} \cdots (m_n + \alpha_n)^{p_n}} \end{aligned} \quad (7)$$

for $p_1, p_2, \dots, p_n \in \mathbb{Z}_{\geq 1}$ with $p_n \geq 2$ and $\alpha_1, \alpha_2, \dots, \alpha_n > 0$. Let $\{\alpha\}^n = (\underbrace{\alpha, \dots, \alpha}_n)$. In

particular, we have $\zeta^*(p_1, \dots, p_n; \{1\}^n) = \zeta^*(p_1, \dots, p_n)$.

Secondly, we recall multi-poly-Bernoulli numbers. Imatomi, Kaneko and Takeda defined

$$\frac{\text{Li}_{k_1, k_2, \dots, k_r}(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1, k_2, \dots, k_r)} \frac{t^n}{n!} \quad (8)$$

$$\frac{\text{Li}_{k_1, k_2, \dots, k_r}(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_n^{(k_1, k_2, \dots, k_r)} \frac{t^n}{n!} \quad (9)$$

for $k_1, k_2, \dots, k_r \in \mathbb{Z}$ in [11], where

$$\text{Li}_{k_1, k_2, \dots, k_r}(z) = \sum_{0 < m_1 < m_2 < \dots < m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \quad (|z| < 1) \quad (10)$$

is the multiple polylogarithm. In particular, we have $B_n^{(1)} = (-1)^n C_n^{(1)}$ and these are “ordinary” Bernoulli numbers. Kaneko and Tsumura defined another type of multi-poly-Bernoulli numbers by

$$\begin{aligned} & \sum_{a=0}^{r-1} (-1)^a \binom{r-1}{a} \sum_{l_1, \dots, l_r \geq 1} \frac{\prod_{j=1}^r (1 - e^{-\sum_{v=j}^r x_v})^{l_j-1}}{(l_1 + \cdots + l_r - a)^s} \\ &= \sum_{m_1, \dots, m_r \geq 0} \mathfrak{B}_{m_1, \dots, m_r}^{(s)} \frac{x_1^{m_1} \cdots x_r^{m_r}}{m_1! \cdots m_r!} \end{aligned} \quad (11)$$

for $s \in \mathbb{C}$ in [13]. The case of $r = 1$ gives $\mathfrak{B}_m^{(k)} = B_m^{(k)}$ for $k \in \mathbb{Z}$.

We see that (1), (2), (3) and (4) can be analytically continued as entire functions. The values of these functions at nonpositive integers are given by

$$\xi(k_1, k_2, \dots, k_r; -m) = (-1)^m C_m^{(k_1, k_2, \dots, k_r)}, \quad (12)$$

$$\eta(k_1, k_2, \dots, k_r; -m) = B_m^{(k_1, k_2, \dots, k_r)}, \quad (13)$$

$$\eta(-k_1, -k_2, \dots, -k_r; -m) = B_m^{(-k_1, -k_2, \dots, -k_r)}, \quad (14)$$

$$\tilde{\xi}(-k_1, -k_2, \dots, -k_r; -m) = C_m^{(-k_1, -k_2, \dots, -k_r)} \quad (15)$$

for $m \in \mathbb{Z}_{\geq 0}$. Arakawa-Kaneko [1] and Kaneko-Tsumura [13] constructed relations for the values of ξ and η with MZVs and MZSVs. For example, the following relations are important. For $m \in \mathbb{Z}_{\geq 0}$, $r, k \in \mathbb{Z}_{\geq 1}$, they proved

$$\xi(\{1\}^{r-1}, k; m+1) = \sum_{\substack{a_1+\dots+a_k=m \\ \forall a_j \geq 0}} \binom{a_k+r}{r} \zeta(a_1+1, \dots, a_{k-1}+1, a_k+r+1), \quad (16)$$

$$\eta(\{1\}^{r-1}, k; m+1) = (-1)^{r-1} \sum_{\substack{a_1+\dots+a_k=m \\ \forall a_j \geq 0}} \binom{a_k+r}{r} \zeta^*(a_1+1, \dots, a_{k-1}+1, a_k+r+1). \quad (17)$$

Further, Kaneko and Tsumura obtained the following result which is a kind of the duality formula. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, we have

$$\eta(-k_1, -k_2, \dots, -k_r; s) = \mathfrak{B}_{k_1, k_2, \dots, k_r}^{(s)} \quad (s \in \mathbb{C}). \quad (18)$$

Therefore, for $m \in \mathbb{Z}_{\geq 0}$, we obtain

$$B_m^{(-k_1, -k_2, \dots, -k_r)} = \mathfrak{B}_{k_1, k_2, \dots, k_r}^{(-m)} \quad (19)$$

(see [13, Theorem 4.7]).

Using these results, we define the special types of η , ξ and $\tilde{\xi}$ functions (see § 2-4). We explicitly give the relations for the values of these functions with MZVs, MZSVs and the multiple Hurwitz zeta star values (see Theorems 2.2, 2.4, 2.6, 3.2, 4.2).

2. η function

In this section, we consider η function, whose indices consist of positive integers and nonpositive integers.

2.1. $\eta(k, -n; s)$

DEFINITION 2.1. For $k \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C}$ with $\Re(s) > 0$, define

$$\eta(k, -n; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{k, -n}(1 - e^t)}{1 - e^t} dt. \quad (20)$$

The integral on the right-hand side converges absolutely in the domain $\Re(s) > 0$, as is seen from Lemma 2.1 and Remark 2.2.

LEMMA 2.1. For $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ with $k \geq n$, we have

$$\begin{aligned} \text{Li}_{k,-n}(z) = z & \left\{ \frac{P_{n-1}^{(n)}(z)}{(1-z)^{n+1}} \text{Li}_k(z) + \frac{P_{n-2}^{(n)}(z)}{(1-z)^n} \text{Li}_{k-1}(z) + \cdots \right. \\ & \left. + \cdots + \frac{P_0^{(n)}(z)}{(1-z)^2} \text{Li}_{k-n+1}(z) + \frac{1}{1-z} \text{Li}_{k-n}(z) \right\}, \end{aligned} \quad (21)$$

where $P_i^{(n)}(z)$ is the polynomial defined by

$$P_i^{(n)}(z) = \binom{n}{i+1} \sum_{j=0}^i \sum_{l=0}^{j+1} (-1)^l \binom{i+2}{l} (j-l+1)^{i+1} z^{i-j} \quad (22)$$

for $0 \leq i \leq n-1$.

Proof of Lemma 2.1. We can derive

$$\begin{aligned} \text{Li}_{k,-n}(z) &= \sum_{l,m=1}^{\infty} \frac{z^{l+m}}{l^k} (l+m)^n \\ &= \sum_{j=0}^n \binom{n}{j} \sum_{l,m=1}^{\infty} \frac{z^{l+m}}{l^k} l^j m^{n-j} \\ &= \sum_{j=0}^n \binom{n}{j} \text{Li}_{-n+j}(z) \text{Li}_{k-j}(z). \end{aligned}$$

Further, we know the formula

$$\text{Li}_{-n+j}(z) = \frac{\mathcal{E}_{n-j}(z)}{(1-z)^{n-j+1}} \quad (j \leq n),$$

where $\mathcal{E}_i(z)$ is the Eulerian polynomial given by

$$\mathcal{E}_i(z) = \sum_{j=0}^{i-1} \sum_{l=0}^{j+1} (-1)^l \binom{i+1}{l} (j-l+1)^i z^{i-j}$$

(see [3, (1.2), (1.3)], [5]). Setting $P_i^{(n)}(z) = \binom{n}{i+1} \mathcal{E}_{i+1}(z) \frac{1}{z}$, we can prove this lemma. \square

EXAMPLE 2.1. By Definition 2.1 and Lemma 2.1, we can derive $\eta(1, 0; s) = -s$, and

$$\begin{aligned} \eta(2, 0; 1) &= -\zeta(2), \\ \eta(2, 0; 2) &= -\zeta(2) - 2\zeta(3), \\ \eta(2, -1; 1) &= -\frac{1}{2} - \frac{1}{2}\zeta(2), \\ \eta(3, 0; 1) &= -2\zeta(3), \\ \eta(3, -1; 1) &= -\frac{1}{2}\zeta(2) - \zeta(3). \end{aligned}$$

THEOREM 2.1. *The function $\eta(k, -n; s)$ can be analytically continued as an entire function, and the values at nonpositive integers are given by*

$$\eta(k, -n; -m) = B_m^{(k, -n)} \quad (m \in \mathbb{Z}_{\geq 0}). \quad (23)$$

Proof. Let C be the standard contour, namely the path consisting of the positive real axis from ∞ to ε , a counterclockwise circle C_ε around the origin of radius ε , and the positive real axis from ε to ∞ . Let

$$\begin{aligned} H(k, -n; s) &= \int_C t^{s-1} \frac{\text{Li}_k(1-e^t)}{1-e^t} dt \\ &= (e^{2\pi i s} - 1) \int_\varepsilon^\infty t^{s-1} \frac{\text{Li}_k(1-e^t)}{1-e^t} dt + \int_{C_\varepsilon} t^{s-1} \frac{\text{Li}_k(1-e^t)}{1-e^t} dt. \end{aligned}$$

We can confirm

$$\text{Li}_k(1-e^t) = O(t^k) \quad (t \rightarrow \infty), \quad (24)$$

from [13, (16)] if $k \geq 1$, and from $\text{Li}_0(1-e^t) = e^{-t} - 1$ if $k = 0$. It follows from Lemma 2.1 and (24) that $H(k, -n; s)$ is entire, because the integrand has no singularity on C and the contour integral is absolutely convergent for all $s \in \mathbb{C}$. When we suppose $\Re(s) > 0$, we can see

$$\int_{C_\varepsilon} t^{s-1} \frac{\text{Li}_k(1-e^t)}{1-e^t} dt \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Hence

$$\eta(k, -n; s) = \frac{1}{(e^{2\pi i s} - 1)\Gamma(s)} H(k, -n; s),$$

which can be analytically continued to \mathbb{C} , and is entire. In fact $\eta(k, -n; s)$ is holomorphic for $\Re(s) > 0$, hence has no singularity at any positive integer. Note that

$$\frac{1}{(e^{2\pi i s} - 1)\Gamma(s)} \xrightarrow{s \rightarrow -m} \frac{(-1)^m m!}{2\pi i} \quad (m \in \mathbb{Z}_{\geq 0}).$$

Setting $s = -m \in \mathbb{Z}_{\leq 0}$, we have $\eta(k, -n; -m) = B_m^{(k, -n)}$. This completes the proof. \square

Concerning the values at positive integer arguments, we prove the following theorem.

THEOREM 2.2. *For $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ with $k > n$, we have*

$$\begin{aligned} &\eta(k, -n; m+1) \\ &= - \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} \sum_{\substack{a_{l+1}+\dots+a_k=m \\ \forall a_i \geq 0}} A_{l,j}^{(n)} (a_k+1) \frac{1}{(n-l-j+1)^{a_{l+1}+1}} \\ &\quad \times \zeta^*(a_{l+2}+1, \dots, a_{k-1}+1, a_k+2; \{n-l-j+1\}^{k-l-1}) \\ &\quad - \sum_{\substack{a_{n+1}+\dots+a_k=m \\ \forall a_i \geq 0}} (a_k+1) \zeta^*(a_{n+2}+1, \dots, a_{k-1}+1, a_k+2), \end{aligned} \quad (25)$$

where $A_{l,j}^{(n)}$ is the rational number defined by

$$A_{l,j}^{(n)} = \binom{n}{l} \sum_{b=0}^{n-l-j-1} \sum_{d=0}^{b+1} (-1)^{d+j} \binom{n-l+1}{d} \binom{n-l-b-1}{j} (b-d+1)^{n-l}. \quad (26)$$

REMARK 2.1. We denote the right-hand side of $\eta(k, -n; m+1)$ of (25) by $-S_1 - S_2$. If $n = 0$, we define $S_1 = 0$. If $n = k-1$, we define $S_2 = m+1$.

In order to prove the theorem, we give the following integral expression, which can be similarly proved as [13, Lemma 2.7].

LEMMA 2.2. For $a_{l+1}, a_{l+2}, \dots, a_{k-1}, a_k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} & \frac{1}{\prod_{m=l+1}^k \Gamma(a_m + 1)} \int_0^\infty \cdots \int_0^\infty x_{l+1}^{a_{l+1}} \cdots x_{k-1}^{a_{k-1}} x_k^{a_k+1} \\ & \times \frac{e^{x_k}}{e^{x_k} - 1} \frac{e^{x_{k-1}+x_k}}{e^{x_{k-1}+x_k} - 1} \cdots \frac{e^{x_{l+2}+\cdots+x_k}}{e^{x_{l+2}+\cdots+x_k} - 1} \frac{1}{e^{n(x_{l+1}+\cdots+x_k)}} dx_{l+1} \cdots dx_k \\ & = \frac{a_k + 1}{n^{a_{l+1}+1}} \zeta^*(a_{l+2} + 1, a_{l+3} + 1, \dots, a_{k-1} + 1, a_k + 2; \{n\}^{k-l-1}). \end{aligned}$$

Proof of Theorem 2.2. By setting $P_i^{(n)}(z) = \sum_{j=0}^i A_{n-i-1,j}^{(n)} (1-z)^j$ in Lemma 2.1, we obtain (26). Consider the integral

$$\begin{aligned} I(k, -n; s) &= \frac{1}{\Gamma(s)} \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} A_{l,j}^{(n)} \int_0^\infty \int_0^{u_k} \cdots \int_0^{u_{l+2}} u_k^{s-1} u_{l+1} \frac{e^{u_{l+1}}}{e^{u_{l+1}} - 1} \cdots \frac{e^{u_{k-1}}}{e^{u_{k-1}} - 1} \\ &\quad \times \frac{1}{e^{(n-l-j+1)u_k}} du_{l+1} \cdots du_k \\ &+ \frac{1}{\Gamma(s)} \int_0^\infty \int_0^{v_k} \cdots \int_0^{v_{n+2}} v_k^{s-1} v_{n+1} \frac{e^{v_{n+1}}}{e^{v_{n+1}} - 1} \cdots \frac{e^{v_{k-1}}}{e^{v_{k-1}} - 1} \\ &\quad \times \frac{1}{e^{v_k}} dv_{n+1} \cdots dv_k. \end{aligned}$$

We can transform this formula as follows by using the differential equation in [13, (17)] and $\text{Li}_1(x) = -\log(1-x)$.

$$\begin{aligned} I(k, -n; s) &= \frac{-1}{\Gamma(s)} \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} A_{l,j}^{(n)} \int_0^\infty u_k^{s-1} \int_0^{u_k} \cdots \left\{ \int_0^{u_{l+2}} \frac{\text{Li}_1(1 - e^{u_{l+1}}) e^{u_{l+1}}}{e^{u_{l+1}} - 1} du_{l+1} \right\} \\ &\quad \times \frac{e^{u_{l+2}}}{e^{u_{l+2}} - 1} \cdots \frac{e^{u_{k-1}}}{e^{u_{k-1}} - 1} \frac{1}{e^{(n-l-j+1)u_k}} du_{l+2} \cdots du_k \\ &+ \frac{-1}{\Gamma(s)} \int_0^\infty v_k^{s-1} \int_0^{v_k} \cdots \left\{ \int_0^{v_{n+2}} \frac{\text{Li}_1(1 - e^{v_{n+1}}) e^{v_{n+1}}}{e^{v_{n+1}} - 1} dv_{n+1} \right\} \\ &\quad \times \frac{e^{v_{n+2}}}{e^{v_{n+2}} - 1} \cdots \frac{e^{v_{k-1}}}{e^{v_{k-1}} - 1} \frac{1}{e^{v_k}} dv_{n+2} \cdots dv_k \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{\Gamma(s)} \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} A_{l,j}^{(n)} \int_0^\infty u_k^{s-1} \frac{\text{Li}_{k-l}(1-e^{u_k})}{e^{(n-l-j+1)u_k}} du_k \\
&\quad + \frac{-1}{\Gamma(s)} \int_0^\infty v_k^{s-1} \frac{\text{Li}_{k-n}(1-e^{v_k})}{e^{v_k}} dv_k \\
&= \frac{-1}{\Gamma(s)} \sum_{l=0}^{n-1} \int_0^\infty u_k^{s-1} \frac{P_{n-l-1}^{(n)}(1-e^{u_k})}{e^{(n-l+1)u_k}} \text{Li}_{k-l}(1-e^{u_k}) du_k \\
&\quad + \frac{-1}{\Gamma(s)} \int_0^\infty v_k^{s-1} \frac{1}{e^{v_k}} \text{Li}_{k-n}(1-e^{v_k}) dv_k \\
&= -\eta(k, -n; s).
\end{aligned}$$

On the other hand, we make the change of variables $u_{l+1} = x_k, u_{l+2} = x_{k-1} + x_k, \dots, u_k = x_{l+1} + \dots + x_k$ and $v_{n+1} = y_k, v_{n+2} = y_{k-1} + y_k, \dots, v_k = y_{n+1} + \dots + y_k$. Then, it follows from Lemma 2.2 that for $m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
I(k, -n; m+1) &= \frac{1}{\Gamma(m+1)} \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} A_{l,j}^{(n)} \int_0^\infty \cdots \int_0^\infty \sum_{\substack{a_{l+1}+\dots+a_k=m \\ \forall a_i \geq 0}} \frac{m!}{a_{l+1}! \cdots a_k!} \\
&\quad \times x_{l+1}^{a_{l+1}} \cdots x_k^{a_k} x_k \\
&\quad \times \frac{e^{x_k}}{e^{x_k} - 1} \cdots \frac{e^{x_{l+2}+\dots+x_k}}{e^{x_{l+2}+\dots+x_k} - 1} \frac{1}{e^{(n-l-j+1)(x_{l+1}+\dots+x_k)}} dx_{l+1} \cdots dx_k \\
&\quad + \frac{1}{\Gamma(m+1)} \int_0^\infty \cdots \int_0^\infty \sum_{\substack{a_{n+1}+\dots+a_k=m \\ \forall a_i \geq 0}} \frac{m!}{a_{n+1}! \cdots a_k!} y_{n+1}^{a_{n+1}} \cdots y_k^{a_k} y_k \\
&\quad \times \frac{e^{y_k}}{e^{y_k} - 1} \cdots \frac{e^{y_{n+2}+\dots+y_k}}{e^{y_{n+2}+\dots+y_k} - 1} \frac{1}{e^{y_{n+1}+\dots+y_k}} dy_{n+1} \cdots dy_k \\
&= \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} \sum_{\substack{a_{l+1}+\dots+a_k=m \\ \forall a_i \geq 0}} A_{l,j}^{(n)} \frac{1}{\Gamma(a_{l+1}+1) \cdots \Gamma(a_k+1)} \\
&\quad \times \int_0^\infty \cdots \int_0^\infty x_{l+1}^{a_{l+1}} \cdots x_{k-1}^{a_{k-1}} x_k^{a_k+1} \\
&\quad \times \frac{e^{x_k}}{e^{x_k} - 1} \cdots \frac{e^{x_{l+2}+\dots+x_k}}{e^{x_{l+2}+\dots+x_k} - 1} \frac{1}{e^{(n-l-j+1)(x_{l+1}+\dots+x_k)}} dx_{l+1} \cdots dx_k \\
&\quad + \sum_{\substack{a_{n+1}+\dots+a_k=m \\ \forall a_i \geq 0}} \frac{1}{\Gamma(a_{n+1}+1) \cdots \Gamma(a_k+1)} \\
&\quad \times \int_0^\infty \cdots \int_0^\infty y_{n+1}^{a_{n+1}} \cdots y_{k-1}^{a_{k-1}} y_k^{a_k+1} \\
&\quad \times \frac{e^{y_k}}{e^{y_k} - 1} \cdots \frac{e^{y_{n+2}+\dots+y_k}}{e^{y_{n+2}+\dots+y_k} - 1} \frac{1}{e^{y_{n+1}+\dots+y_k}} dy_{n+1} \cdots dy_k
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} \sum_{\substack{a_{l+1}+\dots+a_k=m \\ \forall a_i \geq 0}} A_{l,j}^{(n)}(a_k+1) \frac{1}{(n-l-j+1)^{a_{l+1}+1}} \\
&\quad \times \zeta^*(a_{l+2}+1, \dots, a_{k-1}+1, a_k+2; \{n-l-j+1\}^{k-l-1}) \\
&\quad + \sum_{\substack{a_{n+1}+\dots+a_k=m \\ \forall a_i \geq 0}} (a_k+1) \zeta^*(a_{n+2}+1, \dots, a_{k-1}+1, a_k+2; \{1\}^{k-n-1}).
\end{aligned}$$

□

REMARK 2.2. If $k \leq n$, we can obtain similar formulas by using the same method. For example, we have

$$\begin{aligned}
\eta(k, -k; m+1) &= - \sum_{l=0}^{k-2} \sum_{j=0}^{k-l-1} \sum_{\substack{a_{l+1}+\dots+a_k=m \\ \forall a_i \geq 0}} A_{l,j}^{(k)}(a_k+1) \frac{1}{(k-l-j+1)^{a_{l+1}+1}} \\
&\quad \times \zeta^*(a_{l+1}, \dots, a_{k-1}+1, a_k+2; \{k-l-j+1\}^{k-l-1}) \\
&\quad - k(m+1) \frac{1}{2^{m+2}} + \frac{1}{2^{m+1}} + 1
\end{aligned}$$

if $k = n$ with $k \neq 1$, and

$$\begin{aligned}
\eta(k, -k-1; m+1) &= - \sum_{l=0}^{k-2} \sum_{j=0}^{k-l} \sum_{\substack{a_{l+1}+\dots+a_k=m \\ \forall a_i \geq 0}} A_{l,j}^{(k+1)}(a_k+1) \frac{1}{(k-l-j+2)^{a_{l+1}+1}} \\
&\quad \times \zeta^*(a_{l+1}, \dots, a_{k-1}+1, a_k+2; \{k-l-j+2\}^{k-l-1}) \\
&\quad - (m+1) \frac{1}{3^{m+2}} (k+1)k + (m+1) \frac{1}{2^{m+3}} (k+1)k \\
&\quad - (k+3) \left(\frac{1}{3^{m+1}} + \frac{1}{2^{m+1}} \right)
\end{aligned}$$

if $k = n-1$ with $k \neq 1$. If $k \leq n-2$ with $k \neq 1$, we can derive

$$\begin{aligned}
\eta(k, -n; m+1) &= - \sum_{l=0}^{k-2} \sum_{j=0}^{n-l-1} \sum_{\substack{a_{l+1}+\dots+a_k=m \\ \forall a_i \geq 0}} A'_{l,j}^{(n)}(a_k+1) \frac{1}{(n-l-j+1)^{a_{l+1}+1}} \\
&\quad \times \zeta^*(a_{l+2}, \dots, a_{k-1}+1, a_k+2; \{n-l-j+1\}^{k-l-1}) \\
&\quad - \sum_{j=0}^{n-k} A'_{k-1,j}^{(n)} \frac{m+1}{(n-k-j+2)^{m+2}} \\
&\quad - \sum_{j=0}^{n-k-1} A'_{k,j}^{(n)} \left\{ \frac{1}{(n-k-j+1)^{m+1}} - \frac{1}{(n-k-j+2)^{m+1}} \right\},
\end{aligned}$$

where the rational numbers $\{A'_{l,j}^{(n)}\}$ can be determined similar to $\{A_{l,j}^{(n)}\}$ and will be used in Theorem 4.2, but their explicit expressions are omitted here.

EXAMPLE 2.2. If $k = 3$ and $n = 0$ in Theorem 2.2, we have

$$\eta(3, 0; m+1) = - \sum_{\substack{a_1+a_2+a_3=m \\ \forall a_i \geq 0}} (a_3 + 1) \zeta^*(a_2 + 1, a_3 + 2).$$

Setting $m = 0$, we obtain $\eta(3, 0; 1) = -\zeta^*(1, 2)$, which implies $\zeta^*(1, 2) = 2\zeta(3)$ by Example 2.1.

EXAMPLE 2.3. If $k = 2$ and $n = 1$ in Theorem 2.2, we have

$$\eta(2, -1; m+1) = - \sum_{a_2=0}^m (a_2 + 1) \frac{1}{2^{m-a_2+1}} \zeta^*(a_2 + 2; 2) - (m+1).$$

Setting $m = 0$, we obtain $\eta(2, -1; 1) = -\frac{1}{2}\zeta(2) - \frac{1}{2}$. This result corresponds to Example 2.1.

EXAMPLE 2.4. We have $A_{0,0}^{(1)} = 1$. Hence, if $k = 3$ and $n = 1$ in Theorem 2.2, we have

$$\begin{aligned} \eta(3, -1; m+1) &= - \sum_{a_1+a_2+a_3=m} (a_3 + 1) \frac{1}{2^{a_1+1}} \zeta^*(a_2 + 1, a_3 + 2; \{2\}^2) \\ &\quad - \sum_{a_2+a_3=m} (a_3 + 1) \zeta^*(a_3 + 2). \end{aligned}$$

Setting $m = 0$, we obtain $\zeta(3, -1; 1) = -\frac{1}{2}\zeta^*(1, 2; \{2\}^2) - \zeta(2)$, which implies $\zeta^*(1, 2; \{2\}^2) = -\zeta(2) + 2\zeta(3)$ by Example 2.1.

2.2. $\eta(-n, k; s)$

DEFINITION 2.2. For $k \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C}$ with $\Re(s) > 0$, define

$$\eta(-n, k; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{-n,k}(1 - e^t)}{1 - e^t} dt. \quad (27)$$

The integral on the right-hand side converges absolutely in the domain $\Re(s) > 0$, as is seen from the following lemma.

LEMMA 2.3. For $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$\text{Li}_{-n,k}(z) = \sum_{l=0}^{n+1} D_l^{(n)} \text{Li}_{k-l}(z). \quad (28)$$

Here, $D_l^{(n)}$ is the rational number defined by

$$D_l^{(n)} = \begin{cases} -\frac{\delta_{n,0}}{n+1} & (l = 0) \\ \frac{1}{n+1} (-1)^{n-l+1} \binom{n+1}{l} B_{n-l+1} & (l \neq 0) \end{cases} \quad (29)$$

for $l \in \mathbb{Z}$ with $0 \leq l \leq n+1$.

Proof. For $m_2 \geq 2, n \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}$ with $0 \leq l \leq n+1$, we can show

$$\sum_{m_1=1}^{m_2-1} m_1^n = D_0^{(n)} + D_1^{(n)} m_2 + D_2^{(n)} m_2^2 + \cdots + D_{n+1}^{(n)} m_2^{n+1}$$

as follows. By Faulhaber's formula (see [6]), we have

$$\begin{aligned} \sum_{m_1=1}^{m_2-1} m_1^n &= \frac{1}{n+1} (B_{n+1}(m_2) - B_{n+1}(1)) \\ &= \frac{1}{n+1} \left\{ \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} B_j m_2^{n-j+1} - B_{n+1} \right\} \\ &= \frac{1}{n+1} \left\{ \sum_{j=0}^{n+1} (-1)^{n-j+1} \binom{n+1}{j} B_{n-j+1} m_2^j - B_{n+1} \right\} \\ &= \frac{1}{n+1} \left\{ \left((-1)^{n+1} B_{n+1} - B_{n+1} \right) + \sum_{j=1}^{n+1} (-1)^{n-j+1} \binom{n+1}{j} B_{n-j+1} m_2^j \right\} \\ &= \frac{1}{n+1} \left\{ -\delta_{n,0} + \sum_{j=1}^{n+1} (-1)^{n-j+1} \binom{n+1}{j} B_{n-j+1} m_2^j \right\}, \end{aligned}$$

where $B_n(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} B_k x^{k-n}$ is the n th Bernoulli polynomial for $n \geq 0$. By the definition of $D_l^{(n)}$, we can show

$$\begin{aligned} \text{Li}_{-n,k}(z) &= \sum_{1 \leq m_1 < m_2} \frac{m_1^n z^{m_2}}{m_2^k} \\ &= \sum_{m_2=2}^{\infty} \frac{z^{m_2}}{m_2^k} \left(D_0^{(n)} + D_1^{(n)} m_2 + D_2^{(n)} m_2^2 + \cdots + D_{n+1}^{(n)} m_2^{n+1} \right) \\ &= \sum_{l=0}^{n+1} D_l^{(n)} \text{Li}_{k-l}(z) - z \sum_{l=0}^{n+1} D_l^{(n)}, \end{aligned} \tag{30}$$

where

$$\sum_{l=0}^{n+1} D_l^{(n)} = \frac{1}{n+1} (B_{n+1}(1) - B_{n+1}(1)) = 0$$

for $n > 0$ and $n = 0$. Hence we obtain the proof. \square

EXAMPLE 2.5. By Definition 2.2 and Lemma 2.3, we can derive

$$\eta(0, 1; s) = -s\zeta(s+1) + 1,$$

$$\eta(-1, 3; s) = -\frac{1}{4}(s^2 + s + 2)\zeta(s+2) - \frac{1}{2}\zeta(s, 2) + \frac{1}{2}s\zeta(1, s+1) + \frac{1}{2}s\zeta(s+1),$$

$$\eta(-1, 1; s) = \frac{1}{4} \frac{1}{2^{s-1}} - \frac{1}{2}.$$

Setting $s = 1$, we have

$$\begin{aligned}\eta(0, 1; 1) &= -\zeta(2) + 1, \\ \eta(-1, 3; 1) &= -\zeta(3) + \frac{1}{2}\zeta(2), \\ \eta(-1, 1; 1) &= -\frac{1}{4}.\end{aligned}$$

THEOREM 2.3. *The function $\eta(-n, k; s)$ can be analytically continued as an entire function, and the values at nonpositive integers are given by*

$$\eta(-n, k; -m) = B_m^{(-n, k)} \quad (m \in \mathbb{Z}_{\geq 0}). \quad (31)$$

Proof. Similar to Theorem 2.1, we can obtain the proof. \square

THEOREM 2.4. *For $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ with $k > n + 1$, we have*

$$\eta(-n, k; m + 1) = \sum_{l=0}^{n+1} D_l^{(n)} \sum_{\substack{a_1 + \dots + a_{k-l} = m \\ \forall a_j \geq 0}} (a_{k-l} + 1) \zeta^*(a_1 + 1, \dots, a_{k-l-1} + 1, a_{k-l} + 2), \quad (32)$$

and with $k \leq n + 1$, we have

$$\begin{aligned}\eta(-n, k; m + 1) &= \sum_{l=0}^{k-1} D_l^{(n)} \sum_{\substack{a_1 + \dots + a_{k-l} = m \\ \forall a_j \geq 0}} (a_{k-l} + 1) \zeta^*(a_1 + 1, \dots, a_{k-l-1} + 1, a_{k-l} + 2) \\ &\quad + \sum_{l=k}^{n+1} D_l^{(n)} B_{-k+l}^{(m+1)}.\end{aligned} \quad (33)$$

Proof. By Lemma 2.3, we can see that

$$\begin{aligned}\eta(-n, k; m + 1) &= \frac{1}{\Gamma(m + 1)} \int_0^\infty t^m \frac{\text{Li}_{-n, k}(1 - e^t)}{1 - e^t} dt \\ &= \frac{1}{\Gamma(m + 1)} \int_0^\infty t^m \frac{\sum_{l=0}^{n+1} D_l^{(n)} \text{Li}_{k-l}(1 - e^t)}{1 - e^t} dt \\ &= \sum_{l=0}^{n+1} D_l^{(n)} \eta_{k-l}(m + 1).\end{aligned}$$

In the case $k > n + 1$, using (17), we obtain

$$\eta(-n, k; m + 1) = \sum_{l=0}^{n+1} D_l^{(n)} \sum_{\substack{a_1 + \dots + a_{k-l} = m \\ \forall a_j \geq 0}} (a_{k-l} + 1) \zeta^*(a_1 + 1, \dots, a_{k-l-1} + 1, a_{k-l} + 2).$$

On the other hand, in the case $k \leq n + 1$, using (17), (18) and (19), we derive

$$\begin{aligned} \eta(-n, k; m+1) &= \sum_{l=0}^{k-1} D_l^{(n)} \eta_{k-l}(m+1) + \sum_{l=k}^{n+1} D_l^{(n)} \eta_{k-l}(m+1) \\ &= \sum_{l=0}^{k-1} D_l^{(n)} \sum_{\substack{a_1+\dots+a_{k-l}=m \\ \forall a_j \geq 0}} (a_{k-l}+1) \zeta^*(a_1+1, \dots, a_{k-l-1}+1, a_{k-l}+2) \\ &\quad + \sum_{l=k}^{n+1} D_l^{(n)} \mathfrak{B}_{-k+l}^{(m+1)}, \end{aligned}$$

where $\mathfrak{B}_{-k+l}^{(m+1)} = B_{-k+l}^{(m+1)}$. \square

REMARK 2.3. In Theorem 2.4, we denote the right-hand side of (32) by $\sum_{l=0}^{n+1} S_l$. If $n = k - 1$, we define $S_{n+1} = S_k = D_k^{(n)}$.

EXAMPLE 2.6. We have $D_0^{(0)} = -1$ and $D_1^{(0)} = 1$. Hence, if $n = 0$ and $k = 1$ in Theorem 2.4, we have

$$\begin{aligned} \eta(0, 1; m+1) &= \sum_{l=0}^1 D_l^{(0)} \sum_{\substack{a_1+\dots+a_{1-l}=m \\ \forall a_j \geq 0}} (a_{1-l}+1) \zeta^*(a_1+1, \dots, a_{-l}+1, a_{-l+1}+2) \\ &= -\zeta(m+2) + 1. \end{aligned}$$

In particular $\eta(0, 1; 1) = -\zeta(2) + 1$, which corresponds to the result in Example 2.5.

2.3. $\eta(1, \dots, 1, -n; s)$

In this section, we construct the formula similar to (16) and (17).

DEFINITION 2.3. For $r \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C}$ with $\Re(s) > 0$, define

$$\eta(\underbrace{1, \dots, 1}_{r-1}, -n; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{1-e^t} \text{Li}_{\underbrace{1, \dots, 1}_{r-1}, -n}(1-e^t) dt. \quad (34)$$

The integral on the right-hand side converges absolutely in the domain $\Re(s) > 0$, as is seen from the following lemma.

LEMMA 2.4. For $r \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ with $r > n + 1$, we have

$$\begin{aligned} \text{Li}_{\underbrace{1, \dots, 1}_{r-1}, -n}(z) &= z \left\{ \frac{Q_1^{(n)}(z)}{(1-z)^{n+1}} \text{Li}_{\underbrace{1, \dots, 1}_{r-1}}(z) + \frac{Q_2^{(n)}(z)}{(1-z)^{n+1}} \text{Li}_{\underbrace{1, \dots, 1}_{r-2}}(z) + \dots \right. \\ &\quad \left. + \dots + \frac{Q_n^{(n)}(z)}{(1-z)^{n+1}} \text{Li}_{\underbrace{1, \dots, 1}_{r-n}}(z) + \frac{Q_{n+1}^{(n)}(z)}{(1-z)^{n+1}} \text{Li}_{\underbrace{1, \dots, 1}_{r-n-1}}(z) \right\}, \end{aligned} \quad (35)$$

where $Q_i^{(n)}(z)$ is the polynomial defined by

$$Q_i^{(n)}(z) = \sum_{k=i}^{n+1} \sum_{l=0}^{n-k+1} (-1)^l \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} k \\ i \end{Bmatrix} \binom{n-k+1}{l} z^{k+l-1} \quad (36)$$

for $1 \leq i \leq n+1$, and $\begin{Bmatrix} k \\ m \end{Bmatrix}$ is the Stirling number of the first kind and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ is that of the second kind.

In order to prove the lemma, we show the following.

LEMMA 2.5. For $r, k \in \mathbb{Z}_{\geq 1}$ with $r > k+1$, we have

$$\left(\frac{d}{dz}\right)^k \underbrace{\text{Li}_{1,\dots,1}(z)}_r = \sum_{m=1}^k \frac{1}{(1-z)^k} \begin{Bmatrix} k \\ m \end{Bmatrix} \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-m}.$$

Proof. We prove this lemma by induction. If $k=1$, then we have

$$\frac{d}{dz} \underbrace{\text{Li}_{1,\dots,1}(z)}_r = \frac{1}{1-z} \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-1}.$$

For $k \geq 1$, we assume the case of k holds and consider the case of $k+1$ by using the relational expression $\begin{Bmatrix} n+1 \\ m \end{Bmatrix} = \begin{Bmatrix} n \\ m-1 \end{Bmatrix} + n \begin{Bmatrix} n \\ m \end{Bmatrix}$. By induction hypothesis, we have

$$\begin{aligned} \left(\frac{d}{dz}\right)^{k+1} \underbrace{\text{Li}_{1,\dots,1}(z)}_r &= \frac{d}{dz} \left(\sum_{m=1}^k \frac{1}{(1-z)^k} \begin{Bmatrix} k \\ m \end{Bmatrix} \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-m} \right) \\ &= \frac{1}{(1-z)^{k+1}} \left(k \sum_{m=1}^k \begin{Bmatrix} k \\ m \end{Bmatrix} \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-m} + \sum_{m=2}^{k+1} \begin{Bmatrix} k \\ m-1 \end{Bmatrix} \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-m} \right) \\ &= \frac{1}{(1-z)^{k+1}} \left(k \begin{Bmatrix} k \\ 1 \end{Bmatrix} \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-1} \right. \\ &\quad \left. + \sum_{m=2}^k \left(k \begin{Bmatrix} k \\ m \end{Bmatrix} + \begin{Bmatrix} k \\ m-1 \end{Bmatrix} \right) \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-m} + \begin{Bmatrix} k \\ k \end{Bmatrix} \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-k-1} \right) \\ &= \sum_{m=1}^{k+1} \frac{1}{(1-z)^{k+1}} \begin{Bmatrix} k+1 \\ m \end{Bmatrix} \underbrace{\text{Li}_{1,\dots,1}(z)}_{r-m}. \end{aligned}$$

□

Proof of Lemma 2.4. When $r > n+1$, we have

$$\underbrace{\text{Li}_{1,\dots,1,-n}(z)}_{r-1} = \left(z \frac{d}{dz} \right)^{n+1} \underbrace{\text{Li}_{1,\dots,1}(z)}_r$$

$$= \sum_{k=1}^{n+1} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} z^k \left(\frac{d}{dz} \right)^k \text{Li}_{\underbrace{1, \dots, 1}_r}(z).$$

By using Lemma 2.5, we derive

$$\begin{aligned} \text{Li}_{\underbrace{1, \dots, 1}_r, -n}(z) &= \sum_{k=1}^{n+1} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} z^k \frac{1}{(1-z)^k} \sum_{i=1}^k \begin{bmatrix} k \\ i \end{bmatrix} \text{Li}_{\underbrace{1, \dots, 1}_{r-i}}(z) \\ &= \frac{z}{(1-z)^{n+1}} \sum_{i=1}^{n+1} \sum_{k=i}^{n+1} z^{k-1} (1-z)^{n-k+1} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \text{Li}_{\underbrace{1, \dots, 1}_{r-i}}(z). \end{aligned}$$

Therefore we can see that (35) holds and $\mathcal{Q}_i^{(n)}(z)$ is the polynomial written as

$$\begin{aligned} \mathcal{Q}_i^{(n)}(z) &= \sum_{k=i}^{n+1} z^{k-1} (1-z)^{n-k+1} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \\ &= \sum_{k=i}^{n+1} \sum_{l=0}^{n-k+1} (-1)^l \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \binom{n-k+1}{l} z^{k+l-1}. \end{aligned}$$

□

EXAMPLE 2.7. By Definition 2.3 and Lemma 2.4, we can derive

$$\eta(1, 1, -1; s) = \frac{(s+1)s}{2^{s+3}} - \frac{s}{2^{s+1}} + s.$$

Setting $s = 1$, we have

$$\eta(1, 1, -1; 1) = \frac{7}{8}.$$

THEOREM 2.5. *The function $\eta(1, \dots, 1, -n; s)$ can be analytically continued as an entire function, and the values at nonpositive integers are given by*

$$\eta(\underbrace{1, \dots, 1}_{r-1}, -n; -m) = B_m^{(\underbrace{1, \dots, 1}_{r-1}, -n)} \quad (m \in \mathbb{Z}_{\geq 0}). \quad (37)$$

Proof. Similar to Theorem 2.1, we can obtain this theorem. □

THEOREM 2.6. *For $m \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ with $r > n+1$, we have*

$$\eta(\underbrace{1, \dots, 1}_{r-1}, -n; m+1) = \sum_{l=1}^{n+1} \sum_{j=0}^n \binom{m+r-l}{m} (-1)^{r-l} E_{l,j}^{(n)} \frac{1}{(n-j+1)^{m+r-l+1}}, \quad (38)$$

where $E_{l,j}^{(n)}$ is the rational number defined by

$$E_{l,j}^{(n)} = \sum_{M=j+1}^{n+1} \sum_{k=l}^M (-1)^{M-k+j} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} \binom{n-k+1}{M-k} \binom{M-1}{j}. \quad (39)$$

Proof. Suppose $r > n + 1$. By setting $Q_i^{(n)}(z) = \sum_{j=0}^n E_{i,j}^{(n)}(1-z)^j$ in Lemma 2.4, we obtain

$$E_{i,j}^{(n)} = \sum_{M=j+1}^{n+1} \sum_{k=i}^M (-1)^{M-k+j} \binom{n+1}{k} \binom{k}{i} \binom{n-k+1}{M-k} \binom{M-1}{j}.$$

We can transform (34).

$$\begin{aligned} \eta(\underbrace{1, \dots, 1}_{r-1}, -n; s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{1-e^t} \text{Li}_{\underbrace{1, \dots, 1}_{r-1}, -n}(1-e^t) dt \\ &= \frac{1}{\Gamma(s)} \sum_{l=1}^{n+1} \sum_{j=0}^n E_{l,j}^{(n)} \int_0^\infty t^{s-1} e^{-t(n-j+1)} \text{Li}_{\underbrace{1, \dots, 1}_{r-l}}(1-e^t) dt \\ &= \frac{1}{\Gamma(s)} \sum_{l=1}^{n+1} \sum_{j=0}^n (-1)^{r-l} E_{l,j}^{(n)} \frac{1}{(r-l)!} \Gamma(s+r-l) \frac{1}{(n-j+1)^{s+r-l}}. \end{aligned}$$

Setting $s = m + 1$, we have

$$\begin{aligned} \eta(\underbrace{1, \dots, 1}_{r-1}, -n; m+1) &= \frac{1}{m!} \sum_{l=1}^{n+1} \sum_{j=0}^n (-1)^{r-l} E_{l,j}^{(n)} \frac{1}{(r-l)!} (m+r-l)! \frac{1}{(n-j+1)^{m+r-l+1}} \\ &= \sum_{l=1}^{n+1} \sum_{j=0}^n \binom{m+r-l}{m} (-1)^{r-l} E_{l,j}^{(n)} \frac{1}{(n-j+1)^{m+r-l+1}}. \end{aligned}$$

This completes the proof. \square

REMARK 2.4. If $r \leq n + 1$, we can show a certain formula by using same method.

EXAMPLE 2.8. We have $E_{1,0}^{(1)} = 1$, $E_{1,1}^{(1)} = 0$, $E_{2,0}^{(1)} = 1$, $E_{2,1}^{(1)} = -1$. Hence, if $r = 3$ and $n = 1$ in Theorem 2.6, we have

$$\begin{aligned} \eta(1, 1, -1; m+1) &= \sum_{l=1}^2 \sum_{j=0}^1 \binom{m-l+3}{m} (-1)^{-l+3} E_{l,j}^{(1)} \frac{1}{(-j+2)^{m-l+4}} \\ &= \frac{(m+2)(m+1)}{2} \frac{1}{2^{m+3}} - (m+1) \frac{1}{2^{m+2}} + m+1. \end{aligned}$$

In particular $\eta(1, 1, -1; 1) = \frac{7}{8}$, which corresponds to the result in Example 2.7.

3. ξ function

In this section, we consider ξ function, whose indices consist of positive integers and nonpositive integers. We can give the following definition by Lemma 2.3.

DEFINITION 3.1. For $k \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ with $k > n + 1$ and $s \in \mathbb{C}$ with $\Re(s) > 0$, define

$$\xi(-n, k; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{-n,k}(1 - e^{-t})}{e^t - 1} dt. \quad (40)$$

EXAMPLE 3.1. By Definition 3.1 and Lemma 2.3, we can derive

$$\begin{aligned} \xi(0, 2; s) &= -\zeta(s, 2) - \zeta(s+2) + s\zeta(1, s+1) + s\zeta(s+1), \\ \xi(-1, 3; s) &= -\frac{1}{2}\zeta(s, 2) - \frac{1}{2}\zeta(s+2) + \frac{1}{2}s\zeta(1, s+1) + \frac{1}{2}s\zeta(s+1). \end{aligned}$$

Setting $s = 1$, we have

$$\begin{aligned} \xi(0, 2; 1) &= -\zeta(3) + \zeta(2), \\ \xi(-1, 3, 1) &= -\frac{1}{2}\zeta(3) + \frac{1}{2}\zeta(2). \end{aligned}$$

THEOREM 3.1. When $k > n + 1$, the function $\xi(-n, k; s)$ can be analytically continued as an entire function, and the values at nonpositive integers are given

$$\xi(-n, k; -m) = (-1)^m C_m^{(-n, k)} \quad (m \in \mathbb{Z}_{\geq 0}). \quad (41)$$

Proof. Similar to Theorem 2.1, we can obtain this theorem. \square

THEOREM 3.2. For $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ with $k > n + 1$, we have

$$\xi(-n, k; m+1) = \sum_{l=0}^{n+1} D_l^{(n)} \sum_{\substack{a_1 + \dots + a_{k-l} = m \\ \forall a_j \geq 0}} (a_{k-l} + 1) \zeta(a_1 + 1, \dots, a_{k-l-1} + 1, a_{k-l} + 2). \quad (42)$$

Proof. We see

$$\begin{aligned} \xi(-n, k; s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{-n,k}(1 - e^{-t})}{e^t - 1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{e^t - 1} \sum_{l=0}^{n+1} D_l^{(n)} \text{Li}_{k-l}(1 - e^{-t}) dt \\ &= \sum_{l=0}^{n+1} D_l^{(n)} \xi(k-l; s). \end{aligned} \quad (43)$$

For $s = m + 1$, we obtain

$$\xi(-n, k; m+1) = \sum_{l=0}^{n+1} D_l^{(n)} \xi(k-l; m+1).$$

Hence, using (16), we can prove this theorem. \square

EXAMPLE 3.2. We have $D_0^{(0)} = -1$ and $D_1^{(0)} = 1$. Hence, if $n = 0$ and $k = 2$ in Theorem 3.2, we have

$$\xi(0, 2; m+1) = - \sum_{a_1+a_2=m} (a_2+1)\zeta(a_1+1, a_2+2) + (m+1)\zeta(m+2).$$

In particular $\xi(0, 2; 1) = -\zeta(1, 2) + \zeta(2)$, we implies $\zeta(1, 2) = \zeta(3)$ by Example 3.1.

EXAMPLE 3.3. We have $D_0^{(1)} = 0$, $D_1^{(1)} = -\frac{1}{2}$, $D_2^{(1)} = \frac{1}{2}$. Hence, if $n = 1$ and $k = 3$ in Theorem 3.2, we have

$$\xi(-1, 3; m+1) = -\frac{1}{2} \sum_{a_1+a_2=m} (a_2+1)\zeta(a_1+1, a_2+2) + \frac{1}{2}(m+1)\zeta(m+2).$$

In particular $\xi(-1, 3; 1) = -\frac{1}{2}\zeta(1, 2) + \frac{1}{2}\zeta(2)$, we implies $\zeta(1, 2) = \zeta(3)$ by Example 3.1.

REMARK 3.1. For $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$, $\xi(k, -n; s)$ seems to be unable to be defined by the same method as stated in the previous section. In fact, for example, if $k = 1$ and $n = 0$, we see that

$$\xi(1, 0; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Li}_1(1 - e^{-t}) dt = \frac{1}{\Gamma(s)} \int_0^\infty t^s dt$$

which is not convergent for any $s \in \mathbb{C}$. If $k = n = 1$, we see that

$$\begin{aligned} \xi(1, -1; s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} \{e^{2t} \text{Li}_1(1 - e^{-t}) + e^t \text{Li}_0(1 - e^{-t})\} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (te^t + e^t - 1) dt \end{aligned}$$

which is also not convergent for $s \in \mathbb{C}$. For the definition of $\xi(k, -n; s)$, it might be necessary to use another technique (see, for example, [15]). When $k \leq n+1$, $\xi(-n, k; s)$ might be also unable to be defined.

4. $\tilde{\xi}$ function

In this section, we consider $\tilde{\xi}$ function, whose indices consist of positive integers and nonpositive integers.

REMARK 4.1. For $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$, $\tilde{\xi}(-n, k; s)$ might be unable to be defined. We can consider in the same way as Remark 3.1.

On the other hand, we can give the following definition by Lemma 2.1.

DEFINITION 4.1. For $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ with $k < n$ and $s \in \mathbb{C}$ with $\Re(s) > 0$, define

$$\tilde{\xi}(k, -n; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{k,-n}(1 - e^t)}{e^{-t} - 1} dt. \quad (44)$$

REMARK 4.2. The definition of $\tilde{\xi}(k, -n; s)$ ($k \geq n$) is also unclear (see Remark 3.1).

EXAMPLE 4.1. By Definition 4.1 and Lemma 2.1, we can derive $\tilde{\xi}(1, -2; s) = -\frac{s-3}{2^s} + s - 3$, and

$$\begin{aligned}\tilde{\xi}(1, -2; 1) &= -1, \\ \tilde{\xi}(2, -3; 1) &= -1, \\ \tilde{\xi}(3, -4; 1) &= -1.\end{aligned}$$

THEOREM 4.1. When $k < n$, the function $\tilde{\xi}(k, -n; s)$ can be analytically continued as an entire function. And the values at nonpositive integers are given by

$$\tilde{\xi}(k, -n; -m) = C_m^{(k, -n)} \quad (m \in \mathbb{Z}_{\geq 0}). \quad (45)$$

Proof. Similar to Theorem 2.1, we can obtain the proof. \square

THEOREM 4.2. For $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ with $k < n$, we have

$$\begin{aligned}\tilde{\xi}(k, -n; m+1) &= - \sum_{l=0}^{k-2} \sum_{j=0}^{n-l-1} \sum_{\substack{a_{l+1}+\dots+a_k=m \\ \forall a_j \geq 0}} A'_{l,j}^{(n)} (a_k + 1) \frac{1}{(n-l-j)^{a_{l+1}+1}} \\ &\quad \times \zeta^*(a_{l+2} + 1, \dots, a_{k-1} + 1, a_k + 2; \{n-l-j\}^{k-l-1}) \\ &\quad - \sum_{j=0}^{n-k} A'_{k-1,j}^{(n)} \frac{m+1}{(n-k-j+1)^{m+2}} \\ &\quad - \sum_{j=0}^{n-k-1} A'_{k,j}^{(n)} \left\{ \frac{1}{(n-k-j)^{m+1}} - \frac{1}{(n-k-j+1)^{m+1}} \right\}, \quad (46)\end{aligned}$$

where $A'_{l,j}^{(n)}$ is a certain integer.

Proof. When $k < n$, we have

$$\begin{aligned}\text{Li}_{k,-n}(z) &= z \left\{ \frac{P'_{n-1}^{(n)}(z)}{(1-z)^{n+1}} \text{Li}_k(z) + \frac{P'_{n-2}^{(n)}(z)}{(1-z)^n} \text{Li}_{k-1}(z) + \dots \right. \\ &\quad \left. + \dots + \frac{P'_{n-k}^{(n)}(z)}{(1-z)^{n-k+2}} \text{Li}_1(z) + \frac{P'_{n-k-1}^{(n)}(z)}{(1-z)^{n-k+1}} \text{Li}_0(z) \right\}, \quad (47)\end{aligned}$$

where $P'_i^{(n)}(z)$ is a certain polynomial of coefficients of integers. (When $k \geq n$, we have Lemma 2.1.) By setting $P'_i^{(n)}(z) = \sum_{j=0}^i A'_{n-i-1,j}^{(n)} (1-z)^j$, we have $P'_i^{(n)}(1-e^t) = \sum_{j=0}^i A'_{n-i-1,j}^{(n)} e^{tj}$. Consider the integral

$$\begin{aligned}\tilde{I}(k, -n; s) &= \frac{1}{\Gamma(s)} \sum_{l=0}^{k-2} \sum_{j=0}^{n-l-1} A'_{l,j}^{(n)} \int_0^\infty \int_0^{u_k} \cdots \int_0^{u_{l+2}} u_k^{s-1} u_{l+1} \\ &\quad \times \frac{e^{u_{l+1}}}{e^{u_{l+1}} - 1} \cdots \frac{e^{u_{k-1}}}{e^{u_{k-1}} - 1} \frac{1}{e^{(n-l-j)u_k}} du_{l+1} \cdots du_k\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{n-k} A'^{(n)}_{k-1,j} \frac{s}{(n-k-j+1)^{s+1}} \\
& + \sum_{j=0}^{n-k-1} A'^{(n)}_{k,j} \left\{ \frac{1}{(n-k-j)^s} - \frac{1}{(n-k-j+1)^s} \right\}.
\end{aligned}$$

We can transform this formula as follows.

$$\begin{aligned}
\tilde{I}(k, -n; s) & = -\frac{1}{\Gamma(s)} \sum_{l=0}^{k-2} \sum_{j=0}^{n-l-1} A'^{(n)}_{l,j} \int_0^\infty u_k^{s-1} \int_0^{u_k} \cdots \left\{ \int_0^{u_{l+2}} \frac{\text{Li}_1(1-e^{u_{l+1}})e^{u_{l+1}}}{e^{u_{l+1}}-1} du_{l+1} \right\} \\
& \quad \times \frac{e^{u_{l+2}}}{e^{u_{l+2}}-1} \cdots \frac{e^{u_{k-1}}}{e^{u_{k-1}}-1} \frac{1}{e^{(n-l-j)u_k}} du_{l+2} \cdots du_k \\
& + \frac{1}{\Gamma(s)} \sum_{j=0}^{n-k} A'^{(n)}_{k-1,j} \int_0^\infty u_k^s \frac{1}{e^{u_k(n-k-j+1)}} du_k \\
& + \frac{1}{\Gamma(s)} \sum_{j=0}^{n-k-1} A'^{(n)}_{k,j} \int_0^\infty u_k^{s-1} \frac{e^{u_k}-1}{e^{u_k(n-k-j+1)}} du_k \\
& = -\frac{1}{\Gamma(s)} \sum_{l=0}^k \int_0^\infty u_k^{s-1} \frac{\sum_{j=0}^{n-l-1} A'^{(n)}_{l,j} e^{u_{kj}}}{e^{u_k(n-l)}} \text{Li}_{k-l}(1-e^{u_k}) du_k \\
& = -\frac{1}{\Gamma(s)} \sum_{l=0}^k \int_0^\infty u_k^{s-1} \frac{P'^{(n)}_{n-l-1}(1-e^{u_k})}{e^{u_k(n-l)}} \text{Li}_{k-l}(1-e^{u_k}) du_k \\
& = -\tilde{\xi}(k, -n; s).
\end{aligned}$$

We make the change of variables $u_{l+1} = x_k, u_{l+2} = x_{k-1} + x_k, \dots, u_k = x_{l+1} + \cdots + x_k$. Then, it follows from Lemma 2.2 that for $m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
\tilde{I}(k, -n; m+1) & = \frac{1}{\Gamma(m+1)} \sum_{l=0}^{k-2} \sum_{j=0}^{n-l-1} A'^{(n)}_{l,j} \int_0^\infty \cdots \int_0^\infty (x_{l+1} + \cdots + x_k)^m x_k \\
& \quad \times \frac{e^{x_k}}{e^{x_k}-1} \cdots \frac{e^{x_{l+2}+\cdots+x_k}}{e^{x_{l+2}+\cdots+x_k}-1} \frac{1}{e^{(n-l-j)(x_{l+1}+\cdots+x_k)}} dx_{l+1} \cdots dx_k \\
& + \sum_{j=0}^{n-k} A'^{(n)}_{k-1,j} \frac{m+1}{(n-k-j+1)^{m+2}} \\
& + \sum_{j=0}^{n-k-1} A'^{(n)}_{k,j} \left\{ \frac{1}{(n-k-j)^{m+1}} - \frac{1}{(n-k-j+1)^{m+1}} \right\} \\
& = \sum_{l=0}^{k-2} \sum_{j=0}^{n-l-1} \sum_{\substack{a_{l+1}+\cdots+a_k=m \\ \forall a_i \geq 0}} A'^{(n)}_{l,j} \frac{1}{\Gamma(a_{l+1}+1) \cdots \Gamma(a_k+1)}
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^\infty \cdots \int_0^\infty x_{l+1}^{a_{l+1}} \cdots x_{k-1}^{a_{k-1}} x_k^{a_k+1} \\
& \times \frac{e^{x_k}}{e^{x_k} - 1} \cdots \frac{e^{x_{l+2} + \cdots + x_k}}{e^{x_{l+2} + \cdots + x_k} - 1} \frac{1}{e^{(n-l-j)(x_{l+1} + \cdots + x_k)}} dx_{l+1} \cdots dx_k \\
& + \sum_{j=0}^{n-k} A'^{(n)}_{k-1, j} \frac{m+1}{(n-k-j+1)^{m+2}} \\
& + \sum_{j=0}^{n-k-1} A'^{(n)}_{k, j} \left\{ \frac{1}{(n-k-j)^{m+1}} - \frac{1}{(n-k-j+1)^{m+1}} \right\} \\
& = \sum_{l=0}^{k-2} \sum_{j=0}^{n-l-1} \sum_{\substack{a_{l+1} + \cdots + a_k = m \\ \forall a_i \geq 0}} A'^{(n)}_{l, j} (a_k + 1) \frac{1}{(n-l-j)^{a_{l+1}+1}} \\
& \times \xi^*(a_{l+2} + 1, \dots, a_{k-1} + 1, a_k + 2; \{n-l-j\}^{k-l-1}) \\
& + \sum_{j=0}^{n-k} A'^{(n)}_{k-1, j} \frac{m+1}{(n-k-j+1)^{m+2}} \\
& + \sum_{j=0}^{n-k-1} A'^{(n)}_{k, j} \left\{ \frac{1}{(n-k-j)^{m+1}} - \frac{1}{(n-k-j+1)^{m+1}} \right\}.
\end{aligned}$$

□

REMARK 4.3. Using the Eulerian polynomial $\mathcal{E}_i(z)$, we obtain

$$\begin{aligned}
P'^{(n)}_{n-m-1}(z) &= \frac{1}{z} \binom{n}{m} \mathcal{E}_{n-m}(z) \quad (0 \leq m \leq k-1), \\
P'^{(n)}_{n-k-1}(z) &= \frac{1}{z^2} \sum_{j=k}^n \binom{n}{j} \mathcal{E}_{n-j}(z) \mathcal{E}_{j-k}(z).
\end{aligned}$$

Further $A'^{(n)}_{l, j}$ can be explicitly written. However, this is complicated.

EXAMPLE 4.2. We have $A'^{(2)}_{0,0} = 2$, $A'^{(2)}_{0,1} = -1$ and $A'^{(2)}_{1,0} = 3$. Hence, by Theorem 4.2, if $k = 1$ and $n = 2$,

$$\tilde{\xi}(1, -2; m+1) = -2 \frac{m+1}{2^{m+2}} + (m+1) - 3 \left(1 - \frac{1}{2^{m+1}} \right).$$

Setting $m = 0$, we obtain $\tilde{\xi}(1, -2; 1) = -1$. This result corresponds to Example 4.1.

EXAMPLE 4.3. We have $A'^{(4)}_{0,0} = 24$, $A'^{(4)}_{0,1} = -36$, $A'^{(4)}_{0,2} = 14$, $A'^{(4)}_{0,3} = -1$, $A'^{(4)}_{1,0} = 24$, $A'^{(4)}_{1,1} = -24$, $A'^{(4)}_{1,2} = 4$, $A'^{(4)}_{2,0} = 12$, $A'^{(4)}_{2,1} = -6$ and $A'^{(4)}_{3,0} = 5$. Hence, by Theorem 4.2, if $k = 3$ and $n = 4$,

$$\tilde{\xi}(3, -4; m+1) = - \sum_{a_1+a_2+a_3=m} 24(a_3+1) \frac{1}{4^{a_1+1}} \zeta^*(a_2+1, a_3+2; \{4\}^2)$$

$$\begin{aligned}
& + \sum_{a_1+a_2+a_3=m} 36(a_3+1) \frac{1}{3^{a_1+1}} \zeta^*(a_2+1, a_3+2; \{3\}^2) \\
& - \sum_{a_1+a_2+a_3=m} 14(a_3+1) \frac{1}{2^{a_1+1}} \zeta^*(a_2+1, a_2+2; \{2\}^2) \\
& + \sum_{a_1+a_2+a_3=m} (a_3+1) \zeta^*(a_2+1, a_3+2; \{1\}^2) \\
& - \sum_{a_2+a_3=m} 24(a_3+1) \frac{1}{3^{a_2+1}} \zeta^*(a_3+2; 3) \\
& + \sum_{a_2+a_3=m} 24(a_3+1) \frac{1}{2^{a_2+1}} \zeta^*(a_3+2; 2) \\
& - \sum_{a_2+a_3=m} 4(a_3+1) \zeta^*(a_3+2; 1) \\
& - 12 \frac{m+1}{2^{m+2}} + 6(m+1) - 5 \left(1 - \frac{1}{2^{m+1}} \right).
\end{aligned}$$

Setting $m=0$, we obtain $\tilde{\xi}(3, -4; 1) = -6\zeta^*(1, 2; \{4\}^2) + 12\zeta^*(1, 2; \{3\}^2) - 7\zeta^*(1, 2; \{2\}^2) + 6\zeta^*(1, 2; \{1\}^2) - 8\zeta^*(2; 3) + 12\zeta^*(2; 2) - 4\zeta^*(2; 1) + \frac{1}{2}$, which implies $-6\zeta^*(1, 2; \{4\}^2) + 12\zeta^*(1, 2; \{3\}^2) - 7\zeta^*(1, 2; \{2\}^2) + 6\zeta^*(1, 2; \{1\}^2) - 8\zeta^*(2; 3) + 12\zeta^*(2; 2) - 4\zeta^*(2; 1) = -\frac{3}{2}$ by Example 4.1.

REMARK 4.4. More generally it seems possible to construct $\eta(k_1 \dots, k_r; s)$ for $k_1, \dots, k_r \in \mathbb{Z}$, and $\xi(k_1 \dots, k_r; s)$ and $\tilde{\xi}(k_1 \dots, k_r; s)$ for $k_1, \dots, k_r \in \mathbb{Z}$ under certain conditions in a similar manner. However, these procedures will be remarkably complicated.

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