The MDP Procedure Revisited: Is It Possible to Attain Non-Samuelsonian Pareto Optima?

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Abstract. The purpose of this paper is twofold. The first is to extend the Generalized MDP Procedure presented in Fujigaki and Sato(1981) to an economy with many public goods. The second is to propose a version of the MDP Procedures satisfying the *Generalized Optimality Condition* propounded by Campbell and Truchon(1988), which includes the familiar Samuelson Condition as a special case and is valid for all boundary optima in an economy with public goods and one private good. I show that the procedures presented in this paper can attain an open subset of Pareto optima which the original MDP Procedure fails to attain and where the Samuelson Condition does not hold. It is shown that all of the Pareto optima, including boundary *Non-Samuelsonian Pareto optima,* can be reached via the *ν MDP Procedures*. The issues of incentives and normative properties are examined. With Aggregate Correct Revelation, it is possible to characterize the *ν MDP Procedures*. There is in the class of the *ν MDP Procedures* a member whose solutions converge to the core. Finally, three figures are presented to show the trajectories of the *ν MDP Procedures* to reach Pareto optima, including boundary ones.

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Key Words: The MDP Procedure, The ν MDP Procedures, Generalized Optimality Condition, Non-Samuelsonian Pareto Optima.

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1. INTRODUCTION

1.1. For many years since the appearance of the seminal paper by Samuelson(1954), there prevailed a gloomy pessimism that the free rider problem was inevitable in the provision of public goods1[1]. This skeptical unanimity, however, was alleviated by the accumulated literature on the incentive compatible planning procedures for providing public goods.

The last three decades have witnessed numerous attempts to resolve the free rider problem or the problem of incentives by designing iterative planning procedures to efficiently supply public goods. Typically, these procedures involve asking participants to provide information on their preferences to a planning centre in charge of allocating resources among individual agents.

The incentive problem associated with planning procedures to supply public goods may be summarized as follows : the agents might have an incentive to purposely misstate their private information about their preferences in the hope of distorting to their advantage the outcome that process yields. The fundamental problem is how to elicit the unobservable information that is necessary to implement the planning rules.

The idea of employing game-theoretic concepts in solving the incentive problem associated with planning paths of the procedure was first formally introduced into the literature by Drèze and de la Vallée Poussin(1971). Malinvaud (1970-71) introduced planning-theoretic concepts into the incentive problem with public goods. They showed that their procedures converge monotonically to an individually rational Pareto optimum, and that true revelation of preferences for public goods is a minimax strategy for each individual. The processes established by three pioneers have become one of the most important contributions in planning theory with public goods, and in public economics. They have come to be termed the *MDP Procedure,* and subsequently spawned numerous papers with fruitful results. The existence of solutions to the MDP Procedure was proved by Henry(1972).

As for modelling incentives of the players, however, a minimax strategy is weaker and less attractive than the Nash strategy, Roberts(1979) and Henry(1979) worked with Nash equilibria for local games associated with solution paths of the MDP Process, by substituting myopic Nash behaviour for minimax behaviour at its each iteration. Roberts showed that the resulting path converges to an individually rational Pareto optimum even under strategic revelation. As well, Henry refined Roberts' results on the incentive properties in the MDP Procedure by restricting individuals to report nonnegative messages.

Subsequently, Fujigaki and Sato(1981) and (1982) demonstrated that there exists a "satisfactory" planning procedure generalizing the MDP Process, which assures that truth-telling is a dominant strategy for each player. They showed that the path arising from local incentive games converges monotonically to the unique individually rational Pareto optimal allocation. They also proved that any quantity-guided continuous planning procedure satisfying certain axioms is characterized by the very procedure they established. The distributional implications of this characterization were also deduced.

The design of the planning procedures with public goods might be said to have fully developed to reach the acme in 1983. Initiated by three great pioneers—Malinvaud (1970-71), and Drèze and de la Vallée Poussin(1971)—this field of research made remarkable progress in these three decades. The analysis of incentives in planning procedures began in late sixties and was mathematically refined by the characterization theorems of Champsaur and Rochet(1983), theorems that furthermore generalized the previous results of Fujigaki and Sato(1981) and (1982), as well as Laffont and Maskin (1983) who also treated the coalition incentive compatibility.

Most of these procedures can be characterized by the set of axioms:

(i) Feasibility

(ii) Monotonicity

(iii) Pareto Efficiency

(iv) Local Strategy Proofness

Formal definitions of these properties are given in Section 3.

1.2. Samuelson's optimality condition for public goods, first propounded in 1954, has acquired universal familiarity. This condition, however, implicitly assumed interior optima and ignored boundary ones. Campbell and Truchon(1988) generalized the optimality condition which holds for all boundary optima, and thereby showed the existence of an open subset of Pareto optima where the Samuelson condition is not satisfied.

My objective in this paper is to show that a slight modification of the MDP Procedure delivers the property that every Pareto optimal allocation, not just interior ones, is the limit point of the Procedure for some choice of parameters. We do this by using the Campbell-Truchon theorem on the characterization of Pareto optimal allocations in public good economies.

Five sections follow. Section 2 presents the model and introduces the generalized optimality condition(GO). Section 3 describes the procedures based on GO, and Section 4 confirms that the normative conditions are fulfilled by them: i.e., feasibility, monotonicity, stability, neutrality, and incentive properties pertaining to minimax and Nash strategies. It is verified that all of the Pareto optima including boundary ones can be reached by a choice of a vector of weights attached to each individual's marginal rate of substitution calculated by the agents. Existence and stability of the solutions are demonstrated for these procedures. Then the procedures are generalized, and their incentive properties are examined. A proof of the main theorem is given in Section 5. Finally, concluding remarks follow. Three figures are presented in the Appendix to show some trajectories of the procedures to reach Pareto optima.

2. MODEL

2.1. *Notation*

Consider an economy with many public goods and one private good whose quantities are denoted by x^k, $k=1,..,K$, and y, respectively. Let $\boldsymbol{K}$ be the set of public goods and $\boldsymbol{N}=\{1,\ldots,N\}$ be the set of individuals. $H_i \subset \boldsymbol{R}_+^{K+1}$ denotes the consumption set of consumer i. I assume that the preferences of every agent i is numerically represented by some real valued utility function $u_i: \boldsymbol{R}_+^{\mathrm{K}+1} \to \boldsymbol{R}$. Assumptions on H_i and u_i are to follow in the next subsection. Suppose that the initial endowment of every agent comprizes the private good only: $w_i>0$ denotes the quantity of the pivate good that agent i possesses at the outset. I represent the production technology by the cost function, $g:\boldsymbol{R}_+^{\boldsymbol{K}} \to \boldsymbol{R}_+$. That is, for eveny vector $x \in \boldsymbol{R}_+^K$ of public goods, g(x) denotes the minimum amount of the private good for producing x. It is assumed as usual that there is no production of private good.

I assume that the planning centre possesses the production unit so that it has a complete knowledge on the cost function g. Therefore, the centre can compute the marginal cost $\gamma^k(x)$ represented as:

$$\gamma^k(x)=\partial g(x)/\partial x^k,\ \ \forall k \in \boldsymbol{K}.$$

The centre asks each individual i to report his/her marginal rate of substitution (MRS) between each public good and the private numéraire:

$$\pi_i^k(x, y_i) = \frac{\partial u_i(x, y_i)/\partial x^k}{\partial u_i(x, y_i)/\partial y_i}, \quad \forall k \in \boldsymbol{K}.$$

Different from the usual arguments, our analysis throughout this paper does not bypass the possibility of boundary problem, which should not be avoided, since the initial allocation $z_0 = (0, \omega_1, .., \omega_N)$, including only private endowments belongs to the boundary of $\boldsymbol{R}_+^{K+N}$. An assumption to avoid this problem is an innocuous one in the single public good case, because its quantity must be always increasing. For several public goods, however, some difficulties arise[2]. This paper deals with them in the context of planning procedures.

2.2. *Assumptions*

Let me give some assumptions.

Assumption 1. For any $i \in \boldsymbol{N}$, $H_i = \{(x, y_i) \in \boldsymbol{R}_+^{K+1} \mid x \geqq 0,\ y_i \geqq \beta_i\}$.

Assumption 2. For any $i \in \boldsymbol{N}$, $u_i(\cdot, \cdot)$ is strictly quasiconcave and at least twice continuously differentiable.

Assumption 3. For any $i \in \boldsymbol{N}$, $u_i^k(x, y_i) \equiv \partial u_i(x, y_i)/\partial x^k \geqq 0$, and $u_i^y(x, y_i) \equiv \partial u_i(x, y_i)/\partial y_i > 0$, for all (x, y_i).

Assumption 4. For any $i \in \boldsymbol{N}$, $\partial u_i(x, 0)/\partial x^k = 0$ for all $x \in \boldsymbol{R}_+^K$ and for any $k \in \boldsymbol{K}$.

Assumption 5. $\beta \leqq \omega$ and $\beta \neq \omega$, where $\beta = (\beta_1, .., \beta_N)$ and $\omega = (\omega_1, .., \omega_N)$.

Assumption 6. $g(x)$ is convex and twice continuously differentiable.

Assumption 7. For any $k \in \boldsymbol{K}$, there exists x^k in $\boldsymbol{R}_+^K$ such that $\partial g(x)/\partial x^k \neq 0$.

Remark 1. Assumption 1 is the same as Assumption 2′ in Drèze and de la Vallée Poussin (1971), where they considered β_i to be equal to the "otherwise given income" of individual *i*. Campbell and Truchon (1988) employed the same consumption set as in Assumption 1, where they regard β_i as the fixed lower bound, which may be zero, some positive subsistance level, or a negative number to allow for the possibility of borrowing and lending. They themselves acknowledged that they derived their characterization theorem at the cost of making these restictive assumptions. However, Conley and Diamantaras (1996) removed those restrictive assumptions to give a characterization of Pareto optimal allocations in general public good economics. The original MDP Process ruled out convergence to boundary optima by making the assumption of no bankrupty. This paper proposes a variant of the MDP Process which can also converge to boundary optima. The issue of incentives and normative properties are considered. β_i may be called an individual boundary point assigned beforehand by the planner who knows each ω_i. Note that in the original MDP Procedure the planning centre has only to know $\omega \equiv \sum_i \omega_i$, not each ω_i.

2.3. *Definitions*

Following difinitions are used.

Definition 1. An allocation z is feasible if and only if

$$z \in \boldsymbol{Z} = \{(z, \; y_1,.., y_N) \in \boldsymbol{R}_+^{K+N} \mid \sum_i y_i + g(x) = \sum_i \omega_i\}.$$

Definition 2. An allocation z is individually rational if and only if

$$(\forall i \in \boldsymbol{N})[u_i(x, y_i) \geqq u_i(0, \; \omega_i)].$$

Definition 3. A Pareto optimum for this economy is an allocation $z^* \in \boldsymbol{Z}$ such that there exists no feasible allocation z with

$$(\forall i \in \boldsymbol{N})[u_i(x, y_i) \geqq u_i(x^*, y_i^*)]$$
$$(\exists j \in \boldsymbol{N})[u_j(x, y_j) > u_j(x^*, y_j^*)].$$

2.4. *The Generalized Optimality Condition*

Campbell and Truchon(1988) showed under the above assumptions that for each Pareto optimal allocation $z^* \in \boldsymbol{Z}$ there exist $\lambda \in \boldsymbol{R}_+^N \backslash \{0\}$, Lagrangian multipliers $\mu \in \boldsymbol{R}_+^N$, and $\mu_0 \in \boldsymbol{R}_+^N$ such that z^* is a solution of the following nonlinear program:

$$Max \sum_i \lambda_i u_i(x, y_i)$$

subject to

$$\sum_i (\omega_i - y_i) \geqq g(x)$$
$$y_i \geqq \beta_i, \forall i \in N. \tag{1}$$

Eq.(1) is an additional constraint. The first order optimality conditions of this optimization problem imply that any $z \in Z$ maximizes $L(z) = \sum_i \lambda_i u_i(x, y_i)$ if and only if z satisfies the equations:

$$\lambda_i u_i^y - \mu_0 + \mu_i = 0 \quad \text{and} \quad \mu_i (y_i - \beta_i) = 0, \ \forall i \in \boldsymbol{N} \tag{2}$$

$$\sum_i \lambda_i u_i^k - \mu_0 \gamma^k \leqq 0 \quad \text{and} \quad \left(\sum_i \lambda_i u_i^k - \mu_0 \gamma^k\right) x^k = 0, \ \forall k \in \boldsymbol{K}. \tag{3}$$

Eq.(2) must hold for the private good, and (3) has to be valid for all public goods. Following Campbell and Truchon(1988), if one could set $\nu_i = \lambda_i u_i^y / \mu_0$, one can easily derive

$$\nu_i = 1 - \mu_i / \mu_0, \ \ \forall i \in \boldsymbol{N}.$$

If an interior solution is assumed, one observes that $\nu_i = 1, \forall i \in \boldsymbol{N}$. Whereas, if a boundary solution is also taken into consideration, ν_i can vary in the interval $[0, 1]$. The above discussions together give us a generalized condition for all types of Pareto optima in our economy. Campbell and Truchon(1988) verified under Assumptions 1～7 that a necessary and sufficient condition for a generalized Pareto optimality, which includes the Samuelson condition as a special one.

Condition* GO. *Generalized Optimality:

There is some $\nu \in \boldsymbol{R}^N \setminus \{0\}$ such that $\nu \leqq (1, \ldots, 1)$

$$\sum_i \nu_i \pi_i^k \leqq \gamma^k \text{ and } \left(\sum_i \nu_i \pi_i^k - \gamma^k\right) x^k = 0, \ \forall k \in \boldsymbol{K}$$

$$(1 - \nu_i)(y_i - \beta_i) = 0, \ \forall i \in \boldsymbol{N}.$$

In terms of Condition **GO**, one may distinguish two kinds of Pareto optima; namely, Samuelsonian and Non-Samuelsonian. Here I introduce some definitions relevant for my analysis.

Definition 4. An allocation is *interior* if $y_i > \beta_i, \forall i \in \boldsymbol{N}$ and $x^k > 0, \forall k \in \boldsymbol{K}$.

Definition 5. An allocation is on the *boundary* if there exists at least $i \in \boldsymbol{N}$ such that $y_i = \beta_i$ and/or $x^k = 0$ for at least one $k \in \boldsymbol{K}$.

Definition 6. A *Samuelsonian Pareto Optimum* is an allocation such that $\nu_i = 1, \forall i \in \boldsymbol{N}$, holds in Condition **GO**.

Definition 7. A *Non-Samuelsonian Pareto Optimum* is an allocation where there exists $\nu_i \neq 1$ for some $i \in \boldsymbol{N}$ in Condition **GO**.

Our procedures presented below, can attain both Samuelsonian and Non-Samuelsonian Pareto optima. Let **P**, $\mathbf{P}_0$, and **B** be the sets of Pareto, individually rational

Pareto, and boundary optima, respectively. I assume $\mathbf{P}_0 \cap \mathbf{B} \neq \phi$. To reach a point in $\mathbf{P} \setminus \mathbf{P}_0$ is not a task given to the MDP Procedure, so that I may confine myself to focus on the set $\mathbf{P}_0$. Conventional mathematical notation is used throughout in the same manner as in Sato(1983). Hereafter all variables are assumed to be functions of time, however, the argument t is omitted unless confusion could arise.

3. THE νMDP PROCEDURES

3.1. *A Description of the Procedure*

The MDP Procedure is the best-known member belonging to the family of the quantity-guided procedures, in which the planning centre asks individual agents their MRSs between each public good and a private numéraire. Then the centre revises the allocation according to the reported MRSs and the MRT. The relevant information exchanged between the centre and the periphery is in the form of quantity.

Let us describe a generic model of our planning procedures for public goods and a private good used as a numéraire:

$$\begin{cases} dx^k/dt \equiv X^k(s^k(t)), & \text{if } x^k(t) > 0, \quad \forall k \in \boldsymbol{K} \\ dx^k/dt \equiv Max\{0,\ X^k(s^k(t))\}, & \text{if } x^k(t) = 0, \quad \forall k \in \boldsymbol{K} \\ dy_i/dt \equiv Y_i(s(t)), & \forall i \in \boldsymbol{N} \end{cases}$$

where $s^k(t) = (s_1^k(t), \dots, s_N^k(t)) \in \boldsymbol{R}_+^N, \forall k \in \boldsymbol{K}$, and $s(t) = (s_1(t), \dots, s_N(t)) \in \boldsymbol{R}_+^{NK}$ is a vector of ν-parameterized MRSs announced at iteration $t \in [0, \infty)$: i.e., $s_i(t) = (\nu_i \phi_i^1(t), \dots, \nu_i \phi_i^K(t)), \forall i \in \boldsymbol{N}$.

For any $t \in [0, \infty)$, the procedure reads:

$$\begin{cases} X^k(t) = \sum_i \nu_j \phi_j^k(t) - \gamma^k(t), & \text{if } x^k(t) > 0, \quad \forall k \in \boldsymbol{K} \\ X^k(t) = Max\{0,\ X^k(s^k(t))\}, & \text{if } x^k(t) = 0, \quad \forall k \in \boldsymbol{K} \\ Y_i(t) = -\sum_k \nu_i \phi_i^k(t) X^k(t) + \delta_i \sum_k \left\{ \sum_j \nu_j \phi_j^k(t) - \gamma^k(t) \right\} X^k(t), & \forall i \in \boldsymbol{N} \end{cases}$$

where $\delta_i > 0, \forall i \in \boldsymbol{N}$, and $\sum_i \delta_i = 1$.

$\delta = (\delta_1, \dots, \delta_N)$ is a vector of distributional coefficients determined by the planner prior to the beginning of the operation of the procedure. Its role is to share among individuals the surplus, $\sum_k \left\{ \sum_j \nu_j \phi_j^k(t) - \gamma^k(t) \right\} X^k(t)$, which is always positive except at the equilibrium.

Remark 2. $\delta_i > 0$ was posited by Drèze and de la Vallée Poussin (1971), and followed by Roberts (1979), whereas $\delta_i \geqq 0$ was assumed by Champsaur (1976) who advocated a notion of neutrality to be explained below.

Let me call the procedures defined by the above equations the ν*MDP* Procedures since they are parameterized by the vector ν. There could be a large class of the ν*MDP* Processes via the choice of δ. The original MDP Procedure appears if $\nu_i = 1, \forall i \in \boldsymbol{N}$, which reduces to an achievement of the Samuelson Condition at an equilibrium of the process. The νMDP Processes preserve the properties that the MDP Procedure enjoys; viz. feasibility, monotonicity, stability, and neutrality, incentive properties pertaining to minimax and Nash strategies, as was proved by Drèze and de la Vallée Poussin (1971) and Roberts (1979).

Next, I compare the original MDP and the νMDP Procedures. The MDP algorithms evolve in the allocation space and stops when the Samuelson condition is met so that the public good quantity is optimal, and simultaneously the private good is allocated in a Pareto optimal way: i.e., (x, y) is Pareto optimal. The νMDP Procedures generate, ν given, in the allocation space and stops when the **GO** holds; i.e., the allocation at that point is Pareto optimal in the generalized sense. In the νMDP Processes the centre must acquire $\nu_i \phi_i^k$ instead of ϕ_i^k as a relevant information, since ν_i depends on the functional form of his/her utility function. Hence, ν_i is called a "preferece parameter."

There are possibilities of boundary problems due to $x^k = 0$ for at least one k and/or $y_i = \beta_i$ for at least one individual. The case with $x^k = 0$ was solved by Henry (1972) by using the max operator as in the above formulation to avoid a public good adjustment to the negative direction. Other cases were treated by Campbell and Truchon (1988). The νMDP Procedures therefore dispense with any assumption to avoid boundary problems.

3.2. *The Local Incentive Game*

Let me examine the incentive properties of the νMDP Procedures in this section. Now the assumption of truthful revelation of preferences is relaxed: i.e., each agent's announcement, ϕ_i^k, is not necessarily equal to its true marginal rate of substitution, π_i^k.

A local incentive game associated with each iteration of the process is formally defined as the normal form game $(\boldsymbol{N}, \boldsymbol{\Psi}, \boldsymbol{V})$; $\boldsymbol{N}$ is the set of players, $\boldsymbol{\Psi} = \times_{i \in N} \boldsymbol{\Psi}_i \subset \boldsymbol{R}_+$

is the Cartesian product of the individual strategy sets, and $\boldsymbol{V} = (V_1, \ldots, V_N)$ is the N-tuple of payoff functions. Given ν_i calculated by each player at each iteration of the procedure, the time derivative of an individual's utility is defined as

$$du_i / dt = \sum_k u_i^k X^k(s^k(t)) + u_i^y Y_i(s(t)),$$

which is proportional to the payoff in the local incentive game along the procedure given by

$$V_i(s(t)) = \sum_k \pi_i^k X^k(s^k(t)) + Y_i(s(t)). \tag{4}$$

The behavioural hypothesis underlying the above equations is the following myopia assumption : i.e., in order to maximize his/her instantaneous utility increment $V_i(s(t))$, each player determines his/her *dominant strategy* $\phi_i^* \in \boldsymbol{\Psi}_i$ such that

$$(\forall \phi \in \boldsymbol{\Psi})(\forall \phi_i^k \in \boldsymbol{R}_+)(\forall i \in \boldsymbol{N})$$
$$\left[\sum_k \pi_i^k X^k(\phi_i^{k*}, \phi_{-i}^k) + Y_i(\phi_i^*, \phi_{-i}) \geqq \sum_k \pi_i^k X^k(\phi_i^k, \phi_{-i}^k) + Y_i(\phi_i, \phi_{-i})\right]$$

where $\phi = (\phi_1, \ldots, \phi_N)$, $\phi_i = (\phi_i^1, \ldots, \phi_i^K)$, $\phi_{-i}^k = (\phi_1^k, \ldots, \phi_{i-1}^k, \phi_{i+1}^k, \ldots, \phi_N^k)$, and $\phi_{-i} = (\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_N)$.

3.3. Normative Conditions for the ν *MDP* Procedures

The conditions that I have presented in INTRODUCTION are in order. The differences from the usual definitions here are such that $s_i^k \equiv \nu_i \phi_i^k$ instead of ϕ_i^k and that there are many public goods. Note that the conditions are based on Condition **GO**. Letting $\zeta_i^k \equiv \nu_i \pi_i^k$, then I have the following conditions:

Condition* F. *Feasibility :

$$(\forall s \in \boldsymbol{\Psi})(\forall t \in [0, \infty))\left[\sum_k \gamma^k X^k(s^k) + \sum_i Y_i(s) = 0\right].$$

Condition* M. *Monotonicity :

$$(\forall s \in \boldsymbol{\Psi})(\forall i \in \boldsymbol{N})(\forall t \in [0, \infty))\left[V_i(s) = \sum_k \pi_i^k X^k(s^k) + Y_i(s) \geqq 0\right].$$

Condition* PE. *Pareto Efficiency :

$$(\forall s \in \boldsymbol{\Psi})(\forall k \in \boldsymbol{K})\left[X^k(s^k) = 0 \iff \sum_i s_i^k = \gamma^k\right].$$

Condition* LSP. *Local Strategy Proofness:

$$(\forall s \in \boldsymbol{\Psi})(\forall s_i^k \in \boldsymbol{R}_+)(\forall i \in \boldsymbol{N})(\forall t \in [0, \infty))$$

$$\left[\sum\nolimits_k \pi_i^k X^k(\zeta_i^k,\ s_{-i}^{\ k}) + Y_i(\zeta_i,\ s_{-i}) \geqq \sum\nolimits_k \pi_i^k X^k(s^k) + Y_i(s)\right]$$

where $\zeta_i = (\zeta_i^{\ 1}, \ldots, \zeta_i^k)$, $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$, and $s_{-i}^{\ k} = (s_1^k, \ldots, s_{i-1}^k, s_{i+1}^k, \ldots, s_N^k)$.

Remark 3. Conditions except **PE** must be fulfilled for any $t \in [0, \infty)$. Champsaur and Rochet(1983) gave a systematic study on the family of planning procedures which are asymptotically efficient and locally strategy proof. Now I know that three subclasses belong to this class: the Bowen procedure, the Generalized MDP Procedure, and the Generalized Wicksell Procedure as classified in Sato(1986). I introduce in the present paper the class of the Generalized νMDP Procedures defined later, which enjoy all of the above conditions. The MDP Process does not satisfy Condition **LSP**, except for a two-person economy.

4. PROPERTIES OF THE νMDP PROCEDURES

4.1. *Conditions **F**, **M**, and **PE***

Now I examine the properties of the νMDP Procedures just defined above. Condition **F** is easily checked to be satisfied, since it has been already used to formulate the νMDP Procedures. Condition **M** is verified under correct revelation as follows. This is simply derived from the fact that

$$V_i(s) = \sum\nolimits_k \pi_i^k X^k(s^k) + Y_i(s) = \sum\nolimits_k u_i^{\ y} \delta_i (x^k)^2 \geqq 0.$$

Thus, I have the following:

Theorem 1. *The νMDP Procedures satisfy Condition **M** for $\delta_i > 0$, $\forall i \in \boldsymbol{N}$.*

Condition **PE**, of course, comes from Condition **GO**. The statement of the former should be regarded as one of the desiderata the procedures have to possess, and the latter concerns the existence of the vector ν in the definition of the generalized optimality condition.

4.2. *Minimax and Nash Strategies*

What about then the incentive properties of the νMDP Procedures? The results are the same as in the original MDP Process, as seen below.

i) *Minimax strategy*

Theorem 2. *Revealing preferences truthfully in the νMDP Procedures is a minimax strategy for any $i \in \boldsymbol{N}$. It is the only minimax strategy for any $i \in \boldsymbol{N}$, when $x^k > 0$ for any $k \in \boldsymbol{K}$.*

Proof: Each agent aims at minimizing his opponent's payoff, then

$$\partial V_i(s)/\partial s_j^k = \sum\nolimits_k \Big[\zeta_i^k - s_i^k + 2\delta_i \Big(\sum\nolimits_{h \neq j} s_h^k + s_j^k - \gamma^k\Big)\Big] = 0$$

where $s_i^k \equiv \nu_i \phi_i^k$ which can differ from $\zeta_i^k \equiv \nu_i \pi_i^k$. Hence, I have

$$s_j^k = (\zeta_i^k - s_i^k)/2\delta_i + \gamma^k - \sum\nolimits_{h \neq j} s_h^k.$$

When the agents $j \neq i$, $j \in \boldsymbol{N}$, use this strategy, the payoff to agent i is obtained as

$$V_i(s) = -\sum\nolimits_k (\zeta_i^k - s_i^k)^2/4\delta_i \geqq 0.$$

In conclusion, only $s_i^k = \zeta_i^k, \forall k \in \boldsymbol{K}$, i.e., correct revelation assures $V_i(s)$ to be maximized irrespective of the strategies followed by the others. Q.E.D.

ii) *Nash strategy*

Next I examine the Nash strategy of the νMDP Procedures. By the results of Roberts(1979), I obtain the Nash strategy as follows:

$$\nu_i \phi_i^k = \nu_i \pi_i{}^k - \frac{1-2\delta_i}{N-1}\Big(\sum\nolimits_j \nu_j \pi_j{}^k - \gamma^k\Big), \forall k \in \boldsymbol{K}.$$

Generally, $\phi_i^k \neq \pi_i^k$ holds for any $i \in \boldsymbol{N}$, and any $k \in \boldsymbol{K}$, since $\sum_j \nu_j \pi_j^k - \gamma^k \neq 0$ for any $k \in \boldsymbol{K}$. However, $\sum_j \nu_j \pi_j{}^k - \gamma^k = 0$ holds for any $k \in \boldsymbol{K}$ at any equilibrium, $\phi_i^k = \pi_i^k$, $\forall i \in \boldsymbol{N}$, $\forall k \in \boldsymbol{K}$, which results for each individual. The posibility that $\nu_i = 0$ for some i should not be overlooked. Thus I present the following.

Theorem 3. $\phi_i^k = \pi_i^k$ *holds for any* indiridual with $\nu_i \neq 0$ *and for any* $k \in \boldsymbol{K}$ *at amy equilibrium of the νMDP Procedures.*

4.3. *Neutrality of the νMDP Procedures*

Champsaur (1976) advocated the notion of neutrality for the MDP Procedure, and Cornet (1983) generalized it by omitting two restrictive assumptions imposed by Champsaur: i.e., (i) uniqueness of solution and (ii) concavity of the utility functions.[3)] Neutrality depends on the distributional coefficient vector δ. Remember that the role of δ is to attain any individually rational Pareto optima (IRPO) by redistributing the social surplus generated during the operation of the procedure: δ varies the trajectory to reach every IRPO. In other words, the centre can guide the allocation via the choice of δ, however, it cannot predetermine the final allocation to be reached.

***Condition* N. Neutrality**:

For every efficient point, $z^* \in \boldsymbol{Z}$ and for any initial point, $z_0 \in \boldsymbol{Z}$, there exists δ and $z(t, \delta)$, a trajectory starting at z_0, such that $z^* = z(\infty, \delta)$.

Remark 4. Neutrality is autonomous, while Champsaur and Rochet's (1983) local neutrality is not autonomous. See Hirsch and Smale (1974) for the concept of autonomous. For the other concepts of neutrality associated with planning procedures, see Sato (1983) and (1986). See also D'Aspremont and Drèze (1979) for an alternative version of neutrality which is valid in general contexts.

The crucial underpinning of Champsaur-Cornet's neutrality is the nonnegativity requirement of δ. Once dropping this, they cannot prove their neutrality theorems. Originally, Drèze and de la Vallée Poussin (1971) imposed the hypothesis of positive δ; with this assumption, they could demonstrate their Theorem 3 on minimax strategy. At any rate, provided that positivity of δ must be kept, an open subset of IRPO cannot be reached by the MDP Process, as shown below by the **Figure 1** in the Appendix. Successors except Roberts (1979) imposed the nonnegativity of δ to obtain some fruitful results. Hence, Campbell and Truchon (1988) precipitated the design of neutral procedures based on Condition **GO**.

Clearly, the MDP Procedure satisfies Condition **N** which, however, assumed interior optima and nonnegative δ. In order to be able to reach also the Non-Samuelsonian optima, I have to generalize the Champsaur-Cornet's neutrality theorem as follows:

Theorem 4. *Under Assumptions 1～7, for every individually rational Pareto optimum*

z^*, there exists δ and a trajectory $z(\cdot):[0,\infty)\to \boldsymbol{Z}$ of the differential equations defining the ν MDP Procedures such that $u_i(z^*) = lim_{t\to\infty} u_i(x(t), y_i(t)), \forall i \in \boldsymbol{N}$.

Proof: The Cornet's neutrality theorem [**Theorem 5.2**, 1983] tells us that the original MDP Procedure is neutral. However, the solutions of the differential equation system of the MDP Process were assumed to be interior. The νMDP Procedures which can also attain boundary solutions, so that the dynamical system must be replaced by that of the νMDP procedures in the proof to the Cornet's neutrality theorem. It can be checked that the assumptions imposed by Cornet are all satisfied in the statement of the Theorem. Q. E. D.

4.4. *Existence and Stability of Solutions*

This subsection considers the issues on the existence and stability of solutions.

Theorem 5. *For the ν MDP Procedures and for $z_0 \in Z$, there exists a unique solution $z(\cdot):[0,\infty]\to \boldsymbol{Z}$, which is such that $lim_{t\to\infty} z(t)$ exists and is a Pareto optimum.*

Proof: (i) Existence.

In order to verify the existence of solutions to the procedure, I have to modify it. Clearly, the differential equations defining our νMDP Procedures are not globally Lipschitzian continuous, so that possible discontinuities must be dealt with. For that purpose, I modify the procedure as follows:

$$\begin{cases} X^k = \sum_i \nu_j\ \phi_j{}^k(\chi(x), y_j) - \gamma^k(\chi(x)),\ \forall k \in \boldsymbol{K} \\ Y_i = -\sum_k \nu_i \phi_i^k(\chi(x), y_i) X^k + \delta_i \sum_k \left\{ \sum_j \nu_j \phi_i^k(\chi(x), y_i) - \gamma^k(\chi(x)) \right\} X^k \end{cases}$$

where $\chi(x) = (Max\{0, x^1\},\ldots, Max\{0, x^K\})$, and $\delta_i > 0, \forall i \in \boldsymbol{N}$, and $\sum_i \delta_i = 1$.

Now $X^k, \forall k \in \boldsymbol{K}$, and $Y_i, \forall i \in \boldsymbol{N}$, are all Lipschitzian, so that starting from such a point z_0 there exists a locally unique solution path $z(t, z_0)$, by **Theorem 1** in Hirsch and Smale(1974, Ch. 15). Discontinuity has been, as it were, included or "internalized" in $\chi(x)$.

(ii) Convergence.

So as to verify the second part of the Theorem, recollect as an immediate corollary to **Theorem 6.1** in Champsaur, Drèze and Henry(1977), that if a dynamic system has a unique solution for every $z_0 \in Z$, say $z(t, z_0)$, which varies continuously with z_0 remains in Z for all t, and that if there is a Lyapunov function, then the system is quasi-stable.

As a function pertinently chosen for a Lyapunov function, let us take the weighted sum of the utility functions:

$$L(z(t)) = \sum_i \lambda_i u_i(t).$$

Differentiating this gives

$$dL(z)/dt = \sum_i \lambda_i (du_i/dt) = \sum_i \nu_i \mu_0 \delta_i \left(\sum_i \nu_i \pi_i{}^k - \gamma^k\right)^2 \geqq 0.$$

Clearly, any equilibrium of the νMDP Procedures is a Pareto efficient allocation, z^*, one has only to verify that $lim_{t\to\infty} z(t, z_0)$ exists for any $z_0 \in Z$. This is immediate, however, because of our convexity assumptions, there is only one Pareto optimum z^* such that $L(z^*) = lim_{t\to\infty} L(z(t))$. Q.E.D.

4.5. *LSP ν MDP Procedures*

Substituting $v_i \phi_i$ for ϕ_i seems to bring no difference in terms of the preference revelation, since one may regard ν_i as a weight to each individual's MRS and ν plays no essential role in a preference revelation context. Calculated by each player to maximize their payoff in a local incentive game, the vector ν leads to every allocation including any boundary point with $x_k = 0$ for some $k \in \boldsymbol{K}$ and/or $y_i = \beta_i$ for some $i \in \boldsymbol{N}$. In order to be able to choose the appropriate values of the parameter ν, the planner ought to have information about agents' true preferences. In the local incentive game, the planner is assumed to know the true information of individuals, if the νMDP Procedures are locally strategy proof. Our accumulated knowledge of incentives therefore can be immediately used to nonlinearize the νMDP Procedures in such a parallel manner as in Fujigaki and Sato(1981).

The LSP ν MDP Procedure reads:

$$\begin{cases} X^k = \alpha^k \left(\sum_j \nu_j \phi_j{}^k - \gamma^k\right) \left|\sum_j \nu_j \phi_j{}^k - \gamma^k\right|^{N-2}, \ \alpha^k \in \boldsymbol{R}_{++}, \ \forall k \in \boldsymbol{K} \\ Y_i = -\sum_k \nu_i \phi_i{}^k X^k + (1/N) \sum_k \left(\sum_j \nu_i \phi_j{}^k - \gamma^k\right) X^k, \forall i \in \boldsymbol{N} \end{cases}$$

where $\phi_j{}^k \equiv \phi_j{}^k(\chi(x), y_j)$, $\gamma^k \equiv \gamma^k(\chi(x))$, and α^k is an adjustment speed of the kth public good.

In our context, as one of the planner's tasks is to achieve an optimal allocation of public goods given the value of ν, he or she has to collect the relevant information from the periphery so as to meet the conditions presented above. Fortunately, the necessary information is available if the procedure is locally strategy proof. It was already shown by Fujigaki and Sato(1982), however, that the locally strategy proof generalized MDP Procedures cannot preserve neutrality, since δ was concluded to be fixed ; i.e., $1/N$, to accomplish LSP, keeping the other conditions fulfilled. $\delta_i = 1/N \neq 0$, since N cannot be zero. Instead, our LSP νMDP Procedures may reach any limit point, either an interior or a boundary allocation.

Theorem 6. *The LSP νMDP Procedures fulfill Conditions **F**, **M**, **PE**, and **LSP**.*

Proof is postponed to Section 5.

Remark 5. In Fujigaki and Sato (1981) and (1982), we called our procedure as the *Generalized MDP Procedure*. Certainly, the public decision function was generalized to include that of the MDP Procedure, the distributional vector was fixed to a specific value, i.e., $1/N$. Thus, in order to be more precise, we have called the procedure the *LSP νMDP Procedure*. The genuine Generalized νMDP Procedures are presented below.

4.6 *Aggregate Correct Revelation*

The operation of the Generalized MDP Procedure we proposed in(1981) does not even require truthfulness of each player to be a Nash equilibrium strategy, but it needs only aggregate correct revelation to be a Nash equilibrium, as was verified in Sato(1983). It is easily seen from the discussion in the previous subsections that the LSP νMDP procedure is not neutral at all, which means that local strategy proofness impedes the attainment of neutrality. Hence, Sato(1983) proposed another version of neutrality, and Condition *Aggregate Correct Revelation* which is weaker than **LSP**.
It can be stated in our context as follows:

***Condition* ACR. *Aggregate Correct Revelation*:**

$$(\forall \pi^k \in \mathbf{\Pi}^k)(\forall k \in \boldsymbol{K})(\forall t \in [0, \infty))\left[\sum\nolimits_j \nu_j \phi_j^{\,k}(\pi^k) = \sum\nolimits_j \nu_j \pi_j^{\,k}\right]$$

where $\pi^k = (\pi_1^k,\ldots, \pi_N^k) \in \mathbf{\Pi}^k$: the set of MRS vectors for public goods.

Remark 6. Condition **ACR** means that the sum of the *Nash* strategies $\phi_i, \forall i \in \boldsymbol{N}$, always coincides with the aggregate value of the correct MRSs. Clearly, **ACR** claims only truthfulness in the aggregate.

***Condition* TN. *Transfer Neutrality*:**

$$(\forall z^* \in \mathbf{P}_0 \cap \mathbf{B})(\exists T \in \mathbf{\Delta})(\exists z(\cdot) \in \boldsymbol{S})[z^* = lim_{t\to\infty} z(t)]$$

where $\mathbf{\Delta}$ is the set of transfer rules: $T=\{T_1,\ldots, T_N\}$, $z(\cdot)$ is a solution of the procedure, and $\boldsymbol{S}$ is the class of solutions.

I need also the following conditions.

***Condition* TA. *Transfer Anonymity*:**

$$(\forall s \in \boldsymbol{R}_+^{NK})(\forall i \in N)[T_i(s) = T_i(\rho(s))],$$

where $\rho : \boldsymbol{R}_+^N \to \boldsymbol{R}_+^N$ is a permutation function.

Remark 7. Condition **TA** says that the agent i's transfer in private good is invariant under the permutation of its arguments: i.e., the order of strategies does not affect the value of $T_i(s), \forall i \in \boldsymbol{N}$.

Keeping the same nonlinear public good decision function as derived from Condition **LSP**, I can state the characterization theorem.

Theorem 7. *The Generalized ν MDP Procedures fulfill Conditions **ACR, F, M, PE, TA** and **TN**. Conversely, any planning procedure satisfying these conditions is characterized to:*

$$\begin{cases} X^k = \alpha^k \left(\sum_j \nu_j \phi_j{}^k - \gamma^k\right) \left| \sum_j \nu_j \phi_j{}^k - \gamma^k \right|^{N-2}, \ \alpha^k \in \boldsymbol{R}_{++}, \ \forall k \in \boldsymbol{K} \\ Y_i = -\sum_k \nu_i \phi_i{}^k X^k + T_i \sum_k \left(\sum_j \nu_j \phi_j{}^k - \gamma^k\right) X^k, \ \forall i \in \boldsymbol{N}. \end{cases} \tag{5}$$

Proof: Follows immediately from **Theorem 2** in Sato (1983) by replacing $\nu_j \phi_j{}^k$ for $\phi_j{}^k$. Chander (1993) verified the incompatibility between core convergence property and local strategy proofness. It is possible to escape from his "impossibility theorem" by weakening incentive requirement from **LSP** to **ACR**, hence, I can present the following:

Corollary 1. *There exists a Generalized νMDP Procedure such that the solution path converges to some core allocation.*

Proof: Obviously, the family of the Generalized νMDP Procedures involves as its member the Process with

$$T_i = \delta_i\left(\sum_j \nu_j \phi_j^k - \gamma^k\right)$$

and

$$\delta_i = \frac{\sum_k \nu_i \phi_i^k}{\sum_j \sum_k \nu_j \phi_j^k}.$$

Some calculation leads to observe

$$Y_i = -\frac{\sum_k \nu_i \phi_i^k}{\sum_j \sum_k \nu_j \phi_j^k} \sum_k X^k. \tag{6}$$

From **Theorem 3.4** in Chander(1993), the procedure defined by Eqs. (5) and (6) clearly belongs to the class of processes he proposed, whose solutions converge to some core allocation. Q.E.D.

Let me propose here a new concept of neutrality.

Condition νN. νNeutrality:

$$(\forall z^* \in \mathbf{P}_0 \cap \mathbf{B})(\exists \nu \in \mathbf{\Omega})(\exists z(\cdot) \in \boldsymbol{S})[z^* = \lim_{t\to\infty} z(t)]$$

where $\mathbf{\Omega}$ is the class of $\nu = \{\nu_1, \ldots, \nu_N\}$.

It was already shown by Fujigaki and Sato(1982) that the Generalized MDP Procedures could not preserve neutrality, since δ was concluded to be fixed to accomplish LSP. Instead, our Generalized νMDP Procedures dispense with δ, and relies on ν to achieve νNeutrality. I am not yet able to give a rigorous proof here, but I may make the following conjecture. See **Figure 3** in the Appendix for a graphical presentation of the proof.

Conjecture: There exist procedures which can attain whatever limit point in the core according to the change of the vector ν on the part of the individual agents. In other words, there exist processes which are “neutralized” by the choice of ν instead of δ.

Almost all MDP-type planning procedures designed so far share a common characteristic that a social surplus in numéraire appears at each iteration during the working of the process, and that this surplus is distributed among all individuals according to the distributional coefficients specified by a constant N-dimensional vector δ. All these planning procedures assume that this vector is determined exogenously by the planner and prior to the beginning of the procedure, without resort to any knowledge about the periphery, which has often been criticized.

Let me show an internalization of the distributlonal coefficients of surplus sharing in procedures and tried to give a possible solution for endogenous determination of surplus share by specifying a transfer function which has the role of distributing the surplus.

Representative candidates for internally determining δ would be as follows:

$$\text{i)}\ \delta_i = \frac{\omega_i}{\sum_j \omega_j},\ \text{ii)}\ \delta_i = \frac{\pi_i}{\sum_j \pi_j},\ \text{iii)}\ \delta_i = \frac{\nu_i \pi_i}{\sum_j \nu_j \pi_j},$$

and

$$\text{iv)}\ \delta_i = \frac{\sum_{j \neq i} \nu_j \pi_j}{(N-1)\sum_j \nu_j \pi_j}.$$

Remark 8. i) and ii) have obvious implications respectively, so l consider the others; iv) signifies that the smaller $\nu_i \pi_i$, the larger δ_i, which may give agents an incentive to purposely misstate their preferences for public goods. Hence, I have chosen iii) to obtain the above result. Crucial difference between δ and ν is that the latter can partially determine the adjustment speeds of public goods, but the former cannot. Given the amount of surplus, δ can "neutralize" the procedure.

5. PROOF OF THE THEOREM

Here I give the proof to the **Theorem 6**.

It is easy to see that Conditions **F** and **M** are satisfied, and that **F** and **M** entail **PE**. As regards **LSP**, let me modify the proof to **Theorem 1** in Sato(1983) to the case with many public goods. Denote $\theta^k = \sum_j s_j^k - \gamma^k$.

Consider the following procedure:

$$\begin{cases} X^k = G(\theta^k), & \forall k \in \boldsymbol{K} \\ Y_i = -\sum_k s_i^k G(\theta^k) + \sum_k \delta_i(\theta^k)\theta^k G(\theta^k), & \forall i \in \boldsymbol{N}. \end{cases}$$

With this procedure, differentaiation of Eq.(4) with respect to s_i^k gives

$$\frac{\partial V_i}{\partial s_i^k} = \sum_k \Big[-G(\theta^k) + \delta_i{}'(\theta^k)\theta^k G(\theta^k) + \delta_i(\theta^k)\{G(\theta^k) + \theta^k G'(\theta^k)\} \Big] = 0$$

which is a necessary condition for the truth-telling to be a dominant strategy. Hence, one has

$$\delta_i{}'(\theta^k) + \left\{(\theta^k)^{-1} + \frac{G'(\theta^k)}{G(\theta^k)}\right\}\delta_i(\theta^k) = (\theta^k)^{-1}.$$

By the formula of inhomogeneous linear differential equation, one observes

$$\delta_i(\theta^k) = exp(-\Theta^k)\sum_k \left\{ \int (\theta^k)^{-1} exp(\Theta^k) d\theta^k + C_i(s_{-i}^k) \right\}, \quad \forall i \in \boldsymbol{N}$$

where $\Theta^k = \int \left\{ (\theta^k)^{-1} + \frac{G'(\theta^k)}{G(\theta^k)} \right\} d\theta^k$, and C_i is a real-valued function independent of s_i^k.

The equations

$$exp(-\Theta^k) = [\theta^k G(\theta^k)]^{-1} = [T(\theta^k)]^{-1}$$

and

$$exp(\Theta^k) = T(\theta^k)$$

yield

$$\delta_i(\theta^k) = [T(\theta^k)]^{-1} \sum_k \left\{ \int G(\theta^k) d\theta^k + C_i(s_{-i}^k) \right\}, \ \forall i \in \boldsymbol{N}.$$

Defining $\delta_i(\theta^k)T(\theta^k) \equiv T_i(\theta^k)$, one obtains

$$T_i(\theta^k) = \sum_k \left\{ \int G(\theta^k) d\theta^k + C_i(s_{-i}^k) \right\}, \ \forall i \in \boldsymbol{N}. \tag{7}$$

Rewriting (7) in a definite integral form gives

$$T_i(\theta^k)=\sum_k\left\{\int_{\gamma^k-\sum_{j\neq i}s_j^k}^{s_i^k}G(\theta_i^k,\sum_{j\neq i}s_j^k-\gamma^k)d\theta_i^k+C_i(s_{-i}^k)\right\},\ \ \forall i\in\boldsymbol{N}$$

which can be written as

$$T_i(\theta^k)=\sum_k\left\{\int_0^{\theta^k}G(\xi^k)d\xi^k+C_i(s_{-i}^k)\right\},\ \ \forall i\in\boldsymbol{N}$$

Let $\theta^k=0$, then

$$T_i(0)=\sum_k C_i(s_{-i}^k),\ \ \forall i\in\boldsymbol{N}.$$

But, from Conditions **F** and **M**, one gets

$$T_i(0)=0,\ \ \forall i\in\boldsymbol{N}$$

which implies that

$$\sum_k C_i(s_{-i}^k)=0.$$

Consequently

$$T_i(\theta^k)=\sum_k\int_0^{\theta^k}G(\xi^k)d\xi^k,\ \ \forall i\in\boldsymbol{N} \tag{8}$$

leads to

$$\sum_k\theta^kG(\theta^k)=\sum_i T_i(\theta^k)=\sum_i\sum_k\int_0^{\theta^k}G(\xi^k)d\xi^k. \tag{9}$$

Differentiating (9) with respect to θ^k yields

$$G(\theta^k)+\theta^k dG(\theta^k)/d\theta^k=NG(\theta^k)$$

thus

$$\frac{dG(\theta^k)/d\theta^k}{G(\theta^k)}=\frac{N-1}{\theta^k},\ \ \forall k\in\boldsymbol{K}. \tag{10}$$

Solving (10) for $G(\theta^k)$, one obtains

$$G(\theta^k) = \alpha^k(\theta^k)^{N-1},\ \alpha^k \in \mathrm{R}_{++},\ \ \forall k \in \boldsymbol{K}.$$

Since $G(\theta^k)$ is sign-preserving from **Lemma 2** in Sato(1983), finally I have

$$G(\theta^k) = \alpha^k \theta^k \mid \theta^k \mid^{N-2},\ \alpha^k \in R_{++},\ \ \forall k \in \boldsymbol{K}.$$

As can be easily seen from (8)

$$T_1(\theta^k) = T_2(\theta^k) = \cdots = T_N(\theta^k)$$

which reduces to

$$\textstyle\sum_k \theta^k G(\theta^k) = NT_i\,.$$

Hence, I can conclude that

$$T_i = (1/N) \textstyle\sum_k \theta^k G(\theta^k),\ \ \forall i \in \boldsymbol{N}.$$

This completes the proof. Q.E.D.

6. FINAL REMARKS

The issue of the present paper has been to design planning procedures which can attain both Samuelsonian and Non-Samuelsonia Pareto optimal allocations. To this end, I have adopted the generalized optimality condition propounded by Campbell and Trllclion(1988), that is valid for all Pareto optima including boundary ones.

The νMDP Procedures are capable of achieving all Pareto optima, some of which do not satisfy the Samuelson condition, so that the original MDP Process cannot reach them. Whereas, the νMDP Procedures are able to achieve all Pareto optima, including boundary ones, via a weight vector ν attached to each individual's marginal rate of substitution.

What I have wished to show in this paper is that the scope of the foregoing analyses in the literature of the MDP Procedure is much longer and one can go further than we might have expected, by presenting the family of the νMDP Procedures that are convergent, neutral, and incentive compatible.

APPENDIX

A.1. *Impossibility of the MDP Procedure to Reach an Open Subset of IRPO*

Here I show open subsets unachievable via the MDP Procedure under the following example with $\delta = (\delta_1, \delta_2) \neq 0$, $\omega = (\omega_1, \omega_2)$, $\omega' = (\omega_1', 0)$, $\omega'' = (0, \omega_2'')$, $z = (z_1, z_2)$, $\sum_i \omega_i = \sum_i \omega_i' = \sum_i \omega_i''$, $u_i(\omega) = (u_1(0, \omega_1), u_2(0, \omega_2))$, and $u(z) = (u_1(x^*, y_1^*), u_2(x^*, y_2^*))$. See **Figure 1**.

$[\alpha\,\eta]$: Set of IRPO
$[\varepsilon\,\zeta]$: Set of IRPO achievable via the MDP Procedure with $\delta \neq 0$
$[\alpha\,\varepsilon)$, $(\zeta\,\eta]$: Sets of IRPO unattainable by the MDP Procedure with δ kept positive.

A.2. *Graphical Illustrations of the Optima Reached by the MDP and νMDP Procedures*

Malinvaud (1971) reproduced Serge-Christophe Kolm's elaborately devised triangle less familiar than the Edgeworth box diagram with two goods, however, one of which is a public good. By making use of his helpful diagram I illustrate in **Figures 2** and **3** possible optima that are either Samuelsonian or Non-Samuelsonian reached by the trajectories of the νMDP Procedure based on the **Condition GO**.[4]

In a two-person economy the core coincides with the set of IRPO. I can draw a Kolm's equilateral triangle $\mathbf{O}_1\mathbf{O}_2\mathbf{Q}$ which is the simplex in $\boldsymbol{R}^3$ where $x + y_1 + y_2 = 2$. The straight line **BF** is the set of Samuelsonian PO, where $x = 1$. Whereas, **AB** and **FH** are both Non-Samuelsonian PO which are boundary optima. Point $\boldsymbol{\omega}$ corresponds to the equal initial endowment $(x_0, \omega_1, \omega_2) = (0, 1, 1)$, where x_0 is an initial amount of the public good. The trajectories of the MDP Procedure is determined by the distributional coefficient vector δ. If $\delta = (0.5, 0.5)$, the time path is $\boldsymbol{\omega}$ **D**. The trajectory $\boldsymbol{\omega}$ **F** occurs when $\delta = (1, 0)$ and coincides with agent 2's indifference curve. $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are sample indifference curves of individuals 1 and 2.

Trajectories of the MDP Procedure in the Example in **Figure 2** with:
$u_i(x, y_i) = 1\mathrm{n}x + y_i$, $i = 1, 2$, $x = g(x) = y$, $\omega = (\omega_1, \omega_2)$, $\omega_1 = \omega_2 = 1$, and $\beta = (0, 0)$.

[B F] : Set of Samuelsonian IRPO achievable via the MDP Process with a Choice of δ
[A B), **(F H]** : Sets of Non-Samuelsonian PO unattainable by the MDP Process via δ

Next, it is gaphically shown in **Figure 3** that the νMDP Procedure can attain an open subset of Pareto optima where the Samuelson condition does not hold, a subset which is beyond the reach of the original MDP Procedure. Let the procedure start from an unequal initial endowment. $\boldsymbol{I_1}'$ and $\boldsymbol{I_2}'$ are sample indifference curves which determine the CORE [**EH**].

Time Paths of the MDP and νMDP Procedures in the Example with :
$u_i(x, y_i) = 1\mathrm{n}x + y_i$, $i = 1, 2$, $x = g(x) = y$, $\omega' = (\omega_1', \omega_2')$, where $\omega_1' = 1.5$, $\omega_2' = 0.5$, and $\beta = (0, 0)$.

[**DF**] : Set of Samuelsonian PO reachable by the MDP Procedure with a choice of δ

[**F I**] : Set of Non-Samuelsonian PO unachievable by the MDP Procedure with δ adjustment

[**D I**] : Set of either Samuelsonian or Non-Samuelsonian PO achievable by the νMDP Procedure

[**EH**] : Set of IRPO (which coincides with the core in this case) attainable via the νMDP Procedure

[**FH**] : Set of Boundary IRPO unattainable via the MDP Procedure but achievable by the νMDP Procedure.

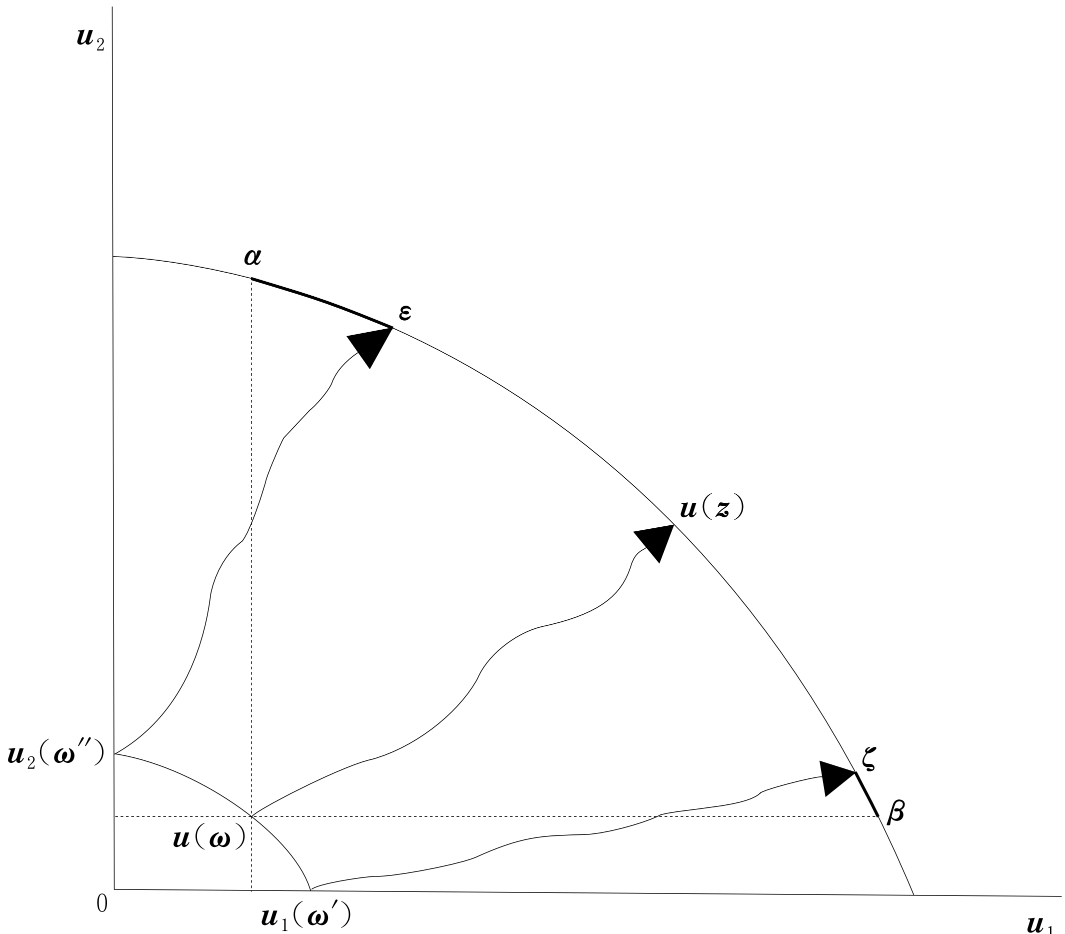
$\boldsymbol{u}_2$
$\boldsymbol{\alpha}$
$\boldsymbol{\varepsilon}$
$\boldsymbol{u}(\boldsymbol{z})$
$\boldsymbol{\zeta}$
$\boldsymbol{\beta}$
$\boldsymbol{u}_2(\boldsymbol{\omega}'')$
$\boldsymbol{u}(\boldsymbol{\omega})$
0
$\boldsymbol{u}_1(\boldsymbol{\omega}')$
$\boldsymbol{u}_1$

Figure 1

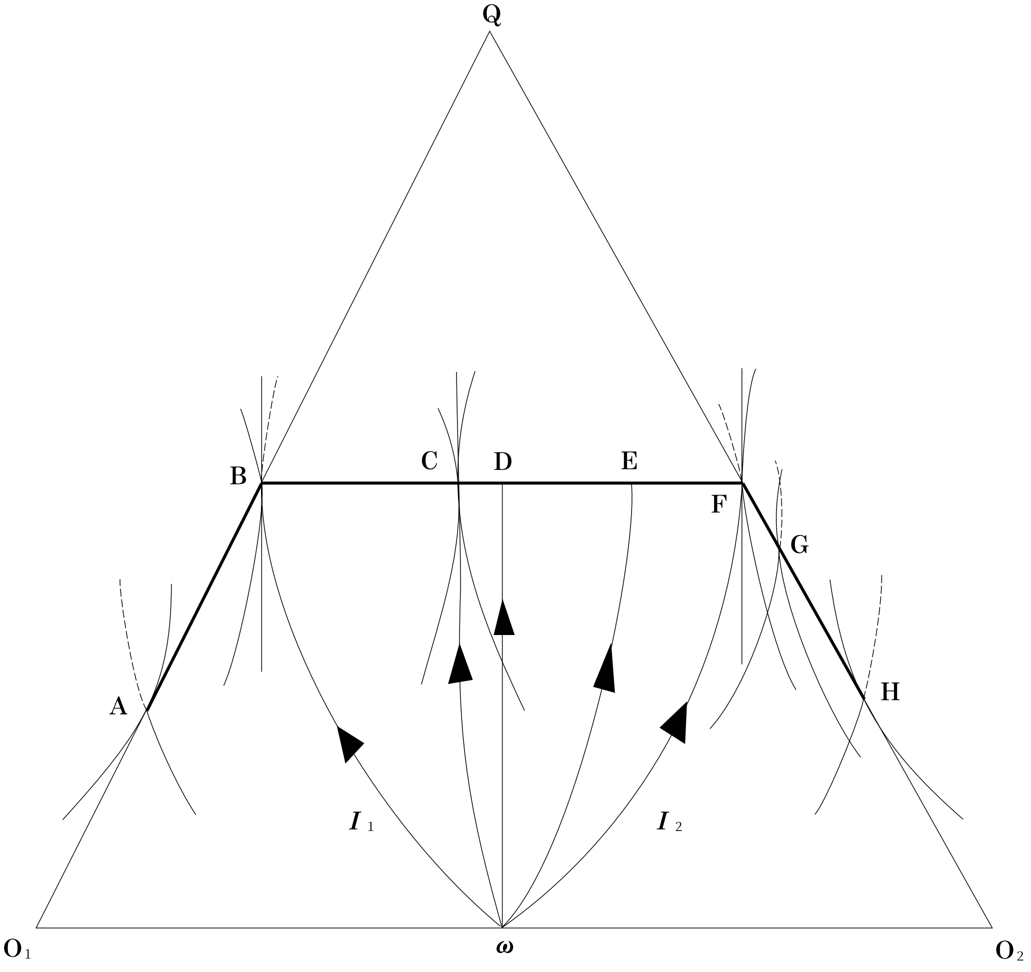

Figure　2

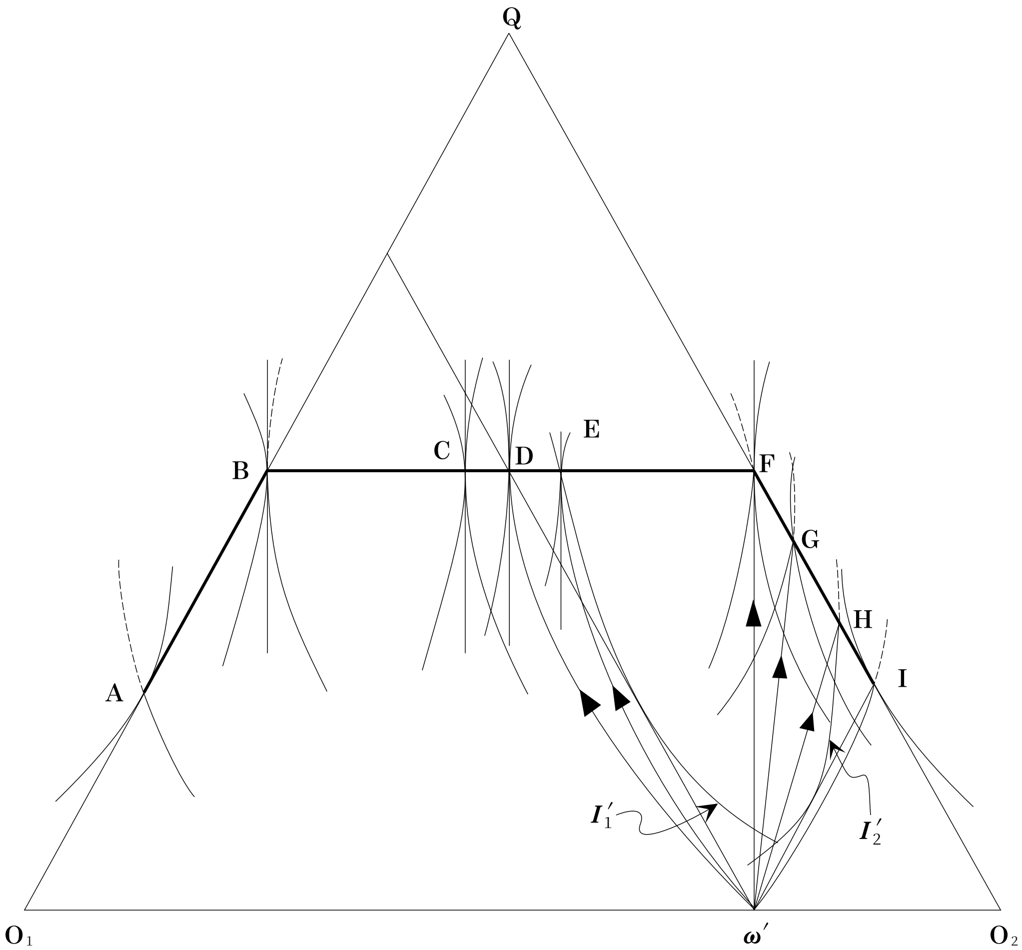

Figure 3

† This is one of the series of papers dedicated to the XXXth anniversary of the MDP Procedure. It was at the Brussels Meeting of the Econometric Society in September 1969 that Drèze and de la Vallée Poussin together, and Malinvaud independently, presented the pepers on planning tâtonnement processes for guiding and financing the optimal provision of public goods. The previous version of this paper was presented at the annual meeting of the Japan Association of the Theoretical Economics and Econometrics held at Osaka University, September 22, 1996. Discussion with Jacques Drèze was very helpful, to whom I express my deep gratitude. Major revisions were made thereafter.

NOTES

1. Jacques Drèze(1995, p.199) lucidly summarized the free rider problem as follows:

“The theory of incentives is concerned with the design of rules or mechanisms that provide individual economic agents with incentives to adopt a nonobservable course of action compatible with overall efficiency, or to reveal truthfully private information that is socially relevant. Information asymmetry is essential. These problems typically belong to the sphere of public economics, because market mechanisms do not provide the required incentives. A standard illustration is the “free rider” problem in choosing an optimal provision of public good, namely a level where the marginal cost provision is equal to the sum of the so-called “marginal willingness-to-pay” of users. If user charges are based on reported preferences, there is an incentive to underreport ; and conversely, if the charges are unrelated to reported preferences. ”

2. See Chapter 16 of Green and Laffont(1979) for the difficulties with several public goods.

3. See also the proofs given by Cornet(1977a,b) and Cornet and Lasry(1976).

4. As for numerical examples showing the existence of Non-Samuelsonian Pareto optima, see Campbell and Truchon(1988). Saijo(1990) exhibited graphical illustrations for two cases : a public good economy in a Kolm’s triangle and a private good economy in an Edgeworth box diagram. He concluded that his observations were not special. Conley and Diamantaras(1996) provided necessary and sufficient conditions for all Pareto optima, including boundary ones, without smoothness and monotonicity.

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