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A Tower of Ramanujan Graphs and a Reciprocity Law of Graph Zeta Functions

by

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Abstract. Let l be an odd prime. We will construct a tower of connected regular Ramanujan graph of degree $l+1$ from modular curves. This supplies an example of a collection of $(l+1)$-regular graphs whose non-zero eigenvalues of the Laplacian are contained in the interval $[(\sqrt{l}-1)^2, (\sqrt{l}+1)^2]$. We also show graph (or Ihara) zeta functions satisfy a certain reciprocity law.

Key words: a Ramanujan graph, the Cheeger constant, an expander, a graph zeta function, a modular curve, a Brandt matrix, a reciprocity law.
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1. Introduction

Let p be a prime satisfying $p \equiv 1 \pmod{12}$ and let us fix an odd prime l different from p. In [25] we have constructed a connected regular Ramanujan graph $G_p^{(l)}(1)$ of degree $l+1$ non-bipartite. The number of vertices $G_p^{(l)}(1)$ is $(p-1)/12$ and the Euler characteristic is

$$\chi(G_p^{(l)}(1)) = \frac{(p-1)(1-l)}{24}.$$

The graph $G_p^{(l)}(1)$ is regarded as a graph of level *one*. In this paper we will construct a connected non-bipartite regular Ramanujan graph of degree $l+1$ of a higher level.

In the following let p be a prime such that $p \equiv 1 \pmod{12}$ and l an odd prime different from p. Let $\mathcal{N}_{p,l}$ be the set of square free positive integers such that every member N is prime to lp. Then to each N of $\mathcal{N}_{p,l}$, a connected non-bipartite $(l+1)$-regular Ramanujan graph $G_p^{(l)}(N)$ of which the number of vertices is $\nu(N) := \frac{(p-1)\sum_{d|N} d}{12}$ will be assigned. Let $\lambda_0(G_p^{(l)}(N)) \leq \lambda_1(G_p^{(l)}(N)) \leq \cdots \leq \lambda_{\nu(N)-1}(G_p^{(l)}(N))$ denote eigenvalues of the Laplacian of $G_p^{(l)}(N)$. Since $G_p^{(l)}(N)$ is connected $\lambda_0(G_p^{(l)}(N)) = 0$ and $\lambda_1(G_p^{(l)}(N))$ is positive. A relationship between the adjacency matrix and the Laplacian (cf. (2)) shows that

$$\rho^i(G_p^{(l)}(N)) := (l+1) - \lambda_i(G_p^{(l)}(N)) \tag{1}$$

is an eigenvalue of the adjacency matrix.

THEOREM 1.1. *(1) For $i \geq 1$.*

$$(\sqrt{l}-1)^2 \leq \lambda_i(G_p^{(l)}(N)) \leq (\sqrt{l}+1)^2, \quad \forall N \in \mathcal{N}_{p,l}.$$

(2) Let M and N be elements of $\mathcal{N}_{p,l}$ satisfying $M|N$. Then $G_p^{(l)}(N)$ is a covering of $G_p^{(l)}(M)$ of degree $\sigma_1(N/M)$ and

$$\rho^1(G_p^{(l)}(N)) \geq \rho^1(G_p^{(l)}(M)), \quad \lambda_1(G_p^{(l)}(N)) \leq \lambda_1(G_p^{(l)}(M)).$$

Here σ_1 is the Euler function defined by

$$\sigma_1(n) = \sum_{d|n} d.$$

Our tower of Ramanujan graphs $\{G_p^{(l)}(N)\}_{N \in \mathcal{N}_{p,l}}$ has an interesting geometric property. In order to explain further we recall *the (discrete) Cheeger constant*. In general let G be a connected d-regular graph of n vertices. The Cheeger constant $h(G)$ of G is defined by

$$h(G) = \min\{\frac{|\partial S|}{|S|} \ : \ S \subset V(G),\ 0 < |S| \leq \frac{n}{2}\},$$

where $V(G)$ denotes the set of vertices and

$$\partial S := \{\{u, v\} \in GE(G) \ : \ u \in S,\ v \in V(G) \setminus S\}.$$

Here $GE(G)$ is the set of geometric edges (i.e. the set of unoriented edges, see §2) and $|\cdot|$ denotes the cardinality. Then the smallest non-zero eigenvalue $\lambda_1(G)$ of the Laplacian satisfies ([2] [26])

$$\frac{\lambda_1(G)}{2} \leq h(G) \leq \sqrt{2d\lambda_1(G)}$$

and the next corollary is an immediate consequence of **Theorem 1.1**.

COROLLARY 1.1. *(A gap theorem)*

$$\frac{(\sqrt{l}-1)^2}{2} \leq h(G_p^{(l)}(N)) \leq \sqrt{2(l+1)}(\sqrt{l}+1)$$

for any $N \in \mathcal{N}_{p,l}$.

In general the graph zeta function (or the Ihara zeta function) $Z(G)(t)$ is defined for a finite connected graph G. Although a priori $Z(G)(t)$ is a power series of t, the Ihara formula tells us that it is a rational function (see **Fact 2.1**). We will show that the zeta functions of our graphs satisfy a reciprocity law.

THEOREM 1.2. *(A reciprocity law) Let p and q be distinct primes satisfying $p \equiv q \equiv 1 \pmod{12}$ and l an odd prime different from p and q. Then*

$$\frac{Z(G_p^{(l)}(q))(t)}{Z(G_p^{(l)}(1))(t)^2} = \frac{Z(G_q^{(l)}(p))(t)}{Z(G_q^{(l)}(1))(t)^2}.$$

In particular

$$Z(G_p^{(l)}(q))(t) \equiv Z(G_q^{(l)}(p))(t) \mod \mathbb{Q}(t)^{\times 2}.$$

Here is an application of **Theorem 1.1** to modular forms. As before let p be a prime satisfying $p \equiv 1 \pmod{12}$ and N a square free positive integer prime to p. Then the spaces of cusp forms $S_2(\Gamma_0(pN))$ and one of p-new forms $S_2(\Gamma_0(pN))_{pN/N}$ of level pN (see §4, especially (21)) have decompositions

$$S_2(\Gamma_0(pN)) = \oplus_\alpha \mathbb{C} f_\alpha, \quad S_2(\Gamma_0(pN))_{pN/N} = \oplus_\chi \mathbb{C} f_\chi,$$

where f_α and f_χ are normalized Hecke eigenforms of character α and χ (cf. **Theorem 4.1** and (22)). Using the result due to Alon-Boppana ([1] [2]) we will show the following.

THEOREM 1.3. *Let p be a prime satisfying $p \equiv 1 \pmod{12}$ and l an odd prime different from p. Let $\{r_i\}_{i=1}^{\infty}$ be a set of mutually distinct primes not dividing lp. Set $N_k = \prod_{i=1}^{k} r_i$ and then*

$$\lim_{k\to\infty} \mathrm{Max}\{a_l(f_\chi) \,:\, S_2(\Gamma_0(pN_k))_{pN_k/N_k} = \oplus_\chi \mathbb{C} f_\chi\} = 2\sqrt{l},$$

where $a_l(f_\chi)$ denotes the l-th Fourier coefficient of f_χ. In particular

$$\lim_{k\to\infty} \mathrm{Max}\{a_l(f_\alpha) \,:\, S_2(\Gamma_0(pN_k)) = \oplus_\alpha \mathbb{C} f_\alpha\} = 2\sqrt{l}.$$

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2. Basic facts of the zeta function of a graph

A (finite) graph G consists of a finite set of vertices $V(G)$ and a finite set of oriented edges $E(G)$, which satisfy the following property: there are *end point maps*,

$$\partial_0, \quad \partial_1 : E(G) \to V(G),$$

and *an orientation resersal*,

$$J : E(G) \to E(G), \quad J^2 = \text{identity},$$

such that $\partial_i \circ J = \partial_{1-i}$ $(i = 0, 1)$. The quotient $E(G)/J$ is called *the set of geometric edges* and is denoted by $GE(G)$. We regard an element of $e \in GE(G)$ as an unoriented edge and if its end-points are u and v we write $e = \{u, v\}$. For $x \in V(G)$ we set

$$E_j(x) = \{e \in E(G) \mid \partial_j(e) = x\}, \quad j = 0, 1.$$

Thus $JE_j(x) = E_{1-j}(x)$. Intuitively $E_0(x)$ (resp. $E_1(x)$) is the set of edges departing from (resp. arriving at) x. The *degree* of x, $d(x)$, is defined by

$$d(x) = |E_0(x)| + |E_1(x)|.$$

$E(G)$ is naturally divided into two classes, *loops* and *passes*. An edge $e \in E(G)$ is called *a loop* if $\partial_0(e) = \partial_1(e)$ (i.e. the two ends points of e coincide) and is called *a pass* otherwise. Let $p(x)$ be the number of passes starting from x. On the other hand $l(x)$ denotes the half of the number of loops at x, that is $l(x)$ is the number of *geometric* loops. Note that, because of the involution J, if we replace "departing" by "arriving" these number does not change. By definition, it is clear that

$$d(x) = 2l(x) + p(x).$$

We set $q(x) := d(x) - 1$. Let $C_0(G)$ be the free $\mathbb{Z}$-module generated by $V(G)$ with vertices as the natural basis. We define endomorphisms Q and A of $C_0(G)$ by

$$Q(x) = q(x)x, \quad x \in V(G),$$

and

$$A(x) = \sum_{e \in E(G), \partial_0(e) = x} \partial_1(e), \quad x \in V(G),$$

respectively. Note that because of the involution J,

$$A(x) = \sum_{e \in E(G), \partial_1(e) = x} \partial_0(e).$$

The operator A will be called *the adjacency operator*. We sometimes identify it with the representing matrix with respect to the basis $\{x\}_{x \in V(G)}$. Thus the yx-entry A_{yx} of A is the number of edges departing from x and arriving at y. The orientation reversing involution J implies

$$A_{xy} = A_{yx}.$$

Note that $A_{xx} = 2l(x)$ and $p(x) = \sum_{y \neq x} A_{yx}$. If $d(x) = k$ for all $x \in V(G)$, G is called *k-regular*.

Connecting distinct vertices x and y by geometric A_{xy}-edges and drawing $\frac{1}{2}A_{xx}$-loops at x, the adjacency matrix A determines an unoriented 1-dimensional simplicial complex. We call it *the geometric realization* of G, and denote it by G again. We say that G is connected if the geometric realization is. The Euler characteristic $\chi(G)$ is equal to $|V(G)| - |GE(G)|$, hence if G is connected, the fundamental group is a free group of rank $1 - |V(G)| + |GE(G)|$. For a later purpose, we summarize the relationship between a graph and its adjacency matrix.

PROPOSITION 2.1. *Let $A = (a_{ij})_{1 \leq i, j \leq m}$ be an $m \times m$-matrix satisfying the following conditions.*

(1) The entries $\{a_{ij}\}_{ij}$ are non-negative integers and satisfy

$$a_{ij} = a_{ji}, \quad \forall i \text{ and } j.$$

(2) a_{ii} is even for every i.

Then there is a unique graph G whose adjacency matrix is A. Moreover, G is k-regular if and only if one of the following equivalent condition satisfied :

(a)

$$\sum_{i=1}^{m} a_{ij} = k, \quad \forall j$$

(b)

$$\sum_{j=1}^{m} a_{ij} = k, \quad \forall i.$$

In the following, a graph G is always assumed to be *connected*. *A path of length m* is a sequence $c = (e_1, \cdots, e_m)$ of edges such that $\partial_0(e_i) = \partial_1(e_{i-1})$ for all $1 < i \leq m$ and the path is *reduced* if $e_i \neq J(e_{i-1})$ for all $1 < i \leq m$. The path is *closed* if $\partial_0(e_1) = \partial_1(e_m)$, and the closed path has *no tail* if $e_m \neq J(e_1)$. A closed path of length one is nothing but a loop. Two closed paths are *equivalent* if one is obtained from the other by a cyclic shift of the edges. Let $\mathfrak{C}(G)$ be the set of equivalence classes of reduced and tail-less closed paths of G. Since the length depends only on the equivalence class, the length function descends to the map;

$$l : \mathfrak{C}(G) \to \mathbb{N}, \quad l([c]) = l(c),$$

where $[c]$ is the class determined by c. We define a reduced and tail-less closed path C to be primitive if it is not obtained by going r (≥ 2) times some another closed path. Let $\mathfrak{P}(G)$ be the subset of $\mathfrak{C}(G)$ consisting of the classes of primitive closed paths (which are reduced and tail-less by definition). The graph zeta function (or *Ihara zeta function*) of G is defined to be

$$Z(G)(t) = \prod_{[c]\in\mathfrak{P}(G)} \frac{1}{1 - t^{l([c])}}.$$

Although this is an infinite product, it is a rational function.

FACT 2.1 ([4], [11], [12], [16], [24]).

$$Z(G)(t) = \frac{(1-t^2)^{\chi(G)}}{\det[1 - At + Qt^2]}.$$

FACT 2.2 ([25]). Let G be a k-regular graph with m vertices. Then the Euler characteristic $\chi(G)$ is

$$\chi(G) = \frac{m(2-k)}{2}.$$

REMARK 2.1. Note that the Euler characteristic does not depend on the number of loops.

Let $E_{or}(G) \subset E(G)$ be a section of the natural projection $E(G) \to GE(G)$. In other word we choose an orientation on geometric edges and make the geometric realization into an oriented one dimensional simplicial complex. Let $C_1(G)$ be the free $\mathbb{Z}$-module generated by $E_{or}(G)$. Then the boundary map

$$\partial : C_1(G) \to C_0(G)$$

is naturally defined. Let ∂^t be the dual of ∂ and the *Laplacian* Δ of G is defined to be $\Delta = \partial\partial^t$. It is known (and easy to check) that ([27], [11]),

$$\Delta = 1 - A + Q. \tag{2}$$

Now let G be a connected k-regular graph. Since 0 is an eigenvalue of Δ with multiplicity one, (2) shows that k is an eigenvalue of A with multiplicity one. Because of semi-positivity of Δ we find that

$$|\lambda| \leq k \quad \text{for any eigenvalue } \lambda \text{ of } A$$

and that $-k$ is an eigenvalue of A if and only if G is bipartite ([27], **Chapter 3**). Here G is called *bipartite* if the set of vertices $V(G)$ can be divided into disjoint subset V_0 and

V_1 such that every edge connects points in V_0 and V_1, namely there is no edge whose end points are simultaneously contained in V_i $(i = 0, 1)$.

DEFINITION 2.1. Let G be a k-regular graph. We say that it is Ramanujan, if all eigenvalues λ of A with $|\lambda| \neq k$ satisfy

$$|\lambda| \leq 2\sqrt{k-1}.$$

See [15], [16], [19] and [28] for detailed expositions of Ramanujan graphs.

A map f from a graph G' to G is defined to be a pair $f = (f_V, f_E)$ of maps

$$f_V : V(G') \to V(G), \quad f_E : E(G') \to E(G)$$

satisfying

$$\partial_i f_E = f_V \partial_i, \quad i = 0, 1.$$

Suppose that G and G' are connected. If there is a positive integer d such that $|f_V^{-1}(v)| = |f_E^{-1}(e)| = d$ for any $v \in V(G)$ and $e \in E(G)$, f is mentioned as *a covering map of degree* d.

3. A construction of a Ramanujan graph

Although there are several ways to construct a Ramanujan graph (eg. [17] [18]), we adopt the construction due to Pizer([20]), which is most suited to our program. Let p be a prime, and B the quaternion algebra over $\mathbb{Q}$ ramified at two places p and ∞. Let R be a fixed maximal order in B and $\{I_1, \cdots, I_n\}$ be the set of left R-ideals representing the distinct ideal classes. We choose $I_1 = R$ and say n *the class number* of B. For $1 \leq i \leq n$, R_i denotes the right order of I_i, and let w_i be the order of $R_i^\times/\{\pm 1\}$. The product

$$W = \prod_{i=1}^{n} w_i \tag{3}$$

is independent of the choice of R and is equal to the exact denominator of $\frac{p-1}{12}$ ([8] p.117) and Eichler's mass formula states that

$$\sum_{i=1}^{n} \frac{1}{w_i} = \frac{p-1}{12}.$$

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. There are n distinct isomorphism classes $\{E_1, \cdots, E_n\}$ of supersingular elliptic curves over $\mathbb{F}$ such that $\mathrm{End}(E_i) \simeq R_i$.
Now we assume that $p-1$ is divisible 12. Then $\frac{p-1}{12}$ is an integer and $W = \prod_{i=1}^{n} w_i = 1$, namely $w_i = 1$ for all i. Hence by Eichler's mass formula

$$n = \frac{p-1}{12}. \tag{4}$$

We fix an odd prime l different from p and let $\mathcal{N}_{p,l}$ denote the set of square free positive integers prime to lp. For $N \in \mathcal{N}_{p,l}$, *an enhanced supersingular elliptic curve of level* N is defined to be a pair $\mathbf{E} = (E, C_N)$ of a supersingular elliptic curve E and its cyclic subgroup C_N of order N. A homomorphism ϕ from $\mathbf{E} = (E, C_N)$ to $\mathbf{E}' = (E', C'_N)$ is defined by a homomorphism $\phi : E \to E'$ satisfying

$$\phi(C_N) = C'_N.$$

Let Σ_N be the set of isomorphism classes of enhanced supersingular elliptic curve of level N defined over $\mathbb{F}$. Then the cardinality $\nu(N)$ of Σ_N is

$$\nu(N) = \frac{(p-1)\sigma_1(N)}{12}, \quad \sigma_1(N) = \sum_{d|N} d. \tag{5}$$

Here $\sigma_1(N)$ counts the number of cyclic subgroups of E of order N. Let Hom$(\mathbf{E}_i,\ \mathbf{E}_j)(l)$ denote the set of homomorphisms from $\mathbf{E}_i$ to $\mathbf{E}_j$ of degree l. We define *the Brandt matrix* $B_p^{(l)}(N)$ is defined to be a $\nu(N) \times \nu(N)$-matrix whose (i, j)-entry is

$$b_{ij} = \frac{1}{2}|\mathrm{Hom}(\mathbf{E}_j,\ \mathbf{E}_i)(l)|. \tag{6}$$

The following result is proved in [25] **Proposition 3.1**.

PROPOSITION 3.1. *Let $N \in \mathcal{N}_{p,l}$. Then the Brandt matrix $B_p^{(l)}(N) = (b_{ij})_{1 \le i, j \le \nu(N)}$ satisfies the following.*

(1) Every entry is a non-negative integer and $B_p^{(l)}(N)$ is symmetric;

$$b_{ij} = b_{ji}.$$

(2) The diagonal entires $\{b_{ii}\}_i$ are even for all i.

(3) For any $i = 1, \cdots, \nu(N)$,

$$\sum_{j=1}^{n} b_{ij} = l + 1.$$

By **Proposition 2.1** there is a regular graph $G_p^{(l)}(N)$ of degree $l+1$ whose adjacency matrix is $B_p^{(l)}(N)$. In **Theorem 5.1** we will show that it is a connected non-bipartite Ramanujan graph.

REMARK 3.1. We compare our construction of a Ramanujan graph with another one (eg. [12][16][17][28]). Let Z be the center of $B^\times$ and set $D = B^\times/Z$. Let l be a prime different from p. Since B is unramified at l, $D(\mathbb{Q}_l)/D(\mathbb{Z}_l) = \mathrm{PGL}_2(\mathbb{Q}_l)/\mathrm{PGL}_2(\mathbb{Z}_l)$ which is the $(l+1)$-regular tree with the action of the Hecke operator T_l as adjacency operator. Let $\mathcal{K}$ be an open subgroup of $\prod_{q \neq l,\infty} D(\mathbb{Z}_q)$ and let

$$\Gamma_{\mathcal{K}} = D(\mathbb{Q}) \cap D(\mathbb{R})D(\mathbb{Q}_l)\mathcal{K}.$$

Then

$$G_p^{(l)}(\mathcal{K}) = \Gamma_{\mathcal{K}} \backslash D(\mathbb{Q}_l)/D(\mathbb{Z}_l)$$

is an $(l+1)$-regular Ramanujan graph (possibly with multiple edges). For example let us take $N \in \mathcal{N}_{p,l}$ and we choose $\mathcal{K} = \prod_{q \neq l,\infty} \mathcal{K}_q$ to be

$$\mathcal{K}_q = \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} & a, b, c, d \in \mathbb{Z}_q,\ c \equiv 0 \pmod{q}, \quad if \quad q \mid N \\ \text{a maximal compact subgroup of } D(\mathbb{Q}_q) & if \quad q \nmid lN, \end{cases}$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{Z}_q)$ is the equivalent class of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q)$. We let $\mathcal{K}_0(N)$ denote the defined compact subgroup. Then $G_p^{(l)}(\mathcal{K}_0(N))$ is $G_p^{(l)}(N)$ under our notation. Hashimoto shows that $G_p^{(l)}(\mathcal{K})$ has no multiple edge if $p \equiv 1 (\mathrm{mod}\, 12)$, which coincides with **Proposition 3.1** (cf. (7.14) of [9]). More precisely, in (7.14) of [9], he has shown that $G_p^{(l)}(1)$ is a quotient of the regular tree of degree $l+1$ by a torsion free group. He also proved that the class number of the function field of the modular curve $X_0(p)_{\mathbb{F}_l}$ over $\mathbb{F}_l$ equals $(g+1)\kappa(G_p^{(l)}(1))$. Here g is the genus of $X_0(p)_{\mathbb{F}_l}$ and $\kappa(G_p^{(l)}(1))$ denotes the complexity of $G_p^{(l)}(1)$.

THEOREM 3.1. *Let M and N be elements of $\mathcal{N}_{p,l}$ such that M is a divisor of N. Then there is a covering map*

$$\pi_{N/M} \,:\, G_p^{(l)}(N) \to G_p^{(l)}(M)$$

of degree $\sigma_1(N/M)$

Proof. Although this is clear from **Remark 3.1** since $\mathcal{K}_0(N)$ is a subgroup of $\mathcal{K}_0(M)$ with index $\sigma_1(N/M)$, we will show another geometric (and elementary) proof. Since N is square free M and N/M are coprime. Thus a cyclic subgroup C_N is written by

$$C_M = C_M \oplus C_{N/M}$$

and we define

$$(\pi_{N/M})_V \,:\, V(G_p^{(l)}(N)) \to V(G_p^{(l)}(M)), \quad (\pi_{N/M})_V(E, C_M \oplus C_{N/M}) = (E, C_M).$$

Since the number of cyclic subgroups of E of order N/M is $\sigma_1(N/M)$, $|\pi_{N/M}^{-1}(v)| = \sigma_1(N/M)$ for any $v \in V(G_p^{(l)}(M))$. By definition an edge of $G_p^{(l)}(N)$ from $\mathbf{E} = (E, C_M \oplus C_{N/M})$ to $\mathbf{E}' = (E', C'_M \oplus C'_{N/M})$ is a homomorphism f from E to E' satisfying

$$f(C_M) = C'_M, \quad f(C_{N/M}) = C'_{N/M}.$$

Forget the homomorphism of cyclic subgroups of order N/M and we have

$$\mathrm{Hom}(\mathbf{E}, \mathbf{E}')(l)/\{\pm 1\} \to \mathrm{Hom}(\pi_{N/M}(\mathbf{E}), \pi_{N/M}(\mathbf{E}'))(l)/\{\pm 1\},$$

which defines a map of the set of edges

$$(\pi_{N/M})_E \,:\, E(G_p^{(l)}(N)) \to E(G_p^{(l)}(M))$$

satisfying

$$\partial_i \circ (\pi_{N/M})_E = (\pi_{N/M})_V \circ \partial_i, \quad i = 0, 1.$$

One finds that this map has degree $\sigma_1(N/M)$. In fact let g be an element of $\mathrm{Hom}(\pi_{N/M}(\mathbf{E}), \pi_{N/M}(\mathbf{E}'))(l)$. Thus g is a homomorphism from E to E' of degree l satisfying

$$g(C_M) = C'_M.$$

Let $C_{N/M}$ be a cyclic subgroup of E of order N/M and we set $C'_{N/M} = g(C_{N/M})$. Then we have a homomorphism of enhanced supersingular elliptic curve of level N

$$g : (E, C_M \oplus C_{N/M}) \to (E', C'_M \oplus C'_{N/M})$$

which defines an edge of $G_p^{(l)}(N)$. The number of cyclic subgroups of order N/M (i.e. choices of $C_{N/M}$) is $\sigma_1(N/M)$ and the claim has been proved. $\square$

4. A spectral decomposition of the character group

In this section we will establish a geometric version of Jacquet-Langlands correspondence (**Theorem 4.2**) and its real version (**Theorem 4.3**).

For a positive integer N, let $S_2(\Gamma_0(N))$ denote the space of cusp forms of weight 2 for the Hecke congruence subgroup

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \quad c \equiv 0 \,(\mathrm{mod}\, N)\}.$$

Let $Y_0(N)$ be the modular curve which parametrizes isomorphism classes of a pair $\mathbf{E} = (E, C_N)$ of an elliptic curve E and its cyclic subgroup C_N of order N. It is a smooth curve defined over $\mathbb{Q}$ and the set of $\mathbb{C}$-valued points is the quotient of the upper half plane by $\Gamma_0(N)$. Let $X_0(N)$ be the compactification of $Y_0(N)$. It is a smooth projective curve defined over $\mathbb{Q}$ and has the canonical model over $\mathbb{Z}$ which has been studied by [7] and [13] in detail. The space of cusp forms $S_2(\Gamma_0(N))$ is naturally identified with the space of holomorphic 1-forms $H^0(X_0(N), \Omega)$ and in particular with the cotangent space $\mathrm{Cot}_0(J_0(N))$ at the origin of the Jacobian variety $J_0(N)$ of $X_0(N)$.

For a prime p with $(p, N) = 1$, $X_0(N)$ is furnished with the p-th Hecke operator defined by

$$T_p(E, C_N) := \sum_C (E/C, (C_N + C)/C), \tag{7}$$

where C runs through all cyclic subgroup schemes of E of order p. If p is a prime divisor of N, an operator U_p is defined by

$$U_p(E, C_N) := \sum_{C \neq D} (E/C, (C_N + C)/C) \tag{8}$$

where D is the cyclic subgroup of C_N of order p. By the functoriality, Hecke operators act on $J_0(N)$ and $\mathrm{Cot}_0(J_0(N)) = S_2(\Gamma_0(N))$ and the resulting action coincides with the usual one on $S_2(\Gamma_0(N))$ (see [23]). We define the Hecke algebra as $\mathbb{T}_0(N) := \mathbb{Z}[\{T_p\}_{(p,N)=1}, \{U_p\}_{p|N}]$, which is a commutative subring of $\mathrm{End}(J_0(N))$. The effects of T_p and U_p on $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$ are

$$f|U_p = \sum_{n=1}^{\infty} a_{pn} q^n \tag{9}$$

and

$$f|T_p = \sum_{n=1}^{\infty} (a_{pn} + p a_{n/p}) q^n. \tag{10}$$

Here $a_{n/p} = 0$ if n/p is not an integer.

DEFINITION 4.1. For a positive integer M, we define a subalgebra $\mathbb{T}_0(N)^{(M)}$ of $\mathbb{T}_0(N)$ to be the omitting of Hecke operators from $\mathbb{T}_0(N)$ whose indices are prime divisors of M, that is

$$\mathbb{T}_0(N)^{(M)} = \mathbb{Z}[[\{T_p\}_{(p,NM)=1}, \{U_p\}_{p|N,(p,M)=1}\}.$$

We call an algebraic homomorphism from $\mathbb{T}_0(N)^{(M)}$ to $\mathbb{C}$ as *a character*. If the image is contained in $\mathbb{R}$ it is referred as *real*.

Let M be a positive integer and f an element of $S_2(\Gamma_0(M))$. For a positive integer d we set

$$f^{(d)}(z) = f(dz) \in S_2(\Gamma_0(dM)).$$

DEFINITION 4.2. Let N be a square free positive integer and let M be a divisor of N. For a divisor d of N/M we define

$$S_2(\Gamma_0(M))^{(d)} = \{f^{(d)}(z) \mid f \in S_2(\Gamma_0(M))\} \subset S_2(\Gamma_0(N)).$$

The space of old forms of level N is defined to be

$$S_2(\Gamma_0(N))_{old} = \sum_{M|N, M\neq N} \sum_{d|(N/M)} S_2(\Gamma_0(M))^{(d)} \subset S_2(\Gamma_0(N))$$

and the orthogonal complement of $S_2(\Gamma_0(N))_{old}$ for the Petersson product is called by the space of *new forms* and denoted by $S_2(\Gamma_0(N))_{new}$.

Let N be a square free positive integer and let q be a prime not dividing N. Since the action of T_q on $S_2(\Gamma_0(N))$ is self-adjoint for the Petersson product and since $S_2(\Gamma_0(N))_{old}$ is stable by T_q, $S_2(\Gamma_0(N))_{new}$ is stable by $\mathbb{T}_0(N)^{(N)}$. This implies that $S_2(\Gamma_0(N))_{new}$ admits a spectral decomposition by $\mathbb{T}_0(N)^{(N)}$. We will show that $S_2(\Gamma_0(N))$ has an irreducible decomposition of multiplicity one by the action of the full Hecke algebra $\mathbb{T}_0(N)$ (**Theorem 4.1**), which plays a key role in our story. Although this fact should be fairly well-known, we will show a proof since we could not find appropriate references. In proving the theorem, a key fact is the following, which is mentioned as *multiplicity one* ([3] [14]).

FACT 4.1. Let N be a positive integer (which may not be square free) and $f = \sum_{n=1}^{\infty} a_n q^n$ an element of $S_2(\Gamma_0(N))$. Suppose that $a_n = 0$ for all n with $(n, t) = 1$, where t is a fixed positive integer. Then $f \in S_2(\Gamma_0(N))_{old}$.

This fact yields an irreducible decomposition as a $\mathbb{T}_0(N)$-module

$$S_2(\Gamma_0(N))_{new} = \oplus_\alpha S_2(\Gamma_0(N))_{new}(\alpha)$$

by real characters such that every irreducible component has dimension one. Here $S_2(\Gamma_0(N))_{new}(\alpha)$ denotes the isotypic component of α

$$S_2(\Gamma_0(N))_{new}(\alpha) = \{f \in S_2(\Gamma_0(N))_{new} \mid f|T = \alpha(T)f, \quad \forall T \in \mathbb{T}_0(N)\},$$

which is spanned by the normalized Hecke eigenform. Moreover since N is square free $\alpha_q = \pm 1$ for $q \mid N$ ([10] **Lemma 3.2**). By the definition of the space of new forms we have

$$S_2(\Gamma_0(N)) = \oplus_{M|N}(\oplus_{d|(N/M)} S_2(\Gamma_0(M))^{(d)}_{new}). \tag{11}$$

Fix a divisor M of N and let us consider the subspace

$$\mathbb{S}_M = \oplus_{d|(N/M)} S_2(\Gamma_0(M))_{new}^{(d)}.$$

Let $N/M = l_1 \cdots l_m$ be the prime decomposition. Then there is an isomorphism as vector spaces

$$\mathbb{S}_M \simeq S_2(\Gamma_0(M))_{new}^{\oplus 2^m}. \tag{12}$$

We will explicitly describe this isomorphism.

PROPOSITION 4.1. *Let N be a square free positive integer and M a divisor of N. Let $f \in S_2(\Gamma_0(M))_{new}$ be a normalized Hecke eigenform. Then for $\epsilon = (\epsilon_{l_1}, \cdots, \epsilon_{l_m})$ $(\epsilon_{l_i} = \pm)$ there is a normalized Hecke eigenform f_ϵ of level N satisfying the following conditions.*

(1) If q is a prime not dividing N/M

$$a_q(f_\epsilon) = a_q(f).$$

(2)

$$a_{l_i}(f_\epsilon) = \alpha_{l_i}^{\epsilon_{l_i}}$$

where

$$\alpha_{l_i}^{+} = \frac{a_{l_i}(f) + \sqrt{\Delta_i}}{2}, \quad \alpha_{l_i}^{-} = \frac{a_{l_i}(f) - \sqrt{\Delta_i}}{2}, \quad \Delta_i = a_{l_1}(f)^2 - 4l_i (< 0).$$

Moreover the 2^m complex numbers $\{\alpha_{l_1}^{(\pm)}, \cdots, \alpha_{l_m}^{(\pm)}\}$ are mutually different.

Proof. In general let p be a prime and F a square free positive integer prime to p. We have two degeneracy maps $\alpha_p, \beta_p : X_0(pF) \to X_0(F)$ defined by

$$\alpha_p(E, C_p \oplus C_F) = (E, C_F), \quad \beta_p(E, C_p \oplus C_F) = (E/C_p, (C_p \oplus C_F)/C_p),$$

which induces linear maps

$$\alpha_p^*, \beta_p^* : S_2(\Gamma_0(F)) \to S_2(\Gamma_0(pF)) \tag{13}$$

whose effects on $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(F))$ are

$$\alpha_p^*(f) = f = \sum_{n=1}^{\infty} a_n q^n, \quad \beta_p^*(f) = f^{(p)} = \sum_{n=1}^{\infty} a_n q^{pn}. \tag{14}$$

Let T be T_r $(r \nmid pF)$ or U_l $(l \mid F)$. Then T commutes with α_p and β_p and

$$\begin{array}{ccc} S_2(\Gamma_0(F)) \oplus S_2(\Gamma_0(F)) & \xrightarrow{\alpha_p^* + \beta_p^*} & S_2(\Gamma_0(pF)) \\ {\scriptstyle (T,\,T)}\downarrow & & \downarrow{\scriptstyle T} \\ S_2(\Gamma_0(F)) \oplus S_2(\Gamma_0(F)) & \xrightarrow{\alpha_p^* + \beta_p^*} & S_2(\Gamma_0(pF)). \end{array} \tag{15}$$

Using (14) and (15) we will inductively construct f_ϵ by the number of prime divisors m. We set $M_m = Ml_1 \cdots l_m$ $(m \geq 1)$ and $M_0 = M$. Suppose that we have constructed a desired normalized Hecke eigenform $f_\epsilon \in S_2(\Gamma_0(M_{m-1}))$ of character χ_ϵ. For a prime r

different from l_m, we let T be T_r or U_r according to $r \nmid M_m$ or $r \mid M_{m-1}$, respectively. Then (15) implies

$$\text{(16)} \qquad \begin{array}{ccc} S_2(\Gamma_0(M_{m-1})) \oplus S_2(\Gamma_0(M_{m-1})) & \xrightarrow{\alpha^*_{l_m} + \beta^*_{l_m}} & S_2(\Gamma_0(M_m)) \\ {\scriptstyle (T,\,T)}\big\downarrow & & {\scriptstyle T}\big\downarrow \\ S_2(\Gamma_0(M_{m-1})) \oplus S_2(\Gamma_0(M_{m-1})) & \xrightarrow{\alpha^*_{l_m} + \beta^*_{l_m}} & S_2(\Gamma_0(M_m)). \end{array}$$

Hence

$$\alpha^*_{l_m}(f_\epsilon)|T = \alpha^*_{l_m}(f_\epsilon|T) = \chi_\epsilon(T)\alpha^*_{l_m}(f_\epsilon)$$

and

$$\beta^*_{l_m}(f_\epsilon)|T = \beta^*_{l_m}(f_\epsilon|T) = \chi_\epsilon(T)\beta^*_{l_m}(f_\epsilon).$$

Define a character

$$\chi_\epsilon^{(l_m)} : \mathbb{T}_0(M_m)^{(l_m)} \to \mathbb{C}$$

by

$$\chi_\epsilon^{(l_m)}(T) = \chi_\epsilon(T),$$

and $\alpha^*_{l_m}(f_\epsilon)$ and $\beta^*_{l_m}(f_\epsilon)$ are $\mathbb{T}_0(M_m)^{(l_m)}$-eigenforms of the same character $\chi_\epsilon^{(l_m)}$. Let us investigate the action of U_{l_m}. By (9), (10) and (14)

$$\begin{pmatrix} \alpha^*_{l_m}(f_\epsilon)|U_{l_m} \\ \beta^*_{l_m}(f_\epsilon)|U_{l_m} \end{pmatrix} = \begin{pmatrix} a_{l_m}(f_\epsilon) & -l_m \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^*_{l_m}(f_\epsilon) \\ \beta^*_{l_m}(f_\epsilon) \end{pmatrix}.$$

Use the assumption (1) and the characteristic polynomial of U_{l_m} is

$$\Phi(t) = t^2 - a_{l_m}(f_\epsilon)t + l_m = t^2 - a_{l_m}(f)t + l_m.$$

Since f is a normalized $\mathbb{T}_0(M)$-eigenform which is new, the discriminant $\Delta_m = a_{l_m}(f)^2 - 4l_m$ is negative ([5]). Therefore the eigenvalue of U_{l_m} are mutually distinct and contained in $\mathbb{C} \setminus \mathbb{R}$. Set

$$\text{(17)} \qquad \alpha^+_{l_m} = \frac{a_{l_m}(f) + \sqrt{\Delta_m}}{2}, \quad \alpha^-_{l_m} = \frac{a_{l_m}(f) - \sqrt{\Delta_m}}{2}$$

and let f_ϵ^+ and f_ϵ^- be the corresponding normalized cusp form of level M_m satisfying

$$f_\epsilon^+ \mid U_{l_m} = \alpha^+_{l_m} f_\epsilon^+, \quad f_\epsilon^- \mid U_{l_m} = \alpha^-_{l_m} f_\epsilon^-.$$

Extend $\chi_\epsilon^{(l_m)}$ to a character χ_ϵ^+ and χ_ϵ^- of $\mathbb{T}_0(M_m) = \mathbb{T}_0(M_m)^{(l_m)}[U_{l_m}]$ by

$$\chi_\epsilon^+(U_{l_m}) = \alpha^+_{l_m}, \quad \chi_\epsilon^-(U_{l_m}) = \alpha^-_{l_m}.$$

Then f_ϵ^+ and f_ϵ^- are $\mathbb{T}_0(M_m)$-eigenforms whose characters are χ_ϵ^+ and χ_ϵ^-, respectively. Observe that $\alpha^+_{l_m}$ and $\alpha^-_{l_m}$ are different from each of $\{\alpha^+_{l_i}, \alpha^-_{l_i}\}_{1 \le i \le m-1}$, where

$$\alpha^+_{l_i} = \frac{a_{l_i}(f) + \sqrt{\Delta_i}}{2}, \quad \alpha^-_{l_i} = \frac{a_{l_i}(f) - \sqrt{\Delta_i}}{2}, \quad \Delta_i = a_{l_i}(f)^2 - 4l_i.$$

In fact if $\alpha^+_{l_m} = \alpha^+_{l_i}$ $(1 \le i \le m-1)$, comparing their real and imaginary part we conclude

$$a_{l_m}(f) = a_{l_i}(f), \quad \Delta_m = \Delta_i$$

which implies $l_m = l_i$. Thus we have constructed normalized 2^m Hecke eigenforms of level M_m from f whose characters are mutually different. $\square$

Proposition 4.1 yields a spectral decomposition of multiplicity one

$$\mathbb{S}_M = \oplus_\beta \mathbb{C} f_\beta \tag{18}$$

where f_β is the normalized Hecke eigenform of character β. Let M' be a divisor of N different from M and we consider the decomposition (18) for M',

$$\mathbb{S}_{M'} = \oplus_\beta \mathbb{C} f_{\beta'}. \tag{19}$$

The following lemma shows that every character β in (18) is different from each of β' in (19).

LEMMA 4.1. *Let $f \in S_2(\Gamma_0(N_f))_{new}$ (resp. $g \in S_2(\Gamma_0(N_g))_{new}$) be a normalized Hecke eigenform. If there is a positive integer t such that*

$$a_l(f) = a_l(g)$$

for any prime l with $l \nmid t$, then $f = g$.

Proof. Let K_f (resp. K_g) be the number field generated by Fourier coefficients of f and (resp. g) over $\mathbb{Q}$ and let K be the minimal extension of $\mathbb{Q}$ that contains K_f and K_g. We fix a prime l satisfying $l \nmid N_f N_g$ and that completely splits in K. Corresponding to f and g, there are absolutely irreducible representations

$$\rho_{f,l} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_l), \quad \rho_{g,l} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_l)$$

of the conductor N_f and N_g respectively which satisfy

$$\det(t - \rho_{f,l}(Frob_q)) = t^2 - a_q(f)t + q, \quad (q, lN_f) = 1$$

and

$$\det(t - \rho_{g,l}(Frob_q)) = t^2 - a_q(g)t + q, \quad (q, lN_g) = 1.$$

([6] **Theorem 3.1**). Here $Frob_q$ is the Frobenius at a prime q. Let S be a finite set of primes. Since a semi-simple representation $\rho_l : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_l)$ is determined by values $\mathrm{Tr}\rho_l(Frob_q)$ on the primes $q \notin S$ at which ρ_l is unramified ([6] **Proposition 2.6 (3)**), the assumption implies that $\rho_{f,l} = \rho_{g,l}$ and in particular $N_f = N_g$. Now we deduce that $f = g$ from **Fact 4.1**. $\square$

REMARK 4.1. Here is an another way to see that any β in (18) is different from each of β' in (19). If necessary changing M and M', let r be a prime divisor of M' not dividing M. By the construction $\beta'(U_r) \in \mathbb{R}$ and $\beta(U_r) \in \mathbb{C} \setminus \mathbb{R}$ and therefore β and β' are different.

For a character α of $\mathbb{T}_0(N)$, let $S_2(\Gamma_0(N))(\alpha)$ denote the isotypic component of α,

$$S_2(\Gamma_0(N))(\alpha) = \{f \in S_2(\Gamma_0(N)) \mid f|T = \alpha(T)f, \quad \forall T \in \mathbb{T}_0(N)\}.$$

THEOREM 4.1. *(Strong multiplicity one) Let N be a square free positive integer. Then there is an isomorphism as $\mathbb{T}_0(N)$-modules*

$$S_2(\Gamma_0(N)) = \oplus_\alpha S_2(\Gamma_0(N))(\alpha)$$

such that every irreducible component has dimension one and is spanned by the normalized Hecke eigenform f_α. The index α in the decomposition runs through the set of closed points $\mathrm{Spec}(\mathbb{T}_0(N))(\mathbb{C})$ *and there is an isomorphism*

$$\Phi : \mathbb{T}_0(N) \otimes \mathbb{C} \simeq \prod_{\alpha \in \mathrm{Spec}(\mathbb{T}_0(N))(\mathbb{C})} \mathbb{C}$$

such that the composition with the projection π_α to the α-factor is α :

$$\pi_\alpha \circ \Phi = \alpha.$$

Proof. The previous argument and (11) show that $S_2(\Gamma_0(N))(\alpha)$ is a $\mathbb{C}$-linear space generated by a normalized Hecke eigenform f_α and we have an irreducible decomposition of multiplicity one

$$S_2(\Gamma_0(N)) = \oplus_\alpha S_2(\Gamma_0(N))(\alpha). \tag{20}$$

The linear isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{T}_0(N), \mathbb{C}) \simeq S_2(\Gamma_0(N)), \quad \rho \mapsto \sum_{m=1}^{\infty} \rho(T_m) q^m$$

implies that $\{\alpha\}$ in the right hand side of (20) is the set of closed points $\mathrm{Spec}(\mathbb{T}_0(N))(\mathbb{C})$ and $\{f_\alpha\}_{\alpha \in \mathrm{Spec}(\mathbb{T}_0(N))(\mathbb{C})}$ is a basis of $S_2(\Gamma_0(N))$. Now the desired decomposition of $\mathbb{T}_0(N) \otimes \mathbb{C}$ is obvious. $\square$

Let p be any prime (*not necessary* $p \equiv 1 (\mathrm{mod}\, 12)$) and N a square free positive integer prime to p. We define *the space of p-new forms* $S_2(\Gamma_0(pN))_{pN/N}$ to be the orthogonal complement of $\alpha_p^*(S_2(\Gamma_0(N))) + \beta_p^*(S_2(\Gamma_0(N)))$ in $S_2(\Gamma_0(pN))$ for the Petersson inner product. Then (11) and (14) imply

$$S_2(\Gamma_0(pN))_{pN/N} = \oplus_{M|N} \oplus_{d|(N/M)} S_2(\Gamma_0(pM))^{(d)}_{new} \tag{21}$$

and by **Theorem 4.1** we have a decomposition of $\mathbb{T}_0(N)$-module of multiplicity one

$$S_2(\Gamma_0(pN))_{pN/N} = \oplus_\chi \mathbb{C} f_\chi. \tag{22}$$

Here f_χ is a normalized Hecke eigenform whose character is χ. Let $\mathbb{T}_0(pN)_{pN/N}$ be the restriction of $\mathbb{T}_0(N)$ to this space. Then the set of characters in (22) coincides with $\mathrm{Spec}(\mathbb{T}_0(pN)_{pN/N})(\mathbb{C})$ and there is an isomorphism

$$\Phi : \mathbb{T}_0(pN)_{pN/N} \otimes \mathbb{C} \simeq \prod_{\chi \in \mathrm{Spec}(\mathbb{T}_0(pN)_{pN/N})(\mathbb{C})} \mathbb{C} \tag{23}$$

such that the composition with the projection π_χ to χ-factor is χ :

$$\pi_\chi \circ \Phi = \chi.$$

Using [21] we will clarify a relationship between $S_2(\Gamma_0(pN))_{pN/N}$ and the Ramanujan graph $G_p^{(l)}(N)$.

By the functoriality α_p and β_p induce a homomorphism

$$\alpha_p^*, \beta_p^* : J_0(N) \to J_0(pN) \tag{24}$$

and we define a subvariety

$$J_0(pN)_{p-old} = \alpha_p^* J_0(M) + \beta_p^* J_0(N) \subset J_0(pN)$$

which is called as *p-old subvariety*. We define *p-new subvariety* to be the quotient

$$J_0(pN)_{pN/N} = J_0(pN)/J_0(pN)_{p-old}.$$

Now we consider the actions of Hecke operators. Let T be T_r ($r \nmid pN$) or U_l ($l \mid N$). Then T commutes with α_p and β_p and

$$\begin{tikzcd} J_0(N) \times J_0(N) \arrow[r, "\alpha_p^* \times \beta_p^*"] \arrow[d, "{(T,\,T)}"'] & J_0(pN) \arrow[d, "T"'] \\ J_0(N) \times J_0(N) \arrow[r, "\alpha_p^* \times \beta_p^*"] & J_0(pN). \end{tikzcd} \tag{25}$$

and $J_0(pN)_{p-old}$ is $\mathbb{T}_0(pN)^{(p)}$-stable. By [21] **Remark 3.9** $J_0(pN)_{p-old}$ is also preserved by U_p and it is $\mathbb{T}_0(pN) = \mathbb{T}_0(pN)^{(p)}[U_p]$-stable. Therefore $J_0(pN)_{pN/N}$ admits the action of $\mathbb{T}_0(pN)$ and the image of $\mathbb{T}_0(pN)$ in $\mathrm{End}(J_0(pN)_{pN/N})$ is temporary denoted by $\mathbb{T}'$. Having identified the holomorphic cotangent space of $J_0(pN)_{pN/N}$ at the origin with $S_2(\Gamma_0(pN))_{pN/N}$ let us consider the representation of $\mathrm{End}(J_0(pN)_{pN/N})$ on $S_2(\Gamma_0(pN))_{pN/N}$. Then the image of $\mathbb{T}'$ in $\mathrm{End}(S_2(\Gamma_0(pN))_{pN/N})$ is $\mathbb{T}_0(pN)_{pN/N}$. Since representation of $\mathrm{End}(J_0(pN)_{pN/N})$ on $S_2(\Gamma_0(pN))_{pN/N}$ faithful, $\mathbb{T}'$ and $\mathbb{T}_0(pN)_{pN/N}$ are isomorphic and we identify them.

It is known that the Néron model of $J_0(pN)_{pN/N}$ over $\mathrm{Spec}\mathbb{Z}$ has purely toric reduction $\mathcal{T}$ at p. Let us describe its character group. $X_0(pN)_{\mathbb{F}_p}$ has two irreducible components Z_F and Z_V, which are isomorphic to $X_0(N)_{\mathbb{F}_p}$. Over Z_F (resp. Z_V) the parametrized cyclic group C_p of order p is the kernel of the Frobenius F (resp. the Verschiebung V). Z_F and Z_V transversally intersect at enhanced supersingular points of level N, that is $\Sigma_N = \{\mathbf{E}_1, \cdots, \mathbf{E}_{\nu(N)}\}$. Set

$$X_N = \oplus_{i=1}^{\nu(N)} \mathbb{Z}\mathbf{E}_i$$

and we adopt $\{\mathbf{E}_1, \cdots, \mathbf{E}_{\nu(N)}\}$ as a base. We define the action of Hecke operators on X_N by (7) and (8) and let $\mathbb{T}$ denote a commutative subring of $\mathrm{End}_{\mathbb{Z}}(X_N)$ generated by Hecke operators. Let us consider the boundary map of the dual graph of $X_0(pN)_{\mathbb{F}_p}$,

$$\partial : X_N \to \mathbb{Z}Z_F \oplus \mathbb{Z}Z_F, \quad \partial(\mathbf{E}_i) = Z_F - Z_V.$$

Being $X_N^{(0)}$ the kernel of ∂, we have the exact sequence of Hecke modules

$$0 \to X_N^{(0)} \to X_N \xrightarrow{\partial} \mathbb{Z}\epsilon \to 0, \quad \epsilon = Z_F - Z_V. \tag{26}$$

For brevity let us write E_i by $[i]$. Then

$$\partial([i]) = \epsilon, \quad 1 \le \forall i \le n$$

and

$$X_N^{(0)} = \{\sum_{i=1}^{n} a_i[i] \mid a_i \in \mathbb{Z}, \sum_{i=1}^{n} a_i = 0\}.$$

The the restriction $\mathbb{T}_0$ of $\mathbb{T}$ to $X_N^{(0)}$ has the following description. By [21] **Proposition 3.1**, $X_N^{(0)}$ is the character group of the connected component of the torus $\mathcal{T}$. By the Néron property, $\mathcal{T}$ admits the action of $\mathbb{T}_0(pN)_{pN/N}(=\ \mathbb{T}')$ and the induced action on $X_N^{(0)}$ is $\mathbb{T}_0$. Therefore $\mathbb{T}_0$ is the image of $\mathbb{T}_0(pN)_{pN/N}$ in $\mathrm{End}_{\mathbb{Z}}(X_N^{(0)})$. Since the action of $\mathbb{T}_0(pN)_{pN/N}$ on $X_N^{(0)}$ is faithful ([21] **Theorem 3.10**), $\mathbb{T}_0$ and $\mathbb{T}_0(pN)_{pN/N}$ are isomorphic and they will be identified from now on. The following theorem is a geometric version of Jacquet-Langlands correspondence.

THEOREM 4.2. *Let N be a square free positive integer. There is an isomorphism as $\mathbb{T}_0(pN)_{pN/N}$-modules*

$$X_N^{(0)} \otimes \mathbb{C} \simeq S_2(\Gamma_0(pN))_{pN/N}.$$

Proof. As we have mentioned before, the action of $\mathbb{T}_0(pN)_{pN/N}$ on $X_N^{(0)}$ is faithful ([21] **Theorem 3.10**). Since the characters $\{\chi\}$ in (22) are mutually different and by (23) we see every irreducible component of (22) should appear as irreducible factor of $X_N^{(0)} \otimes \mathbb{C}$. Thus $S_2(\Gamma_0(pN))_{pN/N}$ is contained in $X_N^{(0)} \otimes \mathbb{C}$. On the other hand the rank of $X_N^{(0)}$ is equal to $\dim \mathcal{T} = \dim J_0(pN)_{pN/N}$. Since the holomorphic cotangent space of $J_0(pN)_{pN/N}$ at the origin is $S_2(\Gamma_0(pN))_{pN/N}$,

$$\dim X_N^{(0)} \otimes \mathbb{C} = \dim S_2(\Gamma_0(pN))_{pN/N},$$

and the claim is proved. □

Let us state a real version of **Theorem 4.2**. Since the character of a normalized Hecke-eigen newform is real, using (15) and (20), **Theorem 4.2** yields an decomposition as a $\mathbb{T}_0(pN)_{pN/N} \otimes \mathbb{R}$-module

$$X_N^{(0)} \otimes \mathbb{R} = \oplus_\gamma V(\gamma),$$

where

$$V(\gamma) = \{v \in X_N^{(0)} \otimes \mathbb{R} \mid T(v) = \gamma(T)v \quad \forall T \in \mathbb{T}_0(pN)_{pN/N}^{(pN)}\}.$$

Here γ is the real character of $\mathbb{T}_0(pN)_{pN/N}^{(pN)}$ which is the restriction of the character of the normalized Hecke eigen newform f_γ whose level N_γ satisfies

$$N_\gamma = pM, \quad M|N.$$

Lemma 4.1 shows that $\{\gamma\}$ are mutually different. Let $N/M = l_1 \cdots l_m$ be the prime decomposition. We write

$$\mathbb{T}_0(pN)_{pN/N} \otimes \mathbb{R} = (\mathbb{T}_0(pN)_{pN/N}^{(N/M)} \otimes \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}[U_{l_1}, \cdots, U_{l_m}] \tag{27}$$

and $V(\gamma)$ is a $\mathbb{R}[U_{l_1}, \cdots, U_{l_m}]$-module. As we have seen in the proof of **Proposition 4.1**, the characteristic polynomial of U_{l_i} is $P_{l_i}(U_{l_i}) = U_{l_i}^2 - a_{l_i}(f_\gamma)U_{l_i} + l_i$ and $\dim_{\mathbb{R}} V(\gamma) = 2^m$. Therefore

$$V(\gamma) \simeq \mathbb{R}[U_{l_1}, \cdots, U_{l_m}]/I,$$

where I is an ideal of $\mathbb{R}[U_{l_1}, \cdots, U_{l_m}]$ generated by the polynomials $\{P_{l_i}(U_{l_i})\}_{i=1,\cdots,m}$. Viewing $\mathbb{R}f_\gamma$ as a $\mathbb{T}_0(pN)_{pN/N}^{(N/M)} \otimes \mathbb{R}$-module, we write it by $\mathbb{R}f_\gamma^{(N/M)}$. Using (27) we see

$$V(\gamma) \simeq \mathbb{R}f_\gamma^{(N/M)} \otimes_{\mathbb{R}} (\mathbb{R}[U_{l_1}, \cdots, U_{l_m}]/I).$$

as $\mathbb{T}_0(pN)_{pN/N} \otimes \mathbb{R}$-modules. Thus we have proved a real version of **Theorem 4.1** and **Theorem 4.2**.

THEOREM 4.3. *(Weak multiplicity one) There is an irreducible decomposition*

$$X_N^{(0)} \otimes \mathbb{R} = \oplus_\gamma V(\gamma)$$

as a $\mathbb{T}_0(pN)_{pN/N} \otimes \mathbb{R}$-module. Here $\{\gamma\}$ runs through the real characters of normalized Hecke eigen newforms $\{f_\gamma\}_\gamma$ such that the level N_{f_γ} of f_γ satisfies $N_{f_\gamma} = pM$ where M is a divisor of N. Let $N/M = l_1 \cdots l_m$ be the prime decomposition. Then a $\mathbb{T}_0(pN)_{pN/N} \otimes \mathbb{R}$-module $V(\gamma)$ is defined to be

$$V(\gamma) \simeq \mathbb{R} f_\gamma^{(N/M)} \otimes_{\mathbb{R}} (\mathbb{R}[U_{l_1}, \cdots, U_{l_m}]/I).$$

Here the action of $\mathbb{T}_0(pN)_{pN/N} \otimes \mathbb{R}$ is defined via (27) and I is an ideal generated by polynomials $\{P_{l_i}(U_{l_i})\}_{i=1,\cdots,m}$ where

$$P_{l_i}(U_{l_i}) = U_{l_i}^2 - a_{l_i}(f_\gamma)U_{l_i} + l_i.$$

Moreover the characters $\{\gamma\}$ are mutually different.

Let l be an odd prime different from p. Remember that $N \in \mathcal{N}_{p,l}$ is the set of square free positive integers prime to lp.

THEOREM 4.4. *(Monotonicity) For $N \in \mathcal{N}_{p,l}$ let $\rho_l^1(N)$ be the largest eigenvalue of the Hecke operator T_l of $X_N^{(0)} \otimes \mathbb{R}$. Then for $M, N \in \mathcal{N}_{p,l}$ such that $M|N$,*

$$\rho_l^1(N) \geq \rho_l^1(M)$$

Proof. **Theorem 4.2** (or **Theorem 4.3**) shows that, under the decomposition (22), $\rho_l^1(N)$ is the maximum of l-th coefficients of Hecke eigenform $\{f_\chi\}_\chi$. By (21) we find $S_2(\Gamma_0(pM))_{pM/M}$ is contained in $S_2(\Gamma_0(pN))_{pN/N}$ and the claim is obtained. □

5. Properties of the graphs

Let p be a prime satisfying $p \equiv 1 (\mathrm{mod}\, 12)$ and l be an odd prime different from p. Let us take $N \in \mathcal{N}_{p,l}$. For brevity we write $\mathbf{E}_i = (E_i, C_N)$ and let Γ_l be the set of cyclic subgroups of E_i of order l. The bijective correspondence

$$\mathrm{Hom}(\mathbf{E}_i, \mathbf{E}_j)(l)/\pm 1 \simeq \Gamma_l, \quad f \mapsto \mathrm{Ker} f.$$

shows that the Brandt matrix $B_p^{(l)}(N)$ is the representation matrix of T_l. Since $B_p^{(l)}(N)$ is symmetric, the eigenvalues are all real. It is easy to check that $\epsilon = Z_F - Z_V$ (cf. (26)) satisfies

$$T_l(\epsilon) = (l+1)\epsilon$$

and since ∂ in (26) commutes with T_l, $l+1$ is an eigenvalue of $B_p^{(l)}(N)$. Let δ be a corresponding eigenvector. Using the Eichler-Shimura relation and the Weil conjecture, **Theorem 4.2** (or **Theorem 4.3**) implies that the modulus of other eigenvalues are less than or equal to $2\sqrt{l}$ and

$$X_N \otimes \mathbb{R} = (X_N^{(0)} \otimes \mathbb{R}) \hat{\oplus} \mathbb{R}\delta,$$

where $\hat{\oplus}$ denotes an orthogonal direct sum. Moreover if $N \in \mathcal{N}_{p,l}$, **Theorem 4.3** and this decomposition yield a spectral decomposition of $X_N \otimes \mathbb{R}$ in terms of eigenspaces of T_l. **Theorem 4.2** implies that

$$\det[1 - B_p^{(l)}(N)t + lt^2] = (1-t)(1-lt)\det[1 - T_l t + lt^2 | S_2(\Gamma_0(pN))_{pN/N}]. \tag{28}$$

THEOREM 5.1. *For any $N \in \mathcal{N}_{p,l}$, $G_p^{(l)}(N)$ is a connected regular Ramanujan graph of degree $l+1$ not bipartite.*

Proof. By construction $G_p^{(l)}(N)$ is a regular graph of degree $l+1$. Let us investigate the eigenvalues of the adjacency matrix $B_p^{(l)}(N)$. As we have seen, $l+1$ is an eigenvalue of $B_p^{(l)}(N)$ and the modulus of other eigenvalues are less than or equal to $2\sqrt{l}$. Thus $G_p^{(l)}(N)$ is a Ramanujan graph. By the equation (1) (see also (2)), 0 is an eigenvalue of the Laplacian with multiplicity one and we see that $G_p^{(l)}(N)$ is connected. In general a connected finite regular graph of degree d is bipartite if and only if $\pm d$ are eigenvalues of the adjacency matrix ([27]). Therefore $G_p^{(l)}(N)$ is not bipartite. □

Now **Theorem 1.1** is a direct consequence of the equation (1) (see also (2)), **Theorem 4.4** and **Theorem 5.1**.

Proof of Theorem 1.2. Set $N = q$ and we use the decomposition (21). Since $S_2(\Gamma_0(p))^{(q)}$ is isomorphic to $S_2(\Gamma_0(p))$ as a $\mathbb{T}_0(pq)^{(pq)}$-module, we see

$$S_2(\Gamma_0(pq))_{pq/q} = S_2(\Gamma_0(pq))_{new} \oplus S_2(\Gamma_0(p))^{\oplus 2}$$

as $\mathbb{T}_0(pq)^{(pq)}$-modules and

$$\frac{\det(1 - B_p^{(l)}(q)t + lt^2)}{\det(1 - B_p^{(l)}(1)t + lt^2)^2} = \frac{\det(1 - T_l t + lt^2 \mid S_2(\Gamma_0(pq))_{new})}{(1-t)(1-lt)} = \frac{\det(1 - B_q^{(l)}(p)t + lt^2)}{\det(1 - B_q^{(l)}(1)t + lt^2)^2}$$

by (28). On the other hand **Fact 2.2** implies,

$$\chi(G_p^{(l)}(q)) - 2\chi(G_p^{(l)}(1)) = \frac{(p-1)(q-1)(1-l)}{24} = \chi(G_q^{(l)}(p)) - 2\chi(G_q^{(l)}(1))$$

and the claim follows from **Fact 2.1**. □

Proof of Theorem 1.3. Let us recall the decomposition (22)

$$S_2(\Gamma_0(pN))_{pN/N} = \oplus_\chi \mathbb{C} f_\chi,$$

where f_χ is a normalized Hecke eigenform. Then the second largest eigenvalue $\rho_l^1(N)$ of $B_p^{(l)}(N)$ is the maximum of $\{a_l(f_\chi)\}_\chi$ by **Theorem 4.2** and satisfies $\rho_l^1(N) \le 2\sqrt{l}$ by **Theorem 5.1**. Let $\{r_i\}_{i=1}^\infty$ be the set of primes and $N_k = \prod_{i=1}^k r_i$. Then by **Theorem 4.4**, $\rho_l^1(N_k)$ is monotone increasing for k. In general let $\{G_i\}_i$ be an infinite family of connected d-regular graphs satisfying

$$\lim_{i\to\infty} |V(G_i)| = \infty.$$

Then it is known that

$$\liminf_{i\to\infty} \rho^1(G_i) \ge 2\sqrt{d-1}$$

by Alon and Boppana ([1][2][28]). We will use this fact. Since $\{G_p^{(l)}(N_k)\}_k$ is an infinite family of connected regular Ramanujan graphs of degree $l+1$ with

$$\lim_{k\to\infty} |V(G_p^{(l)}(N_k))| = \lim_{k\to\infty} \frac{(p-1)\prod_{i=1}^{k}(1+r_i)}{12} = \infty,$$

we see

$$\lim_{k\to\infty} \rho_l^1(N_k) = 2\sqrt{l},$$

and

$$\lim_{k\to\infty} \mathrm{Max}\{a_l(f_\chi) \ : \ S_2(\Gamma_0(pN_k))_{pN_k/N_k} = \oplus_\chi \mathbb{C} f_\chi\} = 2\sqrt{l}.$$

Since $S_2(\Gamma_0(pN_k))_{pN_k/N_k}$ is a subspace of $S_2(\Gamma_0(pN_k))$, the remaining claim immediately follows from this result and the decomposition in **Theorem 4.1**. □

The proof implies the following corollary.

COROLLARY 5.1. *Let p be a prime satisfying $p \equiv 1 (\mathrm{mod}\, 12)$ and l an odd prime with $l \neq p$. Then for any set of mutually distinct primes $\{r_i\}_{i=1}^{\infty}$ which are different from l and p, there is a sequence of normalized Hecke eigenforms $\{f_i\}_i$ of weight 2 such that $f_i \in S_2(\Gamma_0(pr_1\cdots r_i))_{new}$ and*

$$\lim_{i\to\infty} a_l(f_i) = 2\sqrt{l}.$$

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