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Boundedness of Denominators of Special Values of the L-functions for Modular Forms

by

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Abstract. For a cuspidal Hecke eigenform F for $Sp_n(\mathbb{Z})$ and a Dirichlet character χ let $L(s, F, \chi, \mathrm{St})$ be the standard L-function of F twisted by χ. In [3], Böcherer showed the boundedness of denominators of the algebraic part of $L(m, F, \chi, \mathrm{St})$ at a critical point m when χ varies. In this paper, we give a refined version of his result. We also prove a similar result for the products of Hecke L-functions of primitive forms for $SL_2(\mathbb{Z})$.

1. Introduction

Let $\Gamma^{(n)} = Sp_n(\mathbb{Z})$ be the Siegel modular group of genus n. For a cuspidal Hecke eigenform F for $\Gamma^{(n)}$ and a Dirichlet character χ let $L(s, F, \chi, \mathrm{St})$ be the standard L-function of F twisted by χ. In [3], Böcherer showed the boundedness of denominators of the algebraic part of $L(m, F, \chi, \mathrm{St})$ at a critical point m when χ varies (cf. Remark 2.5). To prove this, Böcherer used congruence of Fourier coefficients of modular forms. In this paper, we give a refined version of the above result without using congruence. We state our main results more precisely. Let $M_k(\Gamma^{(n)})$ be the space of modular forms of weight k for $\Gamma^{(n)}$, and $S_k(\Gamma^{(n)})$ its subspace consisting of cusp forms. We suppose that $k \geq n + 1$. Let $F_1, \ldots, F_e$ be a basis of the space $M_k(\Gamma^{(n)})$ consisting of Hecke eigenforms such that $F_1 = F$. Let $L_{n,k}$ be the composite field of $\mathbb{Q}(F_1), \cdots, \mathbb{Q}(F_{e-1})$ and $\mathbb{Q}(F_e)$. Let $\widetilde{\mathfrak{E}}'_F$ be the ideal of $L_{n,k}$ generated by all $\prod_{i=2}^{e}(\lambda_F(T_{i-1}) - \lambda_{F_i}(T_{i-1}))$'s $(T_1, \ldots, T_{e-1} \in \mathbf{L}'_n)$ and put $\widetilde{\mathfrak{E}}_F = \widetilde{\mathfrak{E}}'_F \cap \mathbb{Q}(F)$, where $\mathbf{L}'_n$ is the Hecke algebra for the Hecke pair $(GSp_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}), \Gamma^{(n)})$. Then, by Theorem 2.2, $\widetilde{\mathfrak{E}}'_F$ is a non-zero ideal, and therefore $\widetilde{\mathfrak{E}}_F$ is a non-zero ideal of $\mathbb{Q}(F)$. Let $\mathfrak{I}(l, F, \chi)$ be a certain fractional ideal of $\mathbb{Q}(F, \chi)$ associated with the value $L(l, F, \chi, \mathrm{St})$ as defined in Section 2, where $\mathbb{Q}(F, \chi)$ is the field generated over the Hecke field $\mathbb{Q}(F)$ of F by all the values of χ. Then we prove that we have

$$\mathfrak{I}(m, F, \chi) \subset \langle (C_{n,k}\widetilde{\mathfrak{E}}_F)^{-1} \rangle_{\mathfrak{O}_{\mathbb{Q}(F,\chi)}[N^{-1}]}$$

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for any positive integer $m \leq k-n$ and primitive character χ mod N satisfying a certain condition, where $C_{n,k}$ is a positive integer depending only on k and n. (For a precise statement, see Theorem 2.3). By this we easily see the following result (cf. Corollary 2.4):

Let $\mathcal{P}_F$ be the set of prime ideals $\mathfrak{p}$ of $\mathbb{Q}(F)$ such that

$$\mathrm{ord}_{\mathfrak{p}}(N_{\mathbb{Q}(F,\chi)/\mathbb{Q}(F)}(\mathfrak{I}(m,F,\chi))) < 0$$

for some positive integer $m \leq k-n$ and primitive character χ with conductor not divisible by $\mathfrak{p}$ satisfying the above condition. Then $\mathcal{P}_F$ is a finite set. Moreover, there exists a positive integer $r = r_{n,k}$ depending only on n and k such that we have

$$\mathrm{ord}_{\mathfrak{q}}(\mathfrak{I}(m,F,\chi)) \geq -r[\mathbb{Q}(F,\chi) : \mathbb{Q}(F)]$$

for any prime ideal $\mathfrak{q}$ of $\mathbb{Q}(F,\chi)$ lying above a prime ideal in $\mathcal{P}_F$ and positive integer $m \leq k-n$ and primitive character χ with conductor not divisible by $\mathfrak{q}$ satisfying the above condition.

We have also similar results for the products of Hecke L functions of primitive forms for $SL_2(\mathbb{Z})$.

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Notation We denote by $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ the set of positive integers and the set of non-negative integers, respectively.

For a commutative ring R, let $M_{mn}(R)$ denote the set of $m \times n$ matrices with entries in R, and especially write $M_n(R) = M_{nn}(R)$. We often identify an element a of R and the matrix (a) of size 1 whose component is a. If m or n is 0, we understand an element of $M_{mn}(R)$ is the *empty matrix* and denote it by $\emptyset$. Let $GL_n(R)$ be the group consisting of all invertible elements of $M_n(R)$, and $\mathrm{Sym}_n(R)$ the set of symmetric matrices of size n with entries in R. Let K be a field of characteristic 0, and R its subring. We say that an element A of $\mathrm{Sym}_n(R)$ is non-degenerate if the determinant $\det A$ of A is non-zero. For a subset S of $\mathrm{Sym}_n(R)$, we denote by S^{nd} the subset of S consisting of non-degenerate matrices. For a subset S of $\mathrm{Sym}_n(\mathbb{R})$ we denote by $S_{\geq 0}$ (resp. $S_{>0}$) the subset of S consisiting of semi-positive definite (resp. positive definite) matrices. We say that an element $A = (a_{ij})$ of $\mathrm{Sym}_n(K)$ is half-integral if a_{ii} $(i = 1, ..., n)$ and $2a_{ij}$ $(1 \leq i \neq j \leq n)$ belong to R. We denote by $\mathcal{H}_n(R)$ the set of half-integral matrices of size n over R. We note that $\mathcal{H}_n(R) = \mathrm{Sym}_n(R)$ if R contains the inverse of 2. For an (m,n) matrix X and an (m,m) matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let G be a subgroup of $GL_n(R)$. Then we say that two elements B and B' in $\mathrm{Sym}_n(R)$ are G-equivalent if there is an element g of G such that $B' = B[g]$. For two square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$. We often write $x \perp Y$ instead of $(x) \perp Y$ if (x) is a matrix of size 1. We denote by 1_m the unit matrix of size m and by $O_{m,n}$ the zero matrix of type (m,n). We sometimes abbreviate $O_{m,n}$ as O if there is no fear of confusion.

Let $\mathfrak{b}$ be a subset of K. We then denote by $\langle \mathfrak{b} \rangle_R$ the R-sub-module of K generated by $\mathfrak{b}$. For a non-zero integer M, we put

$$R[M^{-1}] = \{aM^{-s} \mid a \in R,\ s \in \mathbb{Z}_{\geq 0}\}$$

Let K be an algebraic number filed, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K. For a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$, we denote by $\mathfrak{O}_{(\mathfrak{p})}$ the localization of $\mathfrak{O}$ at $\mathfrak{p}$ in K. Let $\mathfrak{A}$ be a fractional ideal in K. If $\mathfrak{A} = \mathfrak{p}^e\mathfrak{B}$ with a fractional ideal $\mathfrak{B}$ of K such that $\mathfrak{O}_{(\mathfrak{p})}\mathfrak{B} = \mathfrak{O}_{(\mathfrak{p})}$ we write $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{A}) = e$. We make the convention that $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{A}) = \infty$ if $\mathfrak{A} = \{0\}$. We simply write $\mathrm{ord}_{\mathfrak{p}}(c) = \mathrm{ord}_{\mathfrak{p}}((c))$ for $c \in K$. We sometimes say that $\mathfrak{p}$ divides c if $\mathrm{ord}_{\mathfrak{p}}(c) > 0$. For an ideal $\mathfrak{J}$ of K, let $\mathfrak{J}^{-1}$ the inverse ideal of $\mathfrak{J}$.

For a complex number x put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$.

2. Main result

For a subring K of $\mathbb{R}$ put

$$GSp_n^+(K) = \{\gamma \in GL_{2n}(K) \mid J_n[\gamma] = \kappa(\gamma)J_n \text{ with some } \kappa(\gamma) > 0\},$$

and

$$Sp_n(K) = \{\gamma \in GSp_n^+(K) \mid J_n[\gamma] = J_n\},$$

where $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$. In particular, put $\Gamma^{(n)} = Sp_n(\mathbb{Z})$ as in Introduction. We sometimes write an element γ of $GSp_n^+(K)$ as $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_n(K)$. We define subgroups $\Gamma^{(n)}(N)$ and $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$ as

$$\Gamma^{(n)}(N) = \{\gamma \in \Gamma^{(n)} \mid \gamma \equiv 1_{2n} \bmod N\},$$

and

$$\Gamma_0^{(n)}(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv O_n \bmod N\}.$$

Let $\mathbf{H}_n$ be Siegel's upper half space of degree n. We write $\gamma(Z) = (AZ + B)(CZ + D)^{-1}$ and $j(\gamma, Z) = \det(CZ + D)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n^+(\mathbb{R})$ and $Z \in \mathbf{H}_n$. We write $F|_k\gamma(Z) = (\det\gamma)^{k/2} j(\gamma, Z)^{-k} f(\gamma(Z))$ for $\gamma \in GSp_n^+(\mathbb{R})$ and a C^∞-function F on $\mathbf{H}_n$. We simply write $F|\gamma$ for $F|_k\gamma$ if there is no confusion. We say that a subgroup Γ of $\Gamma^{(n)}$ is a congruence subgroup if Γ contains $\Gamma^{(n)}(N)$ with some N. We also say that a character η of a congruence subgroup Γ is a congruence character if its kernel is a congruence subgroup. For a positive integer k, a congruence subgroup Γ and its congruence character η, we denote by $M_k(\Gamma, \eta)$ (resp. $M_k^\infty(\Gamma, \eta)$) the space of holomorphic (resp. C^∞-) modular forms of weight k and character η for Γ. We denote by $S_k(\Gamma, \eta)$ the subspace of $M_k(\Gamma, \eta)$ consisting of cusp forms. If η is the trivial character, we abbreviate $M_k(\Gamma, \eta)$ and $S_k(\Gamma, \eta)$ as $M_k(\Gamma)$ and $S_k(\Gamma)$, respectively. Let dv denote the invariant volume element on $\mathbf{H}_n$ defined by

$$dv = \det(\mathrm{Im}(Z))^{-n-1} \wedge_{1 \leq j \leq l \leq n} (dx_{jl} \wedge dy_{jl}).$$

Here for $Z \in \mathbf{H}_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices (x_{jl}) and (y_{jl}). For two elements F and G of $M_k^{\infty}(\Gamma, \eta)$, we define the Petersson scalar product $\langle F, G\rangle_\Gamma$ of F and G by

$$\langle F, G\rangle_\Gamma = \int_{\Gamma\backslash\mathbf{H}_n} F(Z)\overline{G(Z)}\det(\mathrm{Im}(Z))^k dv,$$

provided the integral converges. For $i = 1, 2$, let Γ_i be a congruence subgroup with a congruence character η_i. Then there exists a congruence subgroup Γ contained in $\Gamma_1 \cap \Gamma_2$ and its congruence character η such that $\eta_1|\Gamma = \eta_2|\Gamma = \eta$. Then we have $M_k^{\infty}(\Gamma, \eta) \supset M_k^{\infty}(\Gamma_i, \eta_i)$. For elements F_1 and F_2 of $M_k^{\infty}(\Gamma, \eta_1)$ and $M_k^{\infty}(\Gamma_2, \eta_2)$, respectively, the value $[\Gamma^{(n)} : \Gamma]^{-1}\langle F_1, F_2\rangle_\Gamma$ does not depend on the choice of Γ. We denote it by $\langle F_1, F_2\rangle$.

Let F be an element of $M_k(\Gamma, \eta)$. Then, F has the following Fourier expansion:

$$F(Z) = \sum_{A\in\mathcal{H}_n(\mathbb{Z})_{\geq 0}} c_F\Big(\frac{A}{N}\Big)\mathbf{e}\Big(\mathrm{tr}\Big(\frac{AZ}{N}\Big)\Big)$$

with some positive integer N, where tr denotes the trace of a matrix. For a subset S of $\mathbb{C}$, we denote by $M_k(\Gamma, \eta)(S)$ the set of elements F of $M_k(\Gamma, \eta)$ such that $c_F(\frac{A}{N}) \in S$ for all $A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$, and put $S_k(\Gamma, \eta)(S) = M_k(\Gamma, \eta)(S) \cap S_k(\Gamma, \eta)$. If R is a commutative ring, and S is an R module, then $M_k(\Gamma, \eta)(S)$ and $S_k(\Gamma, \eta)(S)$ are R-modules.

For a Dirichlet character ϕ modulo N, let $\widetilde{\phi}$ denote the character of $\Gamma_0^{(n)}(N)$ defined by $\Gamma_0^{(n)}(N) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \phi(\det D)$, and we write $M_k(\Gamma_0^{(n)}(N), \phi)$ for $M_k(\Gamma_0^{(n)}(N), \widetilde{\phi})$, and so on.

We denote by $\mathbf{L}_n = \mathbf{L}_{\mathbb{Q}}(GSp_n^+(\mathbb{Q}), \Gamma^{(n)})$ be the Hecke ring over $\mathbb{Q}$ associated with the Hecke pair $(GSp_n^+(\mathbb{Q}), \Gamma^{(n)}))$, and by $\mathbf{L}'_n = \mathbf{L}_{\mathbb{Z}}(GSp_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}), \Gamma^{(n)})$ be the Hecke ring over $\mathbb{Z}$ associated with the Hecke pair $(GSp_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}), \Gamma^{(n)})$. For a Hecke eigenform F, we denote by $\mathbb{Q}(F)$ the field generated over $\mathbb{Q}$ by the eigenvalues of all Hecke operators $T \in \mathbf{L}_n$ with respect to F, and call it the Hecke field of F. For Dirichlet characters $\chi_1, \ldots, \chi_r$, we denote by $\mathbb{Q}(\chi_1, \ldots, \chi_r)$ the field generated over $\mathbb{Q}$ by all the values of $\chi_1, \ldots, \chi_r$, and by $\mathbb{Q}(F, \chi_1, \ldots, \chi_r)$ the composite field of $\mathbb{Q}(F)$ and $\mathbb{Q}(\chi_1, \ldots, \chi_r)$. For a Hecke eigenform F in $S_k(\Gamma_0^{(n)}(N))$ and a Dirichlet character χ let $L(s, F, \mathrm{St}, \chi)$ be the standard L function of F twisted by χ. For a Dirichlet character χ, we put $\delta_\chi = 0$ or 1 according as $\chi(-1) = 1$ or $\chi(-1) = -1$. Assume that χ is primitive, and for any positive integer $m \leq k - n$ such that $m - n \equiv \delta_\chi \bmod 2$ define $\Lambda(m, F, \chi, \mathrm{St})$ as

$$\Lambda(m, F, \chi, \mathrm{St}) = \frac{\chi(-1)^n\Gamma(m)\prod_{i=1}^n \Gamma(2k-n-i)L(m, F, \mathrm{S}t, \chi)}{\langle F, F\rangle\pi^{-n(n+1)/2+nk+(n+1)m}\sqrt{-1}^{m+n}\tau(\chi)^{n+1}}.$$

Here, $\tau(\chi)$ is the Gauss sum of χ. For a Dirichlet character χ let m_χ be the conductor of χ. The following proposition is essentially due to [[4], Appendix, Theorem].

PROPOSITION 2.1. *Let F be a Hecke eigenform in $S_k(\Gamma^{(n)})(\mathbb{Q}(F))$. Let m be a positive integer not greater than $k-n$ and χ a primitive character χ satisfying the following condition:*

(C) *$m - n \equiv \delta_\chi$ mod 2, and $m > 1$ if $n > 1$, $n \equiv 1$ mod 4 and χ^2 is trivial.*

Then $\Lambda(m, F, \chi, \mathrm{St})$ *belongs to* $\mathbb{Q}(F, \chi)$.

Let $\mathcal{V}$ be a subspace of $M_k(\Gamma^{(n)})$. We say that a multiplicity one holds for $\mathcal{V}$ if any Hecke eigenform in $\mathcal{V}$ is uniquely determined up to constant multiple by its Hecke eigenvalues.

THEOREM 2.2. *Suppose that* $k \geq n+1$. *Then a multiplicity one theorem holds for* $S_k(\Gamma^{(n)})$.

Proof. This is essentially due to Chenevier-Lannes [[7], Corollary 8.5.4]. It was proved under a more stronger assumption without using [[7], Conjecture 8.4.22]. As is written in the postface in that book, this conjecture has been proved [1], and the same proof is available at least even when $k \geq n+1$. □

Let F be a Hecke eigenform in $S_k(\Gamma^{(n)})$ with $k \geq n+1$. Then by Theorem 2.2, we have $cF \in S_k(\Gamma^{(n)})(\mathbb{Q}(F))$ with some $c \in \mathbb{C}$. Hence for $A, B \in \mathcal{H}_n(\mathbb{Z})_{>0}$ and an integer l satisfying (C), the value $c_F(A)\overline{c_F(B)}\Lambda(l, F, \mathrm{St}, \chi)$ belongs to $\mathbb{Q}(F)$ and does not depend on the choice of c. For A and B and an integer l put

$$I_{A,B}(l, F, \chi) = c_F(A)\overline{c_F(B)}\Lambda(l, F, \chi, \mathrm{St}).$$

Let $\mathfrak{I}(l, F, \chi)$ be the $\mathfrak{O}_{\mathbb{Q}(F)}$-module generated by all $I_{A,B}(l, F, \chi)$'s. Then $\mathfrak{I}_F(l, F, \chi)$ becomes a fractional ideal in $\mathbb{Q}(F, \chi)$. We note that it is uniquely determined by l and the system of eigenvalues of F. Let $F_1, \ldots, F_d$ be a basis of $S_k(\Gamma^{(n)})$ consisting of Hecke eigenforms such that $F_1 = F$. Let $K_{n,k}$ be the composite filed $\mathbb{Q}(F_1)\cdots\mathbb{Q}(F_d)$ of $\mathbb{Q}(F_1), \ldots, \mathbb{Q}(F_d)$. We denote by $\widetilde{\mathfrak{D}}'_F$ the ideal of $K_{n,k}$ generated by all $\prod_{i=2}^{d}(\lambda_F(T_{i-1}) - \lambda_{F_i}(T_{i-1}))$'s $(T_1, \cdots, T_{d-1} \in \mathbf{L}'_n)$, and put $\widetilde{\mathfrak{D}}_F = \mathfrak{D}'_F \cap \mathbb{Q}(F)$. We make the convention that $\widetilde{\mathfrak{D}}'_F = \mathfrak{O}_{K_{n,k}}$ if $d = 1$. Moreover, let $\widetilde{\mathfrak{E}}_F$ be the ideal of $\mathbb{Q}(F)$ defined in Section 1. Then our first main result is as follows.

THEOREM 2.3. *Let* F *be a Hecke eigenform in* $S_k(\Gamma^{(n)})$. *Then we have*

$$\mathfrak{I}(m, F, \chi) \subset \langle (2^{\alpha(n,k)} A_{n,k}\widetilde{\mathfrak{E}}_F)^{-1} \rangle_{\mathfrak{O}_{\mathbb{Q}(F,\chi)}[N^{-1}]}$$

for any positive integer $m \leq k-n$ *and primitive character* χ *mod* N *satisfying the condition* (C), *where* $\alpha(n,k)$ *is a non-negative integer depending only on* k *and* n, *and* $A_{n,k} = \mathrm{LCM}_{n+1 \leq m \leq k}\{\prod_{i=1}^{n}(2l-2i)(2l-2i+1)!)\}$. *In particular if* $m \leq k-n-1$, *then*

$$\mathfrak{I}(m, F, \chi) \subset \langle (2^{\alpha(n,k)} A_{n,k}\widetilde{\mathfrak{D}}_F)^{-1} \rangle_{\mathfrak{O}_{\mathbb{Q}(F,\chi)[N^{-1}]}}$$

We will prove the above theorem in Section 5.

COROLLARY 2.4. *Let* F *be a Hecke eigenform in* $S_k(\Gamma^{(n)})$. *Let* $\mathcal{P}_F$ *be the set of prime ideals* $\mathfrak{p}$ *of* $\mathbb{Q}(F)$ *such that*

$$\mathrm{ord}_{\mathfrak{p}}(N_{\mathbb{Q}(F,\chi)/\mathbb{Q}(F)}(\mathfrak{I}(m, F, \chi))) < 0$$

for some positive integer $m \leq k-n$ *and primitive character* χ *with conductor not divisible by* $\mathfrak{p}$ *satisfying* (C). *Then* $\mathcal{P}_F$ *is a finite set. Moreover, there exists a positive integer* r *such that we have*

$$\mathrm{ord}_{\mathfrak{q}}(\mathfrak{I}(m, F, \chi)) \geq -r[\mathbb{Q}(F, \chi) : \mathbb{Q}(F)]$$

for any prime ideal $\mathfrak{q}$ *of* $\mathbb{Q}(F,\chi)$ *lying above a prime ideal in* $\mathcal{P}_F$ *and integer* l *and primitive character* χ *with conductor not divisible by* $\mathfrak{q}$ *satisfying the condition* (C).

Proof. By Theorem 2.3, we have $\mathfrak{p}|2^{\alpha(n,k)}A_{n,k}\widetilde{\mathfrak{E}}_F$ if $\mathfrak{p}\in\mathcal{P}_F$. This proves the first assertion. Let $2^{\alpha(n,k)}A_{n,k}\widetilde{\mathfrak{E}}_F=\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_s^{e_s}$ be the prime factorization of $2^{\alpha(n,k)}A_{n,k}\widetilde{\mathfrak{E}}_F$, where $\mathfrak{p}_1,\ldots,\mathfrak{p}_s$ are distinct prime ideals and $e_1,\ldots,e_s$ are positive integers. We note that for any prime ideal $\mathfrak{p}$ of $\mathbb{Q}(F)$ and prime ideal $\mathfrak{q}$ of $\mathbb{Q}(F,\chi)$ lying above $\mathfrak{p}$ we have $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{p})\le[\mathbb{Q}(F,\chi):\mathbb{Q}(F)]$. Hence $r=\max\{e_i\}_{1\le i\le s}$ satisfies the required condition in the second assertion. □

REMARK 2.5. (1) Let

$$\Lambda(F,m,\chi)=\frac{\Gamma(m)\prod_{i=1}^n\Gamma(2k-n-i)L(m,F,\mathrm{St},\chi)}{\langle F,F\rangle\pi^{-n(n+1)/2+nk+(n+1)m}}.$$

Then, if m and χ satisfy the condition (C), $\Lambda(F,m,\chi)$ belongs to $\mathbb{Q}(F,\chi,\zeta_N)$, where $\mathbb{Q}(F,\chi,\zeta_N)$ is the field generated over the Hecke field $\mathbb{Q}(F)$ of F by all the values of χ and the primitive N-th root ζ_N of unity. In [[3], Theorem], a similar result has been proved for $\Lambda(F,m,\chi)$. Our L-value belongs to $\mathbb{Q}(F,\chi)$, which is included in $\mathbb{Q}(F,\chi,\zeta_N)$. Therefore, our result can be regarded as a refinement of Böcherer's.
(2) Böcherer [3] excluded the case $m=k-n$. However, we can include this case. We also note that we can get a sharper result if we restrict ourselves to the case $m<k-n$ as stated in the above theorem.
(3) In [3], the main result was formulated without assuming multiplicity one theorem. However, such a formulation is now unnecessary.

3. Pullback of Siegel Eisenstein series

To prove our main result, first we express a certain modular form as a linear combination of Hecke eigenforms (cf. Theorem 3.7). We have carried out it in [[12], Appendix], and here we treat it in a more general setting. We also correct some inaccuracies in [[12], Appendix] (cf. Remark 3.8). For a non-negative integer m, put

$$\Gamma_m(s)=\pi^{m(m-1)/4}\prod_{i=1}^m\Gamma(s-\frac{i-1}{2}).$$

For a Dirichlet character χ we denote by $L(s,\chi)$ the Dirichlet L-function associated to χ, and put

$$\mathcal{L}_n(s,\chi)=\Gamma_n(s)\pi^{-ns}L(s,\chi)\prod_{i=1}^{[n/2]}L(2s-2i,\chi^2)$$
$$\times\begin{cases}\pi^{n/2-s}\Gamma(s-n/2) & \text{if if } n \text{ is even}\\ 1 & n \text{ is odd.}\end{cases}$$

Let n, l and N be positive integers. For a Dirichlet character ϕ modulo N such that $\phi(-1) = (-1)^l$, we define the Eisenstein series $E^*_{n,l}(Z; N, \phi, s)$ by

$$E^*_{n,l}(Z; N, \phi, s) = \big(\det \mathrm{Im}(Z)\big)^s \mathcal{L}_n(l + 2s, \phi)$$
$$\times \sum_{\gamma \in T^{(n)}(N)_\infty \backslash T^{(n)}(N)} \phi^*(\gamma) j(\gamma, Z)^{-l} |j(\gamma, Z)|^{-2s},$$

where

$$T^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid A \equiv O_n \bmod N \right\},$$
$$T^{(n)}(N)_\infty = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid B \equiv O_n \bmod N, C = O_n \right\},$$

and $\phi^*(\gamma) = \phi(\det C)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in T^{(n)}(N)$. Then $E^*_{n,l}(Z; N, \phi, s)$ converges absolutely as a function of s if the real part of s is large enough. Moreover, it has a meromorphic continuation to the whole s-plane, and it belongs to $M^\infty_l(\Gamma^{(n)}_0(N), \phi)$. Moreover it is holomorphic and finite at $s = 0$, which will be denoted by $E^*_{n,l}(Z; N, \phi)$. In particular, if $E^*_{n,l}(Z; N, \phi)$ belongs to $M_l(\Gamma^{(n)}_0(N), \phi)$, it has the following Fourier expansion:

$$E^*_{n,l}(Z; N, \phi) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} c_{n,l}(A, N, \phi) \mathbf{e}(\mathrm{tr}(AZ)).$$

To see the Fourier coefficient of $E^*_{n,l}(Z; N, \phi)$, we define a polynomial attached to local Siegel series. For a prime number p let $\mathbb{Q}_p$ be the field of p-adic numbers, and $\mathbb{Z}_p$ the ring of p-adic integers. For an element $B \in \mathcal{H}_n(\mathbb{Z}_p)$, we define the Siegel series $b_p(B, s)$ as

$$b_p(B, s) = \sum_{R \in \mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(BR)) \nu(R)^{-s},$$

where $\mathbf{e}_p$ is the additive character of $\mathbb{Z}_p$ such that $\mathbf{e}_p(m) = \mathbf{e}(m)$ for $m \in \mathbb{Z}[p^{-1}]$, and $\nu_p(R) = [R\mathbb{Z}^n_p + \mathbb{Z}^n_p : \mathbb{Z}^n_p]$. We define $\chi_p(a)$ for $a \in \mathbb{Q}^\times_p$ as follows:

$$\chi_p(a) := \begin{cases} +1 & \text{if } \mathbb{Q}_p(\sqrt{a}) = \mathbb{Q}_p, \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is quadratic unramified,} \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is quadratic ramified.} \end{cases}$$

For an element $B \in \mathcal{H}_n(\mathbb{Z}_p)^{\mathrm{nd}}$ with n even, we define $\xi_p(B)$ by

$$\xi_p(B) := \chi_p((-1)^{n/2} \det B).$$

For a nondegenerate half-integral matrix B of size n over $\mathbb{Z}_p$ define a polynomial $\gamma_p(B, X)$ in X by

$$\gamma_p(B, X) := \begin{cases} (1 - X) \prod_{i=1}^{n/2} (1 - p^{2i} X^2)(1 - p^{n/2} \xi_p(B) X)^{-1} & \text{if } n \text{ is even,} \\ (1 - X) \prod_{i=1}^{(n-1)/2} (1 - p^{2i} X^2) & \text{if } n \text{ is odd.} \end{cases}$$

Then it is well known that there exists a unique polynomial $F_p(B, X)$ in X over $\mathbb{Z}$ with constant term 1 such that

$$b_p(B, s) = \gamma_p(B, p^{-s}) F_p(B, p^{-s})$$

(e.g. [9]). More precisely, we have the following proposition.

PROPOSITION 3.1. *Let $B \in \mathcal{H}_m(\mathbb{Z}_p)^{\mathrm{nd}}$. Then there exists a polynomial $H_p(B, x)$ in X over $\mathbb{Z}$ such that*

$$F_p(B, X) = H_p(B, p^{[(m+1)/2]}X).$$

Proof. The assertion follows from [14], Theorem 2. □

For $B \in \mathcal{H}_m(\mathbb{Z})_{>0}$ with m even, let $\mathfrak{d}_B$ be the discriminant of $\mathbb{Q}(\sqrt{(-1)^{m/2}\det B})/\mathbb{Q}$, and $\chi_B = (\frac{\mathfrak{d}_B}{*})$ the Kronecker character corresponding to $\mathbb{Q}(\sqrt{(-1)^{m/2}\det B})/\mathbb{Q}$. We note that we have $\chi_B(p) = \xi_p(B)$ for any prime p. We also note that

$$(-1)^{m/2}\det(2B) = \mathfrak{d}_B\mathfrak{f}_B^2$$

with $\mathfrak{f}_B \in \mathbb{Z}_{>0}$. We define a polynomial $F_p^*(T, X)$ for any $T \in \mathcal{H}_n(\mathbb{Z}_p)$ which is not-necessarily non-degenerate as follows: For an element $T \in \mathcal{H}_n(\mathbb{Z}_p)$ of rank $r \geq 1$, there exists an element $\widetilde{T} \in \mathcal{H}_r(\mathbb{Z}_p)^{\mathrm{nd}}$ such that $T \sim_{\mathbb{Z}_p} \widetilde{T} \bot O_{n-r}$. We note that $F_p(\widetilde{T}, X)$ does not depend on the choice of $\widetilde{T}$. Then we put $F_p^*(T, X) = F_p(\widetilde{T}, X)$. For an element $T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$ of rank $r \geq 1$, there exists an element $\widetilde{T} \in \mathcal{H}_r(\mathbb{Z})_{>0}$ such that $T \sim_{\mathbb{Z}} \widetilde{T} \bot O_{n-r}$. Then $\chi_{\widetilde{T}}$ does not depend on the choice of $\widetilde{T}$. We write $\chi_T^* = \chi_{\widetilde{T}}$ if r is even. For a non-negative integer m and a primitive character ϕ let $B_{m,\phi}$ be the m-th generalized Bernoulli number for ϕ. In the case ϕ is the principal character, we write $B_m = B_{m,\phi}$, which is the m-th Bernoulli number. For a Dirichlet character ϕ we denote by ϕ_0 the primitive character associated with ϕ.

PROPOSITION 3.2. *Let n and l be positive integers such that $l \geq n+1$, and ϕ a primitive character mod N. Then $E_{2n,l}^*(Z; N, \phi)$ is holomorphic and belongs to $M_l(\Gamma_0^{(2n)}(N), \phi)$ except the following case:*

$l = n+1 \equiv 2 \bmod 4$ and $\phi^2 = \mathbf{1}_N$.

In the case that $E_{2n,l}^*(Z; N, \phi)$ is holomorphic we have the following assertion:

(1) *Suppose that $N = 1$ and ϕ is the principal character $\mathbf{1}$, Then for $B \in \mathcal{H}_{2n}(\mathbb{Z})_{\geq 0}$ of rank m, we have*

$$c_{2n,l}(B, 1, \mathbf{1}) = (-1)^{l/2+n(n+1)/2}2^{l-1+[(m+1)/2]}\prod_{p \mid \det(2\widetilde{B})} F_p^*(B, p^{l-m-1})$$
$$\times \begin{cases} \prod_{i=m/2+1}^{n} \zeta(1+2i-2l)L(1+m/2-l, \chi_B^*) & \text{if } m \text{ is even,} \\ \prod_{i=(m+1)/2}^{n} \zeta(1+2i-2l) & \text{if } m \text{ is odd.} \end{cases}$$

Here we make the convention that $F_p^*(B, p^{l-m-1}) = 1$ and $Ł(1+m/2-l, \chi_B^*) = \zeta(1-l)$ if $m = 0$.

(2) *Suppose that $N > 1$. Then, $c_{2n,l}(B, N, \phi) = 0$ if $B \in \mathcal{H}_{2n}(\mathbb{Z})_{\geq 0}$ is not positive definite. Moreover, for any $B \in \mathcal{H}_{2n}(\mathbb{Z})_{>0}$ we have*

$$c_{2n,l}(B, N, \phi) = (-1)^{nl+(l-n-\delta_{(\phi\chi_B)_0})/2}2^{n+l-1}\sqrt{-1}^{-\delta_{(\phi\chi_B)_0}}|\mathfrak{d}_B|^{l-n-1/2}$$
$$\times m_{(\phi\chi_B)_0}^{n-l}\tau((\phi\chi_B)_0)\prod_p F_p(B, p^{l-2n-1}\bar{\phi}(p))L(1-l+n, \overline{(\phi\chi_B)_0})$$

$$\times \prod_{p|N|\mathfrak{d}_B|} (1 - p^{n-l}(\phi\chi_B)_0).$$

Proof. (1) The assertion follows from [[11], Theorem 2.3] remarking that

$$\mathcal{L}_{2n}(l,\chi) = \zeta(1-l)\prod_{i=1}^{n}\zeta(1-2l+2i)(-1)^{(n(n+1)+l)/2}2^{l-1}.$$

(2) The first assertion follows from [[4], Section 5]. Let $B \in \mathcal{H}_{2n}(\mathbb{Z})_{>0}$. Then,

$$c_{2n,l}(B,N,\phi) = (-1)^{nl}2^{2n}\Gamma(l-n)$$
$$\times (\det(2B))^{l-n-1/2}\prod_{p} F_p(B, p^{-l}\phi(p))\frac{L(l-n,\phi\chi_B)}{\pi^{l-n}}.$$

We have

$$L(l-n,\phi\chi_B) = L(l-n,(\phi\chi_B)_0)\prod_{p|N|\mathfrak{d}_B|}(1-p^{n-l}(\phi\chi_B)_0),$$

and

$$\frac{\Gamma(l-n)L(l-n,(\phi\chi_B)_0)}{\pi^{l-n}} = (-1)^{(l-n-\delta_{(\phi\chi_B)_0})/2}2^{l-n-1}m_{(\phi\chi_B)_0}^{n-l}\sqrt{-1}^{-\delta_{(\phi\chi_B)_0}}$$
$$\times L(1-l+n,\overline{(\phi\chi_B)_0}).$$

Moreover, by the functional equation of $F_p(B,X)$ (cf. [9]), we have

$$\mathfrak{f}_B^{2l-2n-1}\prod_p F_p(B,p^{-l}\phi(p)) = \prod_p F_p(p^{l-2n-1}\bar{\phi}(p),B).$$

Thus the assertion is proved remarking that $\det(2B) = |\mathfrak{d}_B|\mathfrak{f}_B^2$. □

COROLLARY 3.3. *Let the notation be as above.*

(1) *Suppose that* $N=1$. *Then,* $c_{2n,l}(B,1,\mathbf{1})$ *belongs to* $\langle(\prod_{i=1}^{n}(2l-2i)(2l-2i+1)!)^{-1}\rangle_{\mathbb{Z}}$ *for any* $B \in \mathcal{H}_{2n}(\mathbb{Z})_{\geq 0}$.

(2) *Suppose that* $N>1$. *Then for* $B \in \mathcal{H}_{2n}(\mathbb{Z})_{\geq 0}$, $c_{2n,l}(B,N,\phi)$ *is an algebraic number. In particular if* GCD$(\det(2B),N)=1$, *then* $\tau(\phi)^{-1}\sqrt{-1}^{-l}c_{2n,l}(B,N,\phi)$ *belongs to* $\langle(l-n)^{-1}\rangle_{\mathfrak{O}_{\mathbb{Q}(\phi)}[N^{-1}]}$.

Proof. (1) By Proposition 3.1, the product $\prod_{p|\det(2\widetilde{B})} F_p^*(B,l-m-1)$ is an integer for any m and $B \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$ with rank m. By Clausen-von-Staudt theorem, $\zeta(1-2l+2i)$ belongs to $\langle((2l-2i)(2l-2i+1)!)^{-1}\rangle_{\mathbb{Z}}$. By [[2], (5.1), (5.2)] and Clausen-von-Staudt theorem, for any positive even integer m and $\widetilde{B} \in \mathcal{H}_m(\mathbb{Z})_{>0}$, $L(1-l+m/2,\chi_{\widetilde{B}})$ belongs to $\langle((2l-m)(2l-m+1)!)^{-1}\rangle_{\mathbb{Z}}$. This proves the assertion.
(2) It is well known that $L(1-l+n,\overline{(\phi\chi_B)_0})$ is algebraic. This proves the first part of the assertion. Suppose that $\det(2B)$ is coprime to N. Then $\phi\chi_B$ is a primitive character of conductor $N|\mathfrak{d}_B|$ and

$$\tau(\phi\chi_B) = \phi(|\mathfrak{d}_B|)\chi_B(N)\tau(\phi)\tau(\chi_B)$$
$$= \phi(|\mathfrak{d}_B|)\chi_B(N)\tau(\phi)|\mathfrak{d}_B|^{1/2}\sqrt{-1}^{\delta_{\chi_B}}.$$

By [6] or [15], $N(l-n)L(1-l+n,\overline{\phi\chi_B})$ belongs to $\mathfrak{O}_{\mathbb{Q}(\phi)}$, and by Proposition 3.1, $\prod_p F_p(p^{l-2n-1}\bar{\phi}(p), B)$ is an element of $\mathfrak{O}_{\mathbb{Q}(\phi)}$. Thus the assertion has been proved remarking that $\sqrt{-1}^{l} = \pm\sqrt{-1}^{\delta_{\chi_B}-\delta_{\phi\chi_B}}$. □

Let $\overset{\circ}{\mathcal{D}}{}^{\nu}_{n,l}$ be the differential operator in [4], which maps $M_l^{\infty}(\Gamma_0^{(2n)}(N))$ to $M_{l+\nu}^{\infty}(\Gamma_0^{(n)}(N)) \otimes M_{l+\nu}^{\infty}(\Gamma_0^{(n)}(N))$. Let χ be a primitive character mod N. For a non-negative integer $\nu \le k$, we define a function $\mathfrak{E}_{2n}^{k,\nu}(Z_1, Z_2, N, \chi)$ on $\mathbf{H}_n \times \mathbf{H}_n$ as

$$\mathfrak{E}_{2n}^{k,\nu}(Z_1, Z_2, N, \chi) = (2\pi\sqrt{-1})^{-\nu}\tau(\chi)^{-n-1}\sqrt{-1}^{-k+\nu}$$
$$\times \overset{\circ}{\mathcal{D}}{}^{\nu}_{n,k-\nu}\left(\sum_{X \in M_n(\mathbb{Z})/NM_n(\mathbb{Z})} \chi(\det X) E^*_{2n,k-\nu}(*, N, \chi)|_{k-\nu}\begin{pmatrix} 1_{2n} & S(X/N) \\ O & 1_{2n}\end{pmatrix}\right)(Z_1, Z_2)$$

for $(Z_1, Z_2) \in \mathbf{H}_n \times \mathbf{H}_n$, where $S(X/N) = \begin{pmatrix} O_n & X/N \\ {}^tX/N & O_n \end{pmatrix}$. Let X be a symmetric matrix of size $2n$ of variables. Then there exists a polynomial $P_{n,l}^{\nu}(X)$ in X such that

$$\overset{\circ}{\mathcal{D}}{}^{\nu}_{n,l}\left(\mathbf{e}\left(\mathrm{tr}\left(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}\begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix}\right)\right)\right)$$
$$= (2\pi\sqrt{-1})^{\nu} P_{n,l}^{\nu}\left(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}\right)\mathbf{e}(\mathrm{tr}(A_1Z_1 + A_2Z_2))$$

for $\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix} \in \mathcal{H}_{2n}(\mathbb{Z})_{\ge 0}$ with $A_1, A_2 \in \mathcal{H}_n(\mathbb{Z})_{\ge 0}$ and $\begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix} \in \mathbf{H}_{2n}$ with $Z_1, Z_2 \in \mathbf{H}_n$.

PROPOSITION 3.4. *Under the above notation and the assumption, for a non-negative integer $l \le k$ write $\mathfrak{E}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi)$ as*

$$\mathfrak{E}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi) = \sum_{A_1, A_2 \in \mathcal{H}_n(\mathbb{Z})_{\ge 0}} c_{\mathfrak{E}_{2n}^{k,k-l}(Z_1,Z_2,N,\chi)}(A_1, A_2)\mathbf{e}(\mathrm{tr}(A_1Z_1 + A_2Z_2)$$

Then we have

$$c_{\mathfrak{E}_{2n}^{k,k-l}(Z_1,Z_2,N,\chi)}(A_1, A_2)$$
$$= \sum_{R \in M_n(\mathbb{Z})} P_{n,l}^{k-l}\left(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}\right) c_{2n,l}\left(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}\right)\bar{\chi}(\det R)\tau(\chi)^{-1}\sqrt{-1}^{-l}$$

COROLLARY 3.5. *For any $A_1, A_2 \in \mathcal{H}_n(\mathbb{Z})_{>0}$, $c_{\mathfrak{E}_{2n}^{k,k-l}(Z_1,Z_2,N,\chi)}(A_1, A_2)$ belongs to $\bar{\mathbb{Q}}$, and in particular if $\det\left(\begin{pmatrix} 2A_1 & R \\ {}^tR & 2A_2 \end{pmatrix}\right)$ is prime to N, then $a_{n,l}c_{\mathfrak{E}_{2n}^{k,k-l}(Z_1,Z_2,N,\chi)}(A_1, A_2)$ belongs $\mathfrak{O}_{\mathbb{Q}(\chi)}[N^{-1}]$, where $a_{n,l} = \prod_{i=1}^n (2l-2i)(2l-2i+1)!$.*

Suppose that $l \le k$. Then $\mathfrak{E}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi)$ can be expressed as

$$\mathfrak{E}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi) = \sum_{A \in \mathcal{L}_n(\mathbb{Z})_{>0}} \mathcal{E}_{2n}^{k,k-l}(Z_1, A, N, \chi)\mathbf{e}(\mathrm{tr}(AZ_2))$$

with $\mathcal{E}_{2n}^{k,k-l}(Z_1, A, N, \chi)$ a function of Z_1. Put

$$\mathcal{G}_{2n}^{k,k-l}(Z_1, A, N, \chi) = \sum_{\gamma \in \Gamma_0^{(n)}(N^2)\backslash \Gamma^{(n)}} (\mathcal{E}_{2n}^{k,k-l})|_k\gamma(Z_1, A, N, \chi).$$

It is easily seen that $\mathcal{E}_{2n}^{k,k-l}(Z_1, A, N, \chi)$ belongs to $M_k(\Gamma_0^{(n)}(N^2))$, and therefore $\mathcal{G}_{2n}^{k,k-l}(Z_1, A, N, \chi)$ belongs to $M_k(\Gamma^{(n)})$. In particular, if $l < k$, then $\mathcal{G}_{2n}^{k,k-l}(Z_1, A, N, \chi)$ belongs to $S_k(\Gamma^{(n)})$.

PROPOSITION 3.6. *Suppose that $l \le k$ and let $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$. Then $a_{n,l}\mathcal{G}_{2n}^{k,k-l}(Z_1, N^2A, N, \chi)$ belongs to $M_k(\Gamma^{(n)})(\mathcal{O}_{\mathbb{Q}(\chi,\zeta_N)}[N^{-1}])$. In particular, if $l < k$, it belongs to $S_k(\Gamma^{(n)})(\mathcal{O}_{\mathbb{Q}(\chi,\zeta_N)}[N^{-1}])$.*

Proof. We have

$$c_{\mathfrak{E}_{2n}^{k,k-l}(Z_1,Z_2,N,\chi)}(B, N^2A) = \sum_{R \in M_n(\mathbb{Z})} P_{n,l}^{k-l}\left(\begin{pmatrix} B & R/2 \\ {}^tR/2 & N^2A \end{pmatrix}\right) c_{2n,l}\left(\begin{pmatrix} B & R/2 \\ {}^tR/2 & N^2A \end{pmatrix}\right) \bar{\chi}(\det R)\tau(\chi)^{-1}\sqrt{-1}^{-l}.$$

We note that $\det \begin{pmatrix} 2B & R \\ {}^tR & 2N^2A \end{pmatrix}$ is prime to N if and only $\det R$ is prime to N. Therefore, by Corollary 3.3, $a_{n,l}\mathcal{E}_{2n}^{k,k-l}(Z_1, N^2A, N, \chi)$ belongs to $M_k(\Gamma_0^{(n)}(N^2))(\mathfrak{O}_{\mathbb{Q}(\chi)}[N^{-1}])$. By q-expansion principle (cf. [8], [13]), for any $\gamma \in \Gamma^{(n)}$, $a_{n,l}\mathcal{E}_{2n}^{k,k-l}|_k\gamma(Z_1, N^2A, N, \chi)$ belongs to $M_k(\Gamma^{(n)}(N^2))(\mathfrak{O}_{\mathbb{Q}(\chi,\zeta_N)}[N^{-1}])$. Hence, $a_{n,l}\mathcal{G}_{2n}^{k,k-l}(Z_1, N^2A, N, \chi)$ belongs to $M_k(\Gamma^{(n)}(N^2))(\mathfrak{O}_{\mathbb{Q}(\chi,\zeta_N)}[N^{-1}]) \cap M_k(\Gamma^{(n)}) = M_k(\Gamma^{(n)})(\mathfrak{O}_{\mathbb{Q}(\chi,\zeta_N)}[N^{-1}])$. This proves the first of the assertion. The latter is similar. □

THEOREM 3.7. *Let $\{F_i\}_{i=1}^d$ be an orthogonal basis of $S_k(\Gamma^{(n)})$ consisting of Hecke eigenforms, and $\{F_i\}_{d+1 \le i \le e}$ be a basis of the orthogonal complement $S_k(\Gamma^{(n)})^{\perp}$ of $S_k(\Gamma^{(n)})$ in $M_k(\Gamma^{(n)})$ with respect to the Petersson product. Then we have*

$$\mathcal{G}_{2n}^{k,k-l}(Z, N^2A, N, \chi) = \sum_{i=1}^{d} c(n,l)N^{nl}\Lambda(l-n, F_i, \chi, \mathrm{St})\overline{c_{F_i}(A)}F_i(Z) + \sum_{i=d+1}^{e} c_iF_i(Z)$$

where $c(n,l) = (-1)^{a(n,l)}2^{b(n,l)}$ with $a(n,l), b(n,l)$ integers, and c_i is a certain complex number. Moreover we have $c_i = 0$ for any $d+1 \le i \le e$ if $l < k$.

Proof. Put

$$\mathfrak{G}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi) = \sum_{\gamma \in \Gamma_0^{(n)}(N^2)\backslash \Gamma^{(n)}} \mathfrak{E}_{2n}^{k,k-l}(|_k \gamma Z_1, Z_2, N, \chi).$$

Then we have

$$\mathfrak{G}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi) = \sum_{A \in \mathcal{L}_n(\mathbb{Z})_{>0}} \mathcal{G}_{2n}^{k,k-l}(Z_1, A, N, \chi)\mathbf{e}(\mathrm{tr}(AZ_2))$$

By [[4],(3.24)], for any $\gamma \in \Gamma^{(n)}$ we have

$$\begin{aligned}
&\langle F_i, \mathfrak{G}_{2n}^{k,k-l}(|_k \gamma *, -\overline{Z_2}, N, \chi)\rangle \\
=&\langle F_i|_k\gamma, \mathfrak{G}_{2n}^{k,k-l}(|_k \gamma *, -\overline{Z_2}, N, \chi)\rangle \\
=&\langle F_i, \mathfrak{G}_{2n}^{k,k-l}(*, -\overline{Z_2}, N, \chi)\rangle \\
=&(-1)^{a'(n,l)}2^{b'(n,l)}N^{nl}\chi(-1)^n[\Gamma^{(n)} : \Gamma_0^{(n)}(N^2)]^{-1}\pi^{(l-k)n-(2n+1)l+n}\pi^{n(n+1)/2} \\
&\times L(l-n, F_i, \bar{\chi}, \mathrm{St})\Gamma(l-n)\tau(\chi)^{-n-1}\sqrt{-1}^{-l}F_i(N^2Z_2) \\
&\times \frac{\Gamma_{2n}(l)\Gamma_n(k-n/2)\Gamma_n(k-(n+1)/2)}{\Gamma_n(l)\Gamma_n(l-n/2)},
\end{aligned}$$

with $a'(n,l), b'(n,l) \in \mathbb{Z}$. We note that we take the normalized Petersson inner product. We also note that

$$\Gamma_{2n}(l) = \pi^{n^2/2}\Gamma_n(l)\Gamma_n(l-n/2),$$

and

$$\Gamma_n(k-n/2)\Gamma_n(k-(n+1)/2) = 2^{\gamma'(n,l)}\pi^{n^2/2}\prod_{i=1}^{n}\Gamma(2k-n-i)$$

with an integer $\gamma'(n,l)$. Hence we have

$$\begin{aligned}
&\langle F_i, \mathfrak{G}_{2n}^{k,k-l}(|_k\gamma *, -\overline{Z_2}, N, \chi)\rangle \\
&= c(n,l)[\Gamma^{(n)} : \Gamma_0^{(n)}(N^2)]^{-1}N^{nl}\Lambda(l-n, F_i, \overline{\chi}, \mathrm{St})\langle F_i, F_i\rangle F_i(N^2Z_2),
\end{aligned}$$

where $c(n,l) = (-1)^{a(n,l)}2^{b(n,l)}$ with $a(n,l), b(n,l)$ integers. On the other hand, we have

$$\langle F_i, \mathfrak{G}_{2n}^{k,k-l}(*, -\overline{Z_2}, N, \chi)\rangle = \sum_{A \in \mathcal{L}_n(\mathbb{Z})_{>0}} \langle F_i, \mathcal{G}_{2n}^{k,k-l}(*, A, N, \chi)\rangle\mathbf{e}(\mathrm{tr}(AZ_2)).$$

Hence we have

$$\langle F_i, \mathcal{G}_{2n}^{k,k-l}(*, A, N, \chi)\rangle = c(n,l)N^{nl}\Lambda(l-n, F_i, \overline{\chi}, \mathrm{St})\langle F_i, F_i\rangle c_{F_i}(N^{-2}A)$$

for any A. Now $\mathcal{G}_{2n}^{k,k-l}(Z, A, N, \chi)$ can be expressed as

$$\mathcal{G}_{2n}^{k,k-l}(Z, A, N, \chi) = \sum_{i=1}^{e} c_i F_i(Z)$$

with $c_i \in \mathbb{C}$. For $1 \le i \le d$ we have

$$\langle F_i, \mathcal{G}_{2n}^{k,k-l}(*, A, N, \chi)\rangle = \overline{c_i}\langle F_i, F_i\rangle.$$

Hence we have

$$c_i = \overline{c(n,l)N^{nl}\Lambda(l-n, F_i, \overline{\chi}, \mathrm{St})\langle F_i, F_i\rangle c_{F_i}(N^{-2}A)}.$$

We note that $\overline{\Lambda(l-n, F_i, \overline{\chi}, \mathrm{St})} = \Lambda(l-n, F_i, \chi, \mathrm{St})$. This proves the assertion. □

REMARK 3.8. There are errors in [[12], Appendix].
(1) The factor $\eta^*(\gamma)$ is missing in $E_{n,l}(Z, M, \eta, s)$ on [[12], page 125], and it should be defined as

$$E_{n,l}(Z, M, \eta, s) = L(1-l-2s, \eta)\prod_{i=1}^{[n/2]} L(1-2l-4s+2i, \eta^2)$$
$$\times \det(\mathrm{Im}(Z)))^s \sum_{\gamma\in\Gamma_\infty^{(n)}\backslash\Gamma_0^{(n)}(M)} j(\gamma, Z)^{-l}\eta^*(\gamma)|j(\gamma, Z)|^{-2s}.$$

Then $E^*_{n,l}(Z, M, \eta, s) = E_{n,l}|_l W_M(Z, M, \eta, s)$ with $W_M = \begin{pmatrix} O & -1_n \\ M1_n & O\end{pmatrix}$ coincides with the Eisenstein series $E^*_{n,l}(Z, M, \eta, s)$ in the present paper up to elementary factor. However, to quote several results in [4] smoothly, we define $E^*_{n,l}(Z, M, \eta, s)$ as in the present paper. Accordingly we define $\mathcal{G}^{k,k-l}_{2n}(Z, A, N, \chi)$ as in our paper. With these changes, Propositions 5.1 and 5.2, and (1) of Theorem 5.3 in [12] should be replaced with Corollary 3.3, Corollary 3.5, and Proposition 3.6, respectively, in the present paper.
(2) In [12], we defined $\mathbf{L}(m, F, \chi, \mathrm{St})$ as

$$\mathbf{L}(m, F, \chi, \mathrm{St}) = \Gamma_{\mathbb{C}}(m)(\prod_{i=1}^{n}\Gamma_{\mathbb{C}}(m+k-i))\frac{L(m, F, \chi, \mathrm{St})}{\tau(\chi)^{n+1}\langle F,\ F\rangle},$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. However, the factor $\sqrt{-1}^{m+n}$ should be added in the denominator on the right-hand side of the above definition. With this correction, [[12], Theorem 2.2] remains valid. Moreover, we have

$$\mathbf{L}(l-n, F, \chi, \mathrm{St}) = \frac{\prod_{i=1}^{n}\Gamma_{\mathbb{C}}(l-n+k-i)}{N^{ln}c(n,l)\prod_{i=1}^{n}\Gamma(2k-n-i)\pi^{-n(n+1)/2+nk+(n+1)m}}\Lambda(l-n, F, \chi, \mathrm{St}).$$

We note that

$$\frac{\displaystyle\prod_{i=1}^{n}\Gamma_{\mathbb{C}}(l-n+k-i)}{\displaystyle N^{ln}c(n,l)\prod_{i=1}^{n}\Gamma(2k-n-i)\pi^{-n(n+1)/2+nk+(n+1)m}}$$

is a rational number, and for a prime number p not dividing $N(2k-1)!$, it is p-unit. Therefore, (2) of Theorem 5.3 in [12] should be corrected as follows:

Put

$$\widetilde{\mathcal{G}}^{k,k-l}_{2n}(Z, N^2A, N, \chi)$$

$$= \frac{\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(l-n+k-i)}{N^{ln} c(n,l) \prod_{i=1}^{n} \Gamma(2k-n-i) \pi^{-n(n+1)/2+nk+(n+1)m}}$$
$$\times \mathcal{G}_{2n}^{k,k-l}(Z, N^2A, N, \chi).$$

Then $\widetilde{\mathcal{G}}_{2n}^{k,k-l}(Z, N^2A, N, \chi)$ *belongs to* $\mathfrak{S}_k(\Gamma^{(n)})(\mathfrak{O}_{\mathbb{Q}(F,\chi,\zeta_N)})_{\mathfrak{P}}$ *for any prime ideal* $\mathfrak{P}$ *of* $\mathbb{Q}(F, \chi, \zeta_N)$ *not dividing* $N(2k-1)!$*, and we have*

$$\widetilde{\mathcal{G}}_{2n}^{k,k-l}(Z, N^2A, N, \chi) = \sum_{i=1}^{d} \mathbf{L}(l-n, F_i, \chi, \mathrm{St}) \overline{c_{F_i}(A)} F_i(Z).$$

4. Proof of the main result

LEMMA 4.1. *Let* $r \geq 2$ *and let* $\{F_1, \ldots, F_r\}$ *be Hecke eigenforms* $M_k(\Gamma^{(n)}; \lambda_i)$ *linearly independent over* $\mathbb{C}$*, and* G *an element of* $M_k(\Gamma^{(n)})$*. Write*

$$F_i(Z) = \sum_{A} c_{F_i}(A) \mathbf{e}(\mathrm{tr}(AZ))$$

for $i = 1, \ldots r$ *and*

$$G(Z) = \sum_{A} c_G(A) \mathbf{e}(\mathrm{tr}(AZ)).$$

Let K *be the composite field of* $\mathbb{Q}(F_1), \ldots, \mathbb{Q}(F_r)$*, and* L *a finite extension of* K*. Let* N *be a positive integer. Assume that*

(1) *there exists an element* $\alpha \in K$ *such that* $c_G(A)$ *belongs to* $\alpha \mathfrak{O}_L[N^{-1}]$ *for any* $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$

(2) *there exist* $c_i \in L$ $(i = 1, \ldots, r)$ *and* $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$ *such that*

$$G(Z) = \sum_{i=1}^{r} c_i F_i(Z).$$

Then for any elements $T_1, \ldots, T_{r-1} \in \mathbf{L}_n'$ *and* $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$ *we have*

$$\prod_{i=1}^{r-1} (\lambda_{F_1}(T_i) - \lambda_{F_{i+1}}(T_i)) c_1 c_{F_1}(A) \in \alpha \mathfrak{O}_L[N^{-1}].$$

Proof. We prove the induction on r. The assertion clearly holds for $r = 2$. Let $r \geq 3$ and suppose that the assertion holds for any r' such that $2 \leq r' \leq r-1$. We have

$$G|T_{r-1}(Z) = \sum_{i=1}^{r} \lambda_{F_i}(T_{r-1}) c_i F_i(Z),$$

and we have

$$G|T_{r-1}(Z) - \lambda_{F_r}(T_{r-1}) G(Z) = \sum_{i=1}^{r-1} (\lambda_{F_i}(T_{r-1}) - \lambda_{F_r}(T_{r-1})) c_i F_i(Z).$$

By Theorem 4.1 and Proposition 4.2 of [10], we have

$$G|T_{r-1}(Z) - \lambda_{T_{r-1}} G(Z) \in \alpha S_k(\Gamma^{(n)})(\mathfrak{O}_L[N^{-1}])$$

Hence, by the induction assumption we prove the assertion. □

Proof of Theorem 2.3. Let $b(n,l)$ be the integer in Theorem 3.7, and put $\alpha(n,k) = \max_{\substack{2\le l\le k-n-2 \\ l\equiv 0 \mod 2}} b(n,l)$. Then, $a_{n,l}\mathcal{G}_{2n}^{k,k-l}(Z,N^2A,N,\chi)\in 2^{-\alpha(n,k)}M_k(\Gamma^{(n)})(\mathfrak{O}_{\mathbb{Q}(\chi,\zeta_N)}[N^{-1}])$. Thus, by Theorem 3.7 and Lemma 4.1, for any $B \in \mathcal{H}_n(\mathbb{Z})_{>0}$, and $T_1, \ldots, T_e \in \mathbf{L}'_n$, the value

$$\prod_{i=1}^{e-1}(\lambda_{F_1}(T_i) - \lambda_{F_{i+1}}(T_i))\Lambda(l-n, F, \chi, \mathrm{St})\bar{c}_F(A)c_F(B)$$

belongs to $(2^{\alpha(n,k)}A_{n,k})^{-1}\mathfrak{O}_{L_{n,k}(\chi,\zeta_N)}[N^{-1}]$, where $e = \dim_{\mathbb{C}} M_k(\Gamma^{(n)})$, and $L_{n,k}$ is the field stated in Section 1. In particular for any $v \in \widetilde{\mathfrak{E}}_F$, the value $v\Lambda(l-n, F, \chi, \mathrm{St})\bar{c}_F(A)$ $c_F(B)$ belongs to $(2^{\alpha(n,k)}A_{n,k})^{-1}\mathfrak{O}_{L_{n,k}(\chi,\zeta_N)}[N^{-1}]$. On the other hand, by Proposition 2.1, the value $\Lambda(l-n, F, \chi, \mathrm{St})\bar{c}_F(A)c_F(B)$ belongs to $\mathbb{Q}(F, \chi)$, and hence we have

$$v\Lambda(l-n, F, \chi, \mathrm{St})\bar{c}_F(A)c_F(B) \in (2^{\alpha(n,k)}A_{n,k})^{-1}\mathfrak{O}_{\mathbb{Q}(F,\chi)}[N^{-1}].$$

This implies that we have

$$\mathfrak{I}(l-n, F, \chi) \subset \langle (2^{\alpha(n,k)}A_{n,k}\widetilde{\mathfrak{E}}_F)^{-1}\rangle_{\mathfrak{O}_{\mathbb{Q}(F,\chi)}[N^{-1}]}.$$

REMARK 4.2. Let the notation be as in Lemma 4.1. Then we have the following.

Let $\mathfrak{p}$ be a prime ideal of K. Assume that $c_1 c_{F_1}(A)$ belongs to K and that $\mathrm{ord}_{\mathfrak{p}}(c_1 c_{F_1}(A)) < 0$ for some $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$. Then there exists $i \neq 2$ such that we have

$$\lambda_{F_i}(T) \equiv \lambda_{F_1}(T) \bmod \mathfrak{p} \quad \text{for any } T \in \mathbf{L}'_n.$$

This is a generalization of [[10], Lemma 5.1], and it can be proved in the same way. Let $K_{n,k}$ be the field defined in Section 2. Then, applying the above result to $L = K_{n,k}(\chi, \zeta_N)$, and using a corrected version of [[12], Theorem 5.3] in Remark 3.8 (2), we can remedy the proof of [[12], Theorem 3.1].

We also remark that the $M(2l-1)!$ in [[12], Theorem 3.1] should be $M(2k-1)!$.

5. Boundedness of special values of products of Hecke L-functions

For an element $f(z) = \sum_{m=1}^{\infty} c_f(m)\mathbf{e}(mz) \in S_k(SL_2(\mathbb{Z}))$ and a Dirichlet character χ, we define Hecke's L function $L(s, f, \chi)$ as

$$L(s, f, \chi) = \sum_{m=1}^{\infty} \frac{c_f(m)}{m^s}.$$

Let f be a primitive form. Then, for two positive integers $l_1, l_2 \le k-1$ and Dirichlet characters χ_1, χ_2 such that $\chi_1(-1)\chi_2(-1) = (-1)^{l_1+l_2+1}$, the value

$$\frac{\Gamma_{\mathbb{C}}(l_1)\Gamma_{\mathbb{C}}(l_2)L(l_1, f, \chi_1)L(l_2, f, \chi_2)}{\sqrt{-1}^{l_1+l_2+1}\tau((\chi_1\chi_2)_0)\langle f, f\rangle}$$

belongs to $\mathbb{Q}(f, \chi_1, \chi_2)$ (cf. [17]). We denote this value by $\mathbf{L}(l_1, l_2;\, f;\, \chi_1, \chi_2)$. In particular, we put

$$\mathbf{L}(l_1, l_2;\, f) = \mathbf{L}(l_1, l_2;\, f;\, \chi_1, \chi_2)$$

if χ_1 and χ_2 are the principal characters.

THEOREM 5.1. *Let f be a primitive form in $S_k(SL_2(\mathbb{Z}))$. Then we have*

$$\mathbf{L}(l_1, l_2;\, f;\, \chi_1, \chi_2) \in \langle (2^{b_k}\zeta(1-k)(k!)^2\widetilde{\mathfrak{D}}_f)^{-1}\rangle_{\mathfrak{O}_{\mathbb{Q}(f,\chi_1,\chi_2)}[(N_1N_2)^{-1}]}$$

with some non-negative integer b_k for any integers l_1 and l_2 and primitive characters χ_1 and χ_2 of conductors N_1 and N_2, respectively, satisfying the following conditions:

(D1) $$(\chi_1\chi_2)(-1) = (-1)^{l_1+l_2+1}.$$

(D2) $$k - l_1 + 1 \le l_2 \le l_1 - 1 \le k-2$$

(D3) *Either $l_1 \ge l_2 + 2$, or $l_1 = l_2 + 1$ and χ_1 or χ_2 is non-trivial*

Proof. The proof will proceed by a careful analysis of the proof of [[17], Theorem 4] combined with the argument in Theorem 2.3. For a positive integer $\lambda \ge 2$ and a Dirichlet character ω mod N such that $\omega(-1) = (-1)^\lambda$ we define the Eisenstein series $G_{\lambda,N}(z, s, \omega)$ ($z \in \mathbf{H}_1$, $s \in \mathbb{C}$) by

$$G_{\lambda,N}(z, s, \omega) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0^{(1)}(N)} \omega(d)(cz+d)^{-\lambda}|cz+d|^{-2s} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \bigm| m \in \mathbb{Z}\}$. It is well known that $G_{\lambda,N}(z, s, \omega)$ is finite at $s = 0$ as a function of s, and put

$$G_{\lambda,N}(z, \omega) = G_{\lambda,N}(z, 0, \omega).$$

$G_{\lambda,N}(z, \omega)$ is a (holomorphic) modular form of weight λ and character $\bar{\omega}$ for $\Gamma_0^{(1)}(N)$ if $\lambda \ge 3$ or ω is non-trivial. In the case $\lambda = 2$ and ω is trivial, $G_{2,N}(z, \omega)$ is a nearly automorphic form of weight 2 for $\Gamma_0^{(1)}(N)$ in the sense of Shimura [18]. We also put

$$\widetilde{G}_{\lambda,N}(z, \omega) = \frac{2\Gamma(\lambda)}{(-2\pi\sqrt{-1})^\lambda \tau(\omega_0)} L_N(\lambda, \omega) G_{\lambda,N}(z, \omega),$$

where $L_N(s, \omega) = L(s, \omega)\prod_{p|N}(1 - p^{-s}\omega(p))$. Now let N_i be the modulus of χ_i for $i = 1, 2$. Then, by [[16], Theorem 4.7.1] there exists a modular form g of weight $l_1 - l_2 + 1$ and character $\chi_1\chi_2$ for $\Gamma_0^{(1)}(N_1N_2)$ such that

$$c_g(0) = \begin{cases} 0 & \text{if } \chi_1 \text{ is non-trivial} \\ \frac{-1(1-N_1N_2)}{24} & \text{if } l_1 - l_2 = 1 \text{ and both } \chi_1 \text{ and } \chi_2 \text{ are trivial} \\ \frac{-B_{l_1-l_2+1,\chi_1\chi_2}}{2(l_1-l_2+1)} & \text{otherwise,} \end{cases}$$

$$c_g(m) = \sum_{0<d|m} \chi_1(m/d)\chi_2(d)d^{l_1-l_2} \quad (m \ge 1),$$

and

$$L(s, g) = L(s, \chi_1)L(s - l_1 + l_2, \chi_2).$$

Since we have $k \geq l_2, l_1$, all the Fourier coefficients of g belong to $(k!)^{-1}\mathfrak{O}_{\mathbb{Q}(\chi_1,\chi_2)}$ $[(N_1N_2)^{-1}]$. Let $\delta_\lambda^{(r)}$ be the differential operator in [17], page 788. Then, [[17], Lemma 7] we have

$$g\delta_{-k+l_1+l_2+1}^{(k-l_1-1)}\widetilde{G}_{-k+l_1+l_2+1,N_1N_2}(z,\chi_1\chi_2) = \sum_{\nu=0}^{r}\delta_{k-2\nu}^{(\nu)}h_\nu(z)$$

with some $r < k/2$, and $h_\nu \in M_{k-2\nu}(\Gamma_0^{(1)}(N_1N_2))$. By [[17], (3.3) and (3.4)] and the assumption, all the Fourier coefficients of $\widetilde{G}_{-k+l_1+l_2+1,N_1N_2}(z,\chi_1\chi_2)$ belongs to $(k!)^{-1}$ $\mathfrak{O}_{\mathbb{Q}(\chi_1\chi_2)}[(N_1N_2)^{-1}]$ if $-k+l_1+l_2+1 \geq 3$, or $\chi_1\chi_2$ is non-trivial. Moreover, by [[17], page 795], $\widetilde{G}_{2,N_1N_2}(z,\chi_1\chi_2)$ is expressed as

$$\widetilde{G}_{2,N_1N_2}(z,\chi_1\chi_2) = \frac{c}{4\pi y} + \sum_{n=0}^{\infty} c_n\mathbf{e}(nz),$$

with $c, c_n \in 2^{-1}\mathfrak{O}_{\mathbb{Q}(\chi_1\chi_2)}[(N_1N_2)^{-1}]$ if $-k+l_1+l_2+1 = 2$ and $\chi_1\chi_2$ is trivial. Hence, by the construction of h_0, all the Fourier coefficients of h_0 belong to $((k!)^2)^{-1}\mathfrak{O}_{\mathbb{Q}(\chi_1,\chi_2)}$ $[(N_1N_2)^{-1}]$. Let $f_1,\ldots,f_d$ be a basis of $S_k(SL_2(\mathbb{Z}))$ consisting of primitive forms such that $f_1 = f$. Then, by [[17], Theorem 2, Lemmas 1 and 7], we have

$$\mathbf{L}(l_1,l_2,f_i;\chi_1,\chi_2)\langle f_i,\ f_i\rangle = d_0[SL_2(\mathbb{Z}) : \Gamma_0^{(1)}(N_1N_2)]\langle f, h_0\rangle$$

for any $i = 1,\ldots,d$, where $d_0 = (-1)^{a(k,l_1,l_2)}2^{b(k,l_1,l_2)}$ with some $a(k,l_1,l_2), b(k,l_1,l_2) \in \mathbb{Z}$. (We note that the Petersson product $\langle *,\ *\rangle$ in our paper is $\frac{\pi}{3}$ times that in [17]). Define $\mathbf{h}_0(z)$ by

$$\mathbf{h}_0 = d_0 \sum_{\gamma\in\Gamma_0^{(1)}(N_1N_1)\backslash SL_2(\mathbb{Z})} h_0|\gamma(z).$$

Then, $\mathbf{h}_0$ belongs to $M_k(SL_2(\mathbb{Z}))$. We have

$$\langle f_i,\ h_0|\gamma\rangle = \langle f_i,\ h_0\rangle,$$

for any $\gamma \in SL_2(\mathbb{Z})$, and hence

$$\mathbf{L}(l_1,l_2,f_i;\chi_1,\chi_2)\langle f_i,\ f_i\rangle = \langle f_i,\mathbf{h}_0\rangle,$$

and hence we have

$$\mathbf{h}_0(z) = \alpha\widetilde{G}_k(z) + \sum_{i=1}^{d}\mathbf{L}(l_1,l_2,f_i;\chi_1,\chi_2)f_i(z)$$

with $\alpha \in \mathbb{C}$ and $\widetilde{G}_k(z) = \widetilde{G}_{k,1}(z,\mathbf{1})$. Put $b_k = \min\{\min_{l_1,l_2} b(k,l_1.l_2),0\}$ and $a_k = 2^{b_k}(k!)^2$, where l_1 and l_2 run over all integers satisfying the conditions (D2) and (D3). By q expansion principle, for any $\gamma \in SL_2(\mathbb{Z})$, $h_0|\gamma$ belongs to
$M_k(\Gamma^{(1)}(N_1N_2))(\langle a_k^{-1}\rangle_{\mathfrak{O}_{\mathbb{Q}(\chi_1,\chi_2,\zeta_N)}[(N_1N_2)^{-1}]})$. Therefore $\mathbf{h}_0$ belongs to
$M_k(\Gamma^{(1)}(N_1N_2))(\langle a_k^{-1}\rangle_{\mathfrak{O}_{\mathbb{Q}(\chi_1,\chi_2,\zeta_N)}[(N_1N_2)^{-1}]}) \cap M_k(SL_2(\mathbb{Z}))$. Put $h = \mathbf{h}_0 - \alpha\widetilde{G}_k$. Then all the Fourier coefficients of h belong to $\langle(2^{b_k}k!^2\zeta(1-k))^{-1}\rangle_{\mathfrak{O}_{\mathbb{Q}(\chi_1,\chi_2,\zeta_N)}[(N_1N_2)^{-1}]}$. We note that $\mathbf{L}(l_1,l_2;\ f;\chi_1,\chi_2)$ belongs to $\mathbb{Q}(f,\chi_1,\chi_2)$. Thus, using Lemma 4.1, we can prove the assertion in the same way as Theorem 2.3. $\square$

COROLLARY 5.2. *Let f be a primitive form in $S_k(SL_2(\mathbb{Z}))$. Let $\mathcal{Q}_f$ be the set of prime ideals $\mathfrak{p}$ of $\mathbb{Q}(f)$ such that*

$$\mathrm{ord}_{\mathfrak{p}}(N_{\mathbb{Q}(f,\chi_1,\chi_2)/\mathbb{Q}(f)}(\mathbf{L}(l_1, l_2; f; \chi_1, \chi_2))) < 0$$

for some positive integers l_1, l_2 and primitive characters χ_1, χ_2 with $\mathfrak{p} \nmid m_{\chi_1}, m_{\chi_2}$ satisfying the condition (D1), (D2), (D3). *Then $\mathcal{Q}_f$ is a finite set. Moreover, there exists a positive integer r such that we have*

$$\mathrm{ord}_{\mathfrak{q}}(\mathbf{L}(l_1, l_2; f; \chi_1, \chi_2)) \geq -r[\mathbb{Q}(f, \chi_1, \chi_2) : \mathbb{Q}(f)]$$

for any prime ideal $\mathfrak{q}$ of $\mathbb{Q}(f, \chi)$ lying above a prime ideal in $\mathcal{Q}_f$ and integer l_1, l_2 and primitive characters χ_1, χ_2 satisfying the above conditions.

For a prime ideal $\mathfrak{p}$ of an algebraic number field, let $p = p_{\mathfrak{p}}$ be a prime number such that $(p_{\mathfrak{p}}) = \mathbb{Z} \cap \mathfrak{p}$. Let K a number field containing $\mathbb{Q}(F)$. Then there exists a semi-simple Galois representation $\rho_f = \rho_{f,\mathfrak{p}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(K_{\mathfrak{p}})$ such that ρ_f is unramified at a prime number $l \neq p$ and

$$\det(1_2 - \rho_{f,\mathfrak{p}}(\mathrm{Frob}_l^{-1})X) = L_l(X, f),$$

where Frob_l is the arithmetic Frobenius at l, and

$$L_l(X, f) = 1 - c_f(l)X + l^{k-1}X^2.$$

For a $\mathfrak{p}$-adic representation ρ let $\bar{\rho}$ denote the mod $\mathfrak{p}$ representation of ρ. To prove our last main result, we provide the following lemma.

LEMMA 5.3. *Let $p = p_{\mathfrak{p}}$. Let k be a positive even integer such that $k < p$. Let f be a primitive form in $S_k(SL_2(\mathbb{Z}))$. Let a, b be integers such that $-p+1 < a < b < p-1$. Suppose that*

$$\bar{\rho}_f^{\mathrm{ss}} = \bar{\chi}^a \oplus \bar{\chi}^b,$$

where χ is the p-cyclotomic character. Then $(a, b) = (1-k, 0)$.

Proof. By [[5], Theorem 1.2] and its remark, $\overline{\rho}_f^{ss}|I_p$ should be

$$\overline{\chi}^{1-k} \oplus 1$$

or

$$\omega_2^{1-k} \oplus \omega_2^{p(1-k)}$$

with ω_2 the fundamental character of level 2, where I_p denotes the inertia group of p in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Thus the assertion holds. □

Let $f_1, \ldots, f_d$ be a basis of $S_k(SL_2(\mathbb{Z}))$ consisting of primitive forms with $f_1 = f$ and let $\mathfrak{D}_f$ be the ideal of $\mathbb{Q}(f)$ generated by all $\prod_{i=2}^d(\lambda_{f_i}(T(m)) - \lambda_f(T(m)))$'s $(m \in \mathbb{Z}_{>0})$.

THEOREM 5.4. *Let f be a primitive form in $S_k(SL_2(\mathbb{Z}))$. Let χ_1 and χ_2 be primitive characters of conductors N_1 and N_2, respectively, and let l_1 and l_2 be positive integers such that $k - l_1 + 1 \leq l_2 \leq l_1 - 1 \leq k-2$. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(f, \chi_1, \chi_2)$ with $p_{\mathfrak{o}} > k$. Suppose that $\mathfrak{p}$ divides neither $\mathfrak{D}_f N_1 N_2$ nor $\zeta(1-k)$. Then $\mathbf{L}(l_1, l_2; f; \chi_1, \chi_2)$ is $\mathfrak{p}$-integral.*

Proof. The assertion follows from Theorem 5.1 if l_1, l_2 and χ_1, χ_2 satisfy the conditions (D1),(D2), (D3). Suppose that $l_1 = l_2 + 1$ and χ_1 and χ_2 are trivial. By Lemma 5.3, there exists a prime number q_0 such that q_0 is $\mathfrak{p}$ unit and

$$1 - c_f(q_0)q_0^{-l_1+1} + q_0^{k-2l_1+1} \not\equiv 0 \bmod \mathfrak{p}.$$

As stated in the proof of Theorem 5.1, there exists a modular form $g \in M_2(\Gamma_0(q_0))(\mathbb{Z}_{(p_\mathfrak{p})})$ such that

$$L(s, g) = \zeta(s)\zeta(s-1)(1 - q_0^{-s+1}).$$

We can construct a modular form $h_0 \in M_k(\Gamma_0^{(1)}(q_0))$ in the same way as in the proof of Theorem 5.1. Then

$$\begin{aligned}&(1 - c_{f_i}(q_0)q_0^{-l_1+1} + q_0^{k-2l_1+1})\mathbf{L}(l_1, l_2; f_i)\langle f_i,\ f_i\rangle \\ &= d_0[SL_2(\mathbb{Z}) : \Gamma_0^{(1)}(q_0)]\langle f_i, h_0\rangle\end{aligned}$$

with some integer d_0 prime to $\mathfrak{p}$ for any $i = 1, \ldots, d$. Then by using the same argument as above, we can prove that

$$\mathrm{ord}_\mathfrak{p}(\mathbf{L}(l_1, l_2; f)(1 - c_f(q_0)q_0^{-l_1+1} + q_0^{k-2l_1+1})) \geq 0.$$

This proves the assertion. □

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