COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 69 2021

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

Some Poisson Formula on Tube Domains

by

Tomoyoshi IBUKIYAMA

(Received October 30, 2020)
(Revised December 21, 2020)

The purpose of this short note is to show a Poisson formula for some function on symmetric tube domains. First we state the theorem. The definitions of terminologies in Jordan algebras used in the statement of the theorem will be reviewed precisely later.

Let $T_\Omega = V + i\Omega$ be a symmetric tube domain, where V is a simple formally real Jordan algebra and Ω is the symmetric cone of V. For a lattice L in V, we denote by L^* the dual lattice of L with respect to the inner metric $(x, y) = tr(xy)$. We put $n = \dim_{\mathbb{R}} V$ and $r = \mathrm{rank}(V)$.

THEOREM 1. *For any complex number $\alpha \in \mathbb{C}$ such that $\Re(\alpha) > \frac{2n}{r} - 1$ and $Z \in T_\Omega$, we have*

$$\sum_{S\in L} \det(Z+S)^{-\alpha} = \mathrm{vol}(L^*)\frac{(-2\pi i)^{r\alpha}}{\Gamma_\Omega(\alpha)} \sum_{T\in\Omega\cap L^*} \det(T)^{\alpha-\frac{n}{r}} e^{2\pi i Tr(TZ)}, \tag{1}$$

where $\Gamma_\Omega(s)$ is the gamma function associated with the symmetric cone Ω and $\mathrm{vol}(L^*)$ *is the volume of V/L^* with respect to the Euclid measure of V. Both sides of* (1) *converge absolutely and uniformly on any compact set of T_Ω for the above range of α.*

This theorem is stated in [8] Lemma 8.4 for sufficiently large α, but any exact condition on α for the convergence has not been written there. The range of α for the convergence is known for several classical domains, for example in [7] and [4]. Classically, this type of theorem is often used to prove the convergence of the Eisenstein series since (1) is a subseries of the Eisenstein series. Nowadays there are several ways to prove the convergence of the Eisenstein series, but they converge for weight $\Re(\alpha) > 2n/r$. The range of α for the subseries (1) is sharper. Sometimes we need the summation (1) with this sharper bound. For example, if we consider the contribution of unipotent elements in the dimension formulas of holomorphic modular forms on tube domains, this estimate of range is sometimes critical for the application of the Selberg trace formula using the Bergman kernel and the zeta functions of prehomogeneous vector spaces (see [6]). This was the first motivation of the author to this topic. Now it seems there are also some unexpected applications, e.g. for non-vanishing of some differential operators on some functions ([1] p. 87). So the above formula (1) might have some independent interest with other potential applications

2010 *Mathematics Subject Classification.* 11F55, 17C35.
Key words and phrases. Poisson formula, Jordan algebra, Tube domain.
This work was supported by JSPS KAKENHI Grant Number JP19K03424.

in future. For the proof, we mostly follow the same argument as that of H. Braun in [2] except for technical details on Jordan algebras. The reason to follow [2] is that her argument seems to give the sharpest bound. This paper is more or less expository in nature, but a unified treatment by Jordan algebras would be of some interest. The referee suggested that Theorem 1 for the case $n = 1$ is sometimes refered to Lipschitz [5].

1. A review on Jordan algebras

In this section, we review from Faraut and Korányi [3] some well-known facts on a formally real Jordan algebra and the symmetric cone associated with it.

In this paper, by an algebra V over $\mathbb{R}$, we mean a vector space over $\mathbb{R}$ such that a bilinear product mapping $V \times V \ni (x, y) \to xy \in V$ is defined. Here we do not assume the associativity of the multiplication. An algebra V over $\mathbb{R}$ is said to be a formally real Jordan algebra if it is finite dimensional with the unit element e and satisfies the following three conditions.

(1) $xy = yx$ for any $x, y \in V$.

(2) $x(x^2y) = x^2(xy)$ for any $x, y \in V$.

(3) $x^2 + y^2 = 0$ if and only if $x = y = 0$.

We assume that J is simple, that is, J does not contain non-trivial ideal. The classification of simple formally real Jordan algebras is well known and there are five different types (see for example [3]), though we do not use this fact in the paper. We denote by n the dimension of V as the vector space over $\mathbb{R}$.

Let Ω be the symmetric cone (open convex self-dual homogeneous cone) in V associated with V. This is given by

$$\Omega = \{x^2; x \in V^{\times}\},$$

where $V^{\times}$ is the set of invertible elements of V. Here an element x of V is called invertible, if there exists an element $y \in \mathbb{R}[x]$ such that $xy = e$, where e is the unit of V.

The closure $\overline{\Omega}$ of Ω is given by

$$\overline{\Omega} = \{x^2; x \in V\}.$$

For x and $y \in V$, we write $x \leq y$ (resp. $x < y$), if $y - x \in \overline{\Omega}$ (resp. $y - x \in \Omega$).

An element $c \in V$ is called idempotent if $c^2 = c$, and called primitive if it is non-zero and cannot be written as a sum of two non-zero idempotents. Idempotents c and d are said to be orthogonal if $cd = 0$. We say that a set of primitive idemponents $c_1, \ldots, c_m$ is a Jordan frame, if $c_j c_k = 0$ for all $j \neq k$ and

$$\sum_{j=1}^{m} c_j = e.$$

The maximum of the degree of minimal polynomials over $\mathbb{R}$ of elements $x \in V$ is called the rank of V, and we write $r = \mathrm{rank}(V)$. Then the number of idempotents in any Jordan frame is equal to the rank of V, so $m = r$ in the above. For any element $x \in V$, there exists a Jordan frame $c_1, \ldots, c_r$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that

$$x = \sum_{i=1}^{r} \lambda_i c_i \,.$$

This expression is called the spectral decomposition of x. Here numbers λ_i are uniquely determined by x (counting multiplicities). These numbers are called eigenvalues of x. The determinant and the trace of x are given by $\det(x) = \prod_{i=1}^{r} \lambda_i$ and $\mathrm{tr}(x) = \sum_{i=1}^{r} \lambda_i$, respectively. These are $\mathbb{R}$ valued polynomial functions on V. We can prolong these to the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$.

A positive definite inner product of V is defined by $(x|y) = \mathrm{tr}(xy)$.

For any $x \in V$, define the left translation $L(x)$ by the linear transformation of V defined by $L(x)y = xy$ for any $y \in V$. Then, denoting by Tr the usual trace of a linear transformation, we have

$$\mathrm{tr}(x) = \frac{r}{n} \mathrm{Tr}(L(x)) \,.$$

An element $x \in V$ is invertible if and only if $\det(x) \neq 0$. For any idempotent c of V, $L(c)$ is always semi-simple and possible eigenvalues of $L(c)$ are 0, 1, and 1/2. We write the eigenspace of c with respect to an eigenvalue λ by $V(c, \lambda)$. Now fix any Jordan frame $c_1, \ldots, c_r$ of V. Put $V_{ii} = V(c_i, 1)$ for $i = 1, \ldots, r$ and put $V_{ij} = V(c_i, 1/2) \cap V(c_j, 1/2)$ for $i \neq j$. Then we have a direct orthogonal decomposition

$$V = \sum_{i \leq j} V_{ij}$$

with respect to the metric $(x|y)$ as a vector space over $\mathbb{R}$. For any i with $1 \leq i \leq r$, we have $\dim_{\mathbb{R}} V_{ii} = 1$. For any pairs (j, k) with $j \neq k$, the dimensions of the spaces V_{jk} are the same, and we denote this dimension by $d = \dim(V_{jk})$. Naturally, we have $n = r + r(r-1)d/2$.

For $x \in V$, we write

$$x = \sum_{j=1}^{r} x_j c_j + \sum_{j<k} x_{jk} \,,$$

where $x_j \in \mathbb{R}$, and $x_{jk} \in V_{jk}$. Then we have $\mathrm{tr}(x_{jk}) = (r/n)\mathrm{Tr}(L(x_{jk})) = 0$ for $j < k$ and $\mathrm{tr}(x) = \sum_{j=1}^{r} x_j$. If $x \in \overline{\Omega}$, then we have

$$||x_{ij}|| \leq 2 x_i x_j \,.$$

(cf. [3] p.80, Exercise 7(b).)

2. A proof of Theorem 1

Notation being the same as in the last section, let L be a lattice in V, that is, a free $\mathbb{Z}$-module such that $L \otimes_{\mathbb{Z}} \mathbb{R} = V$. Define the dual lattice L^* of L by

$$L^* = \{x \in V; \mathrm{tr}(xy) \in \mathbb{Z} \text{ for all } y \in L\} \,.$$

Let T_Ω be the symmetric tube domain defined by

$$T_\Omega = V + i\Omega \,.$$

Using the notation of [3], we denote by $\Gamma_\Omega(s)$ the gamma function associated with symmetric cone Ω defined by

$$\Gamma_\Omega(s) = \int_\Omega e^{-\mathrm{tr}(x)} \det(x)^{s-\frac{n}{r}} dx .$$

Here we denote by dx the Euclid measure of V with respect to the inner metric $(x|y)$. More precisely, for the decomposition $V = \sum_{i=1}^{r} V_{ii} + \sum_{j<k} V_{jk}$ and an orthonormal basis ω_i $(1 \le i \le r)$ of V_{ii} and an orthonormal basis $\omega_{jk}^{(\nu)}$ $(1 \le j < k \le r, 1 \le \nu \le d)$ of V_{jk}, we write $x \in V$ as

$$x = \sum_{i=1}^{r} x_i\omega_i + \sum_{\nu=1}^{d} \sum_{1\le j<k\le r} x_{jk}^{(\nu)}\omega_{jk}^{(\nu)}$$

and define

$$dx = \prod_{i=1}^{r} dx_i \prod_{\nu=1}^{d} \prod_{1\le j<k\le r} dx_{jk}^{(\nu)} .$$

The integral $\Gamma_\Omega(s)$ does not depend on the choice of the orthonormal basis.

Example. Let $V = Sym_r(\mathbb{R})$ be the Jordan algebra of $r \times r$ real symmetric matrices $X = (x_{ij})$. Denote by e_{ij} the $r \times r$ matrix whose (i, j) component is 1 and all the other components are 0. Then we have $Tr((e_{ij} + e_{ji})^2) = 2$. So the measure for x_{ij} part should be $\sqrt{2}dx_{ij}$ and the above measure dx is given by

$$dx = 2^{r(r-1)/4} \prod_{i\le j} dx_{ij} .$$

Classically, the measure $\prod_{i\le j} dx_{ij}$ is often used, so this causes a small difference from the classical formulas.

Going back to our case, it is known that

$$\Gamma_\Omega(s) = (2\pi)^{(n-r)/2} \prod_{j=1}^{r} \Gamma\left(s - \frac{(j-1)d}{2}\right)$$

for $\Re(s) > (r-1)d/2$ ([3] p. 123.)

For $Z \in T_\Omega$ and for $\Re\alpha > (r-1)d/2$, we have

$$\int_\Omega e^{2\pi i \mathrm{tr}(xZ)} \det(x)^{\alpha-\frac{n}{r}} dx = (2\pi)^{-r\alpha}\Gamma_\Omega(\alpha) \det(Z/i)^{-\alpha} . \tag{2}$$

Indeed, if $Z = Yi$ with $Y \in \Omega$, this is nothing but the formula in [3] p. 124, and by holomorphy, both sides are equal also for $Z = X+Yi \in T_\Omega$. Now, prolonging the integrand of LHS of (2) to the function $f(x)$ on V by setting 0 outside Ω, the Fourier transform

$$\widehat{f}(S) = \int_V f(x)e^{-2\pi i Tr(xS)}dx$$

of $f(x)$ for $S \in V$ is obtained by changing Z by $Z - S$ in RHS of (2). So we have the formula (1) in Theorem 1 by the usual Poisson summation formula as far as both sides converge.

Next, we will show that, if $\Re(\alpha) > 2n/r - 1$, then both sides of (1) of Theorem 1 converge absolutely and uniformly on any compact subset of T_Ω.

We prepare several lemmas. We denote by e the unit element of V.

LEMMA 1. *(1) Let K be a compact subset of V. Then, there exists a constant $\epsilon > 0$ which depend only on K such that $S^2 \leq \epsilon^2 e$ and that $-\epsilon e \leq S \leq \epsilon e$ for any $S \in K$.*
(2) If K is a compact subset of Ω, then there exists a positive constant ϵ depending only on K such that $\epsilon e \leq Y$ for all $Y \in K$.

Proof. For any compact set K, the set $\{\mathrm{tr}(S^2) \in \mathbb{R};\, S \in K\}$ is bounded, since trace and multiplication are continuous functions. So there exists a constant $a > 0$ such that $\mathrm{tr}(S^2) \leq a$ for any $S \in K$. For $S \in V$ with eigenvalues $\lambda_1, \ldots, \lambda_r$, we see by the spectral decomposition of S that the eigenvalues of S^2 are $\lambda_1^2, \ldots, \lambda_r^2$. Hence $\lambda_j^2 \leq \mathrm{tr}(S^2) \leq a$ and $-\sqrt{a} \leq \lambda_j \leq \sqrt{a}$ for all j with $1 \leq j \leq r$. By the spectral decomposition, this means that $-\sqrt{a}e \leq S \leq \sqrt{a}e$. So the first part is proved. For $K \subset \Omega$, the set $\{\mathrm{tr}(y);\, y \in K\}$ is bounded, so there is a constant $b_1 > 0$ such that $\lambda < b_1$ for all the eigenvalues λ of any element of K. On the other hand, $\{\det(y);\, y \in K\}$ is also a bounded closed set in positive real numbers, and there is a constant $b_2 > 0$ such that $b_2 < \det(y)$ for any $y \in K$. Since we have $\det(y) < b_1^{r-1}\lambda$ for any eigenvalue λ of $y \in K$, we have $b_1^{1-r} b_2 < \lambda$. Hence, taking $\epsilon = b_1^{1-r} b_2$, the second part is proved. $\square$

LEMMA 2. *Fix any positive number $\epsilon > 0$. For any $Z = X + iY \in T_\Omega$, and $S \in V$ with $S^2 \leq \epsilon^{-2} e$, we have*

$$c^{-1} |\det(Z)| \leq |\det(Z + S)| \leq c |\det(Z)|$$

where $c = 2^{r/2}(1 + \epsilon^{-1}\mathrm{tr}(Y^{-1}))^r$.

Proof. The proof is almost a reproduction of the proof by Krieg in [4] p. 141 Proposition 1.4, except for the points that we need a necessary modification for general Jordan algebras. Since $Y \in \Omega$, there exists $t \in \Omega$ such that $Y = t^2$. For any $u \in V$, define the quadratic representation $P(u)$ by $P(u) = 2L(u)^2 - L(u^2)$. It is known that $\det(P(u)v) = (\det u)^2(\det v)$ for any $u, v \in V$ and $P(x^{-1}) = P(x)^{-1}$ for any invertible element $x \in V^\times$ (See [3] II-3) . Hence

$$\det(Z + S) = (\det t)^2 \det(P(t^{-1})(X + S) + ie)$$

and $\det(Z) = (\det t)^2 \det(P(t^{-1})X + ie)$. If $u, v \in \mathbb{R}[w]$ for some $w \in V$, then $\det(uv) = \det(u)\det(v)$ ([3] p. 30). Hence we have

$$|\det(Z)|^2 = \det(Y)^2 \det((P(t^{-1})X)^2 + e)\,,$$

$$|\det(Z + S)|^2 = \det(Y)^2 \det(\big(P(t^{-1})(X + S)\big)^2 + e)\,.$$

Since $\det(u) < \det(v)$ for any $u, v \in \Omega$ with $v - u \in \Omega$, we have

$$\begin{aligned} |\det(Z + S)|^2 &\leq \det(Y)^2 \det((P(t^{-1})(X + S))^2 + (P(t^{-1})(X - S))^2 + 2e) \\ &= \det(Y)^2 \det(2(P(t^{-1})X)^2 + 2(P(t^{-1})S)^2 + 2e) \\ &= \det(Y)^2 \times 2^r \det((P(t^{-1})X)^2 + (P(t^{-1})S)^2 + e)\,. \end{aligned}$$

Since we assumed $S^2 \leq \epsilon^{-2}e$, we have $-\epsilon^{-1}e \leq S \leq \epsilon^{-1}e$. It is known that Ω is stable under the action of $P(u)$ for any invertible $u \in V$, so we have $-\epsilon^{-1}P(t^{-1})e \leq P(t^{-1})S \leq \epsilon^{-1}P(t^{-1})e$. By definition, we have $P(t^{-1})e = t^{-2} = Y^{-1}$ and we also have $Y^{-1} \leq \mathrm{tr}(Y^{-1})e$. Hence $(P(t^{-1})S)^2 \leq \epsilon^{-2}\mathrm{tr}(Y^{-1})^2 e$ and

$$\begin{aligned}
&|\det(Z+S)|^2 \\
&\leq \det(Y)^2 \times 2^r \det((1+\epsilon^{-2}(tr(Y^{-1}))^2)e + (P(t^{-1})X)^2)) \\
&= \det(Y)^2 \times 2^r (1+\epsilon^{-2}(\mathrm{tr}(Y^{-1}))^2)^r \det(e + (P(t^{-1})X)^2(1+\epsilon^{-2}(\mathrm{tr}(Y^{-1}))^2)^{-1}) \\
&\leq \det(Y)^2 \times 2^r (1+\epsilon^{-2}(\mathrm{tr}(Y^{-1}))^2)^r \det((P(t^{-1})X)^2 + e) \\
&= |\det(Z)|^2 \times 2^r (1+\epsilon^{-2}(\mathrm{tr}(Y^{-1}))^2)^r .
\end{aligned}$$

Here we have

$$\begin{aligned}
&(1+\epsilon^{-2}\mathrm{tr}(Y^{-1})^2)^{r/2} \\
&\qquad\qquad \leq (1+2\epsilon^{-1}tr(Y^{-1}) + \epsilon^{-2}(tr(Y^{-1}))^2)^{r/2} = (1+\epsilon^{-1}\mathrm{tr}(Y^{-1}))^r .
\end{aligned}$$

Hence we have the second inequality. The first inequality is obtained if we replace Z by $Z-S$ in the second. □

For $Z \in T_\Omega$, we put

$$I_\alpha(Z) = \int_V |\det(Z+S)|^{-\alpha} d\,S .$$

Here dS is the Euclid measure on V defined before.

LEMMA 3. *For any $Z \in T_\Omega$ and any $\alpha \in \mathbb{C}$ with $\Re(\alpha) > \frac{2n}{r} - 1$, the integral $I_\alpha(Z)$ converges and we have*

$$I_\alpha(Z) = \det(Y)^{\frac{n}{r}-\alpha} 4^{n-r\alpha/2} \pi^n \frac{\Gamma_\Omega(\alpha - \frac{n}{r})}{\Gamma_\Omega(\alpha/2)^2} .$$

Proof. If $Z = X + iY$, it is obvious that $I_k(Z) = I_k(iY)$, since dS is invariant by the translation by X. Take $t \in \Omega$ such that $Y = t^2$. Then $|\det(iY+S)| = \det(Y)|\det(ie + P(t^{-1})S)|$. Since $\det P(t^{-1}) = (\det t)^{-2n/r}$, we have $I_\alpha(iY) = \det(Y)^{n/r-\alpha} I_\alpha(ie)$. Here the formula for $I_\alpha(ie)$ is known in [3] p. 142 Exercise 5. □

LEMMA 4. *Let $L \subset V$ be a lattice and $k \in \mathbb{R}$ with $k > \frac{2n}{r} - 1$. For each $\epsilon > 0$, there exists a positive constant c which depends only on $r = \mathrm{rank}(V)$ and ϵ such that*

$$c^{-k}\mathrm{vol}(L)^{-1} I_k(Z) \leq \sum_{S \in L} |\det(Z+S)|^{-k} \leq c^k \mathrm{vol}(L)^{-1} I_k(Z)$$

for any $Z = X + iY \in T_\Omega$ with $Y \geq \epsilon e$. In particular, the series $\sum_{S\in L} \det(Z+S)^{-k}$ converges absolutely and uniformly on any compact set $K \subset T_\Omega$.

Proof. For $Y \geq \epsilon e$, we have $\mathrm{tr}(Y^{-1}) \leq r\epsilon^{-1}$. Since the fundamental parallelotope F of L in V is compact, the values $Tr(H^2)$ are bounded for $H \in F$. So there exists a constant $c_2 > 0$ such that $H^2 \leq c_2 e$ for all $H \in F$. Hence by Lemma 2, there exists a positive constant which depends only on r and ϵ such that

$$c^{-1}|\det(Z+S)| \le |\det(Z+H+S)| \le c|\det(Z+S)|$$

for all $Y \ge \epsilon e$ and $H \in F$. Since

$$I_k(Z) = \int_F \left(\sum_{S\in L} |\det(Z+H+S)|^{-k}\right) dH\,,$$

we have the inequality in Lemma 4. For any compact set $K \subset T_\Omega$, the integral $I_k(Z)$ is bounded by Lemma 3, so we have the last assertion of Lemma 4. □

We note that by the above lemma 4, LHS of Theorem 1 does not converge absolutely for $\alpha = 2n/r - 1$.

LEMMA 5. *For $Z \in T_\Omega$, for any $\alpha \in \mathbb{C}$ with $\Re\alpha > n/r$ and any lattice $L \subset V$, the series*

$$\sum_{u\in\Omega\cap L^*} e^{2\pi i \operatorname{tr}(uZ)} \det(u)^{\alpha-n/r} \tag{3}$$

converges absolutely and uniformly on any compact set of T_Ω.

Proof. By Lemma 1, for a compact set K of T_Ω, there exists a positive constant ϵ such that $Y \ge \epsilon e$ for any $Z = X + iY \in K$. It is known that the inner product is associative, that is, $(L(x)u|v) = (u|L(x)v)$ for any u, v, $x \in V$. So if we put $u = t^2$ with $t \in V$, then we have $\operatorname{tr}(uY) = \operatorname{tr}(t^2Y) = \operatorname{tr}(2t(tY) - t^2Y) = \operatorname{tr}(P(t)Y)$. So we have $\operatorname{tr}(P(t)Y) \ge \operatorname{tr}(P(t)\epsilon e) = \epsilon\operatorname{tr}(u)$. Hence, we have $\operatorname{tr}(uY) \ge \epsilon\operatorname{tr}(u)$ for any $u \in \Omega$. Now fix a natural number m. Then there exists a constant c such that the cardinality of $\{u \in L^* \cap \Omega\,;\, \operatorname{tr}(u) \le m\}$ is bounded by $(2cm+1)^n$. Indeed, if we write $u = \sum_{j=1}^r u_j c_j + \sum_{j<k} u_{jk}$ for a Jordan frame $c_1, \ldots, c_r$, then $|u_j| \le \sum_{j=1}^r u_j = \operatorname{tr}(u) = m$ and $||u_{jk}||^2 \le 2u_k u_l \le 2m^2$, and hence

$$||u||^2 = \sum_{j=1}^r |u_j|^2 + \sum_{j<k} ||u_{jk}||^2 \le rm^2 + \frac{r(r-1)}{2} \times 2m^2 = r^2m^2$$

and $||u|| \le rm$. On the other hand, for a basis $e_1, \ldots, e_n$ of L^* as a Z module, we write $u = \sum_{j=1}^n a_j e_j$ with $a_j \in \mathbb{Z}$. Then we have

$$||u||^2 = (u|u) = \sum_{1\le i,j\le r} a_i a_j (e_i|e_j)\,.$$

Since $(x|y)$ is a positive definite metric, the $r \times r$ matrix

$$S = ((e_i, e_j))_{1\le i,j\le r}$$

is positive definite, and for some constant $s > 0$, we have

$$s\sum_{j=1}^r a_j^2 \le ||u||^2\,.$$

So there exists a constant $c_3 > 0$ such that we have

$$|a_j| \le c_3 m \text{ for any } j \text{ and any } u \in L^* \text{ with } \operatorname{tr}(u) \le m\,. \tag{4}$$

The number of integers a_j which satisfy (4) is at most $2c_3m+1$. Also, we have $\det(u) \le m^r$ for u with $\mathrm{tr}(u) \le m$. Now, if we take partial sum of the series (3) for u with $m-1 \le \mathrm{tr}(u) < m$, then for any constant $M > 0$, there exists a constant $c_4 > 0$ depending on ϵ and M but not on m such that

$$e^{-\epsilon \mathrm{tr}(u)} \le e^{\epsilon(1-m)} < \frac{c_4}{m^M}.$$

On the other hand, we have

$$\sum_{\substack{u \in L^* \cap \Omega \\ m-1 \le tr(u) < m}} \det(u)^{\Re(\alpha)-\frac{n}{r}} \le m^{r(\Re(\alpha)-\frac{n}{r})}(2c_3m+1)^r \le c_5 m^{r(\Re(\alpha)-\frac{n}{r}+1)}$$

for some constant $c_5 > 0$. So taking $M > r(\Re(\alpha) - \frac{n}{r} + 1) + 2$, there exists a constant $c_6 > 0$ such that the absolute value of the series (3) is bounded by $c_6\zeta(2)$ for any $Y \ge \epsilon e$. Hence it converges uniformly on K. □

Proof of Theorem 1. Now, since $\frac{2n}{r} - 1 \ge \frac{n}{r}$, the absolute and uniform convergence of both sides of (1) on any compact set is shown by Lemma 4 for LHS and by Lemma 5 for RHS. So Theorem 1 is proved. □

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Department of Mathematics,
Graduate School of Science,
Osaka University, Machikaneyama 1–1,
Toyonaka, Osaka, 560–0043, Japan
e-mail: ibukiyam@math.sci.osaka-u.ac.jp