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On the Functional Equations of the Shintani Double Zeta Functions

by

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Abstract. In this paper, we prove that Shintani double zeta functions possess a group of functional equations isomorphic to the dihedral group D_{12} of order 12.

1. Introduction and Main result

Shintani introduced double Dirichlet series whose coefficients are the numbers of distinct solutions to quadratic congruence equations in [1]. Such Dirichlet series are commonly referred to as the Shintani double zeta functions, and they play an important role for the study of asymptotic behaviors of class numbers of integral binary quadratic forms. He proved that his double zeta functions are meromorphically continued to $\mathbb{C}^2$, and they have two kinds of functional equations. These functional equations were derived from the functional equation of Eisenstein series and the theory of prehomogeneous vector spaces. In addition, these functional equations generate a non-cyclic group of order 4. In later years, various researchers gave different proofs for functional equations. It was suggested by Diamantis and Goldfeld in [5] that there are the other functional equations in terms of multiple Dirichlet series. After that, Kim, Tsuzuki and Wakatsuki in [3] explicitly determined two functional equations, which are satisfied by the generalized Shintani double zeta functions, and it is conjectured that they generate a group isomorphic to the dihedral group D_{12} of order 12. In this paper, we prove this conjecture is correct.

Let us explain some notations to define the Shintani double zeta functions. Suppose that S is a finite set of places of $\mathbb{Q}$ such that $\infty \in S$. Let $\mathbb{Q}_v$ denote the completion of $\mathbb{Q}$ at a place v, and set $\mathbb{Q}_S := \prod_{v\in S} \mathbb{Q}_v$. For $x = (x_v)_{v\in S} \in \mathbb{Q}_S^\times$, we set $|x|_S := \prod_{v\in S} |x|_v$ with $|\ |_v$ being the valuation of $\mathbb{Q}_v$. We denote by $\mathbb{A}$ the adele ring of $\mathbb{Q}$. For each character $\chi = \otimes_v \chi_v$ on $\mathbb{Q}^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times$, let $L^S(s,\chi) := \prod_{p\notin S} L_p(s,\chi_p)$, where $L_p(s,\chi_p) := (1 - \chi_p(p)p^{-s})^{-1}$ if χ_p is unramified, and $L_p(s,\chi_p) := 1$ if χ_p is ramified. We set $\zeta^S(s) := L^S(s,\mathbf{1})$ that equals to the Riemann zeta function without the S-factors. We set $N(\mathfrak{f}_\chi^S) := \prod_{p\notin S} \#(\mathbb{Z}_p/\mathfrak{f}_{\chi_p})$ where $\mathfrak{f}_{\chi_p}$ is the conductor of χ_p. We denote by $\widehat{\mathbb{Q}_S^\times/(\mathbb{Q}_S^\times)^2}$ the set of real characters of $\mathbb{Q}_S^\times$. For any finite set S such that $\infty \in S$, any real character ω_S on $\mathbb{Q}_S^\times$ and

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$\underline{s} = (s_1, s_2) \in \mathbb{C}^2$, Kim-Tsuzuki-Wakatsuki [3] defined $\Xi^S(\underline{s}, \omega_S)$ by

$$\Xi^S(\underline{s}, \omega_S) := \zeta^S(2s_1)\,\zeta^S(2s_1 + 2s_2 - 1)\sum_{\chi} \frac{L^S(s_2, \chi)}{L^S(2s_1 + s_2, \chi)\, N(f_\chi^S)^{s_1}} \tag{1}$$

with χ moving over all real characters $\chi = \otimes_v \chi_v$ of $\mathbb{Q}^\times \mathbb{R}_{>0} \backslash \mathbb{A}^\times$ such that $\otimes_{v \in S} \chi_v = \omega_S$. The series (1) is absolutely convergent for $\mathrm{Re}(s_1) > 1$ and $\mathrm{Re}(s_2) > 1$. They showed the series (1) is meromorphically continued to $\mathbb{C}^2$. By the explicit formula [3, Theorem 4.3], the original Shintani double zeta functions are expressed by linear combinations of these series $\Xi^S(\underline{s}, \omega_S)$ with rational functions of p^{s_1}, p^{s_2}. Therefore $\Xi^S(\underline{s}, \omega_S)$ is viewed as a natural generalization of the Shintani double zeta functions. By the functional equation of $L^S(s, \chi)$, they found the first functional equation

$$\Xi^S(\underline{s}, \omega_S) = \Gamma_S(1 - s_2, \omega_S)\,\Xi^S(s_1 + s_2 - \frac{1}{2}, 1 - s_2, \omega_S)\,, \tag{2}$$

where $\Gamma_S(s, \chi_S) := \prod_{v \in S} \Gamma_v(s, \chi_v)$,

(3)

$$\Gamma_\infty(s, \chi_\infty) := \cos(s\pi/2)^t\ \sin(s\pi/2)^{1-t}\ (2(2\pi)^{-s}\Gamma(s)), \quad t := \begin{cases} 1 & \text{if } \chi_\infty \text{ is trivial}\,, \\ 0 & \text{if } \chi_\infty = \mathrm{sgn}\,, \end{cases}$$

(4)

$$\Gamma_p(s, \chi_p) := N(\mathfrak{f}_{\chi_p})^{s-\frac{1}{2}} \times \begin{cases} (1 - \chi_p(p)p^{-1+s})/(1 - \chi_p(p)p^{-s}) & \text{if } \chi_p \text{ is unramified}\,, \\ 1 & \text{is } \chi_p \text{ is ramified}\,. \end{cases}$$

Moreover, from the theory of prehomogeneous vector spaces, they got the second functional equation

$$\Xi^S(\underline{s}, \omega_S) = \sum_{\chi_S \in \widehat{\mathbb{Q}_S^\times/(\mathbb{Q}_S^\times)^2}} \tilde{G}_S(s_1, \frac{3}{2} - s_1 - s_2, \chi_S, \omega_S)\,\Xi^S(s_1, \frac{3}{2} - s_1 - s_2, \chi_S) \tag{5}$$

under the condition $S \supset \{\infty, 2\}$, where $\tilde{G}_S(\underline{s}, \chi_S, \omega_S) := \prod_{v \in S} \tilde{G}_v(\underline{s}, \chi_v, \omega_v)$ and

(6)

$$\tilde{G}_v(\underline{s}, \chi_v, \omega_v) := \frac{|2|_v^{-1/2}}{\#(\mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2)}\tilde{\gamma}_v(s_2, \chi_v)\,\tilde{\gamma}_v(s_1 + s_2 - \frac{1}{2}, \omega_v) \sum_{\eta \in \mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2} \alpha(-\eta)\chi_v\omega_v(\eta)\,.$$

In Section 2, we explain the local gamma factor $\tilde{\gamma}_v(s, \chi_v)$ and the Weil constant $\alpha(\eta)$.

The following theorem is our main result.

THEOREM 1. *Assume that S contains ∞ and 2. Then, the functional equations* (2) *and* (5) *generate a group isomorphic to D_{12}.*

In Section 3, we will give an exact definition for the product of functional equations, and restate Theorem 1 more precisely.

In [2], Blomer defined double Dirichlet series $Z(\underline{s},\psi,\psi')$ by

$$Z(\underline{s},\psi,\psi') := \zeta^S(2s_1+2s_2-2) \sum_{d>0,\ \mathrm{odd}} \frac{L^S(s_1,\rho_d\psi)\,\psi'(d)}{d^{s_2}},$$

where ψ, ψ' are real Dirichlet characters on $\mathbb{Z}/8\mathbb{Z}$ and $\rho_d(a) := \left(\frac{d}{a}\right)$ with the Legendre symbol. He proved that $Z(\underline{s},\psi,\psi')$ has two functional equations and these generate a group isomorphic to D_{12}. It was proved in [3, Appendix A.3] that $Z(\underline{s},\psi,\psi')$ and $\Xi^S(\underline{s},\omega_S)$ are linear combinations of each other when $S=\{\infty,2\}$. Let us describe their relations. Set

$$T_{\infty,1} := \{\pm1,\pm2\}, \quad T_2 := \{1,\ 3,\ 5,\ 7\}.$$

For each square-free integer n, we define a real character $\omega_{n,S} = \otimes_{v\in S}\,\omega_{n,v}$ on $\mathbb{Q}_S^\times$ by $\omega_{n,v}(u) := \langle u,n\rangle_v$ where $\langle\ ,\ \rangle_v$ denotes the Hilbert symbol over $\mathbb{Q}_v^\times$. For each $j\in T_{\infty,1}$, we also define a real Dirichlet character ψ_j on $\mathbb{Z}/8\mathbb{Z}$ by $\psi_j(a) := \omega_{j,2}(a)$ $(a\in T_2)$. Then, we have

$$\Xi^S(\underline{s},\omega_{kb,S}) = \frac{1}{4}\sum_{j\in T_{\infty,1}} \psi_j(b)\, Z(\underline{s},\psi_j,\psi_{(-1)^{(b-1)/2}k}). \tag{7}$$

Therefore, his study gives a proof of Theorem 1 in the special case $S=\{\infty,2\}$.

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2. Preliminaries

In this section, we explain the gamma factors of (5) and (6). Let v be a place of $\mathbb{Q}$, that is, $v=\infty$ or v is a prime number p. We write $\psi_{\mathbb{Q}_v}$ for the additive character of $\mathbb{Q}_v$ defined by

$$\psi_{\mathbb{Q}_v}(x) := \begin{cases} \exp(2\pi i x) & \text{if } \mathbb{Q}_v=\mathbb{R}, \\ \exp(-2\pi i [x]_p) & \text{if } \mathbb{Q}_v=\mathbb{Q}_p, \end{cases}$$

where $[x]_p := \sum_{i=N}^{-1} a_i p^i$ with p-adic expansion $\mathbb{Q}_p \ni x = \sum_{i=N}^{\infty} a_i p^i$ $(a_i\in\{0,1,\cdots,p-1\})$.

Let $\mathrm{d}x_\infty$ denote the ordinary Lebesgue measure on $\mathbb{R}$ and $\mathrm{d}x_p$ denote the Haar measure on $\mathbb{Q}_p$ normalized by $\int_{\mathbb{Z}_p}\mathrm{d}x_p = 1$. We denote by $\mathrm{d}^\times x_v$ a Haar measure on $\mathbb{Q}_v^\times$ defined as $\mathrm{d}^\times x_\infty = \frac{\mathrm{d}x_\infty}{|x|_\infty}$ and $\mathrm{d}^\times x_p = (1-p^{-1})^{-1}\frac{\mathrm{d}x_p}{|x|_p}$. Then one has $\int_{\mathbb{Z}_p^\times}\mathrm{d}^\times x_p = 1$.

For $\phi_v\in C_c^\infty(\mathbb{Q}_v)$ and a real character χ_v on $\mathbb{Q}_v^\times$, we define the local Tate integral by

$$\zeta_v(\phi_v,s,\chi_v) := \int_{\mathbb{Q}_v^\times} \phi_v(x)\,|x|_v^s\,\chi_v(x)\,\mathrm{d}^\times x.$$

This integral is absolutely convergent for $\mathrm{Re}(s) > 0$, and meromorphically continued to the whole s-plane. We also have the local functional equation

$$\zeta_v(\hat{\phi}_v, s, \chi_v) = \tilde{\gamma}_v(s, \chi_v)\, \zeta_v(\phi_v, 1-s, \chi_v)\,,$$

where

$$\hat{\phi}_v(y) := \int_{\mathbb{Q}_v} \psi_{\mathbb{Q}_v}(xy)\, \phi_v(x)\, \mathrm{d}x\,, \quad y \in \mathbb{Q}_v$$

is the Fourier transform of ϕ_v. The local gamma factor $\tilde{\gamma}_v(s, \chi_v)$ is given explicitly as

$$\tilde{\gamma}_\infty(s, \mathrm{sgn}^\delta) = i^\delta\, \pi^{\frac{1}{2}-s}\, \Gamma(\tfrac{s+\delta}{2}) / \Gamma(\tfrac{1-s+\delta}{2}) \quad \text{for } \chi_\infty = \mathrm{sgn}^\delta\ (\delta = 0 \text{ or } 1)\,,$$

$$\tilde{\gamma}_p(s, \chi_p) = \begin{cases} (1 - \chi_p(p)\, p^{s-1})/(1 - \chi_p(p)\, p^{-s}) & \text{if } \chi_p \text{ is unramified}\,, \\ g_{\chi_p} N(\mathfrak{f}_{\chi_p})^s & \text{if } \chi_p \text{ is ramified}\,, \end{cases}$$

where g_{χ_p} is the Gauss sum for χ_p defined by

$$g_{\chi_p} := N(\mathfrak{f}_{\chi_p})^{-1} \sum_{u \in \mathbb{Z}_p^\times/(1+\mathfrak{f}_{\chi_p})} \chi_p(p^{-f_{\chi_p}} u)\, \psi_p(p^{-f_{\chi_p}} u)\,.$$

Here, an integer f_{χ_p} is defined by $\mathfrak{f}_{\chi_p} = p^{f_{\chi_p}} \mathbb{Z}_p$ and $N(\mathfrak{f}_{\chi_p}) = p^{f_{\chi_p}}$. Note that $g_{\chi_p}^2 = \chi_p(-1) N(\mathfrak{f}_{\chi_p})^{-1}$.

For any $a \in \mathbb{Q}_v^\times$, there exists a constant $\alpha_{\psi_{\mathbb{Q}_v}}(a)$ which satisfies

$$\int_{\mathbb{Q}_v} \phi_v(x)\, \psi_{\mathbb{Q}_v}(ax^2)\, \mathrm{d}x = \alpha_{\psi_{\mathbb{Q}_v}}(a)\, |2a|_v^{-1/2} \int_{\mathbb{Q}_v} \hat{\phi}_v(x)\, \psi_{\mathbb{Q}_v}\left(-\frac{x^2}{4a}\right) \mathrm{d}x$$

for any $\phi_v \in C_c^\infty(\mathbb{Q}_v)$. It is called the Weil constant. By the definition, $\alpha_{\psi_{\mathbb{Q}_v}}(a)$ depends only on the square class $a(\mathbb{Q}_v^\times)^2$ of $a \in \mathbb{Q}_v^\times$. It holds that we have the relations

$$\alpha_{\psi_{\mathbb{Q}_v}}(-a) = \overline{\alpha_{\psi_{\mathbb{Q}_v}}(a)}, \qquad \frac{\alpha_{\psi_{\mathbb{Q}_v}}(a)\alpha_{\psi_{\mathbb{Q}_v}}(b)}{\alpha_{\psi_{\mathbb{Q}_v}}(1)\alpha_{\psi_{\mathbb{Q}_v}}(ab)} = \langle a, b \rangle_v \quad \text{for any } a, b \in \mathbb{Q}_v^\times$$

with the Hilbert symbol $\langle\ ,\ \rangle_v$ on $\mathbb{Q}_v^\times$, see [4, p.3 §1] for its detail. For $v = \infty$, $\phi(x) = e^{-\pi x^2}$ satisfies $\hat{\phi} = \phi$. Hence by using $\alpha_{\psi_\mathbb{R}}(1) = \alpha_{\psi_\mathbb{R}}(2^{-1})$, we have

$$\int_\mathbb{R} e^{-\pi(1-i)x^2}\, \mathrm{d}x = \alpha_{\psi_\mathbb{R}}(1) \int_\mathbb{R} e^{-\pi(1+i)x^2}\, \mathrm{d}x\,.$$

On the other hand, for $v = p$, let ϕ be the characteristic function of $\mathbb{Z}_p$. Then $\hat{\phi} = \phi$ holds, so we obtain

$$\alpha_{\psi_{\mathbb{Q}_p}}(a) = |2a|_p^{-1/2} \int_{\mathbb{Z}_p} \psi_{\mathbb{Q}_p}\left(\frac{y^2}{4a}\right) \mathrm{d}y$$

for any $a \in \mathbb{Z}_p$. Hence, it can be explicitly calculated as

$$\alpha_{\psi_\mathbb{R}}(a) = \exp\!\left(\frac{\pi i}{4}\, a\right) \quad \text{for any } a \in \{\pm 1\}\,,$$

and for any $a \in \mathbb{Z}_p^\times$

$$\alpha_{\psi_{\mathbb{Q}_p}}(a) = \begin{cases} 1 & (p > 2)\,, \\ e^{-\pi i/4} & (p = 2,\ a \equiv 1 \ (\mathrm{mod}\ 4\mathbb{Z}_2))\,, \\ e^{\pi i/4} & (p = 2,\ a \equiv -1 \ (\mathrm{mod}\ 4\mathbb{Z}_2))\,, \end{cases}$$

$$\alpha_{\psi_{\mathbb{Q}_p}}(pa) = \begin{cases} \left(\frac{a}{p}\right) & (p > 2,\ p \equiv 1 \ (\mathrm{mod}\ 4\mathbb{Z}_2))\,, \\ -\left(\frac{a}{p}\right) i & (p > 2,\ p \equiv 3 \ (\mathrm{mod}\ 4\mathbb{Z}_2))\,, \\ \psi_{\mathbb{Q}_2}(a/8) & (p = 2)\,. \end{cases}$$

3. The product of functional equations

In this section, we give a rigorous definition of the product of functional equations, and restate Theorem 1 in a precise way. Fix a finite set S of places of $\mathbb{Q}$ and suppose that S contains ∞ and 2. Set

$$\Lambda_1 := \widehat{\mathbb{Q}_S^\times/(\mathbb{Q}_S^\times)^2}, \quad \Lambda := \Lambda_1 \times \Lambda_1\,,$$

$$\mathfrak{F} := \{(\phi_\lambda(\underline{s}))_{\lambda \in \Lambda} \mid \phi_\lambda : \mathbb{C}^2 \to \mathbb{C}\} \times (M_2(\mathbb{C}) \times \mathbb{C}^2)\,.$$

A product on $\mathfrak{F}$ is defined by

$$((\phi_{(\omega,\chi)}(\underline{s}))_{(\omega,\chi)\in\Lambda}, (A, B))\,((\phi'_{(\omega',\chi')}(\underline{s}))_{(\omega',\chi')\in\Lambda}, (A', B')) := \left(\left(\sum_{\chi\in\Lambda_1} \phi_{(\omega,\chi)}(\underline{s})\,\phi'_{(\chi,\chi')}(\underline{s}A + B)\right)_{(\omega,\chi')\in\Lambda}, (AA', BA' + B')\right).$$

Then, $\mathfrak{F}$ is a monoid, and the unit $\mathbf{1}_{\mathfrak{F}}$ is given by $((\phi_{0,\lambda}(\underline{s}))_{\lambda\in\Lambda}, (I_2, (0, 0)))$, where $\phi_{0,(\omega,\chi)} = 1$ if $\chi = \omega$, $\phi_{0,(\omega,\chi)} = 0$ otherwise, and I_2 denotes the unit matrix of degree 2. Two elements F_σ and F_τ in $\mathfrak{F}$ are defined as

$$F_\sigma := \left((\tilde{H}_S(\underline{s}, \chi, \omega))_{(\omega,\chi)\in\Lambda}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \left(-\frac{1}{2}, 1\right)\right),$$

$$F_\tau := \left((\tilde{J}_S(\underline{s}, \chi, \omega))_{(\omega,\chi)\in\Lambda}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \left(0, \frac{3}{2}\right)\right),$$

where

$$\tilde{H}_S((s_1, s_2), \chi, \omega) := \begin{cases} \Gamma_S(1 - s_2, \omega) & \text{if } \chi = \omega, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tilde{J}_S((s_1, s_2), \chi, \omega) := \tilde{G}_S((s_1, \frac{3}{2} - s_1 - s_2), \chi, \omega)\,.$$

Now, we consider a submonoid $\mathfrak{G}$ of $\mathfrak{F}$ generated by F_σ and F_τ, that is,

$$\mathfrak{G} := \langle F_\sigma,\ F_\tau \rangle\,.$$

For each $F = ((\phi_\lambda(\underline{s}))_{\lambda\in\Lambda}, (A, B)) \in \mathfrak{F}$, the equation

$$\Xi^S(\underline{s}, \omega) = \sum_{\chi\in\Lambda_1} \phi_{(\omega,\chi)}(\underline{s})\, \Xi^S(\underline{s}A + B, \chi) \quad (\omega \in \Lambda_1)$$

is called the functional equation F.

LEMMA 2. *$\Xi^S(\underline{s}, \omega)$ satisfies the functional equation F for any $F \in \mathfrak{G}$.*

Proof. The functional equations F_σ and F_τ are equivalent to (2) and (5) respectively. In addition, if $\Xi^S(\underline{s}, \omega)$ satisfies the functional equations F_1 and F_2, where $F_1, F_2 \in \mathfrak{G}$, it can be easily proved that the functional equation F_1F_2 is also satisfied. □

By this lemma, we can identify the element $F(\in \mathfrak{G})$ with the functional equation F. The dihedral group D_{12} of order 12 is generated by two elements σ and τ, which satisfy $\sigma^2 = \tau^2 = (\tau\sigma)^6 = 1$. Therefore, Theorem 1 is precisely restated as follows:

THEOREM 3. *A mapping $\eta : D_{12} \to \mathfrak{G}$ is defined by $\eta(\sigma) = F_\sigma$ and $\eta(\tau) = F_\tau$. Then, η is well-defined and a group isomorphism.*

We will give a proof of Theorem 3 in the next section.

4. Proof of Theorem 3

Let $\rho_j = \tau$ or σ, and $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$. Define an element $(M_{\rho,S}(\underline{s}), (A_\rho, B_\rho))$ in $\mathfrak{G}$ by

$$(M_{\rho,S}(\underline{s}), (A_\rho, B_\rho)) := F_{\rho_1}F_{\rho_2}\cdots F_{\rho_m}\,.$$

To simplify the description, $(\rho_1, \rho_2, \ldots, \rho_m)$ is written as $\rho_1\rho_2\cdots\rho_m$, so for example $M_{(\sigma,\tau,\sigma,\tau),S}(\underline{s})$ is written as $M_{(\sigma\tau)^2,S}(\underline{s})$.

It is easy to prove $F_\sigma F_\sigma = F_\tau F_\tau = \mathbf{1}_{\mathfrak{F}}$, $(A_{(\tau\sigma)^j}, B_{(\tau\sigma)^j}) \neq (I_2, (0, 0))$ $(j = 1, 2, \ldots, 5)$, and $(A_{(\tau\sigma)^6}, B_{(\tau\sigma)^6}) = (I_2, (0, 0))$. Hence, it is sufficient to prove

$$M_{(\sigma\tau)^6,S}(\underline{s}) = (\phi_{0,\lambda}(\underline{s}))_{\lambda\in\Lambda}$$

in order to obtain Theorem 3.

First, we explain that the proof is reduced to computations over $\mathbb{Q}_v$ for each $v \in S$. Take a place v of $\mathbb{Q}$, and set

$$\Lambda_{1,v} := \widehat{\mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2}, \quad \Lambda_v := \Lambda_{1,v} \times \Lambda_{1,v}\,.$$

In the previous section, we supposed $\{\infty,\ 2\} \subset S$, but we can formally define $M_{\rho,S}(\underline{s})$ even if S consists of a single place v. Hence, we have

$$M_{\tau,v}(\underline{s}) = (\tilde{H}_v((s_1, s_2), \chi_v, \omega_v))_{(\omega_v,\chi_v)\in\Lambda_v}\,,$$

$$\tilde{H}_v((s_1, s_2), \chi_v, \omega_v) := \begin{cases} \Gamma_v(1 - s_2, \omega_v) & \text{if } \chi_v = \omega_v, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_{\sigma,v}(\underline{s}) = (\tilde{G}_v((s_1, \frac{3}{2} - s_1 - s_2), \chi_v, \omega_v))_{(\omega_v,\chi_v)\in\Lambda_v}\,,$$

and for any $\rho = \rho_1\cdots\rho_m$ $(\rho_j \in \{\sigma,\ \tau\})$, $M_{\rho,v}(\underline{s})$ is inductively obtained from the product on $\mathfrak{F}$ $(S = \{v\})$. Take an order on $\Lambda_{1,v}$ (resp. Λ_1), then we can identify $M_{\rho,v}(\underline{s})$ (resp.

$M_{\rho,S}(\underline{s})$) with a square matrix of degree $\#\Lambda_{1,v}$ (resp. $\#\Lambda_1$). Hence, for $\rho = \rho_1 \cdots \rho_m$, $\rho' = \rho'_1 \cdots \rho'_l$ ($\rho_j, \rho'_j \in \{\tau, \sigma\}$), we have the relations

$$M_{\rho\rho',v}(\underline{s}) = M_{\rho,v}(\underline{s}) M_{\rho',v}(\underline{s}A_\rho + B_\rho)\,, \tag{8}$$

$$M_{\rho\rho',S}(\underline{s}) = M_{\rho,S}(\underline{s}) M_{\rho',S}(\underline{s}A_\rho + B_\rho) \tag{9}$$

by the product of matrices.

LEMMA 4. *We set* $M_{\rho,v}(\underline{s}) := \left(\phi^{\rho}_{\lambda_v}(\underline{s})\right)_{\lambda_v \in \Lambda_v}$ *for any* $v \in S$ *and any* $\rho = \rho_1\rho_2 \cdots \rho_m$ ($\rho_j \in \{\sigma, \tau\}$). *Then, we have*

$$M_{\rho,S}(\underline{s}) = \left(\prod_{v\in S} \phi^{\rho}_{\lambda_v}(\underline{s})\right)_{\lambda=(\lambda_v)_{v\in S}\in\Lambda=\prod_{v\in S}\Lambda_v}\,. \tag{10}$$

Proof. We prove this assertion by an induction for $m \geq 1$. When $m = 1$, we get $\rho = \sigma$ or τ, so (10) obviously holds. Suppose that the assertion holds for ρ and ρ'. Hence, if we set

$$M_{\rho,v}(\underline{s}) := \left(\phi^{\rho}_{\lambda_v}(\underline{s})\right)_{\lambda_v\in\Lambda_v}, \quad M_{\rho',v}(\underline{s}) := \left(\phi^{\rho'}_{\lambda_v}(\underline{s})\right)_{\lambda_v\in\Lambda_v},$$

then we get by (9)

$$\begin{aligned} M_{\rho\rho',S}(\underline{s}) &= \left(\sum_{\otimes_v\chi_v\in\Lambda_1}\prod_{v\in S}\phi^{\rho}_{(\omega_v,\chi_v)}(\underline{s})\,\phi^{\rho'}_{(\chi_v,\chi'_v)}(\underline{s}A_\rho + B_\rho)\right)_{(\otimes_v\omega_v,\,\otimes_v\chi'_v)\in\Lambda} \\ &= \left(\prod_{v\in S}\left(\sum_{\chi_v\in\Lambda_{1,v}}\phi^{\rho}_{(\omega_v,\chi_v)}(\underline{s})\,\phi^{\rho'}_{(\chi_v,\chi'_v)}(\underline{s}A_\rho + B_\rho)\right)\right)_{(\otimes_v\omega_v,\,\otimes_v\chi'_v)\in\Lambda}. \end{aligned}$$

Hence, (10) for $\rho\rho'$ is proved by (8). Thus, the assertion follows from the induction on m. □

From this lemma, we have only to calculate $M_{(\sigma\tau)^6,v}(\underline{s})$ for each $v \in S$.

Next we set an order for the set $\widehat{\mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2}$ $(= \Lambda_{1,v})$. For each place $v \in S$, we choose the following order on $\mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2$;

$$\begin{aligned} &\mathbb{Q}_\infty^\times/(\mathbb{Q}_\infty^\times)^2 : \ 1 \prec -1\,, \\ &\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 : \ 1 \prec 3 \prec 5 \prec 7 \prec 2 \prec 6 \prec 10 \prec 14\,, \\ &\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 : \ 1 \prec u \prec p \prec pu,\ (p \text{ is an odd prime number, } u \in \mathbb{Z}_p^\times\backslash(\mathbb{Z}_p^\times)^2)\,, \end{aligned} \tag{11}$$

where the notation $\prec$ means the order relation. Now, we define a real character on $\mathbb{Q}_v^\times$ by $\chi_{\delta_v}(a_v) := \langle a_v, \delta_v\rangle_v$ with the Hilbert symbol. Then, the bijection

$$\mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2 \ni \delta_v \mapsto \chi_{\delta_v} \in \widehat{\mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2}$$

gives an order on $\widehat{\mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2}$.

We consider the three cases

$$\text{(i) } v = \infty, \quad \text{(ii) } v = 2, \quad \text{(iii) } v = p\ (> 2)$$

and use the order specified above. The case $S = \{\infty, 2\}$ was already solved by Blomer, but we give an alternative proof by our method, since it gives decompositions of his matrices of functional equations into two smaller matrices. **Case (i) and (ii).** In these cases, we consider matrices $M_{\sigma\tau\sigma,v}(\underline{s})$ and $M_{\tau\sigma,v}(\underline{s})$. From (8) we have

$$M_{\sigma\tau\sigma,v}(\underline{s}) = \Big(\Gamma_v(1-s_2,\omega_v)\tilde{G}_v(s_1+s_2-\tfrac{1}{2}, 1-s_1, \chi_v, \omega_v)\Gamma_v(s_1,\chi_v)\Big)_{\omega_v,\chi_v},$$

$$M_{\tau\sigma,v}(\underline{s}) = \Big(\tilde{G}_v(s_1, \tfrac{3}{2}-s_1-s_2, \chi_v, \omega_v)\,\Gamma_v(s_1+s_2-\tfrac{1}{2},\chi_v)\Big)_{\omega_v,\chi_v}.$$

By (3),(4),(6) and direct calculations, we have

$$M_{\sigma\tau\sigma,\infty}(\underline{s}) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \qquad M_{\tau\sigma,\infty}(s_2,s_1) = \frac{1}{2}\pi^{s_1-\frac{1}{2}}\begin{pmatrix}\frac{\Gamma(\frac{1-s_1}{2})}{\Gamma(\frac{s_1}{2})} & \frac{\Gamma(\frac{1-s_1}{2})}{\Gamma(\frac{s_1}{2})}\\ \frac{\Gamma(\frac{2-s_1}{2})}{\Gamma(\frac{1+s_1}{2})} & -\frac{\Gamma(\frac{2-s_1}{2})}{\Gamma(\frac{1+s_1}{2})}\end{pmatrix},$$

$$M_{\sigma\tau\sigma,2}(\underline{s}) = \frac{1}{2}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1\\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1
\end{pmatrix},$$

$$M_{\tau\sigma,2}(s_2,s_1) = \frac{1}{2}\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}$$

where

$$B_{11} = \begin{pmatrix}b_1 & b_1 & b_1 & b_1\\ b_2 & -b_2 & b_2 & -b_2\\ b_3 & b_3 & b_3 & b_3\\ b_2 & -b_2 & b_2 & -b_2\end{pmatrix}, \quad B_{12} = \begin{pmatrix}b_1 & b_1 & b_1 & b_1\\ -b_2 & b_2 & -b_2 & b_2\\ -b_3 & -b_3 & -b_3 & -b_3\\ b_2 & -b_2 & b_2 & -b_2\end{pmatrix},$$

$$b_1 = \frac{1-2^{-s_1}}{1-2^{-1+s_1}}, \quad b_2 = 4^{\frac{1}{2}-s_1}, \quad b_3 = \frac{1+2^{-s_1}}{1+2^{-1+s_1}},$$

$$B_{21} = 8^{\frac{1}{2}-s_1}\begin{pmatrix}1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\end{pmatrix}, \quad B_{22} = 8^{\frac{1}{2}-s_1}\begin{pmatrix}1 & -1 & -1 & 1\\ -1 & -1 & 1 & 1\\ -1 & 1 & 1 & -1\\ 1 & 1 & -1 & -1\end{pmatrix}.$$

Note that all the entries of $M_{\sigma\tau\sigma,v}(\underline{s})$ are constants. Therefore, we obtain the matrices $M_{\sigma\tau\sigma,\{\infty,2\}}(\underline{s})$ and $M_{\tau\sigma,\{\infty,2\}}(s_2,s_1)$ by combining the above matrices according to Lemma 4. On the other hand, the relation (7) gives a regular matrix $\mathcal{H}$ such that $\Xi^{\{\infty,2\}}(\underline{s}) = \mathcal{H}\,\mathbf{Z}(\underline{s})$ if we let $\Xi^{\{\infty,2\}}(\underline{s})$ be the column vector $\big(\Xi^{\{\infty,2\}}(\underline{s},\omega)\big)_\omega$ with an order composed (11) and $\infty \prec 2$, and also $\mathbf{Z}(\underline{s})$ be the column vector $(Z(\underline{s},\psi,\psi'))_{(\psi,\psi')}$ with a certain order of (ψ,ψ'). Then, $H^{-1}M_{\sigma\tau\sigma,\{\infty,2\}}(\underline{s})H$ and $H^{-1}M_{\sigma\tau\sigma,\{\infty,2\}}(\underline{s})M_{\tau\sigma,\{\infty,2\}}(s_2,s_1)H$ are equal to the gamma matrices of Blomer's functional equations. And now by direct calculations, we have

$$M_{(\tau\sigma)^6,\infty}(\underline{s}) = \frac{1}{8}\, I_2, \qquad M_{(\tau\sigma)^6,2}(\underline{s}) = 8\, I_8\,.$$

Hence, we obtain an alternative proof for $S = \{\infty, 2\}$. **Case (iii).** By the definition of $\Gamma_p(s, \chi_p)$ from (2), $\tilde{G}_p(\underline{s}, \chi_p, \omega_p)$ from (5) and the Weil constant, we have

$$M_{\sigma\tau\sigma,p}(\underline{s}) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \left(\frac{-1}{p}\right) & -\left(\frac{-1}{p}\right) \\ 1 & -1 & -\left(\frac{-1}{p}\right) & \left(\frac{-1}{p}\right) \end{pmatrix},$$

$$M_{\sigma,p}(\underline{s}) = \mathrm{diag}\left(\frac{1-p^{-s_2}}{1-p^{-1+s_2}}, \frac{1+p^{-s_2}}{1+p^{-1+s_2}}, p^{-s_2+\frac{1}{2}}, p^{-s_2+\frac{1}{2}}\right).$$

We further use the regular matrix $U := \left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{smallmatrix}\right)$ for a simplification of the calculation. Then we can see that

$$U^{-1} M_{\sigma\tau\sigma,p}(\underline{s})\, U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{-1}{p}\right) \end{pmatrix},$$

$$U^{-1} M_{\sigma,p}(\underline{s})\, U = \frac{1}{1-p^{-2+2s_2}} \times$$
$$\begin{pmatrix} 1-p^{-1} & -p^{-s_2}+p^{-1+s_2} & 0 & 0 \\ -p^{-s_2}+p^{-1+s_2} & 1-p^{-1} & 0 & 0 \\ 0 & 0 & p^{-s_2+\frac{1}{2}}(1-p^{-2+2s_2}) & 0 \\ 0 & 0 & 0 & p^{-s_2+\frac{1}{2}}(1-p^{-2+2s_2}) \end{pmatrix},$$

$$U^{-1} M_{\sigma,p}(\underline{s}) M_{\sigma\tau\sigma,p}(s_1+s_2-\tfrac{1}{2}, 1-s_2)\, U = \frac{1}{1-p^{-2+2s_2}} \times$$
$$\begin{pmatrix} 1-p^{-1} & 0 & 1-p^{-1}-p^{-s_2}+p^{-1+s_2} & 0 \\ -p^{-s_2}+p^{-1+s_2} & 0 & 1-p^{-1}-p^{-s_2}+p^{-1+s_2} & 0 \\ 0 & p^{-s_2+\frac{1}{2}}(1-p^{-2+2s_2}) & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{-1}{p}\right) p^{-s_2+\frac{1}{2}}(1-p^{-2+2s_2}) \end{pmatrix}.$$

Finally, by using these matrices we find that

$$\begin{aligned} & U^{-1} M_{\sigma,p}(\underline{s})\, M_{\sigma\tau\sigma,p}(s_1+s_2-\tfrac{1}{2}, 1-s_2)\, U \\ \times\, & U^{-1} M_{\sigma,p}(1-s_2, s_1+s_2-\tfrac{1}{2})\, M_{\sigma\tau\sigma,p}(s_1, \tfrac{3}{2}-s_1-s_2)\, U \\ \times\, & U^{-1} M_{\sigma,p}(\tfrac{3}{2}-s_1-s_2, s_1) M_{\sigma\tau\sigma,p}(1-s_2, 1-s_1)\, U \\ \times\, & U^{-1} M_{\sigma,p}(1-s_1, 1-s_2)\, M_{\sigma\tau\sigma,p}(\tfrac{3}{2}-s_1-s_2, s_2)\, U \\ \times\, & U^{-1} M_{\sigma,p}(s_2, \tfrac{3}{2}-s_1-s_2)\, M_{\sigma\tau\sigma,p}(1-s_1, s_1+s_2-\tfrac{1}{2})\, U \end{aligned}$$

$$\times\, U^{-1} M_{\sigma,p}(s_1 + s_2 - \tfrac{1}{2}, 1 - s_1)\, M_{\sigma\tau\sigma,p}(s_2, s_1)\, U$$

becomes the unit matrix. Hence, this means $U^{-1} M_{(\tau\sigma)^6,\, p}\, U = I_4$, which completes the proof. □

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