Doctoral Thesis

Toward Tests of Modified Gravity with Cosmological and Astrophysical Gravitational Waves

宇宙論的、天体物理学的重力波を用いた修正重力理論の検証に向けて

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Abstract

We study black hole perturbations and cosmological perturbations in modified theories of gravity. In particular, we study odd parity perturbations of spherically symmetric black holes with time-dependent scalar hair in shift-symmetric higher-order scalar-tensor theories. We derive the quadratic Lagrangian for the odd parity mode of gravitational waves without imposing the degeneracy conditions. We find that even though the theory contains higher-order derivatives of the scalar field, the quadratic Lagrangian for the odd parity mode of gravitational waves is free from the Ostrogradsky ghost. From the quadratic Lagrangian, we define the effective metric for the graviton, which determines the causal structure of the gravitational waves. We derive the generalized Regge-Wheeler equation and compute the quasinormal modes of a particular solution of black holes.

We also study cosmological perturbations in an inhomogeneous universe in spatially covariant theories with luminal propagation of gravitational waves in a homogeneous and isotropic universe. We find that some terms in this theory change the propagation speed of gravitational waves. According to the result of GW170817 and GRB170817A, we demand that gravitational waves propagate at the speed of light even in an inhomogeneous universe. From this requirement, we put a severe constraint on the spatially covariant theories.

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Chapter 1

Introduction

General relativity has been established as the standard theory of gravity. However, there are several motivations to modify general relativity.

- Since general relativity is a low-energy effective theory, it should be modified for extreme environments such as the early universe and inside black holes.
- According to cosmological observations, the expansion of the universe is accelerating. To explain the accelerating expansion with general relativity, we must consider the exotic energy component such as dark matter and dark energy. Is this natural?
- Purely theoretical interest. To understand gravity more deeply.

From these motivations, there have been many theoretical and observational studies on modified gravity. Recently, the first detection of gravitational waves (GWs) from binary black holes allows us to test the theory of gravity in a strong field regime [1]. If we are going to test general relativity by observations of GWs, we need to construct a comparative theory and have to understand it deeply.

In this thesis, we study black hole perturbations and cosmological perturbations in modified gravity. The dynamics of GWs around black holes are well described by black hole perturbation theory. In particular, the Quasi Normal Mode(QNM) of a black hole is relevant for the test of the modified theories of gravity. QNM is the solution of the master equation with a physically natural boundary condition and well describes waveforms of GWs emitted just after a merger. In general relativity, QNM is characterized by the mass and the angular momentum of a black hole. In contrast, in modified gravity, QNM may also depend on some additional parameters.

Since the differences in the QNM depend on the theory of gravity, we can test gravity from the observation of the QNM. From the theoretical point of view, we can check whether a black hole solution is stable or not by the black hole perturbation theory. Even if we find a black hole solution in modified gravity, it must be stable for it to exist in the universe. Therefore, black hole perturbation theory in modified gravity is relevant for theoretical and observational tests of gravity.

The cosmological perturbation theory is useful to investigate the propagation of GWs. In particular, one can predict how the GWs propagate in the universe. One can put the constraint on gravitational theories requiring the observational constraint on the propagation speed of GWs. In standard cosmology, spacetime is assumed to be homogeneous and isotropic. We will take into account the inhomogeneities of the universe, which are caused by the galaxies, and will study how inhomogeneities affect the propagations of GWs.

This thesis is organized as follows. In chapter 2, we review the modified theories of gravity. In particular, we introduce the Horndeski theory and Degenerate Higher-Order Scalar-Tensor(DHOST) theories. Horndeski theory is the most general scalar-tensor theory whose equations of motion become up to second order. DHOST theory contains higher derivative terms in the Lagrangian but its equations of motion become up to second order by imposing the degeneracy condition.

In chapter 3, we review the black hole perturbation theory in general relativity. We derive the master equations for odd parity perturbations, the Regge-Wheeler equation, and for even parity perturbations, the Zerilli equation. From the master equations, we can calculate the quasinormal mode of black holes. We see that we can test gravity from observations of ringdown GWs which are characterized by the quasinormal mode of black holes.

In chapter 4, we see black hole solutions in scalar-tensor theories. There is a no-hair theorem of shift-symmetric scalar-tensor theories. This theorem implies that if the spacetime and the scalar field are static and spherically symmetric, black holes cannot have a non-trivial configuration of the scalar field. We also review hairy black hole solutions with a time-dependent scalar field.

In chapter 5, we study odd parity perturbations of black holes with linearly time-dependent scalar hair in shift-symmetric scalar-tensor theories. We consider the higher-order scalar-tensor theories without imposing the degeneracy condition. Even though no degeneracy condition is imposed, the Lagrangian of odd parity perturbations does not contain higher derivatives terms. From the master equation, we can define the effective metric which determines the causal structure of the graviton. We show that the horizon for the graviton can be different from that for the photon in general. From this analysis, we improve the previous result of stability conditions.

In chapter 6, we study cosmological perturbations in an inhomogeneous universe in spatially covariant theories of gravity. We review the spatially covariant theories, which contain a wide class of gravitational theories, e.g., Horndeski theory. We also review cosmological perturbations in a homogeneous and isotropic universe and specify the theories with the propagation speed of GWs being equal to that of light. We investigate how GWs propagate in an inhomogeneous universe in this theory and constrain the theory using the event GW170817 and GRB170817A.

Chapter 2

Modification of gravity

In this chapter, we review some important theories of modified gravity.

2.1 Lovelock's theorem

When we would like to modify gravity from general relativity, we should start with Lovelock's theorem [2].

Theorem 1 *In a (pseudo) Riemannian manifold, a tensor field, $A^{\mu\nu}$, satisfying the following assumptions that*

1. *$A^{\mu\nu}$ is symmetric, i.e. , $A^{\mu\nu} = A^{\nu\mu}$,*
2. *$A^{\mu\nu}$ contains the metric and its first two derivatives, i.e. , $A^{\mu\nu} = A^{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,\lambda}, g_{\mu\nu,\lambda\rho})$,*
3. *$A^{\mu\nu}$ is divergence free, i.e. , ${A^{\mu\nu}}_{;\nu} = 0$,*
4. *the dimension of the (pseudo) Riemannian manifold is 4,*

has the unique form

$$A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu}, \tag{2.1.1}$$

where a, b is a constant.

This theorem states the uniqueness of general relativity under some assumptions. Lovelock's theorem tells us that we have to violate at least one of the assumptions in order to modify gravity. The ways to modify the general relativity are as follows.

- Consider $n(> 4)$-dimensional pseudo-Riemannian manifold.
- Adding new degrees of freedom other than the metric.
- Breaking the symmetry of the diffeomorphism.
- Consider non-Riemannian manifold.

Gravitational theories can be classified according to which assumption of Lovelock's theorem they violate.

There are many theories that add new degrees of freedom to general relativity. One of the most simple and relevant gravitational theories is the scalar-tensor theory which contains a metric and a single scalar field. We will explain this type of theory in the next section.

We will see the theory that breaks four-dimensional diffeomorphism but have three-dimensional diffeomorphism.

2.2 Scalar-tensor theories of gravity

2.2.1 Ostrogradsky's theorem

As an example, let us start with a simple Lagrangian that contains the second derivatives

$$L = \frac{1}{2}\ddot{x}^2. \tag{2.2.1}$$

The Euler-Lagrange equation for this Lagrangian is

$$\ddot{\ddot{x}} = 0. \tag{2.2.2}$$

To solve this fourth-order differential equation, we need four initial conditions unlike in the case of second-order differential equations. This implies this system contains two degrees of freedom because two initial conditions are required to determine the motion of a single degree of freedom. We will see that this redundant degrees of freedom is a ghost. By introducing an auxiliary variable, y, we can rewrite the Lagrangian as

$$L = \ddot{x}y - \frac{1}{2}y^2. \tag{2.2.3}$$

Varying with respect to y yields

$$y = \ddot{x} \tag{2.2.4}$$

and the original Lagrangian is reproduced. Performing the integration by part, the Lagrangian (2.2.3) can be written as

$$L = -\dot{x}\dot{y} - \frac{1}{2}y^2. \tag{2.2.5}$$

We define new fields by

$$w = \frac{1}{\sqrt{2}}(x - y), \tag{2.2.6}$$

$$z = \frac{1}{\sqrt{2}}(x + y), \tag{2.2.7}$$

and rewrite the Lagrangian in terms of these fields. After that, we obtain

$$L = -\frac{1}{2}\dot{z}^2 + \frac{1}{2}\dot{w}^2 - \frac{1}{4}(z - w)^2. \tag{2.2.8}$$

This Lagrangian contains the wrong sign kinetic term $-\dot{z}^2/2$. Therefore the higher derivative terms in the Lagrangian produce the redundant degrees of freedom which becomes a ghost. This ghost due to the existence of the higher derivative term is called "Ostrogradsky's ghost"

Let us consider more general situation. We suppose that the Lagrangian contains second derivative terms,

$$L = L(q, \dot{q}, \ddot{q}). \tag{2.2.9}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0. \tag{2.2.10}$$

Here, we assume that $\frac{\partial L}{\partial \ddot{q}}$ depends on $\ddot{q}$. This is known as "non-degeneracy condition". We choose the canonical variables as

$$Q_1 = q, \quad P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}}, \tag{2.2.11}$$

$$Q_2 = \dot{q}, \quad P_2 = \frac{\partial L}{\partial \ddot{q}}. \tag{2.2.12}$$

From the assumption of non-degeneracy, we can write $\ddot{q}$ in terms of Q_1, Q_2 and P_2. The Hamiltonian is obtained by Legendre transformation,

$$H(Q_1, Q_2, P_1, P_2) = P_1\dot{q}(Q_2) + P_2\ddot{q}(Q_1, Q_2, P_2) - L(q(Q_1), \dot{q}(Q_2), \ddot{q}(Q_1, Q_2, P_2)) \tag{2.2.13}$$

The canonical equations which gives the time evolution are given by,

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i}. \tag{2.2.14}$$

One can easily check that these canonical equations reproduce the original Euler-Lagrange equations and if the Hamiltonian does not depend on time explicitly, it is conserve. The problem is that the Hamiltonian (2.2.13) depends on P_1 linearly. This implies that the Hamiltonian is not bounded below and the motion described by this Hamiltonian is unstable. The origin of this instability is the higher derivative term in the Lagrangian. It has been shown that this instability is not resolved by adding $n(>2)$th-order derivative terms. Ostrogradsky showed a theorem about these instabilities caused by higher derivative terms.

Theorem 2 *If a Lagrangian is non-degenerate, i.e.,* $\frac{\partial L}{\partial \ddot{q}} \neq 0$*, and contains higher derivative terms, the Hamiltonian is unbounded below and the system is unstable.*

Let us consider the Lagrangian for coupled point particles $\phi(t), q^i(t), (i = 1, \cdots, n)$

$$L = \frac{1}{2}a\ddot{\phi}^2 + \frac{1}{2}k_0\dot{\phi}^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + b_i\ddot{\phi}\dot{q}^i + c_i\dot{\phi}\dot{q}^i - V(\phi, q), \tag{2.2.15}$$

where a, b_i, c_i, k_0 and k_{ij} are constant. The Euler-Lagrange equations for ϕ and q^i are, respectively,

$$a\ddot{\phi} - k_0\ddot{\phi} + b_i \dddot{q}^{\,i} - c_i\ddot{q}^i - V_\phi = 0, \tag{2.2.16}$$

$$k_{ij}\ddot{q}^j + b_i\ddot{\phi} + c_i\ddot{\phi} + V_i = 0, \tag{2.2.17}$$

where $V_i = \partial L/\partial q^i$ and $V_\phi = \partial L/\partial \phi$. These equations are a fourth-order differential equations with respect to ϕ and we need $2n+4$ initial conditions to determine the dynamics of $n+1$ variables. This means that there are $n+2$ degrees of freedom and Ostrogradsky' ghost appears due to the higher derivative term of ϕ in the Lagrangian. Here we define the new variables by $Q = \dot{\phi}$, and rewrite the Lagrangian as

$$L = \frac{1}{2}a\dot{Q}^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + \frac{1}{2}k_0Q^2 - V(\phi, q) + \left(b_i\dot{Q} + c_iQ\right)\dot{q}^i - \lambda(Q - \dot{\phi}). \tag{2.2.18}$$

where λ is a Lagrange multiplier. The Euler-Lagrange equations for this new Lagrangian are given by

$$a\ddot{Q} + b_i\ddot{q}^i = c_i\dot{q}^i + k_0Q - \lambda, \tag{2.2.19}$$

$$b_i\ddot{Q} + k_{ij}\ddot{q}^j = -V_i - c_i\dot{Q}, \tag{2.2.20}$$

$$\dot{\phi} = Q, \tag{2.2.21}$$

$$\dot{\lambda} = -V_\phi. \tag{2.2.22}$$

One can easily check that these equations reproduce the original equations. Now we introduce a kinetic matrix, K, defined by

$$K = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{Q} \partial \dot{Q}} & \frac{\partial^2 L}{\partial \dot{Q} \partial \dot{q}_j} \\ \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{Q}} & \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \end{pmatrix} = \begin{pmatrix} a & b_j \\ b_i & k_{ij} \end{pmatrix}. \tag{2.2.23}$$

The Euler-Lagrange equations can be written as

$$K \begin{pmatrix} \ddot{Q} \\ \ddot{q}^i \end{pmatrix} = \begin{pmatrix} c_i \dot{q}^i + k_0 Q - \lambda \\ -V_i - c_i \dot{Q} \end{pmatrix}. \tag{2.2.24}$$

If the kinetic matrix is invertible, i.e., $\det K \neq 0$, we can express $\ddot{Q}$ and $\ddot{q}^i$ in terms of up to first order derivative terms. In this case, we need initial conditions for $\dot{Q}, Q, \dot{q}^i, q^i, \lambda$ and ϕ. Therefore we need $2n+4$ initial conditions if the kinetic matrix is invertible and Ostrogradsky's ghost appears. On the other hand, if the kinetic matrix is degenerate, i.e.,

$$0 = \det K = \det(k) \left(a - b_i b_j \left(k^{-1}\right)^{ij} \right), \tag{2.2.25}$$

the higher derivative terms can be removed from the Euler-Lagrange equations and we can avoid the Ostrogradsky' ghost. When $a = b_i = 0$, $\ddot{Q}$ is removed from the Euler-Lagrange equations (2.2.19) and (2.2.20) and Eq.(2.2.19) becomes a constraint equation. Thus, we need only $2n+2$ initial conditions and the Ostrogradsky's ghost does not appear. When $b_i \neq 0$, it is convenient to introduce the vector

$$v = \begin{pmatrix} v^0 \\ v^i \end{pmatrix} = \begin{pmatrix} -1 \\ \left(k^{-1}\right)^{ij} b_j \end{pmatrix}. \tag{2.2.26}$$

Projecting the Euler Lagrange equations (2.2.24) with the vector v gives

$$v^{\mathrm{T}} K \begin{pmatrix} \ddot{Q} \\ \ddot{q}^i \end{pmatrix} = v^{\mathrm{T}} \begin{pmatrix} c_i \dot{q}^i + k_0 Q - \lambda \\ -V_i - c_i \dot{Q} \end{pmatrix}. \tag{2.2.27}$$

The left hand side of this equation becomes zero and we can write this equation as

$$c_i \left(\dot{q}^i + v^i \dot{Q} \right) + k_0 Q + v^i V_i = \lambda. \tag{2.2.28}$$

From this equation, it is convenient to introduce the new variables $x^i = q^i + v^i Q$ instead of q^i. The Euler-Lagrange equations can be written in terms of x^i as

$$c_i \dot{x}^i + k_0 Q + v^i V_i = \lambda, \tag{2.2.29}$$

$$k_{ij} \ddot{x}^j + c_i \dot{Q} + V_i = 0. \tag{2.2.30}$$

Here we take the time derivative of Eq.(2.2.29) and express Q and $\dot{\lambda}$ in terms of the original variable ϕ. The Euler-Lagrange equations becomes

$$\left(k_0 - v^i v^j V_{ij}\right) \ddot{\phi} + c_i \ddot{x}^i = -\left(v^i V_{ij}\right) \dot{x}^j - \left(v^i V_{i\phi}\right) \dot{\phi} - V_\phi, \tag{2.2.31}$$

$$c_i \ddot{\phi} + k_{ij} \ddot{x}^j = -V_i. \tag{2.2.32}$$

where $V_{ij} = \partial V_i/\partial q^j$ and $V_{\phi i} = V_{i\phi} = \partial V_\phi/\partial q^i$. Since these equations are second order differential equation of ϕ and x^i, we need $2n+2$ initial conditions. Therefore there exits only $n+1$ degrees of freedom and Ostrogradsky's ghost can be removed.

We have seen in this section that the Ostrogradsky's ghosts can be avoided if the Lagrangian is degenerate. We will see that this trick can be used for extension of the scalar-tensor theories.

2.2.2 Horndeski theory

Horndeski theory is the most general single scalar-tensor theory whose Euler-Lagrange equation become at most second-order differential equation [3]. To introduce the Horndeski theory, let us start with the Galileon theory [4]. The Galileon theory is most general theory of a scalar field under the assumptions that (1) the theory have the so called "Galileon symmetry", i.e., $\phi \to \phi + b_\mu x^\mu + c$ where b_μ and c are constant; (2) the equation of motion for the scalar field is up to second order differential equation; (3) the spacetime is fixed Minkowski. The Galileon theory is given by

$$\begin{aligned}\mathcal{L} =& c_1\phi + c_2 X - c_3 X\Box\phi + c_4 X\left[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right] \\ & - \frac{c_5}{3}X\left[(\Box\phi)^3 - 3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi\right],\end{aligned} \tag{2.2.33}$$

where $X = -\partial^\mu\phi\partial_\mu\phi/2$, and c_i is constant. The covariant version of the Galileon theory is called "the covariant Galileon theory" whose action is given by [5]

$$\mathcal{L} = c_1\phi + c_2 X - c_3 X\Box\phi + \frac{c_4}{2}X^2 R + c_4 X\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right] \tag{2.2.34}$$

$$+ c_5 X^2 G^{\mu\nu}\phi_{\mu\nu} - \frac{c_5}{3}X\left[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^\mu_\lambda\right] \tag{2.2.35}$$

This theory has no longer the Galileon symmetry because the Lagrangian contains the first derivative terms of the scalar field, but has the property that the equations of motion becomes up to second order. After the further generalization, the generalized Galileon theory is discovered [6]. The generalized Galileon in four-dimensional spacetime is given by

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \tag{2.2.36}$$

$$\mathcal{L}_2 = G_2(\phi, X), \tag{2.2.37}$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \tag{2.2.38}$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \tag{2.2.39}$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right], \tag{2.2.40}$$

where X is the kinetic term of the scalar field, $X = -\nabla^\mu\phi\nabla_\mu\phi/2$, and $G_i(\phi, X)$ is an arbitrary function of ϕ and X. The generalized Galileon also has the property that their equations of motion becomes up to second order. Note that the generalized Galileon theory has been obtained not only four dimensional spacetime but also in arbitrary dimensional spacetime.

The Horndeski theory is obtained under different assumptions than the generalized Galileon. The assumptions of the Horndeski theory are (1) the Lagrangian contains higher order derivative terms of the metric and the scalar field, i.e.,

$$S = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}\left(g_{\mu\nu}, g_{\mu\nu,\lambda_1}, \cdots, g_{\mu\nu,\lambda_1,\cdots,\lambda_p}, \phi, \phi_{,\lambda_1}, \cdots, \phi_{,\lambda_1,\cdots,\lambda_q}\right), \tag{2.2.41}$$

where $p, q \geq 2$; (2) the equation of motion for the metric and the scalar field is up to second order; (3) the spacetime dimension is four. (Of course, the symmetry of four-dimensional diffeomorphism is also assumed.) Horndeski found the most general Lagrangian that satisfies these assumptions [3]. Nevertheless the original Lagrangian found by Horndeski is quite different from the generalized Galileon in four-dimensional spacetime, the authors of Ref.[7] has shown that these two theories are equivalent. After that, we use the generalized Galileon (2.2.37)-(2.2.40) as the Horndeski theory. Note that the scalar-Gauss-Bonnet term such as

$$f(\phi)\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right), \tag{2.2.42}$$

is contained in Horndeski theory. We can reproduce equation of motion for the above term by choosing the arbitrary functions as

$$G_2 = 8f^{(4)}X^2(3 - \ln X), \tag{2.2.43}$$

$$G_3 = 4f^{(3)}X(7 - 3\ln X) \tag{2.2.44}$$

$$G_4 = 4f^{(2)}X(2 - \ln X), \tag{2.2.45}$$

$$G_5 = -4f^{(1)}\ln X, \tag{2.2.46}$$

where $f^{(n)} \equiv \mathrm{d}^n f/\mathrm{d}\phi^n$.

We will see that the coefficients of the second derivative of the scalar field are tuned to avoid the Ostrogradsky's ghost.

2.2.3 Degenerate Higher Order Scalar-Tensor theories

Even if the Lagrangian contains higher derivative terms, one can avoid the Ostrogradsky's ghost if the Lagrangian is degenerate. From this point of view, the Degenerate Higher Order Scalar-Tensor(DHOST) theories are constructed. The DHOST theories contains the quadratic and cubic terms of second derivatives of the scalar field. The former is called "quadratic DHOST" and the latter is called "cubic DHOST". The action of the cubic DHOST theory is given by

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\Bigg[&F_0(\phi,X) + F_1(\phi,X)\Box\phi + F_2(\phi,X)R \\ &+ \sum_{I=1}^{5} A_I(\phi,X)L_I^{(2)} + F_3(\phi,X)G_{\mu\nu}\phi^{\mu\nu} + \sum_{I=1}^{10} B_I(\phi,X)L_I^{(3)}\Bigg], \end{aligned} \tag{2.2.47}$$

where $X := -\phi_\mu\phi^\mu/2$, $\phi_\mu := \nabla_\mu\phi$, $\phi_{\mu\nu} = \nabla_\nu\nabla_\mu\phi$, R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor,F_I, A_I,andBI are arbitrary functions of ϕ and X. Here, $L_I^{(2)}$ are quadratic in the second derivatives of the scalar field and are written explicitly as

$$\begin{aligned} &L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}, \quad L_2^{(2)} = (\Box\phi)^2, \quad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu, \\ &L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu, \quad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2. \end{aligned} \tag{2.2.48}$$

Similarly, $L_I^{(3)}$ are cubic in the second derivatives of the scalar field and are given by

$$\begin{aligned} L_1^{(3)} &= (\Box\phi)^3, & L_2^{(3)} &= (\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu}, \\ L_3^{(3)} &= \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho, & L_4^{(3)} &= (\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu, \\ L_5^{(3)} &= \Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho, & L_6^{(3)} &= \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma, \\ L_7^{(3)} &= \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma, & L_8^{(3)} &= \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda, \\ L_9^{(3)} &= \Box\phi(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2, & L_{10}^{(3)} &= (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3. \end{aligned} \tag{2.2.49}$$

In order to avoid the Ostrogradsky's ghost, we need a degeneracy condition that leads to constraints between arbitrary functions. To find the degeneracy conditions for DHOST theories, it is convenient to perform ADM decomposition of spacetime [8]. The structure of the Lagrangian (2.2.47) written by ADM variables are similar to the example in the section 2.2.1. One can read off the kinetic matrix and the degeneracy condition from the ADM formulation of (2.2.47). For the quadratic DHOST, the degeneracy conditions are given by [9]

$$D_0 = 0, \quad D_1 = 0, \quad D_2 = 0 \tag{2.2.50}$$

with

$$D_0 = -4\left(A_2 + A_1\right)\left[XF_2\left(2A_1 + XA_4 + 4F_{2X}\right) - 2F_2^2 - 8X^2F_{2X}^2\right], \tag{2.2.51}$$

$$\begin{aligned} D_1 =& 4\left[X^2A_1\left(A_1 + 3A_2\right) - 2F_2^2 - 4XF_2A_2\right]A_4 + 4X^2F_2\left(A_1 + A_2\right)A_5 + 8XA_1^3 \\ & - 4\left(F_2 + 4XF_{2X} - 6XA_2\right)A_1^2 - 16\left(F_2 + 5XF_{2X}\right)A_1A_2 + 4X\left(3F_2 - 4XF_{2X}\right)A_1A_3 \\ & - X^2F_2A_3^2 + 32F_{2X}\left(F_2 + 2XF_{2X}\right)A_2 - 16F_2F_{2X}A_1 - 8F_2\left(F_2 - XF_{2X}\right)A_3 + 48F_2F_{2X}^2 \end{aligned} \tag{2.2.52}$$

$$\begin{aligned} D_2 =& 4\left[2F_2^2 + 4XF_2A_2 - X^2A_1\left(A_1 + 3A_2\right)\right]A_5 + 4A_1^3 + 4\left(2A_2 - XA_3 - 4F_{2X}\right)A_1^2 + 3X^2A_1A_3^2, \\ & - 4XF_2A_3^2 + 8\left(F_2 + XF_{2X}\right)A_1A_3 - 32F_{2X}A_1A_2 + 16F_{2X}^2A_1 + 32F_{2X}^2A_2 - 16F_2F_{2X}A_3. \end{aligned} \tag{2.2.53}$$

There are several conditions for the arbitrary functions to satisfy this degeneracy condition, and DHOST theories can be classified according to what type of degeneracy conditions are imposed. In particular, the subclass which is stable for perturbations around FLRW background is important to cosmology. This subclass of the quadratic DHOST is called class Ia which is described by

$$A_2 = -A_1, \tag{2.2.54}$$

$$\begin{aligned} A_4 =& \frac{1}{2\left(F_2 + 2XA_1\right)^2}\left[8XA_1^3 + \left(3F_2 + 16XF_{2X}\right)A_1^2 - X^2F_2A_3^2 + 2X\left(4XF_{2X} - 3f\right)A_1A_3\right. \\ & \left. + 2F_{2X}\left(3F_2 + 4XF_{2X}\right)A_1 + 2F_2\left(XF_{2X} - F_2\right)A_3 + 3F_2F_{2X}^2\right], \end{aligned} \tag{2.2.55}$$

$$A_5 = -\frac{\left(F_{2X} + A_1 + XA_3\right)\left(2F_2A_3 - F_{2X}A_1 - A_1^2 + 3XA_1A_3\right)}{2\left(F_2 + 2XA_1\right)^2}, \tag{2.2.56}$$

with

$$F_2 + 2XA_1 \neq 0. \tag{2.2.57}$$

From these conditions, one can find that class Ia theory contains five independent arbitrary functions A_1, A_3, F_0, F_1 and F_2. The Horndeski theory with $G_5 = 0$ can be reproduced by choosing the arbitrary functions as

$$F_2 = G_4, \quad A_2 = -A_1 = G_{4X}, \quad A_3 = A_4 = A_5 = 0. \tag{2.2.58}$$

These functions obviously satisfy the degeneracy condition of class Ia. Similarly, the degeneracy condition for the cubic DHOST is studied in [10].

2.3 Spatially covariant theory of gravity

In this section, we review the spatially covariant theory of gravity, which has only three-dimensional diffeomorphism but breaks four-dimensional diffeomorphism. The action is given by [11, 12]

$$S = \int \mathrm{d}t \, \mathrm{d}^3 x N \sqrt{\gamma} \mathcal{L} \left(t, N, \gamma_{ij}, K_{ij}, R_{ij}, \varepsilon_{ijk}, \nabla_i \right). \tag{2.3.1}$$

Here, N is the lapse function, γ_{ij} is the metric of three-dimensional space, R_{ij} is the Ricci tensor of three-dimensional space, ϵ_{ijk} is the Levi-Civita tensor with $\varepsilon_{ijk} = \sqrt{\gamma}\epsilon_{ijk}$ and $\epsilon_{ijk} = 1$, ∇_i is the covariant derivative compatible with γ_{ij} and K_{ij} is the extrinsic curvature which is given by

$$K_{ij} = \frac{1}{2N} \left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i \right), \tag{2.3.2}$$

where $\dot{h}_{ij} = \partial_t h_{ij}$ and N_i is the shift vector. This theory is called a spatially covariant theory. The argument ∇_i means that in general, this theory can contain higher derivatives of the space. On the other hand, this theory only contains the time derivative up to the first order in Lagrangian. From this fact, Gao found that this theory contains up to three degrees of freedom through Hamiltonian analysis [13]. Nevertheless, this theory breaks the four-dimensional diffeomorphism, one can always restore it by introducing new fields. These fields are referred to as Stückelberg fields. Conversely, by gauge fixing, one can always rewrite the four-dimensional covariant theory into the three-dimensional form. For this reason, this theory contains various types of modified gravity which are described in the four-dimensional covariant form. In the next subsection, we show an example contained in the spatially covariant theory.

2.3.1 Horndeski theory in the spatially covariant theory

In this subsection, we explain how Horndeski theory can be written in the spatially covariant theory. The Lagrangian of Horndeski theory is given by

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \tag{2.3.3}$$

$$\mathcal{L}_2 = G_2(\phi, X), \tag{2.3.4}$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \tag{2.3.5}$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \tag{2.3.6}$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right], \tag{2.3.7}$$

where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$ and ∇_μ describes an ordinary four-dimensional covariant derivatives compatible with the spacetime metric $g_{\mu\nu}$. We can use the scalar field ϕ to foliate the spacetime and rewrite the theory by the intrinsic and extrinsic geometric quantities on a hypersurface defined as $\phi = const.$ For this purpose, we assume that the gradient of the scalar field is timelike on all points in the spacetime, i.e., $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi < 0$. The normal vector can be defined as

$$n_\mu = \frac{\partial_\mu\phi}{\sqrt{2X}}. \tag{2.3.8}$$

The extrinsic curvature and the acceleration are defined by the covariant derivative of the normal vector, and these are defined as

$$K_{\mu\nu} = \gamma^\rho_\mu\nabla_\rho n_\nu, \quad a_\mu = n^\nu\nabla_\nu n_\mu, \tag{2.3.9}$$

where $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ is the induced metric on the hypersurface. Note that the second derivative of the scalar field can be written as

$$\nabla_\mu\nabla_\nu\phi = \frac{1}{N}\left(-n_\mu n_\nu \pounds_n \ln N + 2n_{(\mu}a_{\nu)} - K_{\mu\nu}\right), \tag{2.3.10}$$

where $\pounds_n$ is the Lie derivative with respect to the normal vector. Using these equations, one can rewrite the Horndeski theory as [14, 15]

$$S_{\text{Horndeski}} = \int \mathrm{d}^4x\sqrt{-g}\left(a_0K + a_1{}^3RK + a_2{}^3R_{\mu\nu}K^{\mu\nu} + b_1K^2 + b_2K_{\mu\nu}K^{\mu\nu} + c_1K^3 + c_2KK_{\mu\nu}K^{\mu\nu} + c_3K^\mu_\nu K^\nu_\rho K^\rho_\mu + d_0 + d_1{}^3R\right), \tag{2.3.11}$$

where ${}^3R_{\mu\nu}$ and ${}^3R$ are the Ricci tensor and the Ricci scalar on the hypersurface respectively. The coefficients are given by

$$a_0 = \frac{\partial F_3}{\partial N} - 2\frac{1}{N}\frac{\partial G_4}{\partial \phi}, \qquad 2a_1 = -a_2 = \frac{1}{N}F_5, \tag{2.3.12}$$

$$-b_1 = b_2 = \frac{\partial (NG_4)}{\partial N} + \frac{1}{2N^2}\frac{\partial G_5}{\partial \phi}, \qquad c_1 = -\frac{1}{3}c_2 = \frac{1}{2}c_3 = -\frac{1}{6}\frac{\partial G_5}{\partial N}, \tag{2.3.13}$$

$$d_0 = G_2 + \frac{1}{N^2}\frac{\partial F_3}{\partial \phi}, \qquad d_1 = G_4 - \frac{1}{2N^2}\frac{\partial (G_5 - F_5)}{\partial \phi}, \tag{2.3.14}$$

with

$$\frac{\partial}{\partial N}\left(\frac{F_3}{N}\right) = -\frac{G_3}{N^2}, \quad \frac{\partial}{\partial N}\left(\frac{F_5}{N}\right) = \frac{1}{N}\frac{\partial G_5}{\partial N}. \tag{2.3.15}$$

After taking the unitary gauge $\phi = t$, Horndeski action (2.3.11) is contained in the spatially covariant theory (2.3.1). On the other hand, one can always return to the original form (2.3.3) by performing the time coordinate transformation. Here, we show that the Horndeski theory can be contained in the spatially covariant theory. The spatially covariant theory can contain a wider class of theories such as GLPV theory [16] and Horava-Lifshitz theory [17] and so on.

Chapter 3

Black hole perturbations in general relativity

In this chapter, we provide a brief review of the black hole perturbation theory in general relativity.

3.1 Master equations for odd and even parity perturbations

We consider static and spherically symmetric spacetime. In this assumption, the metric can be written as

$$\overline{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + C(r)r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi)^2. \tag{3.1.1}$$

The function $C(r)$ is redundant, however, we introduce this function for later convenience. In GR, it is known that the Schwarzschild metric is the unique solution under static and spherically symmetric spacetime [18]. Therefore we take $A(r) = B(r) = 1 - r_h/r$ in this subsection.

We consider the perturbation of the metric, $h_{\mu\nu}$, as

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}. \tag{3.1.2}$$

In black hole perturbation theory, it is convenient to decompose the perturbations by their symmetry with respect to parity transformations [19]. Odd parity perturbations can be written as

$$h_{tt} = 0, \quad h_{tr} = 0, \quad h_{rr} = 0, \tag{3.1.3}$$

$$h_{ta} = \sum_{\ell,m} h_{0,\ell m}(t,r) E_{ab} \partial^b Y_{\ell m}(\theta,\varphi), \tag{3.1.4}$$

$$h_{ra} = \sum_{\ell,m} h_{1,\ell m}(t,r) E_{ab} \partial^b Y_{\ell m}(\theta,\varphi), \tag{3.1.5}$$

$$h_{ab} = \frac{1}{2} \sum_{\ell,m} h_{2,\ell m}(t,r) \left[E_a{}^c \nabla_c \nabla_b Y_{\ell m}(\theta,\varphi) + E_b{}^c \nabla_c \nabla_a Y_{\ell m}(\theta,\varphi) \right] \tag{3.1.6}$$

where we defined $E_{ab} = \sqrt{\gamma} \epsilon_{ab}$ with complete anti-symmetric tensor satisfying $\epsilon_{\theta\varphi} = 1$. Here, γ_{ab} is the metric of 2-dimensional sphere and ∇_a is a covariant derivative compatible with γ_{ab}. Even parity perturbations can be written as

$$h_{tt} = A(r) \sum_{\ell,m} H_{0,\ell m}(t,r) Y_{\ell m}(\theta,\varphi), \tag{3.1.7}$$

$$h_{tr} = \sum_{\ell,m} H_{1,\ell m}(t,r) Y_{\ell m}(\theta,\varphi), \tag{3.1.8}$$

$$h_{rr} = \frac{1}{B(r)} \sum_{\ell,m} H_{2,\ell m}(t,r) Y_{\ell m}(\theta,\varphi), \tag{3.1.9}$$

$$h_{ta} = \sum_{\ell,m} \beta_{\ell m}(t,r) \partial_a Y_{\ell m}(\theta,\varphi), \tag{3.1.10}$$

$$h_{ra} = \sum_{\ell,m} \alpha_{\ell m}(t,r) \partial_a Y_{\ell m}(\theta,\varphi), \tag{3.1.11}$$

$$h_{ab} = \sum_{\ell,m} K_{\ell m}(t,r) g_{ab} Y_{\ell m}(\theta,\varphi) + \sum_{\ell,m} G_{\ell m}(t,r) \nabla_a \nabla_b Y_{\ell m}(\theta,\varphi). \tag{3.1.12}$$

The ten components of perturbations of the metric $h_{\mu\nu}$ are decomposed into the odd parity perturbations described by three variables (h_0, h_1, h_2), and even parity perturbations described by seven variables $(H_0, H_1, H_2, \beta, \alpha, K, G)$. These perturbations include the gauge modes, we perform the gauge transformation to remove the some of these perturbations. We consider an infinitesimal transformation of the coordinates

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x). \tag{3.1.13}$$

From this transformation, the metric transformation up to linear order of ξ^μ becomes

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} - (D_\mu \xi_\nu + D_\nu \xi_\mu), \tag{3.1.14}$$

where D_μ is a covariant derivative compatible with the background metric $\bar{g}_{\mu\nu}$. Here we assume that $D_\mu \xi_\nu$ is the same order of $h_{\mu\nu}$. The infinitesimal coordinate transformation ξ^μ can also be decomposed into odd parity transformation and even parity transformation. Odd parity transformation is

$$\xi_t = 0, \quad \xi_r = 0 \tag{3.1.15}$$

$$\xi_a = \sum_{\ell,m} \Xi_{\ell m}(t,r) E_a{}^b \nabla_b Y_{\ell m}(\theta\varphi), \tag{3.1.16}$$

and even parity transformation is

$$\xi_t = \sum_{\ell,m} T_{\ell m}(t,r) Y_{\ell m}(\theta,\varphi), \tag{3.1.17}$$

$$\xi_r = \sum_{\ell,m} R_{\ell m}(t,r) Y_{\ell m}(\theta,\varphi), \tag{3.1.18}$$

$$\xi_a = \sum_{\ell,m} \Theta_{\ell m}(t,r) \partial_a Y_{\ell m}(\theta,\varphi). \tag{3.1.19}$$

Substituting the metric perturbation Eqs. (3.1.3)-(3.1.12) and the infinitesimal displacements Eqs.(3.1.15)- (3.1.19) into Eqs. (3.1.14), we can find the gauge transformations of the metric perturbations in terms of the coefficients h_0, h_1, etc. The gauge transformations of the coefficients for odd mode will be

$$h_0 \to h_0 - \dot{\Xi}, \quad h_1 \to h_1 - \Xi' + \frac{2}{r}\Xi, \quad h_2 \to h_2 - 2\Xi, \tag{3.1.20}$$

for even mode will be

$$H_{0,\ell m}(t,r) \to H_{0,\ell m}(t,r) + \frac{2}{A}\dot{T}_{\ell m}(t,r) - \frac{A'B}{A} R_{\ell m}(t,r), \tag{3.1.21}$$

$$H_{1,\ell m}(t,r) \to H_{1,\ell m}(t,r) + \dot{R}_{\ell m}(t,r) + T'_{\ell m}(t,r) - \frac{A'}{A} T_{\ell m}(t,r), \tag{3.1.22}$$

$$H_{2,\ell m}(t,r) \to H_{2,\ell m}(t,r) + 2BR'_{\ell m}(t,r) + B' R_{\ell m}(t,r), \tag{3.1.23}$$

$$\beta_{\ell m}(t,r) \to \beta_{\ell m}(t,r) + T_{\ell m}(t,r) + \dot{\Theta}_{\ell m}(t,r), \tag{3.1.24}$$

$$\alpha_{\ell m}(t,r) \to \alpha_{\ell m}(t,r) + R_{\ell m}(t,r) + \Theta'_{\ell m}(t,r) - \frac{2}{r}\Theta_{\ell m}(t,r), \tag{3.1.25}$$

$$K_{\ell m}(t,r) \to K_{\ell m}(t,r) + \frac{2B}{r} R_{\ell m}(t,r), \tag{3.1.26}$$

$$G_{\ell m}(t,r) \to G_{\ell m}(t,r) + 2\Theta_{\ell m}(t,r), \tag{3.1.27}$$

where the dot and prime means a derivative with respect to t and r respectively. From these transformations, we can remove some of the coefficients to simplify the equations. We choose the gauge such that

$$h_{2\ell m} = 0, \quad \beta_{\ell m} = 0, \quad G_{\ell m} = 0, \quad K_{\ell m} = 0. \tag{3.1.28}$$

For $\ell > 2$, the coefficients of the transformation Ξ, T, R, Θ are determined, thus, this gauge is fixed completely. In what follows, we expand the Einstein-Hilbert action and derive the master equation in this gauge for $\ell > 2$.

We consider the Einstein-Hilbert action,

$$S = \int \mathrm{d}^4 x \sqrt{-g} R \tag{3.1.29}$$

and expand the action up to second order for odd and even parity perturbations.

First, we derive the quadratic action and EoM for the odd parity mode. We substitute the odd parity perturbations Eqs. (3.1.3)-(3.1.6) into the Einstein-Hilbert action, and expand it up to second order in perturbations, we obtain the quadratic action for the odd parity perturbation. After performing the integration for the angular variables, the quadratic action for the odd parity perturbations is given by

$$S^{(2)}_{odd} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \int \mathrm{d}t \, \mathrm{d}r \mathcal{L}^{(2)}_{\ell m}, \tag{3.1.30}$$

$$\mathcal{L}^{(2)}_{\ell m} = \frac{1}{2} \left\{ \left[\frac{2}{r^2} (r a_3)' + a_1 \right] |h_0|^2 + a_2 |h_1|^2 + a_3 \left(\left| \dot{h}_1 \right|^2 - 2 \dot{h}_1^* h_0' + |h_0'|^2 + \frac{4}{r} \dot{h}_1^* h_0 \right) \right\} + \text{ c.c. } . \tag{3.1.31}$$

The coefficients are given by

$$a_1 = \frac{c_\ell}{2r^2 A} \frac{M_{\rm Pl}^2}{2}, \tag{3.1.32}$$

$$a_2 = -\frac{c_\ell}{2} \frac{A}{r^2} \frac{M_{\rm Pl}^2}{2}, \tag{3.1.33}$$

$$a_3 = \frac{l(l+1)}{2} \frac{M_{\rm Pl}^2}{2}, \tag{3.1.34}$$

where $c_\ell = (\ell-1)\ell(\ell+1)(\ell+2)$. Here, the quadratic action contains the two variables h_0 and h_1. However, there must be only one degree of freedom in the odd parity perturbation which describe the odd mode of the GWs. Thus we can rewrite the quadratic action with only one variable. By introducing an auxiliary field $\chi_{\ell m}(t, r)$, we rewrite the quadratic action (3.1.31) as

$$\mathcal{L}^{(2)}_{\ell m} = \frac{1}{2} \left[a_1 |h_0|^2 + a_2 |h_1|^2 + a_4 h_1^* h_0 + 2 a_3 \chi^* \left(-\frac{1}{2} \chi + \dot{h}_1 - h_0' + \frac{2}{r} h_0 \right) \right] + \text{ c.c. } . \tag{3.1.35}$$

From the variation with respect to χ^*, we obtain

$$\chi = \dot{h}_1 - h_0' + \frac{2}{r} h_0. \tag{3.1.36}$$

We can reproduce the original action (3.1.31) by substituting Eq.(3.1.36) into the new action (3.1.36), thus the new action (3.1.36) is equivalent to the original one. From the Variation with respect to h_0^* and h_1^*, respectively, we obtain the relations between h_0, h_1 and χ,

$$a_1 h_0 + (a_3\chi)' + \frac{2a_3}{r}\chi = 0, \tag{3.1.37}$$

$$a_2 h_1 - a_3\dot{\chi} = 0, \tag{3.1.38}$$

and we solve these constraints for h_0 and h_1

$$h_0 = -\frac{2a_3(\chi/r) + (a_3\chi)'}{a_1} \tag{3.1.39}$$

$$h_1 = \frac{a_3\dot{\chi}}{a_2}. \tag{3.1.40}$$

Substituting Eq.(3.1.39) and Eq. (3.1.40) into the new action (3.1.35), we can rewrite the quadratic action in terms of the only one variable, χ,

$$\mathcal{L}^{(2)}_{\ell m} = \frac{\ell(\ell+1)r^2}{4(\ell-1)(\ell+2)}\left\{ b_1|\dot{\chi}|^2 - b_2\left|\chi'\right|^2 - \left[\frac{\ell(\ell+1)}{2}\frac{M_{\rm Pl}^2}{r^2} + \frac{V}{2}\right]|\chi|^2 \right\} + \text{ c.c. }, \tag{3.1.41}$$

where

$$b_1 = \frac{M_{\rm Pl}^2}{2A}, \quad b_2 = \frac{M_{\rm Pl}^2 A}{2}, \quad V = -4M_{\rm Pl^2}\frac{r_h}{r^3}. \tag{3.1.42}$$

Varying the quadratic action (3.1.41) with respect to χ, we obtain the EoM for χ

$$b_1\ddot{\chi} - \frac{1}{r^2}\left(r^2 b_2\chi'\right)' + \left[\frac{\ell(\ell+1)}{2}\frac{\mathcal{H}}{r^2} + \frac{V}{2}\right]\chi = 0. \tag{3.1.43}$$

Here, we define the tortoise coordinate by

$$\mathrm{d}r_* = \frac{\mathrm{d}r}{A}. \tag{3.1.44}$$

After performing the integration, the tortoise coordinate is given by

$$r_* = r + r_h \log\frac{r - r_h}{r_h}. \tag{3.1.45}$$

Note that the infinity $r = +\infty$ is correspond to $r_* = +\infty$ and the horizon $r = r_h$ is correspond to $r_* = -\infty$. The EoM for χ in this coordinate becomes

$$(-\partial_t^2 + \partial_{r*}^2 - \tilde{V})\chi = 0 \tag{3.1.46}$$

where

$$\tilde{V} = A\left[\frac{\ell(\ell+1)}{r^2} - \frac{3r_h}{r^3}\right]. \tag{3.1.47}$$

Eq. (3.1.46) called the "Regge-Wheeler equation" describes how the odd mode of the gravitational wave propagate around the Schwarzschild spacetime [19].

Next, we consider the even parity perturbations in the gauge with $G_{\ell m} = K_{\ell m} = \beta_{\ell m} = 0$. In this gauge, the even parity perturbations have the four variables H_0, H_1, H_2 and α. We substitute the even parity perturbations (3.1.17)-(3.1.19) into the Einstein-Hilbert action and perform the expansion of the action up to second order. After integration over the angular variables, the quadratic action of the even parity perturbations is given by

$$\begin{aligned}\frac{2(\ell+1)}{2\pi}\mathcal{L} =& H_0\left[a_3 H_2' + j^2 a_4 \alpha' + \left(a_7 + j^2 a_8\right) H_2 + j^2 a_4 \alpha\right] + j^2 b_1 H_1^2 + H_1\left(b_4 \dot{H}_2 + j^2 b_5 \dot{\alpha}\right)\\ &+ j^2 b_5 H_2 \alpha + c_6 H_2^2 + j^2 d_1 \dot{\alpha}^2 + j^2 d_4 \alpha^2. \end{aligned} \tag{3.1.48}$$

The coefficients are the function of r and background quantities, and given by

$$a_3 = -M_{\rm Pl}^2 r A, \quad a_4 = M_{\rm Pl}^2 A, \quad a_7 = -M_{\rm Pl}^2 (rA)', \quad a_8 = -\frac{M_{\rm Pl}}{2}, \tag{3.1.49}$$

$$b_1 = \frac{M_{\rm Pl}^2}{2}, \quad b_4 = 2M_{\rm Pl}^2 r, \quad b_5 = -M_{\rm Pl}^2, \tag{3.1.50}$$

$$c_6 = \frac{M_{\rm Pl}^2}{2} r^2\left(\frac{A}{r^2} + \frac{A'}{r}\right), \quad d_1 = \frac{M_{\rm Pl}^2}{2}, \quad d_4 = M_{\rm Pl}^2 \frac{A}{r}. \tag{3.1.51}$$

Since H_0 is a Lagrange multiplier in the quadratic action (3.1.48), variation with respect to H_0 gives the constraint between two variables, H_2 and α,:

$$a_3 H_2' + j^2 a_4 \alpha' + \left(a_7 + j^2 a_8\right) H_2 + j^2 a_4 \alpha = 0. \tag{3.1.52}$$

To simplify this equation, we define a new variable, ψ, as

$$H_2 = \frac{1}{a_3}\left(\psi - j^2 a_4 \alpha\right), \tag{3.1.53}$$

and we use this new variable instead of H_2. Substituting the Eq.(3.1.53) into the constraint (3.1.52), the differential term of α can be removed and we can solve the algebraic constraint equation for α as

$$\alpha = \frac{1}{j^4 a_4 a_8}\left(a_3 \psi' + j^2 a_8 \psi\right). \tag{3.1.54}$$

Since there are no derivative terms of H_1 in the quadratic action, the variation with respect to H_1 gives the constraint equation for H_1 as

$$H_1 = -\frac{1}{2j^2 b_1}\left(b_4 H_2 + j^2 b_5 \alpha\right)^{\cdot} . \tag{3.1.55}$$

From the constraints (3.1.53),(3.1.54) and (3.1.55), we can express all the variables, H_1, H_2 and, α in terms of ψ and its derivatives. Finally we define the new variable, Ψ, as

$$\psi = j(1 + j^2 - 3A)\Psi, \tag{3.1.56}$$

and we can write the quadratic action for even parity perturbations by only one variable, Ψ, as

$$\frac{2\ell+1}{2\pi}\mathcal{L} = \frac{2\left(j^2-2\right)}{M_{\rm Pl}^2}\left[\frac{1}{A}\dot{\Psi}^2 - A\Psi'^2 - \frac{3+j^2-j^4+j^6-3\left(1+j^2\right)^2 A + 9\left(1+j^2\right)A^2 - 9A^3}{r^2\left(j^2+1-3A\right)^2}\Psi^2\right]. \tag{3.1.57}$$

This result is expected intuitively. General Relativity has two degrees of freedom corresponding to the two modes of GWs. One of the degrees of freedom belongs to the odd parity perturbation and the other belongs to the even parity perturbation. Therefore the even parity perturbation should be described by only one variable, even though there are many variables in even parity perturbations. By using of the tortoise coordinate (3.1.44), we obtain the equation of motion for Ψ

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial r_*^2} - \frac{A}{r^2\left(j^2+1-3A\right)^2}\left(3+j^2-j^4+j^6-3\left(1+j^2\right)^2 A + 9\left(1+j^2\right)A^2 - 9A^3\right)\Psi = 0. \tag{3.1.58}$$

This equation called the "Zerilli equation" gives how the even mode of the gravitational wave propagate around the Schwarzschild spacetime [20].

We plot the potential for the odd parity perturbations and for the even parity perturbations in figure 3.1. From figure 3.1, it is obvious that the potential for the odd and even parity mode has almost the same form and property. Therefore we can use the same method to solve the master equation for the both of modes.

3.2 Solution of the master equations and quasi normal mode of a black hole

We would like to solve the master equation in some physically relevant conditions. As mentioned in the previous subsection, the Regge-Wheeler equation and the Zerilli

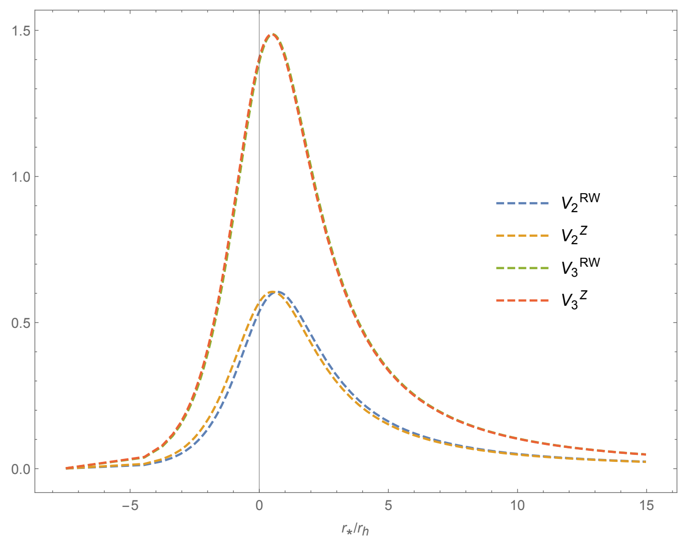

Figure 3.1: RW and Zerilli potentials are plotted in $\ell = 2$ and $\ell = 3$ cases. One can find that the RW and Zerilli potential have almost the same form.

equation are qualitatively equivalent. Thus we only focus on the Regge-Wheeler equation in this subsection.

At first, we have to set the relevant boundary conditions. We observe how the master equation (3.1.46) behave at the infinity and near the horizon. At the infinity, the tortoise coordinate (3.1.45) approximately becomes the original coordinate, $r_* \sim r$. Therefore the RW potential (3.1.47) at the infinity goes to zero as $1/r_*^2$ Near the horizon, $r \sim r_h$, the tortoise coordinate approximately becomes

$$r_* \sim r_h + r_h \log \frac{r - r_h}{r_h}, \tag{3.2.1}$$

and

$$A(r) \sim e^{\frac{r_*}{r_h} - 1}. \tag{3.2.2}$$

Therefore the RW potential vanishes exponentially at the horizon. From the above analysis, the master equation at the infinity and near the horizon becomes a one-dimensional wave equation and the perturbation propagate freely. The solution of the master equation (3.1.46) at the infinity is given by the superposition of plane waves,

$$\tilde{\chi} \sim \tilde{\chi}_{inf} = \int_{-\infty}^{\infty} \mathrm{d}\omega \left[A^{out}(\omega) e^{-i\omega(t - r_*)} + A^{in}(\omega) e^{-i\omega(t + r_*)} \right], \tag{3.2.3}$$

where $A^{out}(\omega)$ is the amplitude of the outgoing wave and $A^{in}(\omega)$ is the amplitude of the ingoing wave. Similarly, the solution of the master equation near the horizon is given by

$$\tilde{\chi} \sim \tilde{\chi}_h = \int_{-\infty}^{\infty} \mathrm{d}\omega \left[B^{out}(\omega)e^{-i\omega(t-r_*)} + B^{in}(\omega)e^{-i\omega(t+r_*)}\right]. \tag{3.2.4}$$

In the most case of the black hole perturbation theory, one imposes the boundary condition as,

$$\tilde{\chi} \sim \int_{-\infty}^{\infty} \mathrm{d}\omega \left[A^{in}(\omega)e^{-i\omega(t+r_*)}\right] \qquad (r_* \to +\infty), \tag{3.2.5}$$

$$\tilde{\chi} \sim \int_{-\infty}^{\infty} \mathrm{d}\omega \left[B^{out}(\omega)e^{-i\omega(t-r_*)}\right] \qquad (r_* \to -\infty). \tag{3.2.6}$$

These conditions says that nothing comes from the infinity and nothing comes out from the horizon.

We assume the time dependence of the master variable as $\tilde{\chi} = Q(r)e^{-i\omega t}$, the master equation becomes Schrödinger equation,

$$\frac{\mathrm{d}^2Q}{\mathrm{d}r_*^2} + \left(\omega^2 - \tilde{V}(r)\right)Q = 0. \tag{3.2.7}$$

Thus how the GWs propagate around the Schwarzschild spacetime is given by solving the eigenvalue problem. As the case of quantum mechanics, the solution of Eq.(3.2.7) exists only for some discrete value of ω. This discrete values of ω are called the "Quasi Normal Modes"(QNMs) of the black hole. We denote the QNMs as ω_{QNM}. Since ω_{QNM} is complex number in general, the time dependence of the master variable can be written as

$$\tilde{\chi} = Q(r)e^{-i\omega t} = Q(r)e^{-i(\mathrm{Re}[\omega_{QNM}]+i\mathrm{Im}[\omega_{QNM}])t} = Q(r)e^{\mathrm{Im}[\omega_{QNM}]t}e^{-i\mathrm{Re}[\omega_{QMN}]t}. \tag{3.2.8}$$

From this equation, we find that the imaginary part of the ω_{QNM} gives the time evolution of the amplitude and the real part of the ω_{QNM} gives the normal frequency. In the boundary conditions (3.2.6), one can expect that GWs finally go to infinity or fall inside of the horizon. Therefore the energy of the GWs will be dissipated and the amplitude will be damped. Since the imaginary part of the ω_{QNM} describes the effect of dissipation, $\mathrm{Im}[\omega_{QNM}]$ takes a negative number in general.

Here, we note the observational importance of the QNM. The merger of the two black holes can be decomposed into three phases, i.e., "inspiral", "merger", and "ringdown" phases. In the inspiral phase, the gravitational field is relatively weak and one can calculate the time evolution of the wave form of the GWs by linearized gravity or Post-Newtonian approach. In the merger phase, the gravitational field is extremely strong and one must include the effects of non-linearity to compute the time evolution of the GWs and take into account the tidal deformation of the neutron stars. According to the high non-linearity, in the merger phase, we need a numerical computation to predict the waveform precisely. After merger of the two black holes or neutron stars, they may form the one black hole. This black hole formed by coalescence is "ringing" for a while. The black hole loses its energy by the GWs emitted by this ringing and finally becomes stationary. This GWs emitted by ringing of a black hole is called the "ringdown GWs". It is known that the ringdown GWs are well described by the QNM of a black hole. The QNM of a black hole is determined only by the mass and the angular momentum of a black hole in general relativity. In contrast, the QNM of a black hole may depend on other parameters in modified gravity. Therefore one can test gravity by observations of the ringdown GWs. In addition to the test of gravity by ringdown GWs, we can use the consistency relation between the waveform of the GWs emitted in the inspiral,merger and ringdown phases [21]. We can determine the mass and the angular momentum of a black hole from the waveform of the inspiral, merger and ringdown phases. If the general relativity is correct, there exist some consistency relations between the three phases. In modified gravity, this consistency relation is different from that of general relativity. Therefore we can test gravity by combining the information from the waveforms in the three phases.

As mentioned above, QNM is damped exponentially in general. It is important for observation of the ringdown GWs to find the least damped mode. This is also called the fundamental mode. The fundamental mode is one of the QNMs which has the minimum value of $|\mathrm{Im}\,[\omega_{QNM}]\,|$. In the chapter 5, we will find the fundamental mode of QNMs in a specific scalar-tensor theory.

There are several methods for solving the master equation (3.2.7) and finding the QNMs such as the numerical calculation which is called "direct integration", Leaver's method [22], and WKB approximation [23, 24]. In the Table 3.2, we show the QNMs in Schwarzschild black hole derived by the Leaver's method in Ref.[22] See Ref.[25] for a review of finding the QNMs. Recent developments of solving the master equation can be found in Ref.[26].

n	$\ell = 2$ $(\mathrm{Re}\left[\omega_n^{QNM}\right], \mathrm{Im}\left[\omega_n^{QNM}\right])$	$\ell = 3$ $(\mathrm{Re}\left[\omega_n^{QNM}\right], \mathrm{Im}\left[\omega_n^{QNM}\right])$
1	$(0.747343, -0.177925)$	$(1.198887, -0.185406)$
2	$(0.693422, -0.547830)$	$(1.165288, -0.562596)$
3	$(0.602107, -0.956554)$	$(1.103370, -0.958186)$
4	$(0.503010, -1.410296)$	$(1.023924, -1.380674)$
5	$(0.415029, -1.893690)$	$(0.940348, -1.831299)$
6	$(0.338599, -2.391216)$	$(0.862773, -2.304303)$
7	$(0.266505, -2.895822)$	$(0.795319, -2.791824)$
8	$(0.185617, -3.407676)$	$(0.737985, -3.287689)$
9	$(0.000000, -3.998000)$	$(0.689237, -3.788066)$
10	$(0.126527, -4.605289)$	$(0.647366, -4.290798)$

Table 3.1: The QNMs for the Schwarzschild black hole [22]

Chapter 4

Black hole solutions in scalar-tensor theories

In general relativity, there exists the no-hair theorem of black holes. The no-hair theorem states that if the spacetime is stationary and axisymmetric, the Kerr metric is a unique vacuum solution, and black holes are determined by only two parameters, the mass and the angular momentum of the black hole [27]. This theorem implies that if we want to study the phenomenology of black holes in the universe, all we need is the Kerr solution. Furthermore, if we can find deviations from the Kerr solution by observation, we can test the theory of gravity. If we consider the modified gravity, black holes may depend on parameters other than the mass and the angular momentum. In particular, in scalar-tensor theories, it is important whether a black hole has scalar hair or not. We will review what kind of scalar-tensor theories can have scalar hair in the next section.

4.1 No-hair theorem for the shift-symmetric scalar-tensor theories

When a scalar-tensor theory have a shift-symmetry, $\phi \to \phi + const.$, there exists a strong theorem. We will review a theorem that if both of the spacetime and the scalar field have static and spherical symmetry, the scalar field cannot have a non-trivial configuration in some classes of the shift-symmetric Horndeski theories.

We consider the shift symmetric Horndeski theory whose Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \tag{4.1.1}$$

$$\mathcal{L}_2 = G_2(X), \tag{4.1.2}$$

$$\mathcal{L}_3 = -G_3(X)\Box\phi, \tag{4.1.3}$$

$$\mathcal{L}_4 = G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \tag{4.1.4}$$

$$\mathcal{L}_5 = G_5(X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right], \tag{4.1.5}$$

where G_2, G_3, G_4, G_5 are arbitrary functions of the kinetic term of the scalar field $X = -\nabla_\mu\phi\nabla^\mu\phi/2$. In Ref. [28], the no-hair theorem for the shift symmetric Horndeski theories has shown.

Theorem 3 *Shift-symmetric Horndeski theories satisfying the assumptions that*

1. *the spacetime is static, spherically symmetric and asymptotically flat,*
2. *the scalar field is also static and spherically symmetric,*
3. *the derivative of the scalar field vanishes at spatial infinity, i.e.,* $\nabla_\mu\phi \to 0$ *at* $r \to \infty$*,*
4. *the Lagrangian has a canonical kinetic term of the scalar field, i.e.,* $\mathcal{L} \supset X$*,*
5. *the norm of the Noether current due to the shift-symmetry is finite on and outside of the horizon,*
6. *the arbitrary functions* $G_i(X)$ *are analytic at* $X = 0$*,*

cannot have a non-trivial configuration of the scalar field.

The Proof of the no-hair theorem for shift symmetric Horndeski theory is as follows. The equation of motion for the scalar field can be written as a conservation law of a Noether current, J^μ, associated with the shift symmetry. Because of the static and spherical symmetry, the current J^μ has the only non-zero component J^r. From the assumption, $J^\mu J_\mu$ must be finite at the horizon. This implies J^μ must be vanish. Since J^μ is conserve, J^μ must vanish everywhere. J^μ can be written as

$$J^\mu = \phi' F[\phi'; g, g'] \tag{4.1.6}$$

with F is a regular function. ϕ'' is absent in J^μ because the Horndeski theory guarantees the equation of motion become up to second order. If the theory contains a canonical kinetic term of the scalar field, the function F approaches to some non-zero constant at spatial infinity which is correspond to $\phi' \to 0$ due to the assumption. Since J^μ must be vanish everywhere, ϕ' must be zero around $\phi' = 0$. This implies that ϕ has to be a constant everywhere and black hole cannot have the scalar hair. Note that if the theory have the scalar-Gauss-Bonnet term such as

$$S = M_{\rm Pl}^2 \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \alpha\phi\mathcal{R}_{\rm BG}^2\right), \tag{4.1.7}$$

with $\alpha = const$, and

$$\mathcal{R}_{\rm GB}^2 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2, \tag{4.1.8}$$

this theory breaks the assumption 6 of no-hair theorem and can have the scalar hair [29, 30].

4.2 Hairy black hole with linearly time dependent scalar field

In the previous section, we have seen that the shift symmetric scalar-tensor theories cannot have scalar hair if the scalar field is static. The authors of Ref. [31] consider the linearly time-dependent scalar field and find black hole solution with scalar hair. This breaks the assumption 2 of the no-hair theorem. Let us consider the action given by

$$S = \int d^4x\sqrt{-g}\left[\zeta R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda\right], \tag{4.2.1}$$

where R is the Einstein-Hilbert term, $G^{\mu\nu}$ is the Einstein tensor, Λ is a cosmological constant, and $\zeta > 0, \eta$, and β are constants. This theory is in a subclass of shift symmetric Horndeski theory. Varying this action with respect to the metric, we obtain

$$\begin{aligned} 0 = \mathcal{E}_{\mu\nu} \equiv \zeta G_{\mu\nu} - \eta\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right) + g_{\mu\nu}\Lambda \\ +\frac{\beta}{2}\left((\partial\phi)^2G_{\mu\nu} + 2P_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi\right. \\ \left.+g_{\mu\alpha}\delta^{\alpha\rho\sigma}_{\nu\gamma\delta}\nabla^\gamma\nabla_\rho\phi\nabla^\delta\nabla_\sigma\phi\right). \end{aligned} \tag{4.2.2}$$

Varying the action with respect to the scalar field gives

$$\nabla_\mu J^\mu = 0, \quad J^\mu = \left(\eta g^{\mu\nu} - \beta G^{\mu\nu}\right)\partial_\nu\phi. \tag{4.2.3}$$

Because of the shift symmetry, the equation of motion for the scalar field can be written as the conservation law of the Noether current. We assume that the spacetime is static and spherically symmetric, i.e.,

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{4.2.4}$$

and the scalar field depends on t and r, i.e.,

$$\phi = \phi(t, r). \tag{4.2.5}$$

In this ansatz, the non-zero components of J^μ are J^t and J^r given by, respectively,

$$J^t = \left(\eta g^{tt} - \beta G^{tt}\right) \dot{\phi}(t, r), \tag{4.2.6}$$

$$J^r = \left(\eta g^{rr} - \beta G^{rr}\right) \phi'(t, r), \tag{4.2.7}$$

where the dot denotes the derivative with respect to t, and the prime denotes the derivative with respect to r. From the ansatz (4.2.4) and (4.2.5), the tr component of Eq.(4.2.2) becomes

$$\mathcal{E}_{tr} = \frac{\beta\phi'}{r^2}\left(\frac{rfh'}{h} + \left(f - 1 - \frac{\eta r^2}{\beta}\right)\dot{\phi} - 2rf\dot{\phi}'\right) = 0. \tag{4.2.8}$$

If $\phi' \neq 0$, $\mathcal{E}_{tr} = 0$ gives

$$\phi(t, r) = \psi(r) + q_1(t)e^{X(r)}, \tag{4.2.9}$$

where

$$X(r) = \frac{1}{2}\int dr \left(\frac{1}{r} - \frac{1}{rf} - \frac{\eta r}{\beta f} + \frac{h'}{h}\right). \tag{4.2.10}$$

Note that this $X(r)$ satisfies the relation,

$$\eta g^{rr} - \beta G^{rr} = 2\beta\frac{f^2 X'}{r}. \tag{4.2.11}$$

Here, we will find the solution in the ansatz such that the metric satisfies

$$\eta g^{rr} - \beta G^{rr} = 0, \tag{4.2.12}$$

and the configuration of the scalar field is

$$\phi(t, r) = qt + \psi(r). \tag{4.2.13}$$

From the ansatz (4.2.12), we obtain

$$f = \frac{(\beta + \eta r^2)\, h}{\beta (rh)'}. \tag{4.2.14}$$

According to the ansatz (4.2.12) and Eq.(4.2.11), X becomes a constant, and the ansatz (4.2.13) is consistent with (4.2.12). One can easily verify that $\mathcal{E}_{tr} = 0$ and the equation of motion for the scalar field are automatically satisfied in these ansatz. Substituting (4.2.13) and (4.2.14) into the $\mathcal{E}_{rr} = 0$, we obtain

$$\psi' = \pm \frac{\sqrt{r}}{h\,(\beta + \eta r^2)} \left(q^2 \beta \left(\beta + \eta r^2\right) h' - \frac{\lambda}{2} \left(h^2 r^2\right)' \right)^{1/2} \tag{4.2.15}$$

where $\lambda = \zeta \eta + \beta \Lambda$. This equation gives the configuration of $\psi(r)$ in terms of $h(r)$. Substituting (4.2.13), (4.2.14) and (4.2.15) into $\mathcal{E}_{tt}$, and we write $h(r)$ as

$$h(r) = -\frac{\mu}{r} + \frac{1}{r} \int \frac{k(r)}{\beta + \eta r^2} dr, \tag{4.2.16}$$

where μ is a constant, $\mathcal{E}_{tt}$ gives the third order algebraic equation for $k(r)$

$$q^2 \beta \left(\beta + \eta r^2\right)^2 - \left(2\zeta\beta + (2\zeta\eta - \lambda) r^2\right) k + C_0 k^{3/2} = 0. \tag{4.2.17}$$

Here we have obtained the black hole solution with time-dependent scalar field which is given by (4.2.13), (4.2.15), (4.2.14), (4.2.16) and (4.2.17).

This fact shows that the shift-symmetric scalar-tensor theories can have the linearly time-dependent scalar-hair. After this solution was discovered, many people have studied on the hairy black hole solution and its perturbation in scalar-tensor theories [32, 33, 34]. We will investigate the black hole perturbations around this type of solution in higher-order scalar-tensor theories.

4.3 Stealth black hole solution in shift-symmetric cubic DHOST theory

In this section, we consider the shift-symmetric cubic DHOST theory whose action is given by (2.2.47) and find the stealth black hole solution. Due to the shift-symmetry $\phi \to \phi + const.$, the arbitrary functions in the action are functions of only X, i.e., $A_i = A_I(X)$ and $B_I = B_I(X)$. We consider static and spherically symmetric spacetime

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + r^2 C(r) \mathrm{d}\sigma^2, \tag{4.3.1}$$

where $\mathrm{d}\sigma^2 := \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$, and linearly time dependent scalar field

$$\phi(t, r) = \mu t + \psi(r), \tag{4.3.2}$$

where μ is constant. Here, we leave the function $C(r)$ in order to obtain all the independent equations. We will set $C(r) = 1$ after varying the action. The kinetic term of the scalar field is

$$X := -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = \frac{1}{2}\left(\frac{\mu^2}{A} - B\psi'^2\right). \tag{4.3.3}$$

We would like to find a stealth Schwarzschild solution defined by

$$A(r) = B(r) = 1 - \frac{r_h}{r}, \tag{4.3.4}$$

and

$$X = X_0 := \frac{\mu^2}{2}. \tag{4.3.5}$$

Varying the action with respect to A, B, C and ψ, and substituting (4.3.4) and (4.3.5), we find that the stealth Schwarzschild solution exists if the arbitrary functions satisfy the following equations [35]:

$$\begin{aligned}
&F_0(X_0) = 0, \quad F_{0X}(X_0) = 0, \quad F_{1X}(X_0) = 0 \\
&A_1(X_0) + A_2(X_0) = 0, \quad A_{1X}(X_0) + A_{2X}(X_0) = 0, \\
&B_2(X_0) = -\frac{1}{2}B_3(X_0) = 9B_1(X_0), \\
&B_4(X_0) + B_6(X_0) - B_{1X}(X_0) - B_{2X}(X_0) - \frac{5}{9}B_{3X}(X_0) \\
&= \frac{6}{X_0}B_1(X_0).
\end{aligned} \tag{4.3.6}$$

Note that these relations are compatible with the degeneracy conditions in the class ^{2}N-I + ^{3}M-I degenerate theories in the terminology of [36]. We will use this type of solution as background in the next chapter.

Chapter 5

Black hole perturbations with time dependent scalar hair in shift-symmetric scalar-tensor theories

The material in this chapter was originally published in "Perturbations and quasi-normal modes of black holes with time-dependent scalar hair in shift-symmetric scalar-tensor theories", Phys. Rev. D 103 (2021) 084041. [37]

5.1 Higher-Order Scalar-Tensor Theories

We consider the action (2.2.47) described in 2.2.3 and write it again for convenience. We consider a scalar-tensor theory given by [9, 38, 36]

$$
\begin{aligned}
S_{\text{grav}} = \int \mathrm{d}^4x\sqrt{-g}\Big[& F_0(X) + F_1(X)\Box\phi + F_2(X)R \\
& + \sum_{I=1}^{5} A_I(X)L_I^{(2)} + F_3(X)G_{\mu\nu}\phi^{\mu\nu} + \sum_{I=1}^{10} B_I(X)L_I^{(3)}\Big],
\end{aligned}
\tag{5.1.1}
$$

where $X := -\phi_\mu\phi^\mu/2$, $\phi_\mu := \nabla_\mu\phi$, $\phi_{\mu\nu} = \nabla_\nu\nabla_\mu\phi$, R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. Here, L^2 is consist of all possible terms which are quadratic in the second derivatives of the scalar field and given by

$$
\begin{aligned}
&L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}, \quad L_2^{(2)} = (\Box\phi)^2, \quad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu, \\
&L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu, \quad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2.
\end{aligned} \tag{5.1.2}
$$

Similarly, $L_I^{(3)}$ are cubic in the second derivatives of the scalar field and are given by

$$
\begin{aligned}
&L_1^{(3)} = (\Box\phi)^3, && L_2^{(3)} = (\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu}, \\
&L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho, && L_4^{(3)} = (\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu, \\
&L_5^{(3)} = \Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho, && L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma, \\
&L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma, && L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda, \\
&L_9^{(3)} = \Box\phi(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2, && L_{10}^{(3)} = (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3.
\end{aligned} \tag{5.1.3}
$$

The arbitrary functions F_0, F_1, F_2, F_3, A_I, and B_I depend only on X, because we impose that this theory has the shift symmetry, $\phi \to \phi + c$.

In this theory, the equation of motion for the metric and the scalar field has higher-order derivatives in general and the Ostrogradsky ghost may appear. The Ostrogradsky ghost can be removed by imposing the degeneracy conditions among the arbitrary functions F_2, F_3, A_I, and B_I [9, 38, 36]. However, one can relax the degeneracy conditions and construct healthy theories. An example, by requiring that the theory degenerates at least in the unitary gauge, the theory can be free from the Ostrogradsky ghost even in general gauges. This theory is called the U-degenerate theories [39]. Another example is given regarding this theory as an effective field theory. In this formalism, this theory can be regarded as some low-energy limits of a UV-complete theory, and one can detune the degeneracy conditions if the ghost will appear only at some high energy scale above a cutoff scale [40].

In this study, we do not impose any particular degeneracy conditions among the arbitrary functions. Nonetheless, we can derive the quadratic Lagrangian for the odd parity mode of the metric perturbations around the static and spherically symmetric background spacetime with a time-dependent scalar field.

5.2 Spherically Symmetric Background

In this section, we explain the background solutions. We consider static and spherically symmetric spacetime given by

$$
\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + r^2C(r)\mathrm{d}\sigma^2, \tag{5.2.1}
$$

where $\mathrm{d}\sigma^2 := \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$. Here we leave the function $C(r)$ to reproduce the $\theta\theta$ component of the equation of motions for the metric. We take $C(r) = 1$ after the variation with respect to $C(r)$ (see, e.g., [41]).

We assume that the scalar field depends linearly on the time,

$$\phi(t, r) = \mu t + \psi(r), \tag{5.2.2}$$

where μ is a constant. Without loss of generality, we assume that $\mu > 0$. This assumption is consistent with the static metric (5.2.1) because the action (5.1.1) depends on ϕ only through its derivatives due to the shift symmetry.

According to the ansatz (5.2.2), one can avoid the assumptions of the no-hair theorem of shift symmetric scalar-tensor theories [28]. Several black hole solutions with spherical symmetry have been found in the context of Horndeski theory [31, 42, 43, 44, 45] and beyond-Horndeski/DHOST theories [46, 47, 48, 35, 49]. Note that in [50, 51], the authors studied the effective field theory of black hole perturbations with a static and spherically symmetric scalar field. However, for the ansatz (5.2.2), one cannot use the effective field theory approach straightforwardly. Therefore, it is interesting to explore a general form of the effective action for black hole perturbations in the presence of the time-dependent scalar field.

Substituting the metric (5.2.1) and the ansatz of scalar field (5.2.2) into the action (5.1.1) and varying it with respect to A, B, C, and ψ, we can derive the background field equations. We write the resultant field equations as $\mathcal{E}_A = 0$, $\mathcal{E}_B = 0$, $\mathcal{E}_C = 0$, and $\mathcal{E}_\psi = 0$. In general, these equations are higher order since we do not impose the degeneracy conditions. Because we only use these background equations to reduce the form of the quadratic Lagrangian, the explicit forms of the background equations are not important. Also, for this reason, it is not important whether the background equations are higher order or not.

Here we show an example of background solutions. An interesting class of solutions often studied in the literature is a stealth Schwarzschild black hole with $X = X_0 = \text{const}$. In this case, the background metric and the kinetic term of the scalar field are given by

$$A = B = 1 - \frac{r_h}{r}, \quad X = X_0 = \frac{\mu^2}{2}. \tag{5.2.3}$$

As we saw in chapter 4.3, the stealth Schwarzschild solution exists if the arbitrary functions satisfy some relations (4.3.6). From $2X = \mu^2 = \mu^2/A - B(\mathrm{d}\psi/\mathrm{d}r)^2$ we have

$$\psi = \pm\mu\left[2\sqrt{r_h r} + r_h \ln\left(\frac{\sqrt{r}-\sqrt{r_h}}{\sqrt{r}+\sqrt{r_h}}\right)\right]. \tag{5.2.4}$$

We choose the "+" branch because we have $\phi \simeq \mu[t \pm r_h \ln(r/r_h - 1)] +$ const near the horizon and it is regular at the horizon only in the "+" branch, as is clear by expressing ϕ in terms of the ingoing null coordinate $v = t + r + r_h \ln(r/r_h - 1)$ [31].

5.3 Odd Parity Perturbations

5.3.1 Derivation of the Quadratic Lagrangian and the Effective Metric

We consider the odd parity mode of the metric perturbations,

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}, \tag{5.3.1}$$

where $\overline{g}_{\mu\nu}$ is the background metric (5.2.1) with $C(r) = 1$. The scalar field does not have an odd mode perturbation. Among the ten components, h_{ta}, h_{ra}, and h_{ab} are concerned with odd parity modes, where $a = \theta, \varphi$. Using the spherical harmonics $Y_{\ell m}(\theta, \varphi)$, we follow the standard procedure and expand the odd mode perturbations as

$$h_{t\theta} = -\frac{1}{\sin\theta}\partial_\varphi \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_0^{(\ell m)}(t,r) Y_{\ell m}(\theta,\varphi), \tag{5.3.2}$$

$$h_{t\varphi} = \sin\theta\partial_\theta \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_0^{(\ell m)}(t,r) Y_{\ell m}(\theta,\varphi), \tag{5.3.3}$$

$$h_{r\theta} = -\frac{1}{\sin\theta}\partial_\varphi \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_1^{(\ell m)}(t,r) Y_{\ell m}(\theta,\varphi), \tag{5.3.4}$$

$$h_{r\varphi} = \sin\theta\partial_\theta \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_1^{(\ell m)}(t,r) Y_{\ell m}(\theta,\varphi). \tag{5.3.5}$$

The odd parity part of h_{ab} can also be expressed using a single pseudo-scalar function, say h_2, but we adopt the Regge-Wheeler gauge in which $h_2 = 0$ and accordingly $h_{ab} = 0$.

We substitute Eqs. (5.3.2)–(5.3.5) into the action (5.1.1) and expand the action up to second order in perturbations. After performing the angular integrations, we get the general action

$$S_{\rm grav} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \int {\rm d}t{\rm d}r \mathcal{L}_{\ell m}^{(2)}, \tag{5.3.6}$$

and

$$\begin{aligned}\mathcal{L}_{\ell m}^{(2)} =& \frac{1}{2}\left\{\left[\frac{2}{r^2}(ra_3)' + a_1\right]|h_0|^2 + a_2|h_1|^2\right. \\ & \left. + a_3\left(|\dot{h}_1|^2 - 2\dot{h}_1^* h_0' + |h_0'|^2 + \frac{4}{r}\dot{h}_1^* h_0\right) + a_4 h_1^* h_0\right\} + \text{c.c.}.\end{aligned} \tag{5.3.7}$$

Here we omit the subscripts ℓm from h_0 and h_1. The coefficients are given by

$$\begin{aligned}a_1 =& \frac{c_\ell}{2r^2\sqrt{AB}}\left\{F_2 + \frac{\mu^2}{A}A_1 + \frac{B\psi' X'}{2}F_{3X} + \frac{\mu^2}{A\psi'}\left[\frac{2B(\psi')^2}{r} - \frac{(AX)'}{A}\right]B_2\right. \\ & \left. + \frac{3\mu^2 B\psi'}{rA}B_3 - \frac{\mu^2 B\psi' X'}{A}B_6\right\},\end{aligned} \tag{5.3.8}$$

$$\begin{aligned}a_2 =& -\frac{c_\ell}{2}\frac{\sqrt{AB}}{r^2}\left\{F_2 - B(\psi')^2 A_1 - \frac{B\psi' X'}{2}F_{3X} - B\psi'\left[\frac{2B(\psi')^2}{r} - \frac{(AX)'}{A}\right]B_2\right. \\ & \left. - \frac{3B^2(\psi')^3}{r}B_3 + B^2(\psi')^3 X' B_6\right\},\end{aligned} \tag{5.3.9}$$

$$\begin{aligned}a_3 =& \frac{\ell(\ell+1)}{2}\sqrt{\frac{B}{A}}\left\{F_2 + 2XA_1 - \frac{B\psi' X'}{2}F_{3X} + \frac{2X}{\psi'}\left[\frac{2B(\psi')^2}{r} - \frac{(AX)'}{A}\right]B_2\right. \\ & \left. + \frac{3X}{\psi'}\left[\frac{B(\psi')^2}{r} - X\frac{A'}{A} - \frac{\mu^2 X'}{2AX}\right]B_3 - 2B\psi' X X' B_6\right\},\end{aligned} \tag{5.3.10}$$

$$a_4 = -\frac{c_\ell}{r^2}\sqrt{\frac{B}{A}}\mu\left\{\psi' A_1 + \frac{X'}{2}F_{3X} + \left[\frac{2B(\psi')^2}{r} - \frac{(AX)'}{A}\right]B_2 + \frac{3B(\psi')^2}{r}B_3 - B(\psi')^2 X' B_6\right\}, \tag{5.3.11}$$

with $c_\ell = (\ell-1)\ell(\ell+1)(\ell+2)$. Here a dot means the derivative of t and a prime means the derivative of$r$. Following Ref. [52], it is convenient to write these coefficients as

$$\begin{aligned}a_1 &= \frac{c_\ell}{4r^2\sqrt{AB}}\mathcal{F}(r), \quad a_2 = -\frac{c_\ell}{4}\frac{\sqrt{AB}}{r^2}\mathcal{G}(r), \\ a_3 &= \frac{\ell(\ell+1)}{4}\sqrt{\frac{B}{A}}\mathcal{H}(r), \quad a_4 = \frac{c_\ell}{2r^2}\sqrt{\frac{B}{A}}\mathcal{J}(r),\end{aligned} \tag{5.3.12}$$

where $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$ have a dimension of (mass)^2. For the Schwarzschild solution in general relativity, we simply have $\mathcal{F} = \mathcal{G} = \mathcal{H} = M_{\rm Pl}^2 = (8\pi G)^{-1}$ and $\mathcal{J} = 0$.

We note that the quadratic Lagrangian (5.3.7) only contains the arbitrary functions F_2, F_3, A_1, B_2, B_3, and B_6. We have dropped the other terms from the

Lagrangian using the background equations. This is expected from the results of [53, 54], in which the quadratic Lagrangian of tensor mode around the cosmological background contains only these arbitrary functions. It is also noted that the quadratic Lagrangian (5.3.7) is obtained without imposing the degeneracy conditions. This is also not surprising because tensorial metric perturbations in the theory (5.1.1) obey second-order equations without regard to the degeneracy conditions. Therefore, our result can be used, for example, to U-degenerate theories [39] and detuned ("scordatura") DHOST theories [40].

Now we rewrite the Lagrangian (5.3.7) in terms of a single master variable. This can be done straightforwardly, following closely Refs. [32, 33, 52]. First, we introduce an auxiliary field $\chi = \chi^{(\ell m)}(t,r)$ and rewrite the Lagrangian (5.3.7) as

$$\begin{aligned}\mathcal{L}^{(2)}_{\ell m} = \frac{1}{2}\Big[& a_1|h_0|^2 + a_2|h_1|^2 + a_4 h_1^* h_0 \\ & + 2a_3\chi^*\left(-\frac{1}{2}\chi + \dot{h}_1 - h_0' + \frac{2}{r}h_0\right)\Big] + \text{c.c.}.\end{aligned} \tag{5.3.13}$$

One can easily confirm that this Lagrangian is equivalent to the original one. Variation with respect to h_0^* and h_1^* leads, respectively, to

$$a_1 h_0 + (a_3\chi)' + \frac{2a_3}{r}\chi + \frac{1}{2}a_4 h_1 = 0, \tag{5.3.14}$$

$$a_2 h_1 - a_3\dot{\chi} + \frac{1}{2}a_4 h_0 = 0, \tag{5.3.15}$$

which can be solved for h_0 and h_1 to express them in terms of χ, $\dot{\chi}$, and χ':

$$h_0 = -\frac{8a_2a_3(\chi/r) + 4a_2(a_3\chi)' + 2a_3a_4\dot{\chi}}{4a_1a_2 - a_4^2}, \tag{5.3.16}$$

$$h_1 = \frac{4a_3a_4(\chi/r) + 2a_4(a_3\chi)' + 2a_1a_3\dot{\chi}}{4a_1a_2 - a_4^2}. \tag{5.3.17}$$

(Here we assumed that $\mathcal{F}\mathcal{G} + (B/A)\mathcal{J}^2 \neq 0$.) Substituting Eqs. (5.3.16) and (5.3.17) back to Eq. (5.3.13), we obtain

$$\begin{aligned}\mathcal{L}^{(2)}_{\ell m} = \frac{\ell(\ell+1)r^2}{4(\ell-1)(\ell+2)}\sqrt{\frac{B}{A}}\Big\{ & b_1|\dot{\chi}|^2 - b_2|\chi'|^2 + b_3\dot{\chi}^*\chi' \\ & - \left[\frac{\ell(\ell+1)}{2}\frac{\mathcal{H}}{r^2} + \frac{V}{2}\right]|\chi|^2\Big\} + \text{c.c.},\end{aligned} \tag{5.3.18}$$

where

$$\begin{aligned} & b_1 = \frac{\mathcal{F}}{2A}\cdot\frac{A\mathcal{H}^2}{A\mathcal{F}\mathcal{G} + B\mathcal{J}^2}, \quad b_2 = \frac{\mathcal{G}B}{2}\cdot\frac{A\mathcal{H}^2}{A\mathcal{F}\mathcal{G} + B\mathcal{J}^2}, \\ & b_3 = \frac{B\mathcal{J}}{A}\cdot\frac{A\mathcal{H}^2}{A\mathcal{F}\mathcal{G} + B\mathcal{J}^2},\end{aligned} \tag{5.3.19}$$

and

$$V = 2\mathcal{H}\left[r^2 b_2\sqrt{\frac{B}{A}}\left(\frac{\sqrt{A/B}}{r^2\mathcal{H}}\right)'\right]' - \frac{2\mathcal{H}}{r^2}. \tag{5.3.20}$$

From this Lagrangian, the equation of motion is given by

$$\begin{aligned} & b_1\ddot{\chi} - \frac{\sqrt{A/B}}{r^2}\left(r^2\sqrt{\frac{B}{A}}b_2\chi'\right)' + \frac{b_3}{2}\dot{\chi}' \\ & + \frac{\sqrt{A/B}}{2r^2}\left(r^2\sqrt{\frac{B}{A}}b_3\dot{\chi}\right)' + \left[\frac{\ell(\ell+1)}{2}\frac{\mathcal{H}}{r^2} + \frac{V}{2}\right]\chi = 0. \end{aligned} \tag{5.3.21}$$

At this stage, it can be seen from the Lagrangian (5.3.18) that we need to impose

$$\mathcal{H} > 0, \tag{5.3.22}$$

as otherwise modes with large ℓ would have large negative energy and make the system unstable quickly.

One notices that Eq. (5.3.21) can be written in the form

$$\mathcal{H}\Omega^2 Z^{\mu\nu}D_\mu D_\nu\chi - V\chi = 0, \tag{5.3.23}$$

where $Z^{\mu\nu}$ is the inverse of the effective metric $Z_{\mu\nu}$ [55],

$$Z_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \Omega^2\left(-\frac{\mathcal{G}}{\mathcal{H}}A\mathrm{d}t^2 - \frac{2\mathcal{J}}{\mathcal{H}}\mathrm{d}t\mathrm{d}r + \frac{\mathcal{F}}{\mathcal{H}}\frac{\mathrm{d}r^2}{B} + r^2\mathrm{d}\sigma^2\right), \tag{5.3.24}$$

with

$$\Omega^2 := \frac{B}{A}\frac{\mathcal{H}^2}{\sqrt{\mathcal{F}\mathcal{G} + (B/A)\mathcal{J}^2}}, \tag{5.3.25}$$

and D_μ is the covariant derivative compatible with the effective metric $Z_{\mu\nu}$. Note here that the metric perturbations have already been expanded in terms of the spherical harmonics and hence the spherical Laplacian in $Z^{\mu\nu}D_\mu D_\nu$ must be replaced with its eigenvalue $-\ell(\ell+1)$. Note also that

$$\zeta^2(r) := \mathcal{F}\mathcal{G} + \frac{B}{A}\mathcal{J}^2 > 0 \tag{5.3.26}$$

must be imposed in order for the effective metric to be well-defined. It is easy to see that one has $Z_{\mu\nu} = M_{\rm Pl}^2\bar{g}_{\mu\nu}$ in general relativity, where $\mathcal{F} = \mathcal{G} = \mathcal{H} = M_{\rm Pl}^2$ and $\mathcal{J} = 0$. However, $Z_{\mu\nu}$ may not be proportional to $\bar{g}_{\mu\nu}$ in modified gravity. This fact itself has already been known in the context of the Horndeski theory [56, 57, 58].

We introduce a new time coordinate τ defined by

$$\mathrm{d}\tau = \mathrm{d}t + \frac{\mathcal{J}}{A\mathcal{G}}\mathrm{d}r. \tag{5.3.27}$$

Using τ, the effective metric (5.3.24) can be written in a diagonal form as

$$Z_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \Omega^2 \left(-\frac{\mathcal{G}}{\mathcal{H}} A\mathrm{d}\tau^2 + \frac{\zeta^2}{\mathcal{G}\mathcal{H}}\frac{\mathrm{d}r^2}{B} + r^2 \mathrm{d}\sigma^2 \right). \tag{5.3.28}$$

It is sometimes more convenient to work in the conformally related effective metric $\widetilde{Z}_{\mu\nu}$ defined as

$$\widetilde{Z}_{\mu\nu} = \Omega^{-2} Z_{\mu\nu}. \tag{5.3.29}$$

In the tilded frame, Eq. (5.3.23) is written as

$$\widetilde{Z}^{\mu\nu}\widetilde{D}_\mu \widetilde{D}_\nu \left(\frac{\widetilde{\chi}}{r} \right) - \left[\frac{V}{\mathcal{H}} + \frac{\widetilde{Z}^{\mu\nu}\widetilde{D}_\mu \widetilde{D}_\nu \Omega}{\Omega} \right] \frac{\widetilde{\chi}}{r} = 0, \tag{5.3.30}$$

where $\widetilde{\chi} := \Omega r \chi$ and $\widetilde{D}_\mu$ is the covariant derivative operator defined in terms of a connection compatible with $\widetilde{Z}_{\mu\nu}$.

Defining the generalized tortoise coordinate by

$$\mathrm{d}r_* = \frac{\zeta}{\mathcal{G}\sqrt{AB}}\mathrm{d}r, \tag{5.3.31}$$

Eq. (5.3.30) can further be rewritten in a more familiar form as

$$\left(-\partial_\tau^2 + \partial_{r_*}^2 - \widetilde{V} \right) \widetilde{\chi} = 0, \tag{5.3.32}$$

where

$$\widetilde{V} = \frac{\mathcal{G}A}{\mathcal{H}} \left\{ \frac{(\ell+2)(\ell-1)}{r^2} + \Omega r \left[\frac{\mathcal{G}\sqrt{AB}}{\zeta} \left(\frac{1}{r\zeta^{1/2}} \right)' \right]' \right\}. \tag{5.3.33}$$

This generalizes the Regge-Wheeler equation known in general relativity [59] to higher-order scalar-tensor theories. In Appendix B, we extend the main result of this section to include the energy-momentum tensor of matter and derive the generalized Regge-Wheeler equation with a matter source term

So far we have focused on the modes with $\ell \geq 2$. The dipole ($\ell = 1$) mode must be treated separately, but here we only comment that the dipole perturbation corresponds to adding a slow rotation, as has been already discussed in detail in the previous literature [60, 32, 33, 52].

5.3.2 Propagation Speed

In theories described by the action (5.1.1), the propagation speed of GWs differs in general from the speed of light. In light of the constraint from GW170817 [61, 62, 63], let us identify the subclass of scalar-tensor theories that admits a luminal speed of GWs at least at large r. This weak requirement was also employed in Ref. [51] (see, however, Refs. [64, 65]).

We assume that the background is given by

$$\begin{aligned} &A = 1 + \mathcal{O}(r^{-1}), \quad B = 1 + \mathcal{O}(r^{-1}), \\ &\psi' = \psi'_\infty + \mathcal{O}(r^{-1}), \end{aligned} \tag{5.3.34}$$

for large r, where ψ'_∞ is a constant. We then find

$$\mathcal{F} = 2\left[F_2(X_\infty) + \mu^2 A_1(X_\infty)\right] + \mathcal{O}(r^{-1}), \tag{5.3.35}$$

$$\mathcal{G} = 2\left[F_2(X_\infty) - (\psi'_\infty)^2 A_1(X_\infty)\right] + \mathcal{O}(r^{-1}), \tag{5.3.36}$$

$$\mathcal{H} = 2\left[F_2(X_\infty) + 2X_\infty A_1(X_\infty)\right] + \mathcal{O}(r^{-1}), \tag{5.3.37}$$

$$\mathcal{J} = -2\mu\psi'_\infty A_1(X_\infty) + \mathcal{O}(r^{-1}), \tag{5.3.38}$$

where $X_\infty := [\mu^2 - (\psi'_\infty)^2]/2$. Thus, if one has

$$A_1(X_\infty) = 0, \tag{5.3.39}$$

Eq. (5.3.32) reduces to $[-\partial_t^2 + \partial_r^2 - \ell(\ell+1)/r^2]\widetilde{\chi} \simeq 0$ for large r, rendering luminal propagation of GWs sufficiently away from a black hole. Note that F_3 and B_I appear only in the $\mathcal{O}(r^{-1})$ terms in Eqs. (5.3.35)–(5.3.38).

5.3.3 Horizons for Photons and Gravitons

Suppose that r_h is the location of the horizon in the metric $\bar{g}_{\mu\nu}$ and the metric components are expanded as

$$A(r) = \sum_{n=1} \alpha_n \epsilon^n, \quad B(r) = \sum_{n=1} \beta_n \epsilon^n, \tag{5.3.40}$$

where $\epsilon := r/r_h - 1 > 0$. We assume that X is regular at the horizon, so that X is of the form

$$X = X_h + \sum_{n=1} X_n \epsilon^n. \tag{5.3.41}$$

Accordingly, one has

$$\psi' = \frac{\mu}{\sqrt{\alpha_1\beta_1}}\frac{1}{\epsilon} + \sum_{n=0}\gamma_n\epsilon^n. \tag{5.3.42}$$

Note that ψ' diverges as $r \to r_h$, but this is not problematic. See the comment below Eq. (5.2.4). Substituting Eqs. (5.3.40)–(5.3.42) to Eqs. (5.3.8)–(5.3.11), we find, in the vicinity of the horizon,

$$\begin{aligned}
&\mathcal{F} = -\frac{d_0}{\epsilon} - d_1 + \mathcal{O}(\epsilon), \quad \mathcal{G} = \frac{d_0}{\epsilon} + d_2 + \mathcal{O}(\epsilon), \\
&\mathcal{H} = d_3 + \mathcal{O}(\epsilon), \quad \sqrt{\frac{B}{A}}\mathcal{J} = \frac{d_0}{\epsilon} + \frac{d_1+d_2}{2} + \mathcal{O}(\epsilon),
\end{aligned} \tag{5.3.43}$$

and hence $\zeta = \text{const} + \mathcal{O}(\epsilon)$, where

$$\begin{aligned}
d_0 = &-\frac{2\mu^2}{\alpha_1}A_1(X_h) + \frac{2\mu}{r_h}\sqrt{\frac{\beta_1}{\alpha_1^3}}\big[(\alpha_1 X_h - 2\mu^2)B_2(X_h) \\
&- 3\mu^2 B_3(X_h) + \mu^2 X_1 B_6(X_h)\big],
\end{aligned} \tag{5.3.44}$$

while the explicit expressions for d_1, d_2, and d_3 are more involved. Hereafter we will consider the case where d_0 is nonvanishing. Thus, at $r \simeq r_h$,

$$\Omega \simeq \text{const}, \quad \widetilde{Z}_{\tau\tau} \simeq \text{const}, \quad \widetilde{Z}_{rr} \simeq \text{const}, \tag{5.3.45}$$

which shows that nothing special happens in the effective metric at the horizon of the metric $\bar{g}_{\mu\nu}$. In particular, this fact implies that $r = r_h$ is not an appropriate place to impose the inner boundary conditions when solving the Regge-Wheeler equation (5.3.32). Rather, the form of the effective metric implies that a possible appropriate boundary will be $r = r_g$, where $\mathcal{G}(r_g) = 0$. To see this more explicitly, let us study some concrete examples.

The first example is given by the special case of the solution in Sec. 5.2, with $A_1(X_0) \neq 0$ and $B_1(X_0) = 0$. Essentially the same solution is also studied in Ref. [52]. This does not satisfy Eq. (5.3.39), but is a good illustrative example. We have

$$\mathcal{G} = 2F_2(X_0)\cdot\frac{1-r_g/r}{1-r_h/r}, \quad \mathcal{H} = 2F_2(X_0)(1+\mathcal{A}), \tag{5.3.46}$$

where

$$r_g := (1+\mathcal{A})r_h, \quad \mathcal{A} := \frac{2X_0 A_1(X_0)}{F_2(X_0)}, \tag{5.3.47}$$

and we assume that $F_2(X_0) > 0$ and $1 + \mathcal{A} > 0$. The conformal factor is a non-vanishing constant, $\Omega^2 = 2F_2(X_0)(1 + \mathcal{A})^{3/2}$, and the components of the (tilded) effective metric are given by

$$\widetilde{Z}_{\tau\tau} = -\frac{1 - r_g/r}{1 + \mathcal{A}}, \quad \widetilde{Z}_{rr} = \frac{1}{1 - r_g/r}, \tag{5.3.48}$$

which shows that the horizon of the effective metric is at $r = r_g$ ($\neq r_h$). In this case, the generalized tortoise coordinate is given by $r_* = (1 + \mathcal{A})^{1/2}[r + r_g \ln(r/r_g - 1)]$ and the potential in Eq. (5.3.32) reads

$$\widetilde{V} = \frac{1 - r_g/r}{1 + \mathcal{A}} \left[\frac{\ell(\ell+1)}{r^2} - \frac{3r_g}{r^3} \right]. \tag{5.3.49}$$

Aside from the constant factor of $(1 + \mathcal{A})^{-1}$, this coincides with the well-known potential in the Regge-Wheeler equation in general relativity with the horizon at $r = r_g$.

In this example, $\mathcal{G}$ is singular at $r = r_h$. One also notices that $\mathcal{G} < 0$ for $r_g < r < r_h$ if $\mathcal{A} < 0$. However, the effective metric and the potential do not depend on r_h explicitly and are free from any pathologies. In particular, the sign of $\mathcal{G}$ does not directly related to the stability of the solution. Indeed, it is now clear that the above solution is stable provided that $F_2(X_0) > 0$ and $1 + \mathcal{A} > 0$ are satisfied.

The second example is again the special case of the solution in Sec. 5.2, but now with $A_1(X_0) = 0$ and $B_1(X_0) \neq 0$. In this case, we have

$$\mathcal{G} = 2F_2(X_0) \cdot \frac{f(r)}{1 - r_h/r}, \quad \mathcal{H} = 2F_2(X_0), \tag{5.3.50}$$

where

$$f(r) = 1 - \frac{r_h}{r} + \mathcal{B} \left(\frac{r_h}{r} \right)^{5/2}, \quad \mathcal{B} := \frac{81}{2} \frac{\mu^3}{r_h} \frac{B_1(X_0)}{F_2(X_0)}. \tag{5.3.51}$$

The conformal factor is given by

$$\Omega^2 = \frac{2F_2(X_0)}{g^{1/2}(r)}, \tag{5.3.52}$$

and the (tilded) effective metric reduces to

$$\widetilde{Z}_{\tau\tau} = -f(r), \quad \widetilde{Z}_{rr} = \frac{g(r)}{f(r)}, \tag{5.3.53}$$

where

$$g(r) = 1 - \mathcal{B} \left(\frac{r_h}{r} \right)^{3/2}. \tag{5.3.54}$$

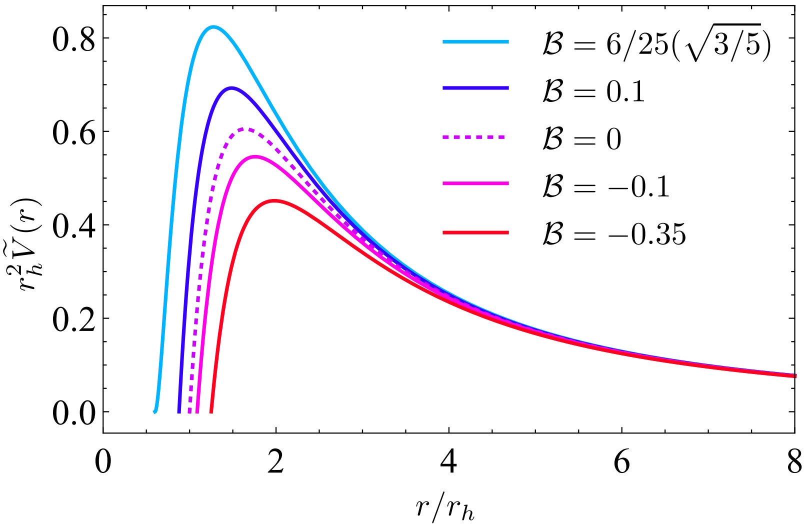

Figure 5.1: Potential $\widetilde{V}$ with $\ell = 2$ as a function of r/r_h.

We see that the horizon of the effective metric is at $r = r_g \neq r_h$, where $f(r_g) = 0$.

Let us investigate the structure of the effective metric (5.3.53) in more detail. For $\mathcal{B} > 6/25(\sqrt{3/5})$ ($\simeq 0.186$), f has no zeros, while $g = 0$ at $r = \mathcal{B}^{2/3} r_h$. We are not interested in this case. For $0 < \mathcal{B} \leq 6/25(\sqrt{3/5})$, we have $f = 0$ at $r = r_g < r_h$. In this case, g remains positive outside the horizon of the effective metric, but $g = 0$ occurs at $r = \mathcal{B}^{2/3} r_h < r_g$. Finally, for $\mathcal{B} < 0$, we have $f = 0$ at $r = r_g > r_h$ and g is always positive for $r > 0$. Therefore, in the latter two cases the solution has an outer horizon of the effective metric at $r = r_g$. It is straightforward to write the potential $\widetilde{V}$, but the expression is messy. The shape of the potential is shown for different values of $\mathcal{B}$ in Fig. 5.1. One can check that $r_* \to -\infty$ as $r \to r_g$.

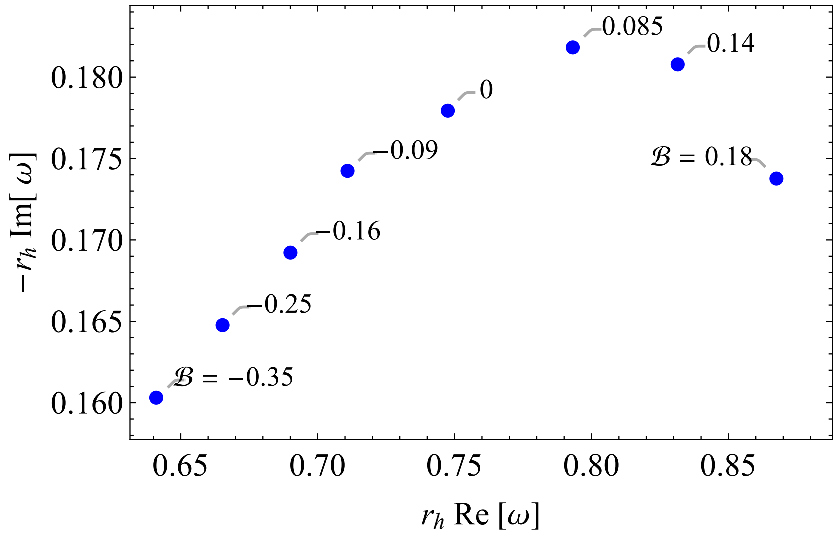

Figure 5.2: Lowest overtone quasi-normal frequencies for $\ell = 2$ and some representative values of $\mathcal{B}$.

5.4 Quasi-Normal Modes

In this section, we compute the QMNs for the second example of the previous section. First, we perform the Fourier transformation as

$$\tilde{\chi} = \int_{-\infty}^{\infty} \mathrm{d}\omega Q(r) e^{-i\omega\tau}, \tag{5.4.1}$$

and the master equation becomes

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}r_*^2} + \left[\omega^2 - \widetilde{V}(r)\right] Q = 0. \tag{5.4.2}$$

where the effective potential $\tilde{V}(r)$ is given by (5.3.33) and (5.3.50). We have to solve this equation with the boundary conditions which are given by

$$Q \propto \begin{cases} e^{-i\omega r_*} & r_* \to -\infty \quad (r \to r_g) \\ e^{+i\omega r_*} & r_* \to \infty \quad (r \to \infty). \end{cases} \tag{5.4.3}$$

Note that we should impose the inner boundary condition at the horizon for gravitons $r = r_g$ rather than for that of photons $r = r_h = 1$. We solve the master equation (5.4.2) by the direct integration method. In a direct integration method, we first expand the master variable as

$$Q = \begin{cases} Q_1(r) \equiv H(r)e^{-i\omega r_*} = \sum_{i=0}^{N} H_i(r-r_g)^i e^{-i\omega r_*} & r \to r_g \\ Q_2(r) \equiv G(r)e^{+i\omega r_*} = \sum_{i=0}^{N} G_i \frac{1}{r^i} e^{+i\omega r_*} & r \to \infty. \end{cases} \tag{5.4.4}$$

Then we substitute the series (5.4.4) into the master equation and obtain

$$a_0 + a_1(r-r_g) + a_2(r-r_g)^2 + \cdots = 0 \tag{5.4.5}$$

Here, a_i are written by the ω and H_i. For example, a_1 can be written by ω, H_0 and H_1, and a_2 can be written by ω, H_0, H_1 and H_2. We solve the (5.4.5) order by order in $r - r_g$. From the linear order, we can write the H_1 in terms of ω and H_0, i.e., $H_1 = H_1(\omega, H_0)$. Similarly, from the quadratic order, we can also write the H_2 in terms of ω, H_1 and H_0, i.e. $H_2 = H_2(\omega, H_1, H_0)$. Substitute the $H_1 = H_1(\omega, H_0)$ into the H_2, we can write the H_2 in terms of ω and H_0. Finally, we choose $H_0 = 1$ just for normalization. Then we can express the all H_i in terms of only the one variable, ω. Similarly, we can also express all G_i in terms of ω.

Next, we perform numerical integration. We use these two solutions for computing the boundary condition numerically. We solve the master equation numerically from the near horizon, $r = r_g + \epsilon$, to the some intermediate radius, $r = r_m$. For this purpose, we use the series solution, $Q_1(r)$, to determine the boundary condition at $r = r_g + \epsilon$ numerically. Using the solution $Q_1(r)$, we can calculate the numerical value of the solution at the near horizon, $Q_1(r_g+\epsilon)$, and its derivative, $\partial_r Q_1(r_g+\epsilon)$. Then we can solve the master equation numerically with these boundary conditions and obtain the solution purely ingoing at the near horizon. We perform a similar procedure for $Q_2(r)$. We use the series solution $Q_2(r)$ and its derivative $\partial_r Q_2(r)$ to find the numerical value at some sufficiently large radius, $r = r_{inf}$. We solve the master equation numerically with this boundary condition at $r = r_{inf}$ and obtain the solution purely outgoing at infinity. Now, we have two solutions. One is purely ingoing at the near horizon but not purely outgoing at infinity, and another is purely outgoing at infinity but not purely ingoing at the near horizon. For a general value of ω, these two solutions do not coincide. Therefore, they cannot satisfy the boundary conditions for QNMs. On the other hand, for some particular values of ω, these two solutions can be the same solution and satisfy the boundary condition for QNMs (5.4.3). In order to satisfy the boundary conditions for QNMs, we demand that these two solutions coincide each other. Hence we impose that the Wronskian becomes zero at some intermediate radius r_m, i.e.,

$$W\left[Q_1(r_m), Q_2(r_m)\right] \equiv \begin{vmatrix} Q_1(r_m) & Q_2(r_m) \\ \partial_r Q_1(r_m) & \partial_r Q_2(r_m) \end{vmatrix} = 0. \tag{5.4.6}$$

From this constraint, we can find the value of QNM frequencies.

We compute the lowest overtone quasinormal frequencies for $\ell = 2$, and the result is Fig. 5.2. This figure shows how the frequencies depend on the theory parameter $\mathcal{B}$. In the calculation, we choose some values, which are required to perform numerical computation, as $\epsilon = 10^{-4}, r_{inf} = 15r_g$ and $r_m = 2r_g$. We confirm that our results do not change even if we use other values of these parameters. The fundamental frequency at $\mathcal{B} = 0$ in this code, which corresponds to GR, agrees with the well-known fundamental frequency of GR to at least the first four digits. The Mathematica notebook we used to compute the QNMs can be available in [66].

Chapter 6

GWs propagating in an inhomogeneous universe

In this chapter, we study the propagation of GWs in spatially covariant theory. In particular, we investigate the propagation of GWs in an inhomogeneous universe using the effective metric for the gravitons. We put severe constraints on the coefficients in the spatially covariant theory.

6.1 Propagation of GWs in a homogeneous and isotropic universe

In this section, we review the propagation of GWs in the spatially covariant theory according to [12]. The action of spatially covariant theory is given by (2.3.1). From the observation of the coalescence of binary neutron stars, the propagation speed of GWs is constrained as [61, 63]

$$-3\times 10^{-15}\leq c_{\mathrm{T}}-1\leq 7\times 10^{-16}, \tag{6.1.1}$$

at the redshift $z\leq 0.009$ and the frequency $f\sim 10-100\mathrm{Hz}$. We can use this constraint to test the spatially covariant theory. For this purpose, we perform cosmological perturbations and investigate how GWs propagate in the universe. While the real universe has some inhomogeneity caused by the galaxies, we consider the propagation of GWs in a homogeneous and isotropic universe as a first approximation. Thus we use the metric of flat FLRW spacetime

$$\mathrm{d}s^2=-\mathrm{d}t^2+a^2(t)\left(\delta_{ij}+h_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j, \tag{6.1.2}$$

where h_{ij} is transverse-traceless tensor perturbations with $\delta^{ij}h_{ij} = 0$ and $\delta^{ij}\partial_i h_{jk} = 0$.

In order to derive the quadratic Lagrangian, we have to specify the concrete form of the Lagrangian. Here we classify terms by the order of differentiation and consider terms up to the fourth order of derivatives which are listed in Table 6.1.

d	(d_t, d_s)	Operators
0	(0,0)	1
1	(1,0)	K
	(0,1)	no terms
2	(2,0)	$K_{ij}K^{ij}, \quad K^2$
	(1,1)	no terms
	(0,2)	R
3	(3,0)	$K_{ij}K^{jk}K^i_k, \quad K_{ij}K^{ij}K, \quad K^3$
	(2,1)	$\varepsilon_{ijk}K^i_l\nabla^j K^{kl}$
	(1,2)	$\nabla^i\nabla^j K_{ij}, \quad \nabla^2 K, \quad R^{ij}K_{ij}, \quad RK$
	(0,3)	no terms
4	(4,0)	$K_{ij}K^{jk}K^i_kK, \quad \left(K_{ij}K^{ij}\right)^2, \quad K_{ij}K^{ij}K^2, \quad K^4$
	(3,1)	$\varepsilon_{ijk}\nabla_m K^i_n K^{jm}K^{kn}, \quad \varepsilon_{ijk}\nabla^i K^j_m K^k_n K^{mn}, \quad \varepsilon_{ijk}\nabla^i K^j_l K^{kl}K$
	(2,2)	$\nabla_k K_{ij}\nabla^k K^{ij}, \quad \nabla_i K_{jk}\nabla^k K^{ij}, \quad \nabla_i K^{ij}\nabla_k K^k_j, \quad \nabla_i K^{ij}\nabla_j K_{ij}$
		$\nabla_i K\nabla^i K, \quad R_{ij}K^i_k K^{jk}, \quad RK_{ij}K^{ij}, \quad R_{ij}K^{ij}K, \quad RK^2$
	(1,3)	$\varepsilon_{ijk}R^{il}\nabla^j K^k_l, \quad \varepsilon_{ijk}\nabla^i R^j_l K^{kl}$
	(0,4)	$\nabla^i\nabla^j R_{ij}, \quad \nabla^2 R, \quad R_{ij}R^{ij}, \quad R^2$

Table 6.1: The terms which can be contained in Lagrangian up to the fourth order of derivatives [12]. d means an order of derivatives in each term. d_t and d_s are time and spatial derivatives respectively.

Then we write the action as

$$S = \int \mathrm{d}t\ \mathrm{d}^3x N\sqrt{\gamma}\left(L^{(0)} + L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}\right). \tag{6.1.3}$$

Here, $L^{(d)}$ means the total order of derivatives where $d = d_t + d_s$ with d_t the order of time derivatives and d_s the order of spatial derivatives. We consider all possible terms up to the fourth order of derivatives such as

$$L^{(0)} = c_1^{(0,0)}, \tag{6.1.4}$$

$$L^{(1)} = c_1^{(1,0)}K, \tag{6.1.5}$$

and

$$L^{(2)} = c_1^{(2,0)}K_{ij}K^{ij} + c_2^{(2,0)}K^2 + c_1^{(0,2)}R, \tag{6.1.6}$$

etc. The coefficients c_i are arbitrary functions of the time and the lapse

$$c_1^{(0,0)} = c_1^{(0,0)}(t, N). \tag{6.1.7}$$

Substituting the metric (6.1.2) into the action, we obtain the quadratic action for the tensor perturbation

$$\begin{aligned} S_2 = \int \mathrm{d}t\ \mathrm{d}^3x \frac{a^3}{2} \Bigg(& \mathcal{G}_0(t)\dot{h}_{ij}\dot{h}^{ij} + \mathcal{G}_1(t)\epsilon^{ijk}\dot{h}_{li}\frac{1}{a}\partial_j\dot{h}^l_k \\ & - \mathcal{G}_2(t)\dot{h}_{ij}\frac{\Delta}{a^2}\dot{h}^{ij} + \mathcal{W}_0(t)h_{ij}\frac{\Delta}{a^2}h^{ij} \\ & + \mathcal{W}_1(t)\epsilon^{ijk}h_{li}\frac{1}{a}\frac{\Delta}{a^2}\partial_j h^l_k - \mathcal{W}_2(t)h_{ij}\frac{\Delta^2}{a^4}h^{ij}\Bigg), \end{aligned} \tag{6.1.8}$$

where $\mathcal{G}_i$ and $\mathcal{W}_i$ are given by

$$\begin{aligned} \mathcal{G}_0(t) =& \frac{1}{2}\left[c_1^{(2,0)} + 3\left(c_1^{(3,0)} + c_2^{(3,0)}\right)H + 3\left(3c_1^{(4,0)} + 2c_2^{(4,0)} + 3c_3^{(4,0)}\right)H^2\right], \\ \mathcal{G}_1(t) =& \frac{1}{2}\left[c_1^{(2,1)} - \left(c_1^{(3,1)} - 2c_2^{(3,1)} - 3c_3^{(3,1)}\right)H\right], \\ \mathcal{G}_2(t) =& \frac{1}{2}c_1^{(2,2)}, \\ \mathcal{W}_0(t) =& \frac{1}{4}\Big[2c_1^{(0,2)} + \partial_t c_3^{(1,2)} + \left(3c_3^{(1,2)} + 6c_4^{(1,2)} + 2\partial_t c_6^{(2,2)} + 3\partial_t c_8^{(2,2)}\right)H \\ & + \left(4c_6^{(2,2)} + 6c_7^{(2,2)} + 9c_8^{(2,2)} + 18c_9^{(2,2)}\right)H^2 + \left(2c_6^{(2,2)} + 3c_8^{(2,2)}\right)\dot{H}\Big], \\ \mathcal{W}_1(t) =& \frac{1}{4}\partial_t\left(c_1^{(1,3)} + c_2^{(1,3)}\right), \\ \mathcal{W}_2(t) =& -\frac{1}{2}c_3^{(0,4)}, \end{aligned} \tag{6.1.9}$$

with $\Delta = \delta^{ij}\partial_i\partial_j$. We define the Fourier component of $h_{ij}(t,\boldsymbol{x})$ as

$$h_{ij}(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{s=\pm2} h^{(s)}(t,\boldsymbol{k})e^{(s)}_{ij}(\hat{\boldsymbol{k}})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{6.1.10}$$

with $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$. Here $e^{(s)}_{ij}(\hat{\boldsymbol{k}})$ is the polarization tensor satisfying

$$\delta^{ij}e^{(s)}_{ij}(\hat{\boldsymbol{k}}) = k^i e^{(s)}_{ij}(\hat{\boldsymbol{k}}) = 0, \tag{6.1.11}$$

and

$$e^{(s)*}_{ij}(\hat{\boldsymbol{k}}) = e^{(-s)}_{ij}(\hat{\boldsymbol{k}}) = e^{(s)}_{ij}(-\hat{\boldsymbol{k}}), \tag{6.1.12}$$

where the asterisk is the complex conjugate. We suppose that the two polarization tensors are normalized by

$$e^{(s)}_{ij}(\hat{\boldsymbol{k}})e^{(-s')ij}(\hat{\boldsymbol{k}}) = \delta^{ss'}. \tag{6.1.13}$$

Using these equations, the quadratic action in Fourier space is given by

$$S_2 = \int \mathrm{d}\tau \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{a^2}{2} \sum_{s=\pm 2} \mathcal{G}^{(s)}(\tau, k) \Bigg(\partial_\tau h^{(s)}(\tau, \boldsymbol{k}) \partial_\tau h^{(s)}(\tau, -\boldsymbol{k}), \\ - k^2 \frac{\mathcal{W}^{(s)}(\tau, k)}{\mathcal{G}^{(s)}(\tau, k)} h^{(s)}(\tau, \boldsymbol{k}) h^{(s)}(\tau, -\boldsymbol{k}) \Bigg). \tag{6.1.14}$$

Here we use the conformal time τ defined by $\mathrm{d}t = a\mathrm{d}\tau$ and $\mathcal{G}^{(s)}(\tau, k)$ and $\mathcal{W}^{(s)}(\tau, k)$ are given by

$$\mathcal{G}^{(s)}(\tau, k) = \sum_{n=0} \mathcal{G}_n(\tau) \left(\frac{s}{2} \frac{k}{a} \right)^n \tag{6.1.15}$$

$$\mathcal{W}^{(s)}(\tau, k) = \sum_{n=0} \mathcal{W}_n(\tau) \left(\frac{s}{2} \frac{k}{a} \right)^n . \tag{6.1.16}$$

From the (6.1.14), the propagation speed of two tensor modes are given by

$$\left(c_{\mathrm{T}}^{(s)} \right)^2 \equiv \frac{\mathcal{W}^{(s)}}{\mathcal{G}^{(s)}} = \frac{\mathcal{W}_0(t) + \mathcal{W}_1(t) \frac{s}{2} \frac{k}{a} + \mathcal{W}_2(t) \frac{k^2}{a^2}}{\mathcal{G}_0(t) + \mathcal{G}_1(t) \frac{s}{2} \frac{k}{a} + \mathcal{G}_2(t) \frac{k^2}{a^2}}. \tag{6.1.17}$$

From this equation, in order for both polarization modes of GWs to propagate with the speed of light, we require

$$\mathcal{W}_i = \mathcal{G}_i, \quad (i = 0, 1, 2.), \tag{6.1.18}$$

with any value of $H(t)$. From these requirements, we obtain 7 constraints

$$\begin{gathered} c_1^{(2,0)} - c_1^{(0,2)} - \frac{1}{2} \partial_t c_3^{(1,2)} = 0, \\ 6c_1^{(3,0)} + 6c_2^{(3,0)} - 3c_3^{(1,2)} - 6c_4^{(1,2)} - 2\partial_t c_6^{(2,2)} - 3\partial_t c_8^{(2,2)} = 0, \\ 18c_1^{(4,0)} + 12c_2^{(4,0)} + 18c_3^{(4,0)} - 4c_6^{(2,2)} - 6c_7^{(2,2)} - 9c_8^{(2,2)} - 18c_9^{(2,2)} = 0, \\ 2c_6^{(2,2)} + 3c_8^{(2,2)} = 0, \\ c_1^{(2,1)} - \frac{1}{2} \partial_t \left(c_1^{(1,3)} + c_2^{(1,3)} \right) = 0, \\ c_1^{(3,1)} - 2c_2^{(3,1)} - 3c_3^{(3,1)} = 0, \\ c_1^{(2,2)} + c_3^{(0,4)} = 0, \end{gathered} \tag{6.1.19}$$

and by solving these constraints, we can remove 7 coefficients:

$$
\begin{aligned}
c_1^{(0,2)} &= c_1^{(2,0)} - \frac{1}{2}\partial_t c_3^{(1,2)}, \\
c_1^{(2,1)} &= \frac{1}{2}\partial_t \left(c_1^{(1,3)} + c_2^{(1,3)}\right), \\
c_4^{(1,2)} &= c_1^{(3,0)} + c_2^{(3,0)} - \frac{1}{2}c_3^{(1,2)}, \\
c_3^{(3,1)} &= \frac{1}{3}\left(c_1^{(3,1)} - 2c_2^{(3,1)}\right), \\
c_8^{(2,2)} &= -\frac{2}{3}c_6^{(2,2)}, \\
c_9^{(2,2)} &= \frac{1}{9}\left(9c_1^{(4,0)} + 6c_2^{(4,0)} + 9c_3^{(4,0)} + c_6^{(2,2)} - 3c_7^{(2,2)}\right), \\
c_3^{(0,4)} &= -c_1^{(2,2)}.
\end{aligned}
\tag{6.1.20}
$$

Note that while there are 21 arbitrary coefficients in the original action (6.1.3), we have 21-7=14 coefficients after imposing the constraints. Substituting the constraints (6.1.20) into the original action (6.1.3), we obtain the theory with $c_T = 1$ which is given by

$$
S_{c_{\mathrm{T}=1}} = \int \mathrm{d}t \ \mathrm{d}^3x N\sqrt{\gamma}\left(L^{(0)} + L^{(1)} + \tilde{L}^{(2)} + \tilde{L}^{(3)} + \tilde{L}^{(4)}\right), \tag{6.1.21}
$$

with

$$
\tilde{L}^{(2)} = c_1^{(2,0)}\left(K_{ij}K^{ij} + R\right) + c_2^{(2,0)}K^2, \tag{6.1.22}
$$

$$
\begin{aligned}
\tilde{L}^{(3)} =& c_1^{(3,0)}\left(K_{ij}K^{jk}K^i_k + RK\right) + c_2^{(3,0)}\left(K_{ij}K^{ij} + R\right)K, \\
&+ c_3^{(3,0)}K^3 + c_1^{(1,2)}\nabla^i\nabla^j K_{ij} + c_2^{(1,2)}\nabla^2 K, \\
&+ c_3^{(1,2)}G^{ij}K_{ij} - \frac{1}{2N}\partial_t c_3^{(1,2)}R,
\end{aligned}
\tag{6.1.23}
$$

and

$$
\begin{aligned}
\tilde{L}^{(4)} =& c_1^{(4,0)} \left(K_{ij}K^{jk}K^i_k + RK\right) K + c_2^{(4,0)} \left(\left(K_{ij}K^{ij}\right)^2 + \frac{2}{3}RK^2\right) \\
&+ c_3^{(4,0)} \left(K_{ij}K^{ij} + R\right) K^2 + c_4^{(4,0)} K^4, \\
&+ c_1^{(3,1)} \varepsilon_{ijk} \left(\nabla_m K^i_n K^{jm} K^{kn} + \frac{1}{3}\nabla^i K^j_l K^{kl} K\right) \\
&+ c_2^{(3,1)} \varepsilon_{ijk} \left(\nabla^i K^j_m K^k_n K^{mn} - \frac{2}{3}\nabla^i K^j_l K^{kl} K\right), \\
&+ c_1^{(2,2)} \left(\nabla_k K_{ij} \nabla^k K^{ij} - R_{ij}R^{ij}\right) + c_2^{(2,2)} \nabla_i K_{jk} \nabla^k K^{ij} \\
&+ c_3^{(2,2)} \nabla_i K^{ij} \nabla_k K^k_j + c_4^{(2,2)} \nabla_i K^{ij} \nabla_j K, + c_5^{(2,2)} \nabla_i K \nabla^i K \\
&+ c_6^{(2,2)} R_{ij} \left(K^i_k K^{jk} - \frac{2}{3}K^{ij}K + \frac{1}{9}h^{ij}K^2\right) + c_7^{(2,2)} R \left(K_{ij}K^{ij} - \frac{1}{3}K^2\right), \\
&+ c_1^{(1,3)} \varepsilon_{ijk} R^{il} \nabla^j K^k_l + c_2^{(1,3)} \varepsilon_{ijk} \nabla^i R^j_l K^{kl} + \frac{1}{2N}\partial_t \left(c_1^{(1,3)} + c_2^{(1,3)}\right) \varepsilon_{ijk} K^i_l \nabla^j K^{kl}, \\
&+ c_1^{(0,4)} \nabla^i \nabla^j R_{ij} + c_2^{(0,4)} \nabla^2 R + c_4^{(0,4)} R^2.
\end{aligned}
\tag{6.1.24}
$$

In the next subsection, we use the subclass of this theory to study GWs propagating in an inhomogeneous universe.

6.2 Propagation of GWs in an inhomogeneous universe

In this section, we study the propagation of GWs in an inhomogeneous universe. In the previous subsection, we obtain the subclass of spatially covariant theory (6.1.21) with $c_T = 1$ in a homogeneous and isotropic universe. Although the constraint $c_T = 1$ in a homogeneous and isotropic universe is efficient to restrict several coefficients, we can further restrict the theory (6.1.21) by taking into account the inhomogeneities of the universe.

The effect of inhomogeneities on the propagation speed of GWs was also studied in [67, 68]. In [67], the authors proposed a theory that can evade the constraints on the propagation speed of GWs. In this theory, the difference between the speeds of GWs and light is proportional to the equation of motion of the scalar field, which allows evading the constraint. However, if we take into account inhomogeneities on the scale of ~ 100Mpc, it affects the speed of GWs and the difference becomes $|c_T - 1| \sim 10^{-3}$. This completely contradicts the observational constraint (6.1.1).

Therefore this theory is severely constrained by considering the effect of inhomogeneities on the propagation of GWs. In [68], the authors considered the effective field theory of dark energy and proposed a broader class of theories than in the previous work, in which the speed of GWs equals the speed of light. However, inhomogeneities affect the speed of GWs and all of these theories are severely constrained as well as the previous study. These studies show that investigating the propagation of GWs in an inhomogeneous universe enables us to severely constrain theories of gravity. In the following, we will consider how the propagation speed of GWs is affected by inhomogeneities and will put the constraint on the spatially covariant theory proposed in [12].

We consider the action given by

$$S = \int \mathrm{d}t\mathrm{d}x^3 N\sqrt{\gamma}\left(L^{(2)} + L^{(3)} + L^{(4)}\right), \tag{6.2.1}$$

$$L^{(2)} = c_1^{(2,0)}\left(K_{ij}K^{ij} + R\right), \tag{6.2.2}$$

$$L^{(3)} = c_1^{(3,0)}\left(K^i_j K^j_l K^l_i + RK\right) + c_2^{(3,0)}\left(K_{ij}K^{ij} + R\right)K, \tag{6.2.3}$$

$$L^{(4)} = c_1^{(4,0)}\left(K^i_j K^j_l K^l_i + RK\right)K + c_2^{(4,0)}\left(\left(K_{ij}K^{ij}\right)^2 + \frac{2}{3}RK^2\right) + c_3^{(4,0)}\left(K_{ij}K^{ij} + R\right)K^2. \tag{6.2.4}$$

This theory is the subclass of the theory (6.1.21). This action consists of only the terms which affect the propagation of GWs in a homogeneous and isotropic universe.

Now we study the propagation of GWs in a slightly inhomogeneous universe. In this case, we take into account inhomogeneity as the perturbation around the homogeneous universe. We consider the perturbations in comoving gauge. The metric is given by,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right), \tag{6.2.5}$$

where

$$N = 1 + \alpha, \quad N_i = \partial_i\chi, \quad \gamma_{ij} = e^{2\zeta}\left(e^h\right)_{ij} = e^{2\zeta}\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h^k_j + \cdots\right). \tag{6.2.6}$$

Here, α, χ and ζ are the scalar perturbations and h_{ij} is the tensor perturbation with $\delta^{ij}h_{ij} = 0$ and $\delta^{ij}\partial_i h_{jk} = 0$. We expand the action up to (scalar) $\times$ (tensor) $\times$ (tensor) order since the effects of the inhomogeneity in propagation appear in this order. We denote (scalar) × (tensor) × (tensor) order as shh just for simplicity. Note that we can recognize this metric as

$$\begin{aligned} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= \left(\overline{g}_{\mu\nu} + h_{\mu\nu}\right)\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\left(1+2\alpha\right)\mathrm{d}t^2 + 2\partial_i\chi \mathrm{d}t\mathrm{d}x^i + a^2\left(1+2\zeta\right)\left(\delta_{ij}+h_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j. \end{aligned} \tag{6.2.7}$$

Here, $\overline{g}_{\mu\nu}$ is background metric which contains only the scalar perturbations, and $h_{\mu\nu}$ is the tensor perturbations. In order to study the propagation of GWs, we expand the action up to quadratic order of h_{ij} using this metric. Since the background metric has linear terms of the scalar perturbations, the quadratic action for h_{ij} becomes the order of shh.

After the expansion, the Lagrangian becomes

$$S_{shh} = \int \mathrm{d}t\mathrm{d}x^3 \left(L^{(2)}_{shh} + L^{(3)}_{shh} + L^{(4)}_{shh}\right). \tag{6.2.8}$$

Here $L^{(2)}_{shh}$, $L^{(3)}_{shh}$ and $L^{(4)}_{shh}$ are given as follows.

$$L^{(2)}_{shh} = \frac{a^3}{4}c_1^{(2,0)}e^{\delta\Omega_1^{(2,0)}}\left[\left(1-\alpha+3\zeta\right)\dot{h}_{ij}^2 - 2a^{-2}\partial_i\chi\dot{h}_{jk}\partial_i h_{jk} - a^{-2}\left(1+\alpha+\zeta\right)\left(\partial_k h_{ij}\right)^2\right], \tag{6.2.9}$$

with

$$\delta\Omega_1^{(2,0)} \equiv \frac{c_{1,N}^{(2,0)}}{c_1^{(2,0)}}\alpha, \quad c_{1,N}^{(2,0)} \equiv \frac{\partial c_1^{(2,0)}}{\partial N}. \tag{6.2.10}$$

$$L^{(3)}_{shh} = \sum_{A=1,2} L^{(3)}_{Ashh}, \tag{6.2.11}$$

with

$$\begin{aligned} L^{(3)}_{1shh} =& \frac{3}{4}a^3Hc_1^{(3,0)}e^{\delta\Omega_1^{(3,0)}}\left[\left(1-\alpha+3\zeta\right)\dot{h}_{ij}^2 - 2a^{-2}\partial_i\chi\dot{h}_{jk}\partial_i h_{jk} - a^{-2}\left(1+\alpha+\zeta\right)\left(\partial_k h_{ij}\right)^2\right] \\ &- \frac{3}{4}ac_1^{(3,0)}\tilde{\chi}_{ij}\dot{h}_{ik}\dot{h}_{jk}, \end{aligned} \tag{6.2.12}$$

and

$$L^{(3)}_{2shh} = \frac{3}{4}a^3Hc_2^{(3,0)}e^{\delta\Omega_2^{(3,0)}}\left[\left(1-\alpha+3\zeta\right)\dot{h}_{ij}^2 - 2a^{-2}\partial_i\chi\dot{h}_{jk}\partial_i h_{jk} - a^{-2}\left(1+\alpha+\zeta\right)\left(\partial_k h_{ij}\right)^2\right], \tag{6.2.13}$$

with

$$\tilde{\chi}_{ij} := \partial_i \partial_j \chi - \frac{1}{3} \partial^2 \chi \delta_{ij}, \tag{6.2.14}$$

$$\delta\Omega_A^{(3,0)} := -\alpha + \frac{c_{A,N}^{(3,0)}}{c_A^{(3,0)}} \alpha - \frac{\dot{\zeta}}{H} + \frac{\partial^2 \chi}{3a^2 H}. \tag{6.2.15}$$

$$L_{shh}^{(4)} = \sum_{A=1,2,3} L_{Ashh}^{(4)}, \tag{6.2.16}$$

with

$$\begin{aligned} L_{1shh}^{(4)} =& \frac{9}{4} a^3 H c_1^{(4,0)} e^{\delta\Omega_1^{(4,0)}} \left[(1 - \alpha + 3\zeta) \dot{h}_{ij}^2 - 2a^{-2} \partial_i \chi \dot{h}_{jk} \partial_i h_{jk} - a^{-2} (1 + \alpha + \zeta) (\partial_k h_{ij})^2 \right] \\ & - \frac{9}{4} a c_1^{(4,0)} \tilde{\chi}_{ij} \dot{h}_{ik} \dot{h}_{jk}, \end{aligned} \tag{6.2.17}$$

and

$$L_{2shh}^{(4)} = \frac{3}{2} a^3 H c_2^{(4,0)} e^{\delta\Omega_2^{(4,0)}} \left[(1 - \alpha + 3\zeta) \dot{h}_{ij}^2 - 2a^{-2} \partial_i \chi \dot{h}_{jk} \partial_i h_{jk} - a^{-2} (1 + \alpha + \zeta) (\partial_k h_{ij})^2 \right], \tag{6.2.18}$$

and

$$L_{3shh}^{(4)} = \frac{9}{4} a^3 H c_3^{(4,0)} e^{\delta\Omega_3^{(4,0)}} \left[(1 - \alpha + 3\zeta) \dot{h}_{ij}^2 - 2a^{-2} \partial_i \chi \dot{h}_{jk} \partial_i h_{jk} - a^{-2} (1 + \alpha + \zeta) (\partial_k h_{ij})^2 \right] \tag{6.2.19}$$

with

$$\delta\Omega_A^{(4,0)} := -2\alpha + \frac{c_{A,N}^{(4,0)}}{c_A^{(4,0)}} \alpha + \frac{2\dot{\zeta}}{H} - \frac{2\partial^2 \chi}{3a^2 H}. \tag{6.2.20}$$

Now we will define the effective metric for the tensor modes and consider their propagation. Temporarily, we focus on (6.2.9) for simplicity. Using the components of the background metric

$$\sqrt{-\overline{g}}\,\overline{g}^{00} = -a^3 (1 - \alpha + 3\zeta), \quad \sqrt{-\overline{g}}\,\overline{g}^{0i} = a \partial_i \chi, \quad \sqrt{-\overline{g}}\,\overline{g}^{ij} = a (1 + \alpha + \zeta) \delta^{ij}, \tag{6.2.21}$$

we can rewrite this Lagrangian as

$$L_{shh}^{(2)} = \frac{1}{4} c_1^{(2,0)} e^{\delta\Omega_1^{(2,0)}} \overline{g}^{\mu\nu} \partial_\mu h_{ij} \partial_\nu h_{ij} =: Z^{\mu\nu} \partial_\mu h_{ij} \partial_\nu h_{ij}. \tag{6.2.22}$$

Here we define the effective metric $Z_{\mu\nu}$ which determines the propagation of GWs. For this Lagrangian, the effective metric is given by

$$Z^{\mu\nu} = \frac{1}{4} c_1^{(2,0)} e^{\delta\Omega_1^{(2,0)}} \overline{g}^{\mu\nu}. \tag{6.2.23}$$

Therefore the effective metric is conformal to the background metric. This result shows that the terms contained in $L^{(2)}$ do not change the propagation speed of GWs even in an inhomogeneous universe. We can perform a similar calculation for $L^{(3)}$ and $L^{(4)}$. For the terms with coefficients $c_2^{(3,0)}, c_2^{(4,0)}$ and $c_3^{4,0}$, the effective metric become also conformal to the background metric. Hence we find that these terms do not change the speed of GWs in an inhomogeneous universe.

On the other hand, we find that the terms with coefficients $c_1^{(3,0)}$ and $c_1^{(4,0)}$ affect the speed of GWs in an inhomogeneous universe and the speed will be different from that of light. The first line of (6.2.12) and (6.2.17) is conformal to the background metric. However, because of the second line of (6.2.12) and (6.2.17), the effective metric is not conformal to the background metric. As a result, we can very strictly restrict these coefficients $c_1^{(3,0)}$ and $c_1^{(4,0)}$ to evade the observational constraint of GW170817/GRB170817A (6.1.1) even in an inhomogeneous universe. We will continue a similar investigation for the terms which contain higher spatial derivatives.

Chapter 7

Conclusions

In this thesis, we have studied black hole perturbations and cosmological perturbations in modified theories of gravity. In particular, we have studied black hole perturbations in shift symmetric higher-order scalar-tensor theories. Also, we have investigated the stability of the background black hole solution and computed the quasinormal mode of a black hole. Finally, we have studied cosmological perturbation in an inhomogeneous universe in spatially covariant theories.

In chapter 2, we have reviewed the modified theories of gravity. According to Ostrogradsky's theorem, the equations of motion have to be up to a second-order differential equation. Horndeski theory is the most general scalar-tensor theory whose equations of motion become up to second order. DHOST theory contains higher derivative terms in the Lagrangian but its equations of motion become up to second order by imposing the degeneracy condition.

In chapter 3, we have reviewed the formalism of black hole perturbations in general relativity. Black hole perturbations can be decomposed into odd and even parity perturbations which describe the odd and even parity mode of GWs. We have derived the master equations for odd mode, the Regge-Wheeler equation, and for even mode, the Zerilli equation. We have explained that we can test gravity from the ringdown GWs which are characterized by quasinormal mode of black holes.

In chapter 4, we have explained black hole solutions in shift-symmetric scalar-tensor theories. The no-hair theorem implies that most subclass of Horndeski theories cannot have scalar hair. However, if the scalar field depends on time, black holes can have scalar hair.

In chapter 5, we have studied odd parity perturbations of black holes with linearly time-dependent scalar hair in shift-symmetric scalar-tensor theories. Although we do not impose the degeneracy conditions, the resultant quadratic Lagrangian for the odd parity mode of metric perturbations does not contain higher-order derivatives and avoids the Ostrogradsky ghost. Then, we have derived the generalized Regge-Wheeler equations and have defined the effective metric which determines the causal structure of the odd parity mode of GWs. We have defined the horizon for gravitons from the component of the effective metric and have found that the ordinary horizon (for photons) is nothing special for gravitons. Finally, we have considered two concrete examples and have computed QNMs for the second example. We have found that the signal of the discrepancy between these two horizons can be observed from the quasinormal frequencies.

In chapter 6, we have studied the effect of inhomogeneities on the speed of propagation of GWs. We reviewed the spatially covariant theory of gravity, which can contain a wide class of theories even though the theories have the symmetry of the four-dimensional diffeomorphism, e.g., the Horndeski theory. We also reviewed the propagation of GWs in a homogeneous and isotropic universe in spatially covariant theory. We required that the propagation speed of GWs is equal to that of light in a homogeneous and isotropic universe, and obtained the constraints on some coefficients in the theory. We have studied how GWs propagate in an inhomogeneous universe in this obtained theory. We have shown that some coefficients in this theory change the propagation speed of GWs. In order to be consistent with the observation GW170817/GRB170817A, these coefficients must be very small with respect to other coefficients. Thus we have put some severe constraints on the spatially covariant theory.

Acknowledgements

I would like to show my greatest appreciation to my supervisor Tsutomu Kobayashi. I have learned his deep insight into physics and his attitude toward research. I am grateful to Tomohiro Harada, Takashi Hiramatsu, Masashi Kimura, Yasuyuki Hatsuda, and Yu Nakayama for fruitful discussions and comments. I also thank Takuya Katagiri, Shingo Akama, Aya Iyonaga, Yosuke Mishima, Tact Ikeda, Takuma Sato, Tomoaki Murata, Jin Saito, Toshiki Takadera, Akihito Katsumata, Tomo Tanaka and Hiroaki Tahara for discussions, conversations, and encouragement. Finally, I would like to express my deepest gratitude to my parents, Yumiko and Masaru, and my grandmother, Sadako for their dedicated support. This work was supported by the Rikkyo University Special Fund for Research (SFR).

Appendix A

Generality of the Quadratic Lagrangian

Starting from the action (5.1.1), we have shown in the main text that the quadratic Lagrangian for the odd parity mode is given by Eq. (5.3.7). Actually, one can show that more general scalar-tensor theories lead to the quadratic Lagrangian for the odd parity modes having the same structure as Eq. (5.3.7) as long as the equation of motion for gravitational-wave degrees of freedom remains of second order.

For example, one may add to the action (5.1.1)

$$\widetilde{F}_3(X)R\Box\phi, \tag{A.0.1}$$

to consider a general derivative coupling of the form $F_3 G_{\mu\nu}\phi^{\mu\nu} + \widetilde{F}_3 R\Box\phi = F_3 R_{\mu\nu}\phi^{\mu\nu} + (\widetilde{F}_3 - F_3/2)R\Box\phi$. This only shifts the coefficients as

$$\mathcal{F},\mathcal{G},\mathcal{H} \to \mathcal{F},\mathcal{G},\mathcal{H} + \left[\frac{B\psi'}{r} - \frac{(AX)'}{A\psi'}\right]\widetilde{F}_3, \tag{A.0.2}$$

$$\mathcal{J} \to \mathcal{J}, \tag{A.0.3}$$

and does not give rise to any new terms in Eq. (5.3.7).

Similarly, one may also add terms quartic in second derivatives of ϕ such as

$$C_1(X)\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho\lambda}\phi^{\lambda\mu}, \quad C_2(X)(\Box\phi)^4, \quad \cdots . \tag{A.0.4}$$

One can verify by direct computation that such quartic terms merely shift the coefficients without altering the structure of the Lagrangian (5.3.7) or have no contribution to the odd parity sector.

We thus conclude that the form of the Lagrangian (5.3.7) is generic to scalar-tensor theories in which gravitational-wave degrees of freedom obey a second-order equation of motion.

Appendix B

Sourced Regge-Wheeler Equation

In this appendix, we generalize our main result to include the source term, which has not been considered in the previous similar studies [60, 32, 33, 52]. Assuming that matter is minimally coupled to gravity, the source term can be obtained from

$$S_{\text{source}} = \frac{1}{2} \int \mathrm{d}^4x \sqrt{-\bar{g}} h^{\mu\nu} T_{\mu\nu}, \tag{B.0.1}$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor. Similarly to the metric perturbations, the odd parity part of the energy momentum tensor can also be expanded as

$$T_{t\theta} = -\frac{1}{\sin\theta}\partial_\varphi \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} S_0^{(\ell m)}(t,r)Y_{\ell m}, \tag{B.0.2}$$

$$T_{t\varphi} = \sin\theta\partial_\theta \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} S_0^{(\ell m)}(t,r)Y_{\ell m}, \tag{B.0.3}$$

$$T_{r\theta} = -\frac{1}{\sin\theta}\partial_\varphi \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} S_1^{(\ell m)}(t,r)Y_{\ell m}, \tag{B.0.4}$$

$$T_{r\varphi} = \sin\theta\partial_\theta \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} S_1^{(\ell m)}(t,r)Y_{\ell m}, \tag{B.0.5}$$

$$T_{\theta\theta} = \frac{2}{\sin\theta}\left(\partial_\theta\partial_\varphi - \cot\theta\partial_\varphi\right) \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} S_2^{(\ell m)}(t,r)Y_{\ell m}, \tag{B.0.6}$$

$$\begin{aligned} T_{\theta\varphi} = &\left(\frac{1}{\sin\theta}\partial_\varphi^2 + \cos\theta\partial_\theta - \sin\theta\partial_\theta^2\right) \\ &\times \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} S_2^{(\ell m)}(t,r)Y_{\ell m}, \end{aligned} \tag{B.0.7}$$

$$T_{\varphi\varphi} = -2\sin\theta\left(\partial_\theta\partial_\varphi - \cot\theta\partial_\varphi\right) \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} S_2^{(\ell m)}(t,r)Y_{\ell m}. \tag{B.0.8}$$

The conservation of the matter energy-momentum tensor, $\nabla_\nu T^{\mu\nu} = 0$, yields

$$\begin{aligned} &-\frac{\dot{S}_0^{(\ell m)}}{A} + \frac{\sqrt{B/A}}{r^2}\left(r^2\sqrt{AB}S_1^{(\ell m)}\right)' \\ &+ \frac{(\ell-1)(\ell+2)}{r^2}S_2^{(\ell m)} = 0. \end{aligned} \tag{B.0.9}$$

It is straightforward to perform the angular integrations in Eq. (B.0.1) to obtain

$$\begin{aligned} S_{\text{source}} = &-\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} \frac{\ell(\ell+1)}{2} \\ &\times \int \mathrm{d}t\mathrm{d}r\left(\frac{h_0^* S_0}{\sqrt{AB}} - \sqrt{AB}h_1^* S_1 + \text{c.c.}\right), \end{aligned} \tag{B.0.10}$$

where we omitted the labels (ℓm) from S_0 and S_1. This is the source action for the odd mode perturbations (see also Ref. [51]). We add the above source action to the gravitational part of the action (5.3.6). Then, Eqs. (5.3.14) and (5.3.15) are generalized to

$$a_1 h_0 + (a_3\chi)' + \frac{2a_3}{r}\chi + \frac{1}{2}a_4 h_1 = \frac{\ell(\ell+1)}{2\sqrt{AB}}S_0, \tag{B.0.11}$$

$$a_2 h_1 - a_3\dot{\chi} + \frac{1}{2}a_4 h_0 = -\frac{\ell(\ell+1)}{2}\sqrt{AB}S_1, \tag{B.0.12}$$

Solving these equations for h_0 and h_1 and removing h_0 and h_1 from the quadratic Lagrangian, we see that the Lagrangian (5.3.18) is generalized to include the source as

$$\mathcal{L}^{(2)}_{\ell m,\mathrm{total}} = \mathcal{L}^{(2)}_{\ell m} - \frac{\ell(\ell+1)r^2}{4(\ell-1)(\ell+2)}\sqrt{\frac{B}{A}}\left(\chi^* S_{\mathrm{odd}} + \mathrm{c.c.}\right), \tag{B.0.13}$$

where $\mathcal{L}^{(2)}_{\ell m}$ in the right-hand side is the same Lagrangian as the one defined as Eq. (5.3.18) and

$$\begin{aligned} S^{(\ell m)}_{\mathrm{odd}}(t,r) := 2\mathcal{H}\left(\frac{\mathcal{G}}{\zeta^2}S^{(\ell m)}_0\right)' - \frac{2\mathcal{F}\mathcal{H}}{\zeta^2}\dot{S}^{(\ell m)}_1 \\ - \frac{2\mathcal{H}\mathcal{J}}{A\zeta^2}\dot{S}^{(\ell m)}_0 - 2\mathcal{H}\left(\frac{B\mathcal{J}}{\zeta^2}S^{(\ell m)}_1\right)'. \end{aligned} \tag{B.0.14}$$

Now Eq. (5.3.23) with the source term reads

$$\mathcal{H}\Omega^2 Z^{\mu\nu} D_\mu D_\nu \chi - V\chi = S_{\mathrm{odd}}, \tag{B.0.15}$$

and, accordingly, Eq. (5.3.32) with the source term is given by

$$\left(-\partial_\tau^2 + \partial_{r_*}^2 - \widetilde{V}\right)\widetilde{\chi} = \frac{\mathcal{G}r\sqrt{AB}}{\mathcal{H}\zeta^{1/2}}S_{\mathrm{odd}}. \tag{B.0.16}$$

This is the generalization of the sourced Regge-Wheeler equation.

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