An Axiomatization of the Gauthier Solution to the Bargaining Problem

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ABSTRACT: This paper attempts to give an axiomatic characterization of the solution advanced by David Gauthier, which extends the Kalai and Smorodinsky solution to the n-person bargaining problem, keeping Pareto Optimality preserved. The n-person Gauthier solution is shown to be characterized by three familiar axioms, Pareto Optimality, Scale Invariance, Symmetry, and new dual axioms, *Minimax Relative Concession*, and *Maximin Relative Benefit*.

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1. INTRODUCTION

In the last two and half decades the bargaining theory has undergone a remarkable upheaval, at the forefront of which seems to have been the concept of perfect equilibrium in the sequential bargaining game. Another line of research, however, has been continued, bringing us fruitful results, one of which is the arrival of the bargaining solution originally due to David Gauthier(1985), (1986) and (1990). By proposing the dual principles of *Minimax Relative Concession* and of *Maximin Relative Benefit*, he extended and generalized the Kalai/Smorodinsky solution to an n-person bargaining games, keeping Pareto optimality preserved.

The Gauthier solution coincides with the familiar Kalai/Smorodinsky solution when there are only two players. The situation would be different when the number of players becomes more than three, i.e., the Kalai/Smorodinsky solution could lose Pareto optimality, whereas the Gauthier solution always satisfies it. Unfortunately, however, the latter has not yet been axiomatically characterized.

The purpose of this paper is to verify that the n-person Gauthier solution is shown to be characterized by the dual axioms, Maximin Relative Benefit and Mimimax Relative Concession, as well as three familiar ones, Pareto Optimality, Scale Invariance and Symmetry.

The composition of the paper is as follows. The next section introduces the preliminaries of the bargaining problem. Section 3 discusses the advanced Gauthier solution. An axiomatic characterization of this solution is given in Section 4. Some concluding remarks follow in the final section.

2. PRELIMINARIES

We shall consider an n-person bargaining problem, $B=(S, d)$, where S is a nonempty, compact, and convex subset of $\boldsymbol{R}^n$ representing the set of utility vectors available to n players belonging to the set $\boldsymbol{N}$, and d, called the ‘disagreement point,’ is a point of S strictly dominated by at least one other point z of S, i.e., $z>d$; it is the outcome that would be achieved if the players disagree.

The set of all bargaining problems will be denoted by $\boldsymbol{B}^n$. For S, we write $P(S)$ for the (strong) Pareto optimal boundary of S:

$$P(S) = \{v \in S,\ w \geqq v \Rightarrow w = v\}.$$

An n-person bargaining solution $f(S, d)$ is a function defined on $\boldsymbol{B}^n$, which associates to each (S, d) in $\boldsymbol{B}^n$ a point of S, and interpreted as the agreement reached by the players. We define player i's ***ideal point***, $I_i(B) \in \boldsymbol{R}^n$ such that $I_i(B) = \max\ \{v_i | d \leqq v \in S\}$.[1] ρ^* denotes the corresponding transformation on $\boldsymbol{R}^n$ for a permutation on $\boldsymbol{N}$. An affine transformation of utility is a function $H_v : \boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$ such that for some $\alpha_i > 0$ and $\beta_i \in \boldsymbol{R}^n$ for $i \in \boldsymbol{N}$, $H_v = (\alpha_i v_i + \beta_i)_{i \in \boldsymbol{N}}$ for each $v \in \boldsymbol{R}^n$.

For the sake of completeness, we first introduce the definition of the Kalai/Smorodinsky solution(1975), proposed earlier by Howard Raiffa(1955). We say that $k(S, d)$ is the Kalai/Smorodinsky solution such that for all bargaining pair (S, d) there exists the unique maximal point with $(u_i - d_i)/(I_i - d_i) = (u_j - d_j)/(I_j - d_j)$ for all $i, j \in \boldsymbol{N}$.

The axiomatic characterization of this solution involves the following axioms :

***Axiom PO*(*Pareto Optimality*).** For any $B \in \boldsymbol{B}^n$, $f(S, d) \in P(S)$.

***Axiom SY*(*Symmetry*).** For any permutation ρ^* on $\boldsymbol{N}$, $\rho^* f(S, d) = f(\rho^*(S), \rho^*(d))$.

***Axiom SI*(*Scale Invariance*).** For any affine transformation H_v, $H_v\,(S, d) = F\,(H_v(S), H_v(d))$.

***Axiom MO*(*Monotonicity*).** For any $B \subset B' \in \boldsymbol{B}^n$ such that $B' = (T, d)$, $I\,(B) = I(B') \Rightarrow f(S,\ d) < f(T,\ d)$.

Remark 1. Although the Axioms ***PO***, ***SI***, and ***SY*** are not so controversial, there has been some debate over ***MO***. Axiom ***MO*** was introduced by Kalai and Smorodinsky (1975) who attempted to correct the insensitivity of the Nash solution to the ideal point. Imposing this axiom, however, may prevent the Kalai/Smorodinsky solution from being Pareto optimal in the n-person bargaining game, as will be examined in detail below.

So much for the preliminaries of the bargaining problem. Now to the Gauthier solution, to which we will give a characterization below.

3. THE GAUTHIER SOLUTION

In "Bargaining and Justice"(1985), David Gauthier proposed a novel bargaining solution similar to that of Kalai and Smorodinsky which satisfies the axiom of monotonicity and is dependent upon the ideal point. Replacing this axiom by that of minimax relative concession, he generalized the Kalai/Smorodinsky solution to the n-person bargaining games. More precisely, Gauthier and Kalai/Smorodinsky solutions coincide in two person games, but they do depart when there are more than three players: the former still satisfies Pareto Optimality, whereas the latter does not necessarily do so.[2]

In his work, *Morals by Agreement*(1986), Gauthier further developed his theory, by presenting more lucidly his original conceptions, and challenges the Zeuthen/Nash/Harsanyi paradigm. One of the Gauthier's task is to verify the existence of a contractarian rationale for morality. It seems that he tries to employ utilitarian technique to develop his contractarian theory of morals. He constructs his own bargaining theory as a pertinent framework to resolve the issues of justice, by advocating the principle of minimax relative concession which "governs the *ex ante* agreement that underlies a fair and rational cooperative venture."(ibid., p. 14). Moreover, in order to answer the question of *compliance*, i. e., the question as to why each party need accept this principle as constraining its actual ex post choices, he introduces the conception of *constrained maximization*. Namely, constraining "maximizing behaviour by internalizing moral principles to govern one's choices"(ibid., p. 15) gives the rationality of compliance. As a basis for such rationality, he needs a cooperative framework for voluntary interaction. For that purpose, Gauthier introduces a conception of proviso, which, constraining an initial bargaining position, is a precondition assuring the possibility of agreement. "The proviso determines the initial endowments of interacting persons, taking account of the real differences among those persons as actors."(ibid., p. 220).

Gauthier analyses in depth the requirements that should be fulfilled by rational bargainings. Here we shall introduce his concepts.

(i) A *claim* is a demand by a prospective cooperative player for a particular joint benefit; it is called an *initial claim* if it is made before the bargaining starts, and a *maximal claim* which affords a player maximum uility, keeping his opponent's utility no less than that of the initial bargaining position or the disagreement point d. (ii) A concession is an offer by a prospective cooperative player to accept a particular utility less than that of his maximal claim. *A relative concession* is defined as follows. Let d_i be i's utility level at a disagreement point. We shall call the proportion $(I_i-u_i)/(I_i-d_i)$ a relative concession. Similarly, the proportion $(u_i-d_i)/(I_i-d_i)$ is referred to as a relative benefit. The sum of a relative concession and a relative benefit is obviously unity.

The Zeuthen's(1930) principle states that the bargainer with a lesser subjective probability of disagreement must concede to the other. With Gauthier's conception, this principle states that the bargainer with a lesser relative concession must concede. The principle of minimax relative concession extends the Zeuthen's one to bargaining among several individuals, which reads as follows. "[G]iven a range of outcomes, each of which requires concessions by some or all bargainers if it is to be selected, then an outcome be selected only if the greatest or maximum relative concession it requires, is as small as possible, or a minimum, that is, no greater than the maximum relative concession required by every other outcome."(ibid., p. 137). Gauthier raises crucial objections to the Zeuthen/Nash/Harsanyi bargaining account: i. e., (i) indeterminacy of the size of one's claim, (ii) arbitrariness of the magnitude of concession, (iii) meaningless character of sequence of mutual concessions, and (iv) unrealistic supposition of making threats.[3]

Here we introduce new dual axioms: maximin relative benefit and minimax relative concession.

***Axiom MRB*(*Maximin Relative Benefit*).** For u, $v \in P(S)$, if u is such that $\min_{i \in N}(u_i-d_i)/(I_i-d_i) > \min_{i \in N}(v_j-d_j)/(I_j-d_j)$,
then $v \neq f(S)$.

***Axiom MRC*(*Minimax Relative Concession*).** For u, $v \in P(S)$, if u is such that $\max_{i \in N}(I_i-u_i)/(I_i-d_i) < \max_{i \in N}(I_j-v_j)/(I_j-d_j)$,
then $v \neq f(S)$.

Remark 2. ***MRB*** amounts to saying that the solution should be the one that maximizes the minimum value among relative benefits of the parties, which is equivalent to ***MRC*** requiring that it should be the one that minimizes the maximum value among relative concessions of the parties. Two axioms play exactly the same roles throughout this paper.

4. A CHARACTERIZATION

We are now in a position to state the theorems.

Theorem 4.1. *Axioms **PO**, **SI**, **SY**, and **MO** are incompatible for bargaining games with three or more players.*

Proof: Follows immediately from the example in Gauthier(1985, pp. 35–6) originally due to Roth(1979, pp. 105–7, Th. 16), which tells us that incompatibility arrives even with ***PO***, ***SY***, and ***MO***.

The same outcome can be reached in two-person bargaining games by different principles: that of monotonicity in the case of Kalai/Smorodinsky solution, and that of minimax relative concession in the case of the Gauthier solution. Gauthier rejected monotonicity and replaced it by minimax relative concession to escape from the impossibility theorem of Roth(1979, Th. 16), stating that the Kalai/Smorodinsky solution is not necessarily Pareto optimal for the bargaining games with more than three players, keeping symmetry and monotonicity preserved.

In order to verify the differences among the Nash, the Kalai/Smorodinsky, and the Gauthier solutions, we give the following:

Example 1. Consider the normalized pair$(S, 0)$ where

$$S=\text{Convex Hull}\{(1, 0, 0), (1, 1/2, 0), (0, 1/2, 0), (0, 0, 1), (0, 1/2, 1), (1/2, 0, 1), (1/2, 1/2, 1)\}.$$

In this three-person game, the Kalai/Smorodinsky solution can no longer satisfy Axiom ***PO***, whereas our Gauthier solution does, as shown in Figure 1.

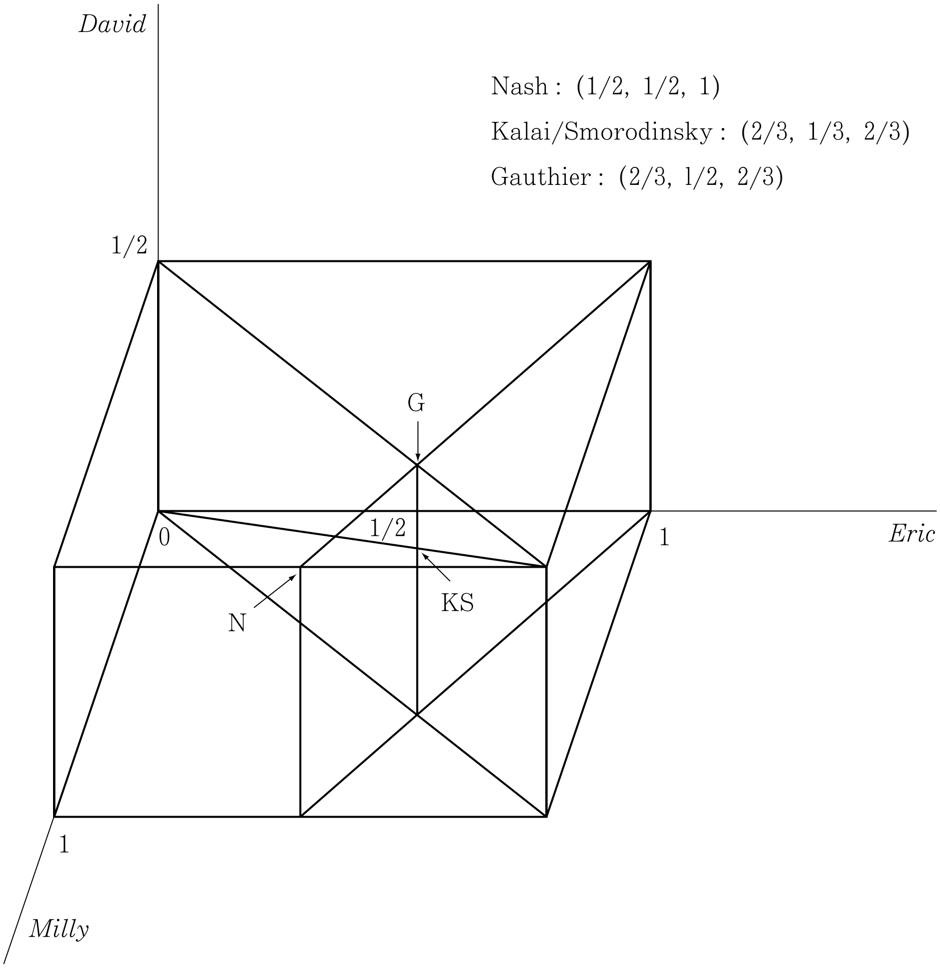

Figure 1 Comparing Bargaining Solutions

Figure 1 clarifies the sharp contrast between the Kalai/Smorodinsky solution and the Gauthier solution. It is not too much to say that this difference stems from the axioms involved in each solution, i. e., ***MO*** in the former, and ***MRB***(or ***MRC***) in the latter. Axiom ***MRB***(***MRC***, resp.) requires equal relative benefits(concessions, resp.) of all players except when some can be better-off by departing from equality without damaging others.

Now we give an axiomatic characterization of the Gauthier solution, which he did not do.

Theorem 4.2. *A solution satisfies Axioms **PO**, **SY**, **SI**, and **MRB**(or **MRC**) if and only if it is the n-person Gauthier solution G^n obtained as* :

$$\forall B \in \boldsymbol{B}^n, \qquad G^n(S,\ d) = Arg\ Max_{u \in S}\ Min_{i \in \boldsymbol{N}}\{(u_i - d_i)/(I_i - d_i)\}.$$ [4]

Proof: The argument that the solution G^n is well-defined, and satisfies all the axioms in the theorem is straightforward. It suffices to show that G^n is the only n-person solution satisfying these axioms. It is an analogue of the proof by Kalai and Smorodinsky (1975).

Let f be any function satisfying ***PO***, ***SY***, ***SI***, and ***MRB*** or ***MRC***. Pick arbitrarily $B \in \boldsymbol{B}^n$ such that $B = (S, d)$, and let $\gamma = G^n(S, d)$. To show that $f(S, d) = \gamma$, let $\eta \in \boldsymbol{R}^n$ be the unique positive linear transformation mapping B into B' such that $B' = (T, d)$ where $I(B) = 1$. It is easy to see that $\omega = \eta(\gamma)$ has equal coordinates. Define now B'' $\in \boldsymbol{B}^n$ such that $B'' = (T', d)$ by the convex hull of η and n unit vectors. In view of ***PO*** and ***SY***, $f(T', d) = \omega$, and by ***MRB*** or ***MRC***, $f(T, d) = \omega$. Finally, ***SI*** gives the desired conclusion: $f(S, d) = \eta^{-1}(\omega) = \gamma$. Q. E. D.

Corollary 4.3. *The Gauthier and the Kalai/Smorodinsky solutions coincide for the two-person bargaining games.*

Proof: Follows directly from the definitions of the two solutions for the two-person bargaining games.

Remark 3. Further properties of the Gauthier solution should be established. What other axioms it can satisfy is an issue of great interest. See, for example, Salonen (1987) for the axioms so far proposed. I conjecture that the Gauthier solution can satisfy the axiom ***MON***(Monotonicity with Respect to Changes in the Number of Agents), which was proposed by Thomson (1983a) and (1983b), and the axiom ***MST*** (Multilateral Stability) introduced by Lensberg (1985). See also Lensberg (1987) for the developments in this field.

The Kalai-Smorodinsky Solution is illustrated in the 2-person economy.

Remark 4. It is also important to study the relationship between the Gauthier solution and other solutions such as the Egalitarian and the Leximin solutions. See Kalai (1985) and Thomson (1985) for these solutions. See Gauthier (1990) for further

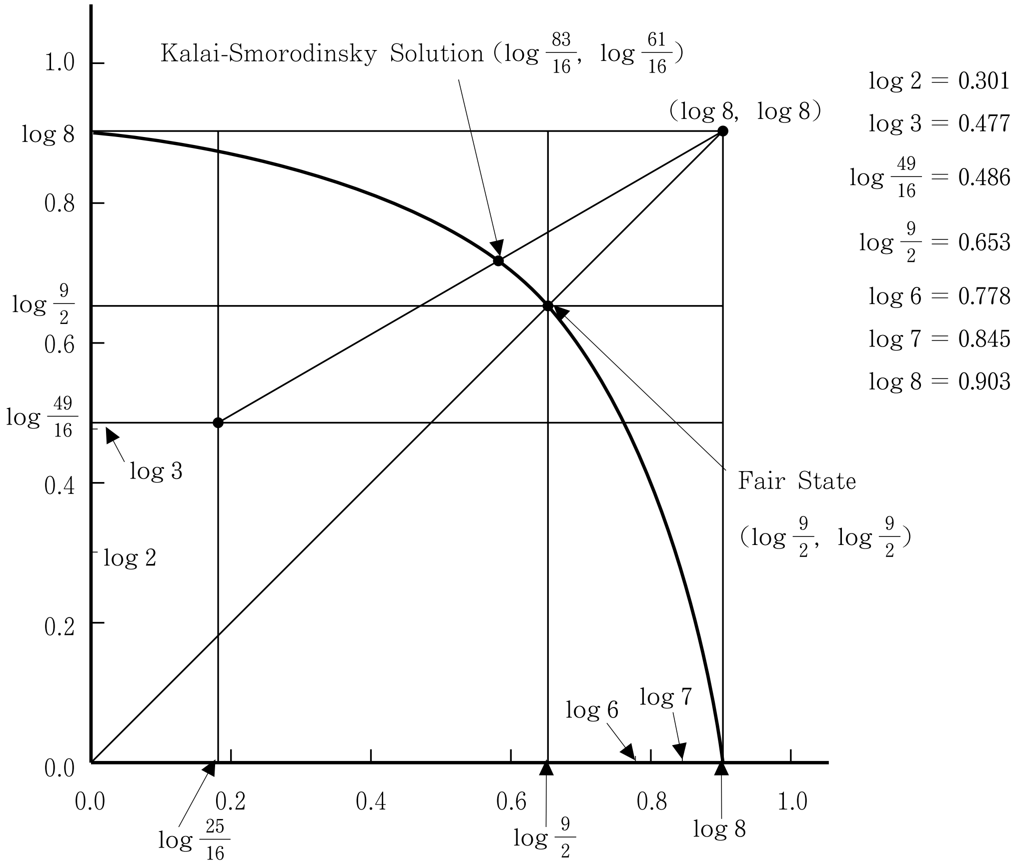

Figure 2 The Gauthier Solution with Two Persons

developments in his thought.

Remark 5. The following graphical observation can be given to the Gauthier solution G^3, which assigns to every $(S, d) \in \boldsymbol{B}^3$ the unique element $\gamma \in \Pi\{\gamma, I(B)\} \in S$, where $\Pi\{\gamma, I(B)\}$ is the parallelepiped with the minimal cubic content as well as the northeast diagonal $\gamma - I(B)$, shown in **Figure 3**.[5] There may be many cases where the Gauthier and the Kalai/Smorodinsky solutions coincide as in Figure 3, however, this result may depend on the curvature of the bargaining region S.

Remark 6. The conceptions of Nash are entirely opposite from those of Kalai/Smorodinsky as well as Gauthier, because Nash views his solution from the disagreement point, while both Kalai/Smorodinsky and Gauthier view theirs from the

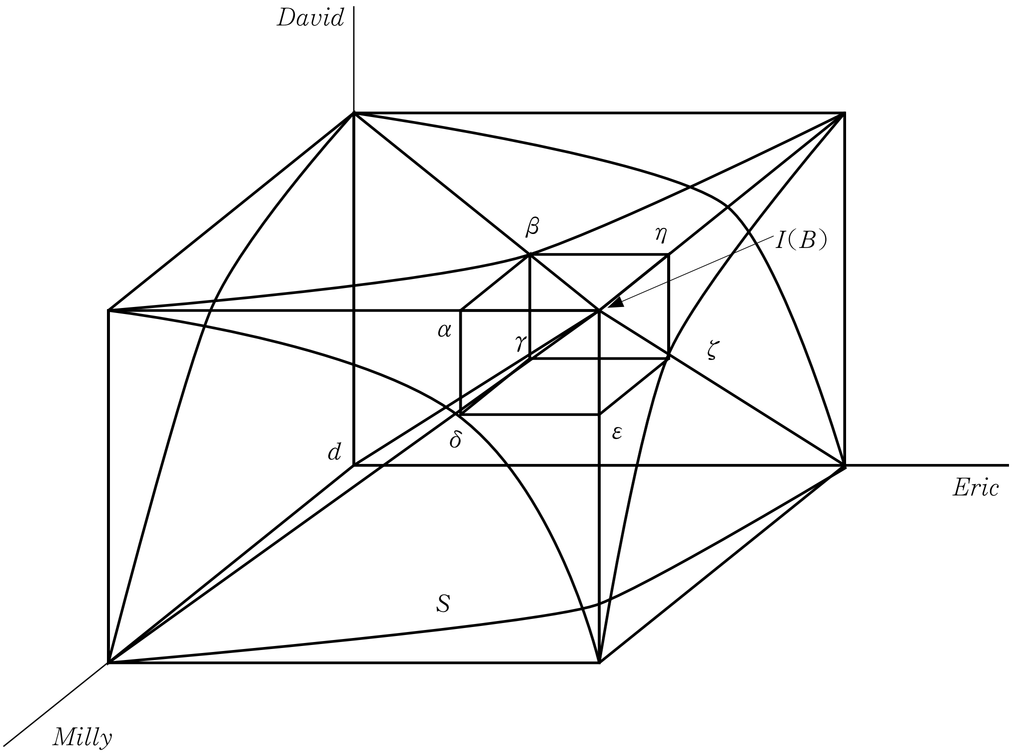

Figure 3 The Gauthier Solution with Three Persons

ideal point. Thomson's(1987) new axiom, ***DMO***(D-Monotonicity or Monotonicity with Respect to the Disagreement Point) should be checked for the Gauthier solution, which is also sensitive to the ideal point.

5. FINAL REMARKS

This paper has led to an axiomatization of the Gauthier solution. The axiom of minimax relative concession requires that the solution minimizes the maximum value among the relative concessions of the parties. As we saw in the preceding subsections, in two-person games the same bargaining outcomes are attainable via either the Gauthier or the Kalai/Smorodinsky solutions on different grounds. The situation, however, could differ when the game involves more than three players, since these solutions do deviate from each other, that is, there is a sharp contrast between the Kalai/Smorodinsky solution and the Gauthier's one, as was seen previously.

This paper has confined itself to the Chapter V(bargaining theory) of David Gauthier's *Morals by Agreement*. The book in its entirety covers broad analysis of justice and fairness. On these issues, the reader should refer to the three papers at the Symposium on *Morals by Agreement* in the issue of *Ethics*, July 1987 and Yi(1992).

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Endnotes

1 Some authors call this a 'utopia point' or an 'aspiration point'.

2 The ways of exodus were surveyed by Kalai(1985, p. 100). There are several attempts to generalize the Kalai/Smorodinsky solution to n-person games satisfying **weakly *PO***, and Gauthier's challenge is one of them, which achieves not weakly ***PO***, but ***PO***, though. Kalai(1985) provided an overview of bargaining solutions.

3 See Gauthier(1985) and (1986, Chapter V, pp. 148–150, and p. 200) for the details of these critics.

4 Alternative version of the theorem reads as follows:

$$\forall (S,\ d) \in \boldsymbol{B}^n,\ G^n(S,\ d) = \text{ Arg Min}_{u \in S} \text{ Max}_{i \in N}\ \{(I_i - u_i)/(I_i - d_i)\}.$$

5 In our **Figure 1**, the parallelpiped reduces to the rectangle.

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