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On Generalized Maass Relations and Their Application to Miyawaki-Ikeda Lifts

by

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Abstract. Some generalizations of the Maass relation for Siegel modular forms of higher degrees have been obtained by several authors. In the present article we first give a new generalization of the Maass relation for Siegel-Eisenstein series of arbitrary degrees. Furthermore, we show that the Duke-Imamoglu-Ibukiyama-Ikeda lifts satisfy this generalized Maass relation with some modifications. As an application of the generalized Maass relation, we give a new computation of the standard L-function of the Miyawaki-Ikeda lift of two elliptic modular forms.

1. Introduction

1.1.

The Maass relation is a relation among Fourier coefficients of Siegel-Eisenstein series of degree two, and the Maass relation characterizes the Saito-Kurokawa lifts (cf. [E-Z 85].) In his article [Ya 86] Yamazaki has obtained a generalization of the Maass relation for Siegel-Eisenstein series of arbitrary degrees. Furthermore, in [Ya 89] Yamazaki obtained a relation among Jacobi-Eisenstein series of arbitrary degrees. Here the Jacobi-Eisenstein series is a Jacobi form which is constructed like the Siegel-Eisenstein series. This relation among Jacobi-Eisenstein series is necessary to obtain a new generalization of the Maass relation, which is different from the generalized Maass relation in [Ya 86]. However, the relation among Jacobi-Eisenstein series in [Ya 89] is not enough to obtain a new generalization of the Maass relation, because in [Ya 89] the Jacobi-Eisenstein series of *index* 1 is treated and we need the relation among the Jacobi-Eisenstein series of *arbitrary index*. One of the aim of the present article is to generalize the relation among Jacobi-Eisenstein series obtained in [Ya 89] for arbitrary index and to give a new generalization of the Maass relation for Siegel-Eisenstein series of general degrees.

On the other hand, a generalization of the Saito-Kurokawa lift for Siegel modular forms of even degrees was conjectured by Duke and Imamoglu, and by Ibukiyama, independently, and the conjecture was solved by Ikeda [Ik 01]. In the present article, we call

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these lifts the Duke-Imamoglu-Ibukiyama-Ikeda lifts. It is known that the Duke-Imamoglu-Ibukiyama-Ikeda lifts satisfy the generalized Maass relations in [Ya 86] by inserting the Satake parameters of the preimage of the Duke-Imamoglu-Ibukiyama-Ikeda lift into the relation (cf. [Ha 11].)

By applying the Duke-Imamoglu-Ibukiyama-Ikeda lift, Ikeda [Ik 06] solved and generalized one of the two conjectures posed by Miyawaki [Mi 92] under a certain assumption. Namely, he obtained lifts from pairs of an elliptic modular form and a Siegel modular form of degree r to Siegel modular forms of degree $2n + r$ under the assumption that the constructed Siegel modular form does not vanish identically. In the present article we call these lifts the Miyawaki-Ikeda lifts. In [Ik 06] Ikeda obtained a conjecture about the relation between the Petersson norm of the Miyawaki-Ikeda lift and a special value of a certain L-function. For more details about the conjecture of non-vanishing of the Miyawaki-Ikeda lift, we refer the reader to [Ik 06].

The purpose of the present article is as follows:

(1) we generalize the relation among Jacobi-Eisenstein series given in [Ya 89] for arbitrary integer-indices and obtain a new generalization of the Maass relation for the Siegel-Eisenstein series of arbitrary degrees (Theorem 1.1),
(2) we show a new generalization of the Maass relation for the Duke-Imamoglu-Ibukiyama-Ikeda lifts (Theorem 1.2),
(3) By using the generalized Maass relation we obtain a new proof of the explicit expression of the standard L-functions of the Miyawaki-Ikeda lift of two elliptic modular forms (Corollary 1.4).

As for generalization of the Maass relation, Kohnen [Ko 02] obtained another kind of generalization of the Maass relation which is related to the Fourier-Jacobi coefficients of *matrix index of size* $2n-1$, while the generalization of the Maass relation in the present article is related to the Fourier-Jacobi coefficients of *integer index*. It is known that the generalized Maass relation in [Ko 02] characterizes the image of the Duke-Imamoglu-Ibukiyama-Ikeda lifts (cf. Kohnen-Kojima [KK 05], Yamana [Ya 10].)

We remark that a certain identity of the spinor L-function of the Miyawaki-Ikeda lift of two elliptic modular forms has been given by Heim [He 12] for the case of degree three and weight twelve. This identity has been generalized in [Ha 13] for any odd degrees $2n-1$ and for any even weights k.

1.2.

We explain our results more precisely. We denote by $\mathfrak{H}_n$ the Siegel upper-half space of size n. For integers n and $k > n + 2$, the Siegel-Eisenstein series of weight k of degree $n + 1$ is defined by

$$E_k^{(n+1)}(Z) := \sum_{M=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_{n+1,0}\backslash \Gamma_{n+1}} \det(CZ + D)^{-k} ,$$

where $\tau \in \mathfrak{H}_{n+1}$, where $\Gamma_{n+1} := \mathrm{Sp}_{n+1}(\mathbb{Z})$ is the symplectic group of size $2n+2$ with entries in $\mathbb{Z}$, and we set $\Gamma_{n+1,0} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{n+1} \mid C = 0 \right\}$. The Fourier-Jacobi expansion of $E_k^{(n+1)}$ is given by

$$E_k^{(n+1)} \left(\begin{pmatrix} \tau & z \\ {}^t z & \omega \end{pmatrix} \right) = \sum_{m=0}^{\infty} e_{k,m}^{(n)}(\tau, z) e^{2\pi i m \omega} ,$$

where $\tau \in \mathfrak{H}_n$, $\omega \in \mathfrak{H}_1$ and $z \in \mathbb{C}^n$. The form $e_{k,m}^{(n)}$ is called the m-th Fourier-Jacobi coefficient of $E_k^{(n+1)}$. We remark that $e_{k,m}^{(n)}$ is a Jacobi form of weight k of index m of degree n (cf. Ziegler [Zi 89].)

We denote by $J_{k,m}^{(n)}$ the space of Jacobi forms of weight k of index m of degree n. For the definition of Jacobi forms of higher degree, we refer the reader to [Zi 89] or Section 2.2 in the present article. We define two kinds of index-shift maps:

$$V_{l,n-l}(p^2) \,:\, J_{k,m}^{(n)} \to J_{k,mp^2}^{(n)} ,$$
$$U(p) \,:\, J_{k,m}^{(n)} \to J_{k,mp^2}^{(n)} .$$

Here the index-shift map $V_{l,n-l}(p^2)$ $(0 \le l \le n)$ is given by the action of the double coset $\Gamma_n \mathrm{diag}(1_l, p1_{n-1}, p^2 1_l, p1_{n-l})\Gamma_n$. For the precise definition of $V_{l,n-1}(p^2)$ see Section 2.4, and we define $(\phi|U(d))(\tau, z) := \phi(\tau, dz)$ for $\phi \in J_{k,m}^{(n)}$ and for any natural number d.

THEOREM 1.1. *Let $e_{k,m}^{(n)}$ be the m-th Fourier-Jacobi coefficient of Siegel-Eisenstein series. Then we obtain the relation*

$$\begin{aligned} & e_{k,m}^{(n)} | \left(V_{0,n}(p^2), \ldots, V_{n,0}(p^2) \right) \\ & = \left(e_{k,\frac{m}{p^2}}^{(n)} | U(p^2), e_{k,m}^{(n)} | U(p), e_{k,mp^2}^{(n)} \right) \begin{pmatrix} 0 & 1 \\ p^{-k} & p^{-k}(-1 + p\,\delta_{p|m}) \\ 0 & p^{-2k+2} \end{pmatrix} A_{2,n+1}^{p,k} , \end{aligned}$$

where the both sides of the above identity are vectors of functions and $A_{2,n+1}^{p,k}$ is a certain matrix with size 2 *times* $(n+1)$ *which depends only on p and k, and where we regard $e_{k,\frac{m}{p^2}}^{(n)}$ as identically* 0 *if $p^2 \nmid m$. Here $\delta_{p|m}$ is defined by 1 or 0, according as $p|m$ or $p \nmid m$. For the precise definition of $A_{2,n+1}^{p,k}$, see Section 2.6.*

The relation in Theorem 1.1 is a new generalization of the Maass relation for Siegel-Eisenstein series of arbitrary degrees. As for the function $e_{k,m}^{(n)}|V_n(p)$, a similar identity has already been given in [Ya 86]. Here the operator $V_n(p)$ is obtained from the double coset $\Gamma_n \mathrm{diag}(1_n, p1_n)\Gamma_n$.

Now we apply the relation in Theorem 1.1 to the Duke-Imamoglu-Ibukiyama-Ikeda lifts. We denote by $S_k(\Gamma_n)$ the space of Siegel cusp forms of weight k of degree n. Let $f \in S_{2k}(\Gamma_1)$ be a normalized Hecke eigenform and let $F \in S_{k+n}(\Gamma_{2n})$ be the Duke-Imamoglu-Ibukiyama-Ikeda lift of f (cf. Ikeda [Ik 06].) We remark that there is no canonical choice of F, however F is determined up to constant multiple. We consider the Fourier-Jacobi expansion of F:

$$F\left(\begin{pmatrix} \tau & z \\ {}^t z & \omega \end{pmatrix}\right) = \sum_{m=1}^{\infty} \phi_m(\tau, z) e^{2\pi i m\omega},$$

where $\tau \in \mathfrak{H}_n$, $\omega \in \mathfrak{H}_1$ and $z \in \mathbb{C}^n$. Then ϕ_m is the m-th Fourier-Jacobi coefficient of F and is a Jacobi cusp form of weight $k+n$ of index m of degree $2n-1$. We denote by $J_{k,m}^{(n)\,cusp}$ the space of Jacobi cusp forms of weight k of index m of degree n. The restriction of the maps $V_{l,n-l}(p^2)$ and $U(p)$ to $J_{k,m}^{(n)\,cusp}$ gives maps from $J_{k,m}^{(n)\,cusp}$ to $J_{k,mp^2}^{(n)\,cusp}$. Let $\alpha_p^{\pm 1}$ be the complex numbers which satisfy

$$(\alpha_p + \alpha_p^{-1}) p^{k-\frac{1}{2}} = a(p),$$

where $a(p)$ is the p-th Fourier coefficient of f.

The following theorem is a generalization of the Maass relation for the Duke-Imamoglu-Ibukiyama-Ikeda lifts, which is different from the ones in [Ko 02] and in [Ha 11].

THEOREM 1.2. *Let $\phi_m \in J_{k+n,m}^{(2n-1)\,cusp}$ be the m-th Fourier-Jacobi coefficient of the Duke-Imamoglu-Ibukiyama-Ikeda lift F as the above. Then we have*

$$\begin{aligned} &\phi_m | \left(V_{0,2n-1}(p^2), \dots, V_{2n-1,0}(p^2)\right) \\ &= p^{-(n-1)(2k-1)} \left(\phi_{\frac{m}{p^2}} | U(p^2), \phi_m | U(p), \phi_{mp^2}\right) \\ &\quad \times \begin{pmatrix} 0 & 1 \\ p^{-k-n} & p^{-k-n}(-1 + p\,\delta_{p|m}) \\ 0 & p^{-2k-2n+2} \end{pmatrix} A'_{2,2n}(\alpha_p), \end{aligned}$$

where $A'_{2,2n}(\alpha_p)$ is a certain matrix with size 2 times $2n$ which depends only on f and p. We regard the form $\phi_{\frac{m}{p^2}}$ as identically zero if $p^2 \nmid m$. The matrix $A'_{2,2n}(\alpha_p)$ is obtained by substituting $X_p = \alpha_p$ into a matrix $A'_{2,2n}(X_p)$. For the precise definition of $A'_{2,2n}(X_p)$, see Section 2.6.

Now we apply the relation in Theorem 1.2 to the Miyawaki-Ikeda lifts of two elliptic modular forms. Let f and F be as above. Let $g \in S_{k+n}(\Gamma_1)$ be a normalized Hecke eigenform. Then one can construct a Siegel cusp form $\mathcal{F}_{f,g}$ of weight $k+n$ of degree $2n-1$:

$$\mathcal{F}_{f,g}(\tau) := \int_{\Gamma_1 \backslash \mathfrak{H}_1} F\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right) \overline{g(\omega)}\, \mathrm{Im}(\omega)^{k+n-2}\, d\omega.$$

The form $\mathcal{F}_{f,g}$ is the Miyawaki-Ikeda lift of g associated to f. It is shown by Ikeda [Ik 06] that if $\mathcal{F}_{f,g}$ is not identically zero, then $\mathcal{F}_{f,g}$ is an eigenfunction for Hecke operators for the Hecke pair $(\Gamma_{2n-1}, \mathrm{Sp}_{2n-1}(\mathbb{Q}))$. Furthermore, the standard L-function of $\mathcal{F}_{f,g}$ is expressed as a certain product of L-functions related to f and g. Now by virtue of Theorem 1.2, we obtain a new proof of these facts by using the generalized Maass relations.

THEOREM 1.3. *Let $\mathcal{F}_{f,g} \in S_{k+n}(\Gamma_{2n-1})$ be the Miyawaki-Ikeda lift of g associated to f. Then*

$$\mathcal{F}_{f,g}|\big(T_{0,2n-1}(p^2), \ldots, T_{2n-1,0}(p^2)\big) = p^{2nk+n-1}\big(p^{-k-n}, p^{-2k-2n+2}\lambda_g(p^2)\big)A'_{2,2n}(\alpha_p)\,\mathcal{F}_{f,g}\,,$$

where $T_{l,2n-1-l}(p^2)$ are Hecke operators (see Section 2.4) and $A'_{2,2n}(\alpha_p)$ is the same matrix in Theorem 1.2. Here $\lambda_g(p^2)$ is the eigenvalue of g for $T_{1,0}(p^2)$.

We denote by $\beta_p^{\pm 1}$ the complex numbers which satisfy:

$$(\beta_p + \beta_p^{-1})p^{\frac{k+n-1}{2}} = b(p)\,,$$

where $b(p)$ is the p-th Fourier coefficient of g. The adjoint L-function of g is defined by

$$L(s, g, \mathrm{Ad}) := \prod_p \big\{(1-p^{-s})(1-\beta_p^2\, p^{-s})(1-\beta_p^{-2}\, p^{-s})\big\}^{-1}\,.$$

COROLLARY 1.4. *If $\mathcal{F}_{f,g}$ is not identically zero, then the Satake parameter of $\mathcal{F}_{f,g}$ at prime p is*

$$\big\{\mu_1^{\pm 1}, \ldots, \mu_{2n-1}^{\pm 1}\big\} = \big\{\beta_p^{\pm 2}, \alpha_p^{\pm 1}p^{-n+\frac{3}{2}}, \alpha_p^{\pm 1}p^{-n+\frac{5}{2}}, \ldots, \alpha_p^{\pm 1}p^{n-\frac{3}{2}}\big\}\,.$$

Furthermore, the standard L-function of $\mathcal{F}_{f,g}$ is

$$L(s, \mathcal{F}_{f,g}, st) = L(s, g, Ad)\prod_{i=1}^{2n-2} L(s+k+n-1-i, f)\,,$$

where $L(s, f)$ is the Hecke L-function of f. (see Section 2.3 for the definition of the standard L-function.)

We remark that Corollary 1.4 has already been shown by Ikeda [Ik 01] for more general case, namely for Siegel modular form $g \in S_{k+n}(\Gamma_r)$. The method in [Ik 01] is based on the theory of automorphic representations. On the other hand, if a Siegel modular form is an eigenform for Hecke operators, the eigenvalues are calculated from the Satake parameters by using the explicit map of the Satake isomorphism. This explicit map is given in [Kr 86]. Hence Theorem 1.3 and Corollary 1.4 are equivalent. Therefore Theorem 1.3 essentially follows from [Ik 01, Proposition 3.1] as a special case of $r = 1$. However, in the present article we obtained a new proof of Theorem 1.3 and Corollary 1.4 by using the generalized Maass relation.

Furthermore, we remark that a certain identity of the spinor L-function of $\mathcal{F}_{f,g}$ has been obtained in [Ha 13] which is a generalization of the case $(n, k) = (2, 12)$ in [He 12].

This paper is organized as follows: In Section 2 we give a notation and review some operators for Jacobi forms, and in Section 3 we shall show a certain relation among Jacobi-Eisenstein series with respect to the index-shift maps. In Section 4 we shall prove Theorem 1.1, while we shall prove Theorem 1.2, Theorem 1.3 and Corollary 1.4 in Section 5.

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2. Operators on Jacobi forms

2.1. Symbols.

We denote by $M_{i,j}(R)$ the set of all i by j matrices with entries in the ring R and put $M_n(R) := M_{n,n}(R)$. For any square matrix $A \in M_n(\mathbb{Z})$ we denote by $\mathrm{rank}_p(A)$ the rank of A in $M_n(\mathbb{Z}/p\mathbb{Z})$. For any two matrices $A \in M_n(\mathbb{Z})$ and $B \in M_{n,m}(\mathbb{Z})$ we write $A[B]$ for ${}^tBAB$. The set of all half-integral symmetric matrices of size n is denoted by Sym_n^*.

We put $J_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and set

$$\mathrm{GSp}_n^+(\mathbb{R}) := \left\{ M \in M_{2n}(\mathbb{R}) \mid M J_n {}^t M = \nu(g) J_n,\ \nu(g) > 0 \right\},$$

where the number $\nu(g)$ is called the similitude of g.

We put $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z}) \subset \mathrm{SL}_{2n}(\mathbb{Z})$. For any square matrix x we set $e(x) := e^{2\pi i\, tr(x)}$, where $\mathrm{tr}(x)$ denotes the trace of x. For any natural number m we put $< m > := \frac{m(m+1)}{2}$.

The symbol $\mathfrak{H}_n$ denotes the Siegel upper-half space of size n. The action of $\mathrm{GSp}_n^+(\mathbb{R})$ on $\mathfrak{H}_n$ is given by $g \cdot \tau := (A\tau + B)(C\tau + D)^{-1}$ for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_n^+(\mathbb{R})$ and for $\tau \in \mathfrak{H}_n$.

The symbol $\mathrm{Hol}(\mathfrak{H}_n \to \mathbb{C})$ (resp. $\mathrm{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \to \mathbb{C})$) denotes the space of all holomorphic function on $\mathfrak{H}_n$ (resp. $\mathfrak{H}_n \times \mathbb{C}^n$.) For any integer k, we define the slash operator $|_k$:

$$(F|_k g)(\tau) := \det(C\tau + D)^{-k} F(g \cdot \tau),$$

where $F \in \mathrm{Hol}(\mathfrak{H}_n \to \mathbb{C})$, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_n^+(\mathbb{R})$ and $\tau \in \mathfrak{H}_n$. By this definition the group $\mathrm{GSp}_n^+(\mathbb{R})$ acts on $\mathrm{Hol}(\mathfrak{H}_n \to \mathbb{C})$.

2.2. Jacobi group.

We define a subgroup of $\mathrm{GSp}_{n+1}^+(\mathbb{R})$:

$$G_n^J := \left\{ \gamma \in \mathrm{GSp}_{n+1}^+(\mathbb{R}) \,\middle|\, \gamma = \begin{pmatrix} A & 0 & B & * \\ * & \nu(\gamma) & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_n^+(\mathbb{R}) \right\}.$$

A bijective map from $\mathrm{GSp}_n^+(\mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}$ to G_n^J is given by

$$\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu), \kappa \right] \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & \nu(g) & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 & \mu \\ {}^t\lambda & 1 & {}^t\mu & {}^t\lambda\mu + \kappa \\ 0 & 0 & 1_n & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_n^+(\mathbb{R})$, $\lambda, \mu \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}$. We identify $\mathrm{GSp}_n^+(\mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}$ and G_n^J. By this bijection the group G_n^J can be regarded as a semi-direct product of $\mathrm{GSp}_n^+(\mathbb{R})$ and $((\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R})$, namely $G_n^J \cong \mathrm{GSp}_n^+(\mathbb{R}) \ltimes ((\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R})$.

Let k and m be integers and let $\phi \in \mathrm{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \to \mathbb{C})$ be a holomorphic function on $\mathfrak{H}_n \times \mathbb{C}^n$. We define the slash operator $|_{k,m}$:

$$(\phi|_{k,m}\gamma)(\tau, z) := ((\phi(\tau, z)e(m\,\omega))|_k\gamma)\, e(-\nu(\gamma)\, m\, \omega),$$

where $\begin{pmatrix} \tau & z \\ {}^t z & \omega \end{pmatrix} \in \mathfrak{H}_{n+1}$, $\tau \in \mathfrak{H}_n$, $\omega \in \mathfrak{H}_1$, $z \in \mathbb{C}^n$ and $\gamma \in G_n^J$. We remark that the RHS of the above definition does not depend on the choice of ω. By this definition, the group G_n^J acts on $\mathrm{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \to \mathbb{C})$.

For $\gamma = [g, (\lambda, \mu), \kappa] \in G_n^J$ we have

$$\begin{aligned}(\phi|_{k,m}\gamma)(\tau, z) = &\det(C\tau + D)^{-k}\, e(-\nu(g)\, m\, ((C\tau + D)^{-1}C)[z + \tau\lambda + \mu]) \\ &\times e(\nu(g)\, m\, ({}^t\lambda\tau\lambda + 2{}^t\lambda z + 2{}^t\lambda\mu + \kappa)) \\ &\times \phi(g \cdot \tau, \nu(g){}^t(C\tau + D)^{-1}(z + \tau\lambda + \mu))\,,\end{aligned}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_n^+(\mathbb{R})$.

We put a discrete subgroup Γ_n^J of G_n^J:

$$\Gamma_n^J := \left\{[M, (\lambda, \mu), \kappa] \in G_n^J \mid M \in \Gamma_n,\ (\lambda, \mu) \in \mathbb{Z}^n \times \mathbb{Z}^n,\ \kappa \in \mathbb{Z}\right\}.$$

We denote by $J_{k,m}^{(n)}$ the space of Jacobi forms of weight k of index m of degree n (cf. Ziegler [Zi 89].) For $n > 1$ the space $J_{k,m}^{(n)}$ is defined by

$$J_{k,m}^{(n)} := \left\{\phi \in \mathrm{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \to \mathbb{C}) \mid \phi|_{k,m}\gamma = \phi \text{ for any } \gamma \in \Gamma_n^J\right\}.$$

2.3. The standard L-functions.

Let $F \in S_k(\Gamma_n)$ be a Siegel cusp form which is an eigenform for all Hecke operators. Let $\{\mu_{0,p}, \mu_{1,p}, \ldots, \mu_{n,p}\}$ be the Satake parameter of F at a prime p. The standard L-function of F is defined by

$$L(s, F, \mathrm{st}) := \prod_p \left\{(1 - p^{-s}) \prod_{i=1}^n (1 - \mu_{i,p}p^{-s})(1 - \mu_{i,p}^{-1}p^{-s})\right\}^{-1}.$$

In our setting we have $\mu_{0,p}^2\mu_{1,p}\cdots\mu_{n,p} = p^{nk-<n>}$.

2.4. Index-shift maps of Jacobi forms.

For any function $\phi \in J_{k,m}^{(n)}$ and for any matrix $g \in \mathrm{GSp}_n^+(\mathbb{R}) \cap M_{2n}(\mathbb{Z})$ we define

$$\phi|V(\Gamma_n g\Gamma_n) := \sum_i \phi|_{k,m}[g_i, (0, 0), 0]\,,$$

where $\Gamma_n g\Gamma_n = \bigcup_i \Gamma_n g_i$ is a coset decomposition. It is known that $\phi|V(\Gamma_n g\Gamma_n)$ is well-defined and belongs to $J_{k,\nu(g)m}^{(n)}$.

For any integer l $(0 \le l \le n)$, we define

$$\phi|V_{l,n-l}(p^2) := \phi|V(\Gamma_n \mathrm{diag}(1_l, p1_{n-l}, p^2 1_l, p1_{n-l})\Gamma_n)\,.$$

For any non-negative integer d we define

$$(\phi|U(d))(\tau, z) := \phi(\tau, dz)\,.$$

Then $\phi|V_{l,n-l}(p^2) \in J^{(n)}_{k,mp^2}$ and $\phi|U(d) \in J^{(n)}_{k,md^2}$.

Let F be a Siegel modular form of weight k of degree n. Let g be an element of $\mathrm{GSp}^+_n(\mathbb{R}) \cap M_{2n}(\mathbb{Z})$. For any double coset $\Gamma_n g \Gamma_n$, the Hecke operator $T(\Gamma_n g \Gamma_n)$ is defined by

$$F|T(\Gamma_n g\Gamma_n) := \nu(g)^{nk-<n>} \sum_i F|_k g_i\,,$$

where $\Gamma_n g \Gamma_n = \bigcup_i \Gamma_n g_i$ is a coset decomposition. For any integer l $(0 \le l \le n)$, we define

$$F|T_{l,n-l}(p^2) := F|T(\Gamma_n \mathrm{diag}(1_l, p1_{n-l}, p^2 1_l, p1_{n-l})\Gamma_n)\,.$$

For any Jacobi form $\phi \in J^{(n)}_{k,m}$, we define the function

$$W(\phi)(\tau) := \phi(\tau, 0)$$

for $\tau \in \mathfrak{H}_n$. From the definition of Jacobi form, it follows that $W(\phi)$ is a Siegel modular form of weight k of degree n.

Furthermore, due to a straightforward calculation, we obtain

$$W(\phi)|T(\Gamma_n g \Gamma_n) = \nu(g)^{nk-<n>} W(\phi|V(\Gamma_n g \Gamma_n)) \tag{2.1}$$

for any Jacobi form $\phi \in J^{(n)}_{k,m}$ and for any $g \in \mathrm{GSp}^+_n(\mathbb{R}) \cap M_{2n}(\mathbb{Z})$.

2.5. Siegel Φ-operator for Jacobi forms.

Let $\phi \in \mathrm{Hol}(\mathfrak{H}_n \times \mathbb{C}^n \to \mathbb{C})$ be a holomorphic function. We define the Siegel Φ-operator:

$$\Phi(\phi)\,(\tau_1, z_1) := \lim_{t\to+\infty} \phi\left(\begin{pmatrix}\tau_1 & 0\\ 0 & it\end{pmatrix}, \begin{pmatrix} z_1 \\ 0\end{pmatrix}\right)\,,$$

where $\tau_1 \in \mathfrak{H}_{n-1}$ and $z_1 \in \mathbb{C}^{n-1}$.

It is known that if $\phi \in J^{(n)}_{k,m}$ is a Jacobi form, then the function $\Phi(\phi)$ is also a Jacobi form which belongs to $J^{(n-1)}_{k,m}$.

2.6. The Satake isomorphism and the Siegel Φ-operator.

Let $\mathcal{H}^n_p$ be the local Hecke ring with respect to the Hecke pair $(\Gamma_n, \mathrm{GSp}^+_n(\mathbb{R}) \cap M_{2n}(\mathbb{Z}[p^{-1}]))$. We denote by $\mathbb{C}[X_0^{\pm1}, \ldots, X_n^{\pm1}]^{W_n}$ the subring of the polynomial ring $\mathbb{C}[X_0^{\pm1}, \ldots, X_n^{\pm1}]$ which is invariant under the action of the Weyl group W_n associated to the symplectic group. The Satake isomorphism $\varphi_n : \mathcal{H}^n_p \to \mathbb{C}[X_0^{\pm1}, \ldots, X_n^{\pm1}]^{W_n}$ is given by

$$\Gamma_n g \Gamma_n = \bigcup_i \Gamma_n \begin{pmatrix} p^l\, {}^tD_i^{-1} & B_i \\ 0 & D_i\end{pmatrix} \mapsto X_0^l \sum_i \prod_j \left(\frac{X_j}{p^j}\right)^{l_{i,j}}\,,$$

where $\nu(g) = p^l$ and $D_i = \begin{pmatrix} p^{l_{i,1}} & * & * \\ & \ddots & * \\ & & p^{l_{i,n}} \end{pmatrix}$ (cf. Andrianov [An 79]).

We write $\varphi = \varphi_n$ for simplicity. In this article we consider the subring of $\mathcal{H}_p^n$ which is generated by $T_{0,n}(p^2)^{\pm 1}$ and $T_{l,n-l}(p^2)$ $(l = 1, \ldots, n)$.

The following proposition follows from [Kr 86, Satz].

PROPOSITION 2.1. *If $n \geq 2$ we have*

$$\begin{aligned}
\varphi(T_{n,0}(p^2)) &= X_n\big\{\big(X_n^{-1} + (p-1)p^{-1} + X_n\big)\varphi(T_{n-1,0}(p^2)) \\
&\quad + (p^2-1)p^{-1}\varphi(T_{n-2,1}(p^2))\big\}, \\
\varphi(T_{1,n-1}(p^2)) &= X_n\big\{p^{1-n}\,\varphi(T_{1,n-2}(p^2)) \\
&\quad + \big(X_n^{-1} + (p-1)p^{-n} + X_n\big)\varphi(T_{0,n-1}(p^2))\big\}, \\
\varphi(T_{0,n}(p^2)) &= X_n\big\{p^{-n}\,\varphi(T_{0,n-1}(p^2))\big\},
\end{aligned}$$

and for $1 < j < n$ we have

$$\begin{aligned}
\varphi(T_{j,n-j}(p^2)) &= X_n\big\{p^{j-n}\varphi(T_{j,n-j-1}(p^2)) \\
&\quad + \big(X_n^{-1} + p^{j-n-1}(p-1) + X_n\big)\varphi(T_{j-1,n-j}(p^2)) \\
&\quad + (p^{2n-2j+2}-1)p^{j-n-1}\varphi(T_{j-2,n-j+1}(p^2))\big\}\,.
\end{aligned}$$

Proof. We obtain this proposition by replacing p^{-r} in [Kr 86, Satz] by $p^{-n}X_n$. For the detail the reader is referred to [Kr 86, Satz]. □

Now for integers l $(2 \leq l)$, t $(0 \leq t \leq l)$, j $(0 \leq j \leq l)$, we put

$$b_{t,j} := b_{t,j,l,p}(X_l) = \begin{cases} (p^{2l-2j+2}-1)p^{j-1-l}X_l & \text{if } t = j-2\,, \\ 1 + p^{j-1-l}(p-1)X_l + X_l^2 & \text{if } t = j-1\,, \\ p^{-l+j}X_l & \text{if } t = j\,, \\ 0 & \text{otherwise,} \end{cases}$$

and we put a matrix

$$B_{l,l+1}(X_l) := \big(b_{t,j}\big)_{\substack{t=0,\ldots,l-1 \\ j=0,\ldots,l}} = \begin{pmatrix} b_{0,0} & \cdots & b_{0,l} \\ \vdots & \ldots & \vdots \\ b_{l-1,0} & \cdots & b_{l-1,l} \end{pmatrix}$$

with entries in $\mathbb{C}[X_l, X_l^{-1}]$. From Proposition 2.1 and from the definition of $B_{l,l+1}(X_l)$, we have the identity:

$$(\varphi(T_{0,l}(p^2)), \ldots, \varphi(T_{l,0}(p^2))) = (\varphi(T_{0,l-1}(p^2)), \ldots, \varphi(T_{l-1,0}(p^2)))B_{l,l+1}(X_l)\,.$$

For Jacobi forms we obtain the following lemma.

LEMMA 2.2. *Let $\phi \in J_{k,m}^{(l)}$ be a Jacobi form such that $\Phi(\phi)$ is not identically zero. Then we have*

$$\Phi(\phi|(V_{0,l}(p^2), \ldots, V_{l,0}(p^2))) = \Big(\Phi(\phi)|(V_{0,l-1}(p^2), \ldots, V_{l-1,0}(p^2))\Big)\,B_{l,l+1}(p^{l-k})\,,$$

where we put $\phi|(V_{0,l}(p^2),\dots,V_{l,0}(p^2)) := (\phi|V_{0,l}(p^2),\dots,\phi|V_{l,0}(p^2))$.

Proof. Let $\gamma = \left[\begin{pmatrix} A & B \\ 0 & D\end{pmatrix}, (0,0), 0\right] \in G_l^J$ with $A = \begin{pmatrix} A^* & 0 \\ \mathfrak{a} & a\end{pmatrix}$, $B = \begin{pmatrix} B^* & \mathfrak{b}_1 \\ \mathfrak{b}_2 & b\end{pmatrix}$, $D = \begin{pmatrix} D^* & \mathfrak{d} \\ 0 & d\end{pmatrix}$, where $A^*, D^* \in \mathrm{GL}_{l-1}(\mathbb{R})$ and $B^* \in M_{l-1}(\mathbb{R})$. Then

$$\Phi(\phi|_{k,m}\gamma) = d^{-k}\Phi(\phi)|_{k,m}\gamma^*\,,$$

where $\gamma^* = \left[\begin{pmatrix} A^* & B^* \\ 0 & D^*\end{pmatrix}, (0,0), 0\right] \in G_{l-1}^J$.

The rest of the proof of this lemma is the same to the case of Siegel modular forms (cf. [Kr 86, Satz].) Thus we conclude this lemma. $\square$

We define a matrix

$$B_{2,n+1}(X_2, X_3, \dots, X_n) := \prod_{l=2}^{n} B_{l,l+1}(X_l)\,,$$

which entries are in $\mathbb{C}[X_2^{\pm},\dots,X_n^{\pm}]$. Then we have

$$(\varphi(T_{0,n}(p^2)),\dots,\varphi(T_{n,0}(p^2))) = (\varphi(T_{0,1}(p^2)),\varphi(T_{1,0}(p^2)))B_{2,n+1}(X_2,\dots,X_n)\,.$$

The precise expression of $\varphi(T_{l,n-l}(p^2))$ by using the elementary symmetric polynomials has been given in [Kr 86, Korollar 2].

To explain our results we define two matrices $A_{2,n+1}^{p,k}$ and $A'_{2,2n}(X_p)$. First we define a $2\times(n+1)$ matrix

$$A_{2,n+1}^{p,k} := B_{2,n+1}(p^{2-k}, p^{3-k},\dots,p^{n-k})\,.$$

We remark that the matrix $A_{2,n+1}^{p,k}$ depends only on the prime p and the integer $k>0$.

We set a $2\times 2n$ matrix

$$B'_{2,2n}(X_2,\dots,X_{2n-1}) := \left(\prod_{i=2}^{2n-1} X_i\right)^{-1} B_{2,2n}(X_2,\dots,X_{2n-1})\,.$$

From the definition of $B_{2,2n}(X_2,\dots,X_{2n-1})$ it is not difficult to see that the entries in the matrix $B'_{2,2n}(X_2,\dots,X_{2n-1})$ belong to $\mathbb{C}[X_2+X_2^{-1},\dots,X_{2n-1}+X_{2n-1}^{-1}]$. We define a $2\times 2n$ matrix

$$A'_{2,2n}(X_p) := B'_{2,2n}(p^{\frac{3}{2}-n}X_p, p^{\frac{5}{2}-n}X_p,\dots,p^{-\frac{3}{2}+n}X_p)\,.$$

In Section 5.3 we will show $A'_{2,2n}(X_p) = A'_{2,2n}(X_p^{-1})$.

3. Jacobi-Eisenstein series

The goal of this section is to show a certain relation among Jacobi-Eisenstein series with respect to the index-shift maps $V_{l,n-l}(p^2)$ $(l=0,\dots,n)$. In Section 4 we shall translate such relation to the relation among Fourier-Jacobi coefficients $e_{k,m}^{(n)}$ and will prove Theorem 1.1.

3.1. Definition of Jacobi-Eisenstein series.

For integers k, m and n, we define the Jacobi-Eisenstein series of weight k of index m of degree n by

$$E^{(n)}_{k,m}(\tau, z) := \sum_{\gamma \in \Gamma^J_{n,0}\backslash \Gamma^J_n} (1|_{k,m}\gamma)\,,$$

where we put

$$\Gamma^J_{n,0} := \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 & \mu \\ 0 & 1 & {}^t\mu & \kappa \\ 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^J_n \;\middle|\; \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n,\, \mu \in \mathbb{Z}^n,\, \kappa \in \mathbb{Z} \right\}.$$

It is known that if $k > n+2$, then $E^{(n)}_{k,m}$ converges and belongs to $J^{(n)}_{k,m}$ (cf. Ziegler [Zi 89].)

The purpose of this section is to show that $E^{(n)}_{k,m}|V_{l,n-l}(p^2)$ is a linear combination of three forms $E^{(n)}_{k,\frac{m}{p^2}}|U(p^2)$, $E^{(n)}_{k,m}|U(p)$ and $E^{(n)}_{k,mp^2}$.

LEMMA 3.1. *Let m and n be positive integers. Then the forms $\left\{E^{(n)}_{k,\frac{m}{d^2}}|U(d)\right\}_d$ are linearly independent, where d runs over all positive integers such that $d^2|m$.*

Proof. Let Φ be the Siegel Φ-operator for Jacobi forms introduced in Section 2.5. It follows from the definition that $\Phi(E^{(n)}_{k,m}) = E^{(n-1)}_{k,m}$. Hence it is enough to show that the forms $\left\{E^{(1)}_{k,\frac{m}{d^2}}|U(d)\right\}_d$ are linearly independent.

Let $E^{(1)}_{k,m}(\tau, z) = \sum_{n',r} c(n', r)\, e(n'\tau + rz)$ be the Fourier expansion of $E^{(1)}_{k,m}$. We call $c(n', r)$ the (n', r)-th Fourier coefficient of $E^{(1)}_{k,m}$. Let $n' > 0$ and $r \geq 0$ be integers such that $4n'm - r^2 > 0$. Then it is known that the (n', r)-th Fourier coefficient of $E^{(1)}_{k,m}$ is not zero (cf. Eichler-Zagier [E-Z 85, p.17–p.20].) On the other hand, for any $d > 1$ such that $d^2|m$, the (n', r)-th Fourier coefficient of $E^{(1)}_{k,\frac{m}{d^2}}|U(d)$ is zero unless $d|r$. Therefore we obtain this lemma. □

3.2. Definition of a form $K^{(n)}_{i,j}$.

We quote some symbols from [Ya 89]. For a fixed prime p and for $0 \leq i \leq j \leq n$, we put

$$\delta_{i,j} := \mathrm{diag}(1_i,\, p1_{j-i},\, p^2 1_{n-j})$$

and

$$\delta_i := \delta_{i,n} = \mathrm{diag}(1_i,\, p1_{n-i})\,.$$

And for $x = \mathrm{diag}(0_i, x_{2,2}, 0_{n-j})$ with $x_{2,2} = {}^t x_{2,2} \in M_{j-i}(\mathbb{Z})$ we set

$$\delta_{i,j}(x) := \begin{pmatrix} p^2\delta^{-1}_{i,j} & x \\ 0_n & \delta_{i,j} \end{pmatrix}.$$

We denote by $\Gamma_{n,0}$ the set of all matrices $\begin{pmatrix} A & B \\ 0_n & D \end{pmatrix}$ in Γ_n. We set

$$\Gamma(\delta_{i,j}) := \left\{ \begin{pmatrix} A & B \\ 0_n & D \end{pmatrix} \in \Gamma_{n,0} \;\middle|\; A \in \delta_{i,j}\mathrm{GL}_n(\mathbb{Z})\,\delta_{i,j}^{-1} \right\},$$
$$\Gamma(\delta_i) := \left\{ \begin{pmatrix} A & B \\ 0_n & D \end{pmatrix} \in \Gamma_{n,0} \;\middle|\; A \in \delta_i\mathrm{GL}_n(\mathbb{Z})\,\delta_i^{-1} \right\}$$

and put a subgroup $\Gamma(\delta_{i,j}(x))$ of $\Gamma(\delta_{i,j})$:

$$\Gamma(\delta_{i,j}(x)) := \Gamma_n \cap (\delta_{i,j}(x)^{-1}\Gamma_{n,0}\,\delta_{i,j}(x)).$$

For $\lambda \in \mathbb{Z}^n$ and for $M \in \mathrm{GSp}_n^+(\mathbb{R}) \cap M_{2n}(\mathbb{Z})$ we put

$$j(k,m;M,\lambda)(\tau,z) := (1|_{k,m}[1_{2n},(\lambda,0),0][M,(0,0),0])(\tau,z).$$

For two matrices $x = \mathrm{diag}(0_i, x_{2,2}, 0_{n-j})$ and $y = \mathrm{diag}(0_j, y_{2,2}, 0_{n-j})$ such that $x_{2,2} = {}^t x_{2,2}$, $y_{2,2} = {}^t y_{2,2} \in M_{j-i}(\mathbb{Z})$, we say they are equivalent and write $[x] = [y]$, if there exists a matrix $u = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ p\,u_{2,1} & u_{2,2} & u_{2,3} \\ p^2 u_{3,1} & p\,u_{3,2} & u_{3,3} \end{pmatrix} \in \delta_{i,j}\mathrm{GL}_n(\mathbb{Z})\delta_{i,j}^{-1} \cap \mathrm{GL}_n(\mathbb{Z})$ which satisfies $u_{2,2}x_{2,2}{}^t u_{2,2} \equiv y_{2,2} \bmod p$, where $u_{2,2} \in M_{j-i}(\mathbb{Z})$, $u_{1,1} \in M_i(\mathbb{Z})$ and $u_{3,3} \in M_{n-j}(\mathbb{Z})$.

We define a function $K_{i,j}^{\alpha}$ on $(\tau,z) \in \mathfrak{H}_n \times \mathbb{C}^n$ by

$$K_{i,j}^{\alpha} := K_{i,j,m,p}^{\alpha}(\tau,z) = \sum_{\substack{[x] \\ rank_p(x)=\alpha}} \sum_{M \in \Gamma(\delta_{i,j}(x))\backslash\Gamma_n} \sum_{\lambda \in \mathbb{Z}^n} j(k,m;\delta_{i,j}(x)M,\lambda)(\tau,z),$$

where in the first summation on the RHS, $[x]$ runs over all equivalence classe which satisfy $\mathrm{rank}_p(x) = \alpha$. A straightforward calculation shows that the function ϕ defined by

$$\phi(\tau,z) = \sum_{M \in \Gamma(\delta_{i,j}(x))\backslash\Gamma_n} \sum_{\lambda \in \mathbb{Z}^n} j(k,m;\delta_{i,j}(x)M,\lambda)(\tau,z)$$

satisfies the transformation formula $\phi|_{k,mp^2}\gamma = \phi$ for any element $\gamma \in \Gamma_n^J$. Moreover, the convergence of ϕ can be shown as in [Zi 89, Theorem 2.1]. Hence $K_{i,j}^{\alpha}$ belongs to $J_{k,mp^2}^{(n)}$. In Lemma 3.8 we will show that the form $K_{i,j}^{\alpha}$ is a linear combination of three forms $E_{k,\frac{m}{p^2}}^{(n)}|U(p^2)$, $E_{k,m}^{(n)}|U(p)$ and $E_{k,mp^2}^{(n)}$.

PROPOSITION 3.2 (Yamazaki [Ya 89]). *The double coset* $\Gamma_n \begin{pmatrix} \delta_l & 0_n \\ 0_n & p^2\delta_l^{-1} \end{pmatrix} \Gamma_n$ *is a disjoint union*

$$\Gamma_n \begin{pmatrix} \delta_l & 0_n \\ 0_n & p^2\delta_l^{-1} \end{pmatrix} \Gamma_n = \bigcup_{\substack{i,j \\ 0 \le i \le j \le n}} \bigcup_{\substack{[x] \\ rank_p(x)=l-n-i+j}} \Gamma_{n,0}\delta_{i,j}(x)\Gamma_n,$$

where in the last union on the RHS, $[x]$ *runs over all equivalence classes which satisfy* $rank_p(x) = l - n - i + j$.

Proof. This proposition has been shown in [Ya 89, Corollary 2.2]. □

LEMMA 3.3. *We obtain*

$$E^{(n)}_{k,m}|V_{l,n-l}(p^2) = \sum_{\substack{i,j\\0\le i\le j\le n}} K^{l-i-n+j}_{i,j}\,.$$

Proof. It follows from Proposition 3.2 and from the definitions of $E^{(n)}_{k,m}$, $V_{l,n-l}(p^2)$ and $K^{\alpha}_{i,j}$. □

LEMMA 3.4. *If $p^2|m$, then*

$$K^{\alpha}_{i,j} = p^{-k(2n-i-j)+(n-j)(n-i+1)} \sum_{\substack{x=diag(0_i,x_{2,2},0_{n-j})\\x_{2,2}={}^tx_{2,2}\in M_{j-i}(\mathbb{Z})\ mod\ p\\rank_p(x_{2,2})=\alpha}} \sum_{M\in\Gamma(\delta_{i,j})\backslash\Gamma_n}$$
$$\times \sum_{\lambda\in(p^2\mathbb{Z})^i\times(p\mathbb{Z})^{j-i}\times\mathbb{Z}^{n-j}} j\left(k,\frac{m}{p^2};\begin{pmatrix}1_n & p^{-1}x\\0 & 1_n\end{pmatrix}M,\lambda\right)(\tau,p^2z)\,.$$

If $p^2 \nmid m$, then

$$K^{\alpha}_{i,j} = p^{-k(2n-i-j)+(n-j)(n-i+1)} \sum_{\substack{x=diag(0_i,x_{2,2},0_{n-j})\\x_{2,2}={}^tx_{2,2}\in M_{j-i}(\mathbb{Z})\ mod\ p\\rank_p(x_{2,2})=\alpha}} \sum_{M\in\Gamma(\delta_{i,j})\backslash\Gamma_n}$$
$$\times \sum_{\lambda\in(p\mathbb{Z})^i\times\mathbb{Z}^{n-i}} j\left(k,m;\begin{pmatrix}1_n & p^{-1}x\\0 & 1_n\end{pmatrix}M,\lambda\right)(\tau,pz)\,.$$

We remark that this lemma has been shown for the case $m=1$ by Yamazaki [Ya 89].

Proof. The proof of this lemma is an analogue to [Ya 89]. If $p^2 \nmid m$, then the proof is similar to the case $m=1$. Hence we assume $p^2|m$ and shall prove this lemma.

We put $U := \left\{\left(\begin{smallmatrix}1_n & s\\0 & 1_n\end{smallmatrix}\right) \mid s={}^t s\in M_n(\mathbb{Z})\right\}$. Then the set

$$U' := \left\{\left(\begin{smallmatrix}1_n & s\\0 & 1_n\end{smallmatrix}\right)\,\middle|\, s=\left(\begin{smallmatrix}0&0&0\\0&0&s_{23}\\0&{}^ts_{23}&s_{33}\end{smallmatrix}\right) \bmod p,\ s_{23}\in M_{j-i,n-j}(\mathbb{Z}),\ s_{33}={}^ts_{33}\in M_{n-j}(\mathbb{Z})\right\}$$

is a complete set of representatives of $\Gamma(\delta_{i,j}(x))\backslash\Gamma(\delta_{i,j}(x))U$. Thus

$$\sum_{\substack{[x]\\rank_p(x)=\alpha}} \sum_{M\in\Gamma(\delta_{i,j}(x))\backslash\Gamma_n} \sum_{\lambda\in\mathbb{Z}^n} j(k,m;\delta_{i,j}(x)M,\lambda)(\tau,z)$$
$$= \sum_{\substack{[x]\\rank_p(x)=\alpha}} \sum_{M\in\Gamma(\delta_{i,j}(x))U\backslash\Gamma_n} \sum_{\lambda\in\mathbb{Z}^n} j(k,m;\delta_{i,j}(x)M,\lambda)(\tau,z)$$
$$\times \sum_{\left(\begin{smallmatrix}1_n & s\\0 & 1_n\end{smallmatrix}\right)\in U'} e(p^2m\,{}^t\lambda\delta^{-1}_{i,j}s\delta^{-1}_{i,j}\lambda)$$

$$= p^{(j-i)(n-j)+(n-j)(n-j+1)} \sum_{\substack{[x] \\ rank_p(x)=\alpha}} \sum_{M\in\Gamma(\delta_{i,j}(x))U\backslash\Gamma_n} \sum_{\lambda\in\mathbb{Z}^n} \times j(k,m;\delta_{i,j}(x)M,\lambda)(\tau,z)\,.$$

We remark

$$j(k,m;\delta_{i,j}(x),\lambda)(\tau,z) = p^{-k(2n-i-j)} e\big(m{}^t\lambda(p^2\delta_{i,j}^{-1}\tau\delta_{i,j}^{-1} + p^{-1}x)\lambda + 2p^2m{}^t\lambda\delta_{i,j}^{-1}z\big)\,.$$

Hence if we put $\lambda' = p^2\delta_{i,j}^{-1}\lambda$, then $\lambda' \in (p^2\mathbb{Z})^i \times (p\mathbb{Z})^{j-i} \times \mathbb{Z}^{n-j}$ and we have

$$j(k,m;\delta_{i,j}(x),\lambda)(\tau,z) = p^{-k(2n-i-j)} j(k,p^{-2}m;\begin{pmatrix} 1_n & p^{-1}x \\ 0 & 1_n \end{pmatrix},\lambda').(\tau,p^2z)\,.$$

Thus

$$K_{i,j}^{\alpha} = p^{-k(2n-i-j)+(n-j)(n-i+1)} \sum_{\substack{[x] \\ rank_p(x)=\alpha}} \sum_{M\in\Gamma(\delta_{i,j}(x))U\backslash\Gamma_n} \times \sum_{\lambda'\in(p^2\mathbb{Z})^i\times(p\mathbb{Z})^{j-i}\times\mathbb{Z}^{n-j}} j(k,p^{-2}m;\begin{pmatrix} 1_n & p^{-1}x \\ 0 & 1_n \end{pmatrix}M,\lambda')(\tau,p^2z)$$

$$= p^{-k(2n-i-j)+(n-j)(n-i+1)} \sum_{\substack{[x] \\ rank_p(x)=\alpha}} \sum_{\begin{pmatrix} u & 0 \\ 0 & {}^tu^{-1} \end{pmatrix}\in\Gamma(\delta_{i,j}(x))U\backslash\Gamma(\delta_{i,j})} \sum_{M\in\Gamma(\delta_{i,j})\backslash\Gamma_n} \times \sum_{\lambda'\in(p^2\mathbb{Z})^i\times(p\mathbb{Z})^{j-i}\times\mathbb{Z}^{n-j}} j(k,p^{-2}m;\begin{pmatrix} 1_n & p^{-1}u^{-1}x{}^tu^{-1} \\ 0 & 1_n \end{pmatrix}M,{}^tu\lambda')(\tau,p^2z)\,.$$

Here, the matrix u in the above summation belongs to $\delta_{i,j}\mathrm{GL}(n,\mathbb{Z})\delta_{i,j}^{-1}\cap\mathrm{GL}(n,\mathbb{Z})$. Hence ${}^tu$ stabilizes the lattice $(p^2\mathbb{Z})^i \times (p\mathbb{Z})^{j-i} \times \mathbb{Z}^{n-j}$. Furthermore, the summation over the equivalence classes $[x]$ and the summation over the representatives of $\Gamma(\delta_{i,j}(x))U\backslash\Gamma(\delta_{i,j})$ turn into the summation over $x = \mathrm{diag}(0, x_{2,2}, 0)$ such that $x_{2,2} = {}^tx_{2,2} \in M_{j-i}(\mathbb{Z}) \bmod p$ and $\mathrm{rank}_p(x) = \alpha$. Therefore we conclude this lemma. $\square$

3.3. Summation $G_j^n(m,\lambda)$.

We define

$$g_p(n,i) := \begin{cases} \prod_{a=1}^{i}(p^{n-a+1}-1)(p^a-1)^{-1} & \text{if } 1\le i\le n\,, \\ 1 & \text{if } i=0\,, \\ 0 & \text{otherwise.} \end{cases}$$

For any $\lambda \in \mathbb{Z}^n$ and for $0 \le j \le n$ we define

$$G_j^n(m,\lambda) := \sum_{\substack{x={}^tx\in M_n(\mathbb{Z}/p\mathbb{Z}) \\ rank_p x=j}} e\left(\frac{m}{p}{}^t\lambda x\lambda\right).$$

PROPOSITION 3.5. *For $m \in \mathbb{Z}$ and for $\lambda \in \mathbb{Z}^n$ we have*

$$G_j^n(m,\lambda) = \begin{cases} p^{\lfloor \frac{j}{2} \rfloor \left(\lfloor \frac{j}{2} \rfloor + 1 \right)} g_p(n,j) \displaystyle\prod_{\substack{\alpha=1 \\ \alpha : odd}}^{j} (p^\alpha - 1) & \text{if } m\lambda \equiv 0 \mod p, \\ (-1)^j p^{\lfloor \frac{j}{2} \rfloor \left(\lfloor \frac{j}{2} \rfloor + 1 \right)} g_p\left(n-1, 2\lfloor \frac{j}{2} \rfloor\right) \displaystyle\prod_{\substack{\alpha=1 \\ \alpha : odd}}^{j-1} (p^\alpha - 1) & \text{if } m\lambda \not\equiv 0 \mod p. \end{cases}$$

Proof. If $p|m$, then $G_j^n(m,\lambda) = G_j^n(1,0)$. And if $p \not| m$, then $G_j^n(m,\lambda) = G_j^n(1,\lambda)$. Hence we need to calculate the case $m = 1$. The calculation of $G_j^n(1,\lambda)$ has already been obtained by [Ya 89, Lemma 3.1]. □

3.4. Some cardinalities.

In this subsection we will give some lemmas to calculate $K_{i,j}^\alpha$.

For $0 \le i \le j \le n$, we put

$$H_i := \delta_i \mathrm{GL}_n(\mathbb{Z}) \delta_i^{-1} \cap \mathrm{GL}_n(\mathbb{Z}),$$
$$H_{i,j} := \delta_{i,j} \mathrm{GL}_n(\mathbb{Z}) \delta_{i,j}^{-1} \cap \mathrm{GL}_n(\mathbb{Z}).$$

We define two sets

$$S_i := \left\{ \begin{pmatrix} * & * \\ p\,{}^t b & * \end{pmatrix}^{-1} \in \mathrm{GL}_n(\mathbb{Z}) \;\middle|\; b \in \mathbb{Z}^i \right\},$$
$$S_{i,j} := \left\{ \begin{pmatrix} * & * & * \\ p^2\,{}^t b_1 & p\,{}^t b_2 & * \end{pmatrix}^{-1} \in \mathrm{GL}_n(\mathbb{Z}) \;\middle|\; b_1 \in \mathbb{Z}^i,\ b_2 \in \mathbb{Z}^{j-i} \right\},$$

where b, b_1 and b_2 in the above sets are column vectors.

LEMMA 3.6. *We have*

$$|H_i \backslash GL_n(\mathbb{Z})| = g_p(n,i),$$
$$|H_i \backslash S_i| = g_p(n-1,i).$$

Furthermore, we have

$$\left| H_{i,j} \backslash GL_n(\mathbb{Z}) \right| = p^{i(n-j)} g_p(n,j)\, g_p(j,i),$$
$$\left| H_{i,j} \backslash S_i \right| = p^{i(n-j)} g_p(n-1,i)\, g_p(n-i,n-j),$$
$$\left| H_{i,j} \backslash S_{i,j} \right| = p^{i(n-1-j)} g_p(n-1,j)\, g_p(j,i).$$

Proof. These are elementary. We leave details to the reader. □

LEMMA 3.7. *Let $B(\lambda)$ be a function on $\lambda \in \mathbb{Z}^n$. We put $L_0 := \left(p^2\mathbb{Z}\right)^i \times (p\mathbb{Z})^{j-i} \times \mathbb{Z}^{n-j}$. We assume that the sum $\displaystyle\sum_{A \in H_{i,j} \backslash GL_n(\mathbb{Z})} \sum_{\lambda \in L_0} B({}^t A \lambda)$ converges absolutely. Then we have*

$$\sum_{A \in H_{i,j} \backslash GL_n(\mathbb{Z})} \sum_{\lambda \in L_0} B({}^t A \lambda) = a_0 \sum_{\lambda \in \mathbb{Z}^n} B(\lambda) + a_1 \sum_{\lambda \in \mathbb{Z}^n} B(p\lambda) + a_2 \sum_{\lambda \in \mathbb{Z}^n} B(p^2\lambda),$$

where a_0, a_1 and a_2 are integers which satisfy

$$a_0 + a_1 + a_2 = \left|H_{i,j}\backslash GL_n(\mathbb{Z})\right| , \ a_0 + a_1 = \left|H_{i,j}\backslash S_i\right| \ \textit{and}\ a_0 = \left|H_{i,j}\backslash S_{i,j}\right| .$$

Proof. For $\lambda \in \mathbb{Z}^n$ we denote by $\gcd(\lambda)$ the greatest common divisor of all entries in λ. Let X be a complete set of representatives of $H_{i,j}\backslash \mathrm{GL}_n(\mathbb{Z})$. For $\lambda \in \mathbb{Z}^n$ we define

$$N(\lambda) := \left|\left\{A \in X \mid \lambda \in {}^tAL_0\right\}\right| .$$

We remark that $N(\lambda)$ does not depend on the choice of X. To show this lemma, it is enough to calculate $N(\lambda)$ for given $\lambda \in \mathbb{Z}^n$.

By the definition of $S_{i,j}$ and S_i, we have

$$S_{i,j} = \left\{A \in \mathrm{GL}_n(\mathbb{Z}) \mid {}^t(0, \ldots, 0, 1) \in {}^tAL_0\right\} ,$$
$$S_i = \left\{A \in \mathrm{GL}_n(\mathbb{Z}) \mid {}^t(0, \ldots, 0, p) \in {}^tAL_0\right\} .$$

Hence we have $N({}^t(0, \ldots, 0, 1)) = |H_{i,j}\backslash S_{i,j}|$ and $N({}^t(0, \ldots, 0, p)) = |H_{i,j}\backslash S_i|$. Furthermore, we have $N({}^t(0, \ldots, 0, p^2)) = |H_{i,j}\backslash \mathrm{GL}_n(\mathbb{Z})|$.

For any $\lambda \in \mathbb{Z}^n$, there exists a matrix $B \in \mathrm{GL}_n(\mathbb{Z})$ such that ${}^tB\lambda = \gcd(\lambda)\, {}^t(0, \ldots, 0, 1)$. Thus we have $N(\lambda) = N(\gcd(\lambda){}^t(0, \ldots, 0, 1))$. Hence $N(\lambda)$ equals to $\left|H_{i,j}\backslash S_{i,j}\right|$, $|H_{i,j}\backslash S_i|$ or $\left|H_{i,j}\backslash \mathrm{GL}_n(\mathbb{Z})\right|$, according as $\gcd(p^2, \gcd(\lambda)) = 1$, p or p^2. Therefore we obtain this lemma. □

3.5. Calculation of the function $K^{\alpha}_{i,j}$.

For simplicity we define

$$G^n_j(m) := G^n_j(m, \lambda) ,$$

where $\lambda \in \mathbb{Z}^n$ is an vector which satisfy $\lambda \not\equiv 0 \mod p$. Due to Proposition 3.5, the value $G^n_j(m)$ does not depend on the choice of λ.

LEMMA 3.8. *If $p^2|m$, then we have*

$$\begin{aligned} K^{\alpha}_{i,j} = {}& p^{-k(2n-i-j)+(n-j)(n-i+1)} G^{j-i}_{\alpha}(0) \\ & \times\left\{a_0 E^{(n)}_{k,\frac{m}{p^2}}(\tau, p^2z) + a_1 E^{(n)}_{k,m}(\tau, pz) + a_2 E^{(n)}_{k,mp^2}(\tau, z)\right\} , \end{aligned}$$

where

$$a_0 + a_1 + a_2 = \left|H_{i,j}\backslash GL_n(\mathbb{Z})\right| , \ a_0 + a_1 = \left|H_{i,j}\backslash S_i\right| \ \textit{and}\ a_0 = \left|H_{i,j}\backslash S_{i,j}\right| .$$

If $p^2 \nmid m$, then we have

$$\begin{aligned} K^{\alpha}_{i,j} = {}& p^{-k(2n-i-j)+(n-j)(n-i+1)} \left\{(G^{j-i}_{\alpha}(0) - G^{j-i}_{\alpha}(m)) \left[\Gamma(\delta_j); \Gamma(\delta_{i,j})\right]\right. \\ & \times\left\{g_p(n-1, j) E^{(n)}_{k,m}(\tau, pz) + p^{n-j} g_p(n-1, j-1) E^{(n)}_{k,mp^2}(\tau, z)\right\} \\ & + G^{j-i}_{\alpha}(m) \left[\Gamma(\delta_i); \Gamma(\delta_{i,j})\right] \\ & \left.\times\left\{g_p(n-1, i) E^{(n)}_{k,m}(\tau, pz) + p^{n-i} g_p(n-1, i-1) E^{(n)}_{k,mp^2}(\tau, z)\right\}\right\} , \end{aligned}$$

where $\Gamma(\delta_{i,j})$ and $\Gamma(\delta_i)$ are groups denoted in Section 3.2.

In particular, the function $K^{\alpha}_{i,j}$ *is a linear combination of* $E^{(n)}_{k,\frac{m}{p^2}}|U(p^2)$, $E^{(n)}_{k,m}|U(p)$ *and* $E^{(n)}_{k,mp^2}$.

Proof. First we assume $p^2|m$. In this case the sum $G^{j-i}_{\alpha}(m,\lambda')$ equals to $G^{j-i}_{\alpha}(0)$ for any $\lambda' \in \mathbb{Z}^{j-i}$. Hence due to Lemma 3.4, we obtain

$$K^{\alpha}_{i,j} = p^{-k(2n-i-j)+(n-j)(n-i+1)} G^{j-i}_{\alpha}(0) \sum_{M \in \Gamma(\delta_{i,j})\backslash\Gamma_n} \times \sum_{\lambda \in (p^2\mathbb{Z})^i \times (p\mathbb{Z})^{j-i} \times \mathbb{Z}^{n-j}} j\left(k, \frac{m}{p^2}; M, \lambda\right)(\tau, p^2 z) .$$

If $\{A_l\}_l$ is a complete set of representatives of $H_{i,j}\backslash \mathrm{GL}_n(\mathbb{Z})$, then the set $\left\{\begin{pmatrix} A_l & 0 \\ 0 & {}^tA_l{}^{-1}\end{pmatrix}\right\}_l$ is a complete set of representatives of $\Gamma(\delta_{i,j})\backslash\Gamma_{n,0}$. Hence we have

$$K^{\alpha}_{i,j} = p^{-k(2n-i-j)+(n-j)(n-i+1)} G^{j-i}_{\alpha}(0) \sum_{M \in \Gamma_{n,0}\backslash\Gamma_n} \sum_{A \in H_{i,j}\backslash GL_n(\mathbb{Z})} \times \sum_{\lambda \in (p^2\mathbb{Z})^i \times (p\mathbb{Z})^{j-i} \times \mathbb{Z}^{n-j}} j\left(k, \frac{m}{p^2}; M, {}^tA\lambda\right)(\tau, p^2 z) .$$

From Lemma 3.7 we obtain

$$K^{\alpha}_{i,j} = p^{-k(2n-i-j)+(n-j)(n-i+1)} G^{j-i}_{\alpha}(0) \sum_{M \in \Gamma_{n,0}\backslash\Gamma_n} \left\{ a_0 \sum_{\lambda \in \mathbb{Z}^n} j\left(k, \frac{m}{p^2}; M, \lambda\right)(\tau, p^2 z) \right.$$
$$\left. + a_1 \sum_{\lambda \in \mathbb{Z}^n} j\left(k, \frac{m}{p^2}; M, p\lambda\right)(\tau, p^2 z) + a_2 \sum_{\lambda \in \mathbb{Z}^n} j\left(k, \frac{m}{p^2}; M, p^2\lambda\right)(\tau, p^2 z) \right\} .$$

Due to the two identities

$$j\left(k, \frac{m}{p^2}; M, p\lambda\right)(\tau, p^2 z) = j(k, m; M, \lambda)(\tau, pz)$$

and

$$j\left(k, \frac{m}{p^2}; M, p^2\lambda\right)(\tau, p^2 z) = j\left(k, mp^2; M, \lambda\right)(\tau, z) ,$$

we have

$$K^{\alpha}_{i,j} = p^{-k(2n-i-j)+(n-j)(n-i+1)} G^{j-i}_{\alpha}(0) \times \left\{ a_0 E^{(n)}_{k,\frac{m}{p^2}}(\tau, p^2 z) + a_1 E^{(n)}_{k,m}(\tau, pz) + a_2 E^{(n)}_{k,mp^2}(\tau, z) \right\} .$$

Thus we showed this lemma for the case $p^2|m$.

We now assume $p^2 \not| m$. In this case the sum $G_\alpha^{j-i}(m,\lambda')$ equals to $G_\alpha^{j-i}(0)$ or $G_\alpha^{j-i}(m)$, according as $\lambda' \in p\mathbb{Z}^{j-i}$ or $\lambda' \notin p\mathbb{Z}^{j-i}$. Thus due to Lemma 3.4 we have

$$K_{i,j}^\alpha = p^{-k(2n-i-j)+(n-j)(n-i+1)} \left\{ (G_\alpha^{j-i}(0) - G_\alpha^{j-i}(m)) \sum_{M\in\Gamma(\delta_{i,j})\backslash\Gamma_n} \sum_{\lambda\in(p\mathbb{Z})^j\times\mathbb{Z}^{n-j}} \right.$$

$$\left. \times j(k,m;M,\lambda)(\tau,pz) + G_\alpha^{j-i}(m) \sum_{M\in\Gamma(\delta_{i,j})\backslash\Gamma_n} \sum_{\lambda\in(p\mathbb{Z})^i\times\mathbb{Z}^{n-i}} j(k,m;M,\lambda)(\tau,pz) \right\}.$$

Here we have

$$\begin{aligned}
&\sum_{M\in\Gamma(\delta_{i,j})\backslash\Gamma_n} \sum_{\lambda\in(p\mathbb{Z})^j\times\mathbb{Z}^{n-j}} j(k,m;M,\lambda)(\tau,pz) \\
&\quad = \left[\Gamma(\delta_j);\Gamma(\delta_{i,j})\right] \sum_{M\in\Gamma(\delta_j)\backslash\Gamma_n} \sum_{\lambda\in(p\mathbb{Z})^j\times\mathbb{Z}^{n-j}} j(k,m;M,\lambda)(\tau,pz) \\
&\quad = \left[\Gamma(\delta_j);\Gamma(\delta_{i,j})\right] \sum_{M\in\Gamma_{n,0}\backslash\Gamma_n} \sum_{A\in H_j\backslash GL_n(\mathbb{Z})} \sum_{\lambda\in(p\mathbb{Z})^j\times\mathbb{Z}^{n-j}} j\big(k,m;M,{}^tA\lambda\big)(\tau,pz) \\
&\quad = \left[\Gamma(\delta_j);\Gamma(\delta_{i,j})\right] \left\{ g_p(n-1,j) E_{k,m}^{(n)}(\tau,pz) + p^{n-j} g_p(n-1,j-1) E_{k,mp^2}^{(n)}(\tau,z) \right\}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{M\in\Gamma(\delta_{i,j})\backslash\Gamma_n} \sum_{\lambda\in(p\mathbb{Z})^i\times\mathbb{Z}^{n-i}} j(k,m;M,\lambda)(\tau,pz) \\
&\quad = \left[\Gamma(\delta_i);\Gamma(\delta_{i,j})\right] \left\{ g_p(n-1,i) E_{k,m}^{(n)}(\tau,pz) + p^{n-i} g_p(n-1,i-1) E_{k,mp^2}^{(n)}(\tau,z) \right\}.
\end{aligned}$$

Hence we showed this lemma also for the case $p^2 \not| m$. □

The following proposition has been shown by Yamazaki [Ya 89, Theorem 3.3] for the case $m = 1$. We generalize it for any positive-integer m.

PROPOSITION 3.9. *For any natural number l $(0 \le l \le n)$, the form $E_{k,m}^{(n)}|V_{l,n-l}(p^2)$ is a linear combination of $E_{k,\frac{m}{p^2}}^{(n)}|U(p^2)$, $E_{k,m}^{(n)}|U(p)$ and $E_{k,mp^2}^{(n)}$ over $\mathbb{C}$.*

Proof. This proposition follows from Lemma 3.3 and Lemma 3.8. □

3.6. Relation among Jacobi-Eisenstein series.

Now we shall calculate the coefficients in the linear combinations in Proposition 3.9. This calculation can be directly done by using the values of $G_{j-i}^\alpha(m)$ and $g_p(a,b)$. However, we will here use the Siegel Φ-operators for simplicity.

We set

$$\begin{pmatrix} a_{0,m,p,k} \\ a_{1,m,p,k} \\ a_{2,m,p,k} \end{pmatrix} := \begin{cases} \begin{pmatrix} p^{-2k+2} \\ p^{-k}(p-1) \\ 1 \end{pmatrix} & \text{if } p^2|m\,, \\ \begin{pmatrix} 0 \\ p^{-2k+2}+p^{-k+1}-p^{-k} \\ 1 \end{pmatrix} & \text{if } p^2\not| m \text{ and } p|m\,, \\ \begin{pmatrix} 0 \\ p^{-2k+2}-p^{-k} \\ p^{-k+1}+1 \end{pmatrix} & \text{if } p \not| m\,. \end{cases}$$

LEMMA 3.10. *For the Jacobi-Eisenstein series $E^{(1)}_{k,m}$ of degree* 1, *we have the identity*

$$E^{(1)}_{k,m}|\left(V_{0,1}(p^2),\, V_{1,0}(p^2)\right) = \left(E^{(1)}_{k,\frac{m}{p^2}}|U(p^2),\, E^{(1)}_{k,m}|U(p),\, E^{(1)}_{k,mp^2}\right) \begin{pmatrix} 0 & a_{0,m,p,k} \\ p^{-k} & a_{1,m,p,k} \\ 0 & a_{2,m,p,k} \end{pmatrix}.$$

Proof. Since $\Gamma_1(p^2 1_2)\Gamma_1 = \Gamma_1(p^2 1_2)$, the relation $E^{(1)}_{k,m}|V_{0,1}(p^2) = p^{-k}E^{(1)}_{k,m}|U(p)$ is obvious.

From Lemma 3.3 we obtain

$$E^{(1)}_{k,m}|V_{1,0}(p^2) = K^0_{0,0} + K^1_{0,1} + K^0_{1,1}\,.$$

Due to Lemma 3.6 and Lemma 3.8, we have

$$K^0_{0,0} = \begin{cases} p^{-2k+2}E^{(1)}_{k,\frac{m}{p^2}}(\tau, p^2 z) & \text{if } p^2|m\,, \\ p^{-2k+2}E^{(1)}_{k,m}(\tau, pz) & \text{if } p^2 \not| m\,, \end{cases}$$

$$K^1_{0,1} = \begin{cases} p^{-k}(p-1)E^{(1)}_{k,m}(\tau, pz) & \text{if } p|m\,, \\ p^{-k+1}E^{(1)}_{k,mp^2}(\tau, z) - p^{-k}E^{(1)}_{k,m}(\tau, pz) & \text{if } p \not| m\,, \end{cases}$$

$$K^0_{1,1} = E^{(1)}_{k,mp^2}(\tau, z)\,.$$

Therefore this lemma follows. □

Let $B_{l,l+1}(X_l)$, $B_{2,n+1}(X_2, \ldots, X_n)$ and $A^{p,k}_{2,n+1}$ be the matrices introduced in Section 2.6. We recall $A^{p,k}_{2,n+1} = B_{2,n+1}(p^{2-k}, p^{3-k}, \ldots, p^{n-k})$ and the matrix $A^{p,k}_{2,n+1}$ has the size 2 times $(n+1)$.

The following proposition has been shown by Yamazaki [Ya 89, Theorem 4.1] for the case $m = 1$. We generalize it for any positive-integer m.

PROPOSITION 3.11. *For any Jacobi-Eisenstein series $E^{(n)}_{k,m}$ of degree n, the following identity holds*

$$E^{(n)}_{k,m}|\left(V_{0,n}(p^2),\ldots,V_{n,0}(p^2)\right)$$
$$= \left(E^{(n)}_{k,\frac{m}{p^2}}|U(p^2),\, E^{(n)}_{k,m}|U(p),\, E^{(n)}_{k,mp^2}\right)\begin{pmatrix} 0 & a_{0,m,p,k} \\ p^{-k} & a_{1,m,p,k} \\ 0 & a_{2,m,p,k}\end{pmatrix} A^{p,k}_{2,n+1}\,.$$

Proof. Let m be a positive-integer. Let Φ be the Siegel Φ-operator for Jacobi forms introduced in Section 2.5. From Lemma 2.2 and from the fact that $\Phi(E^{(n)}_{k,m}) = E^{(n-1)}_{k,m}$, we have

$$\Phi(E^{(n)}_{k,m}|(V_{0,n}(p^2),\ldots,V_{n,0}(p^2))) = E^{(n-1)}_{k,m}|(V_{0,n-1}(p^2),\ldots,V_{n-1,0}(p^2))B_{n,n+1}(p^{n-k})\,.$$

Hence by using Siegel Φ-operator $n-1$ times and by using Lemma 3.10, we have

$$\Phi^{(n-1)}(E^{(n)}_{k,m}|(V_{0,n}(p^2),\ldots,V_{n,0}(p^2)))$$
$$= E^{(1)}_{k,m}|(V_{0,1}(p^2),V_{1,0}(p^2))B_{2,n+1}(p^{2-k},p^{3-k},\ldots,p^{n-k})$$
$$= \left(E^{(1)}_{k,\frac{m}{p^2}}|U(p^2),\, E^{(1)}_{k,m}|U(p),\, E^{(1)}_{k,mp^2}\right)\begin{pmatrix} 0 & a_{0,m,p,k} \\ p^{-k} & a_{1,m,p,k} \\ 0 & a_{2,m,p,k}\end{pmatrix} B_{2,n+1}(p^{2-k},p^{3-k},\ldots,p^{n-k})\,.$$

On the other hand, due to Proposition 3.9, there exists a $3\times(n+1)$ matrix B^k_n which satisfies

$$E^{(n)}_{k,m}|(V_{0,n}(p^2),\ldots,V_{n,0}(p^2)) = \left(E^{(n)}_{k,\frac{m}{p^2}}|U(p^2),\, E^{(n)}_{k,m}|U(p),\, E^{(n)}_{k,mp^2}\right)B^k_n.$$

From this identity we have

$$\Phi^{(n-1)}(E^{(n)}_{k,m}|(V_{0,n}(p^2),\ldots,V_{n,0}(p^2))) = \left(E^{(1)}_{k,\frac{m}{p^2}}|U(p^2),\, E^{(1)}_{k,m}|U(p),\, E^{(1)}_{k,mp^2}\right)B^k_n\,.$$

Because three forms $E^{(1)}_{k,\frac{m}{p^2}}|U(p^2)$, $E^{(1)}_{k,m}|U(p)$ and $E^{(1)}_{k,mp^2}$ are linearly independent (see Lemma 3.1), we obtain

$$B^k_n = \begin{pmatrix} 0 & a_{0,m,p,k} \\ p^{-k} & a_{1,m,p,k} \\ 0 & a_{2,m,p,k}\end{pmatrix} B_{2,n+1}(p^{2-k},p^{3-k},\ldots,p^{n-k}) \;=\; \begin{pmatrix} 0 & a_{0,m,p,k} \\ p^{-k} & a_{1,m,p,k} \\ 0 & a_{2,m,p,k}\end{pmatrix} A^{p,k}_{2,n+1}\,.$$

Thus this proposition follows for any positive-integer m. □

4. Generalized Maass relation for Siegel-Eisenstein series

The purpose of this section is to prove Theorem 1.1. Let $e^{(n)}_{k,m}$ be the m-th Fourier-Jacobi coefficient of Siegel-Eisenstein series $E^{(n+1)}_k$, which is introduced in Section 1.2.

In this section we write $\sum_{d|m}$ for $\sum_{\substack{d>0\\ d|m}}$, and $\sum_{d^2|m}$ for $\sum_{\substack{d>0\\ d^2|m}}$, for simplicity.

4.1. Fourier-Jacobi coefficients.

We define an arithmetic function

$$g_k(m) := \sum_{d^2|m} \mu(d)\, \sigma_{k-1}\left(\frac{m}{d^2}\right) ,$$

where $\mu(d)$ is the Möbius function and we put $\sigma_{k-1}(m) := \sum_{d|m} d^{k-1}$ as usual.

LEMMA 4.1. *We obtain*

$$g_k(mp) = \begin{cases} \left(p^{k-1}+1\right) g_k(m) & \text{if } p \nmid m\, , \\ p^{k-1} g_k(m) & \text{if } p|m\, . \end{cases}$$

Proof. The function $g_k(m)$ is a multiplicative function, namely $g_k(ml) = g_k(m)g_k(l)$ if $\gcd(m,l) = 1$. Hence we obtain the identity $g_k(m) = m^{k-1} \prod_{\substack{q\,:\,prime \\ q|m}} \left(1 + \frac{1}{q^{k-1}}\right)$. This lemma follows from this identity. □

The following proposition is a special case of a result in [Bo 83, Satz 7].

PROPOSITION 4.2 (Boecherer [Bo 83]). *We have*

$$e^{(n)}_{k,m} = \sum_{d^2|m} g_k\left(\frac{m}{d^2}\right)\, E^{(n)}_{k,\frac{m}{d^2}}|U(d)\, .$$

Proof. For the proof of this proposition, the reader is referred to [Ya 86, Theorem 5.5]. □

PROPOSITION 4.3. *For any $n > 0$ and for any $m > 0$ we have the identity*

$$\sum_{d^2|m} g_k\left(\frac{m}{d^2}\right) \left(E^{(n)}_{k,\frac{m}{p^2d^2}}|U(p^2d),\, E^{(n)}_{k,\frac{m}{d^2}}|U(pd),\, E^{(n)}_{k,\frac{mp^2}{d^2}}|U(d)\right) \begin{pmatrix} a_{0,\frac{m}{d^2},p,k} \\ a_{1,\frac{m}{d^2},p,k} \\ a_{2,\frac{m}{d^2},p,k} \end{pmatrix}$$

$$= \left(e^{(n)}_{k,\frac{m}{p^2}}|U(p^2),\, e^{(n)}_{k,m}|U(p),\, e^{(n)}_{k,mp^2}\right) \begin{pmatrix} 1 \\ p^{-k}(-1 + p\,\delta_{p|m}) \\ p^{-2k+2} \end{pmatrix} ,$$

where $\delta_{p|m}$ is 1 or 0, according as $p|m$ or $p \nmid m$.

Proof. Due to Proposition 4.2 and Lemma 4.1, this proposition is obtained by straightforward calculation as follows. We set nine functions

$$Eg_1 := \sum_{\substack{d^2|m \\ \frac{m}{d^2} \equiv 0\,(p^2)}} p^{-2k+2} E^{(n)}_{k,\frac{m}{p^2d^2}}|U(p^2d)\, g\left(\frac{m}{d^2}\right) ,$$

$$Eg_2 := \sum_{\substack{d^2|m \\ \frac{m}{d^2} \equiv 0\,(p^2)}} p^{-k}(p-1) E^{(n)}_{k,\frac{m}{d^2}}|U(pd)\, g\left(\frac{m}{d^2}\right) ,$$

$$Eg_3 := \sum_{\substack{d^2|m \\ \frac{m}{d^2}\equiv 0\,(p^2)}} E^{(n)}_{k,\frac{p^2m}{d^2}}|U(d)\, g\Big(\frac{m}{d^2}\Big),$$

$$Eg_4 := \sum_{\substack{d^2|m \\ \frac{m}{d^2}\equiv 0\,(p) \\ \frac{m}{d^2}\not\equiv 0\,(p^2)}} p^{-2k+2}E^{(n)}_{k,\frac{m}{d^2}}|U(pd)\, g\Big(\frac{m}{d^2}\Big),$$

$$Eg_5 := \sum_{\substack{d^2|m \\ \frac{m}{d^2}\equiv 0\,(p) \\ \frac{m}{d^2}\not\equiv 0\,(p^2)}} (p^{-k+1}-p^{-k})E^{(n)}_{k,\frac{m}{d^2}}|U(pd)\, g\Big(\frac{m}{d^2}\Big),$$

$$Eg_6 := \sum_{\substack{d^2|m \\ \frac{m}{d^2}\equiv 0\,(p) \\ \frac{m}{d^2}\not\equiv 0\,(p^2)}} E^{(n)}_{k,\frac{p^2m}{d^2}}|U(d)\, g\Big(\frac{m}{d^2}\Big),$$

$$Eg_7 := \sum_{\substack{d^2|m \\ \frac{m}{d^2}\not\equiv 0\,(p)}} p^{-2k+2}E^{(n)}_{k,\frac{m}{d^2}}|U(pd)\, g\Big(\frac{m}{d^2}\Big),$$

$$Eg_8 := \sum_{\substack{d^2|m \\ \frac{m}{d^2}\not\equiv 0\,(p)}} (-p^{-k})E^{(n)}_{k,\frac{m}{d^2}}|U(pd)\, g\Big(\frac{m}{d^2}\Big),$$

and

$$Eg_9 := \sum_{\substack{d^2|m \\ \frac{m}{d^2}\not\equiv 0\,(p)}} (p^{-k+1}+1)E^{(n)}_{k,\frac{p^2m}{d^2}}|U(d)\, g\Big(\frac{m}{d^2}\Big).$$

If $\mathrm{ord}_p m \equiv 1\ (2)$, then

$$\sum_{d^2|m} g_k\Big(\frac{m}{d^2}\Big)\left(E^{(n)}_{k,\frac{m}{p^2d^2}}|U(p^2d),\, E^{(n)}_{k,\frac{m}{d^2}}|U(pd),\, E^{(n)}_{k,\frac{mp^2}{d^2}}|U(d)\right)\begin{pmatrix} a_{0,\frac{m}{d^2},p,k} \\ a_{1,\frac{m}{d^2},p,k} \\ a_{2,\frac{m}{d^2},p,k}\end{pmatrix}$$
$$= Eg_1 + Eg_2 + Eg_3 + Eg_4 + Eg_5 + Eg_6,$$

and

$$Eg_1 = e^{(n)}_{k,\frac{m}{p^2}}|U(p^2),$$
$$Eg_2 + Eg_5 = p^{-k}(p-1)e^{(n)}_{k,m}|U(p),$$
$$Eg_3 + Eg_4 + Eg_6 = p^{-2k+2}e^{(n)}_{k,mp^2}.$$

Because of the assumption $\mathrm{ord}_p m \equiv 1\ (2)$, we have $\delta_{p|m} = 1$.

Hence this proposition follows for the case $\mathrm{ord}_p m \equiv 1\ (2)$.

If $\mathrm{ord}_p m \equiv 0\ (2)$, then

$$\sum_{d^2|m} g_k\Big(\frac{m}{d^2}\Big) \left(E^{(n)}_{k,\frac{m}{p^2d^2}}|U(p^2d),\, E^{(n)}_{k,\frac{m}{d^2}}|U(pd),\, E^{(n)}_{k,\frac{mp^2}{d^2}}|U(d)\right)\begin{pmatrix} a_{0,\frac{m}{d^2},p,k} \\ a_{1,\frac{m}{d^2},p,k} \\ a_{2,\frac{m}{d^2},p,k}\end{pmatrix}$$
$$= Eg_1 + Eg_2 + Eg_3 + Eg_7 + Eg_8 + Eg_9,$$

and

$$Eg_1 = \delta_{p^2|m}\left\{ e^{(n)}_{k,\frac{m}{p^2}}|U(p^2) + p^{-k+1}\sum_{\substack{d^2|m \\ \frac{m}{d^2}\not\equiv 0\,(p^2)}} E^{(n)}_{k,\frac{m}{d^2}}|U(pd)\, g\Big(\frac{m}{d^2}\Big)\right\},$$

$$Eg_2 + Eg_8 = \delta_{p^2|m}\left\{ p^{-k+1}e^{(n)}_{k,m}|U(p) - p^{-k+1}\sum_{\substack{d^2|m \\ \frac{m}{d^2}\not\equiv 0\,(p^2)}} E^{(n)}_{k,\frac{m}{d^2}}|U(pd)\, g\Big(\frac{m}{d^2}\Big)\right\}$$
$$-p^{-k}e^{(n)}_{k,m}|U(p),$$

$$Eg_3 + Eg_7 + Eg_9 = p^{-2k+2}e^{(n)}_{k,mp^2}.$$

Here $\delta_{p^2|m}$ is defined by 1 or 0, according as $p^2|m$ or $p^2 \nmid m$. Because of the assumption $\mathrm{ord}_p m \equiv 0\ (2)$, we have $\delta_{p^2|m} = \delta_{p|m}$.

Therefore this proposition follows also for the case $\mathrm{ord}_p m \equiv 0\ (2)$. $\square$

4.2. Proof of Theorem 1.1.

Now we shall prove Theorem 1.1. For any prime p and for any positive-integer d, the operators $V_{l,n-l}(p^2)$ and $U(d)$ are compatible. Hence from Proposition 4.2, Proposition 3.11 and Proposition 4.3, we have

$$e^{(n)}_{k,m}|\big(V_{0,n}(p^2),\ldots,V_{n,0}(p^2)\big)$$
$$= \sum_{d^2|m} g_k\Big(\frac{m}{d^2}\Big)\left(E^{(n)}_{k,\frac{m}{d^2}}|\Big(V_{0,n}(p^2),\ldots,V_{n,0}(p^2)\Big)|U(d)\right)$$
$$= \sum_{d^2|m} g_k\Big(\frac{m}{d^2}\Big)\left(E^{(n)}_{k,\frac{m}{p^2d^2}}|U(p^2d),\, E^{(n)}_{k,\frac{m}{d^2}}|U(pd),\, E^{(n)}_{k,\frac{mp^2}{d^2}}|U(d)\right)$$
$$\times \begin{pmatrix} 0 & a_{0,\frac{m}{d^2},p,k} \\ p^{-k} & a_{1,\frac{m}{d^2},p,k} \\ 0 & a_{2,\frac{m}{d^2},p,k}\end{pmatrix} A^{p,k}_{2,n+1}$$

$$= \left(e^{(n)}_{k,\frac{m}{p^2}}|U(p^2), e^{(n)}_{k,m}|U(p), e^{(n)}_{k,mp^2}\right)\begin{pmatrix} 0 & 1 \\ p^{-k} & p^{-k}(-1+p\,\delta_{p|m}) \\ 0 & p^{-2k+2}\end{pmatrix} A^{p,k}_{2,n+1}\,.$$

Thus we obtain Theorem 1.1.

5. Generalized Maass relation for Siegel cusp forms

In this section we shall show Theorem 1.2, Theorem 1.3 and Corollary 1.4. Let $\phi_m \in J^{(2n-1)}_{k+n,m}$ be the m-th Fourier-Jacobi coefficient of the Duke-Imamoglu-Ibukiyama-Ikeda lift F stated in Theorem 1.2.

In this section the letters p and q are reserved for prime numbers. For example, the symbol $\prod_{p|N}$ denotes the product over primes p such that $p|N$.

5.1. Fourier coefficients of ϕ_m.

We take the Fourier expansion of ϕ_m:

$$\phi_m(\tau, z) = \sum_{N,R} C_m(N,R)e(N\tau)e({}^tRz)\,,$$

where in the summation $N \in \mathrm{Sym}^*_{2n-1}$ and $R \in \mathbb{Z}^{2n-1}$ run over all elements which satisfy $4Nm - R^tR > 0$. We set $M = \begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^tR & m\end{pmatrix}$. We denote by D_M and by f_M the integers such that $(-1)^n\det(2M) = D_M f_M^2$, where D_M is a fundamental discriminant and f_M is a positive integer. Then the (N,R)-th Fourier coefficient $C_m(N,R)$ of ϕ_m is

$$C_m(N,R) = C(|D_M|) f_M^{k-\frac{1}{2}} \prod_{p|f_M} \widetilde{F}_p(M;\alpha_p)\,,$$

where $C(|D_M|)$ is the $|D_M|$-th Fourier coefficient of h which corresponds to g by Shimura correspondence, and $\widetilde{F}_p(M;X_p) \in \mathbb{C}[X_p + X_p^{-1}]$ is a certain Laurent polynomial introduced in [Ik 01, §1].

5.2. Matrix $A^{p,k+n}_{2,2n}$.

Let $A^{p,k}_{2,n+1}$ and $A'_{2,2n}(X_p)$ be the matrices introduced in Section 2.6.

LEMMA 5.1. *For any even integer k we obtain*

$$A^{p,k+n}_{2,2n} = p^{-(n-1)(2k-1)}A'_{2,2n}(p^{-(k-\frac{1}{2})})\,.$$

Proof. From the definition of $A^{p,k}_{2,n+1}$ we get

$$\begin{aligned} A^{p,k+n}_{2,2n} &= B_{2,2n}(p^{2-n-k}, p^{3-n-k},\dots,p^{n-1-k}) \\ &= \left(\prod_{i=2}^{2n-1} p^{i-n-k}\right) B'_{2,2n}(p^{\frac{3}{2}-n-(k-\frac{1}{2})}, p^{\frac{5}{2}-n-(k-\frac{1}{2})},\dots,p^{-\frac{3}{2}+n-(k-\frac{1}{2})}) \\ &= p^{-(n-1)(2k-1)}A'_{2,2n}(p^{-(k-\frac{1}{2})})\,. \end{aligned}$$

□

5.3. Proof of Theorem 1.2.

Let $g \in \mathrm{GSp}^+_{2n-1}(\mathbb{R}) \cap M_{4n-2}(\mathbb{Z})$ be a matrix such that the similitude of g is $\nu(g) = p^2$. We write the coset decomposition $\Gamma_{2n-1} g \Gamma_{2n-1} = \bigcup_i \Gamma_n g_i$ with $g_i = \begin{pmatrix} A_i & B_i \\ 0_{2n-1} & D_i \end{pmatrix}$. We take the Fourier expansion of $\phi_m | V(\Gamma_{2n-1} g \Gamma_{2n-1})$:

$$(\phi_m | V(\Gamma_{2n-1} g \Gamma_{2n-1}))(\tau, z) = \sum_{N,R} C_m(g; N, R)\, e(N\tau)\, e({}^t R z)\,,$$

where in the summation $N \in \mathrm{Sym}^*_{2n-1}$ and $R \in \mathbb{Z}^{2n-1}$ run over all elements which satisfy $4Nmp^2 - R\,{}^t R > 0$.

We now fix $N \in \mathrm{Sym}^*_{2n-1}$ and $R \in \mathbb{Z}^{2n-1}$ such that $4Nmp^2 - R\,{}^t R > 0$. And we set $M_1 = \begin{pmatrix} N & \frac{1}{2p} R \\ \frac{1}{2p} {}^t R & m \end{pmatrix}$.

LEMMA 5.2. *The (N,R)-th Fourier coefficient $C_m(g;N,R)$ of $\phi_m | V(\Gamma_{2n-1} g \Gamma_{2n-1})$ is*

$$C_m(g; N, R) = p^{-(2n-1)(k-\frac{1}{2})} C\left(|D_{M_1}|\right) f_{M_1}^{k-\frac{1}{2}} \sum_i \det D_i^{-n-\frac{1}{2}}$$
$$\times \prod_{q | f_{M_1[diag(p^{-1t} D_i, 1)]}} \widetilde{F}_q\left(M_1[diag(p^{-1t} D_i, 1)]; \alpha_q\right).$$

Here we regard $\widetilde{F}_q \left(M_1[diag(p^{-1t} D_i, 1)]; X_q\right)$ as 0, if $M_1[diag(p^{-1t} D_i, 1)] \notin Sym^*_{2n}$.

Proof. From the definition of $V(\Gamma_{2n-1} g \Gamma_{2n-1})$ the (N, R)-th Fourier coefficient of the form $\phi_m | V(\Gamma_{2n-1} g \Gamma_{2n-1})$ is

$$\sum_i \det D_i^{-(k+n)} C\left(|D_{M_{1,i}}|\right) f_{M_{1,i}}^{k-\frac{1}{2}} \prod_{q | f_{M_{1,i}}} \widetilde{F}_q\left(M_{1,i}; \alpha_q\right)\,,$$

where $M_{1,i} := M_1[diag(p^{-1t} D_i, 1)]$. Thus this lemma follows from the fact that if $M_{1,i}$ is a half-integral symmetric matrix, then $D_{M_{1,i}} = D_{M_1}$ and $f_{M_{1,i}} = p^{-(2n-1)} (\det D_i) f_{M_1}$. □

Now we shall prove Theorem 1.2. In the same manner as in Lemma 5.2 we obtain the fact that the (N, R)-th Fourier coefficient of $e^{(2n-1)}_{k+n,m} | V(\Gamma_{2n-1} g \Gamma_{2n-1})$ is

$$p^{-(2n-1)(k-\frac{1}{2})} h_{k+\frac{1}{2}}\left(|D_{M_1}|\right) f_{M_1}^{k-\frac{1}{2}} \sum_i \det D_i^{-n-\frac{1}{2}}$$
$$\times \prod_{q | f_{M_1[diag(p^{-1t} D_i, 1)]}} \widetilde{F}_q\left(M_1[\mathrm{diag}(p^{-1t} D_i, 1)]; q^{k-\frac{1}{2}}\right),$$

where $h_{k+\frac{1}{2}}(|D_{M_1}|)$ is the $|D_{M_1}|$-th Fourier coefficient of the Cohen type Eisenstein series of weight $k + \frac{1}{2}$ which corresponds to the Eisenstein series of weight $2k$ by the Shimura correspondence.

By virtue of Theorem 1.1, the form $e^{(2n-1)}_{k+n,m}|V(\Gamma_{2n-1}g\Gamma_{2n-1})$ is a linear combination of $e^{(2n-1)}_{k+n,\frac{m}{p^2}}|U(p^2)$, $e^{(2n-1)}_{k+n,m}|U(p)$ and $e^{(2n-1)}_{k+n,mp^2}$. Hence there exist constants u_0, u_1 and u_2, such that

$$e^{(2n-1)}_{k+n,m}|V(\Gamma_{2n-1}g\Gamma_{2n-1}) = u_0\, e^{(2n-1)}_{k+n,\frac{m}{p^2}}|U(p^2) + u_1\, e^{(2n-1)}_{k+n,m}|U(p) + u_2\, e^{(2n-1)}_{k+n,mp^2}\,.$$

We remark that the constants u_0, u_1 and u_2 depend on the choices of p, k, m and n. The (N, R)-th Fourier coefficient of the form of the above RHS is

$$\begin{aligned} &u_0\, h_{k+\frac{1}{2}}\left(|D_{M_1}|\right) p^{-k+\frac{1}{2}} f_{M_1}^{k-\frac{1}{2}} \prod_{q|f_{M_0}} \widetilde{F}_q\left(M_0; q^{k-\frac{1}{2}}\right) \\ &\quad + u_1\, h_{k+\frac{1}{2}}\left(|D_{M_1}|\right) f_{M_1}^{k-\frac{1}{2}} \prod_{q|f_{M_1}} \widetilde{F}_q\left(M_1; q^{k-\frac{1}{2}}\right) \\ &\quad + u_2\, h_{k+\frac{1}{2}}\left(|D_{M_1}|\right) p^{k-\frac{1}{2}} f_{M_1}^{k-\frac{1}{2}} \prod_{q|f_{M_2}} \widetilde{F}_q\left(M_2; q^{k-\frac{1}{2}}\right), \end{aligned}$$

where $M_0 = \begin{pmatrix} N & \frac{1}{2p^2}R \\ \frac{1}{2p^2}{}^tR & \frac{m}{p^2} \end{pmatrix}$ and $M_2 = \begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^tR & mp^2 \end{pmatrix}$. Because $h_{k+\frac{1}{2}}\left(|D_{M_1}|\right) \neq 0$, we obtain

$$\begin{aligned} (5.1)\quad & p^{-(2n-1)(k-\frac{1}{2})} \sum_i \det D_i^{-n-\frac{1}{2}} \prod_{q|f_{M_1[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\left(M_1[\mathrm{diag}(p^{-1t}D_i, 1)]; q^{k-\frac{1}{2}}\right) \\ &= u_0\, p^{-k+\frac{1}{2}} \prod_{q|f_{M_0}} \widetilde{F}_q\left(M_0; q^{k-\frac{1}{2}}\right) + u_1 \prod_{q|f_{M_1}} \widetilde{F}_q\left(M_1; q^{k-\frac{1}{2}}\right) \\ &\quad + u_2\, p^{k-\frac{1}{2}} \prod_{q|f_{M_2}} \widetilde{F}_q\left(M_2; q^{k-\frac{1}{2}}\right). \end{aligned}$$

We denote by $c_0(N, R)$, $c_1(N, R)$ and $c_2(N, R)$ the (N, R)-th Fourier coefficients of $e^{(2n-1)}_{k+n,\frac{m}{p^2}}|U(p^2)$, $e^{(2n-1)}_{k+n,m}|U(p)$ and $e^{(2n-1)}_{k+n,mp^2}$, respectively. We remark $c_0(N, R) = 0$ if $p^2 \nmid m$. Furthermore, we remark that $c_0(N, R) = 0$ if $R \notin p^2\mathbb{Z}^{2n-1}$, and $c_1(N, R) = 0$ if $R \notin p\mathbb{Z}^{2n-1}$.

Because the forms in the set $\left\{E^{(2n-1)}_{\frac{m}{d^2},k}|U(d)\right\}_d$, where d runs over all positive-integers such that $d^2|m$, are linearly independent (see Lemma 3.1) and because of Proposition 4.2, three forms $e^{(2n-1)}_{k+n,\frac{m}{p^2}}|U(p^2)$, $e^{(2n-1)}_{k+n,m}|U(p)$ and $e^{(2n-1)}_{k+n,mp^2}$ are linearly independent.

From now on we assume $p^2|m$ for simplicity. The proof of Theorem 1.2 for the case $p^2 \nmid m$ is similar to the case $p^2|m$.

There exist pairs of matrices (N_j, R_j) $(j = 1, 2, 3)$ such that

$$\det\left(\begin{pmatrix} c_0(N_1, R_1) & c_1(N_1, R_1) & c_2(N_1, R_1) \\ c_0(N_2, R_2) & c_1(N_2, R_2) & c_2(N_2, R_2) \\ c_0(N_3, R_3) & c_1(N_3, R_3) & c_2(N_3, R_3) \end{pmatrix}\right) \neq 0\,.$$

For $j = 1, 2, 3$, we define

$$M_0^{(j)} := \begin{pmatrix} N_j & \dfrac{1}{2p^2}R_j \\ \dfrac{1}{2p^2}R_j & \dfrac{m}{p^2} \end{pmatrix}, \quad M_1^{(j)} := \begin{pmatrix} N_j & \dfrac{1}{2p}R_j \\ \dfrac{1}{2p}R_j & m \end{pmatrix}, \quad M_2^{(j)} := \begin{pmatrix} N_j & \dfrac{1}{2}R_j \\ \dfrac{1}{2}R_j & mp^2 \end{pmatrix},$$

and we put a 3×3 matrix

$$C\big(\{(N_j, R_j)\}_j;\ \{X_q\}_{q:prime}\big) := \left(\prod_{q | f_{M_i^{(j)}}} \widetilde{F}_q\big(M_i^{(j)}; X_q\big) \right)_{\substack{j=1,2,3 \\ i=0,1,2}}.$$

Then from the identity (5.1) we have

$$\begin{aligned} & p^{-(2n-1)(k-\frac{1}{2})} \sum_i (\det D_i)^{-n-\frac{1}{2}} \\ & \quad \times \begin{pmatrix} \prod_{q | f_{M_1^{(1)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(1)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ q^{k-\frac{1}{2}}\big) \\ \prod_{q | f_{M_1^{(2)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(2)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ q^{k-\frac{1}{2}}\big) \\ \prod_{q | f_{M_1^{(3)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(3)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ q^{k-\frac{1}{2}}\big) \end{pmatrix} \\ & = C\big(\{(N_j, R_j)\}_j;\ \{q^{k-\frac{1}{2}}\}_q\big) \begin{pmatrix} u_0\, p^{k-\frac{1}{2}} \\ u_1 \\ u_2\, p^{-k+\frac{1}{2}} \end{pmatrix}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} & \sum_i (\det D_i)^{-n-\frac{1}{2}}\ C\Big(\big\{(N_j, R_j)\big\}_j;\ \{q^{k-\frac{1}{2}}\}_q\Big)^{-1} \\ & \quad \times \begin{pmatrix} \prod_{q | f_{M_1^{(1)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(1)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ q^{k-\frac{1}{2}}\big) \\ \prod_{q | f_{M_1^{(2)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(2)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ q^{k-\frac{1}{2}}\big) \\ \prod_{q | f_{M_1^{(3)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(3)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ q^{k-\frac{1}{2}}\big) \end{pmatrix} \\ & = p^{(2n-1)(k-\frac{1}{2})} \begin{pmatrix} u_0\, p^{k-\frac{1}{2}} \\ u_1 \\ u_2\, p^{-k+\frac{1}{2}} \end{pmatrix}. \end{aligned}$$

The RHS of the above identity does not depend on the choices of (N_j, R_j) $(j = 1, 2, 3)$. Furthermore, the above identity holds for infinitely many integer k. Therefore there exist Laurent polynomials $\Phi_i(X_p) \in \mathbb{C}[X_p + X_p^{-1}]$ $(i = 0, 1, 2)$ which are independent of the choices of (N_j, R_j) $(j = 1, 2, 3)$, such that

$$\sum_i (\det D_i)^{-n-\frac{1}{2}}\ C\Big(\big\{(N_j, R_j)\big\}_j;\ \{X_q\}_q\Big)^{-1}$$

$$\times \begin{pmatrix} \prod_{q|f_{M_1^{(1)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(1)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ X_q\big) \\ \prod_{q|f_{M_1^{(2)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(2)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ X_q\big) \\ \prod_{q|f_{M_1^{(3)}[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1^{(3)}[\mathrm{diag}(p^{-1t}D_i, 1)];\ X_q\big) \end{pmatrix}$$

$$= \begin{pmatrix} \Phi_0(X_p) \\ \Phi_1(X_p) \\ \Phi_2(X_p) \end{pmatrix}.$$

In particular, we have

$$\sum_i \det D_i^{-n-\frac{1}{2}} \prod_{q|f_{M_1[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\big(M_1[\mathrm{diag}(p^{-1t}D_i, 1)];\ X_q\big)$$

$$= \Phi_0(X_p) \prod_{q|f_{M_0}} \widetilde{F}_q\big(M_0;\ X_q\big)$$

$$+ \Phi_1(X_p) \prod_{q|f_{M_1}} \widetilde{F}_q\big(M_1;\ X_q\big) + \Phi_2(X_p) \prod_{q|f_{M_2}} \widetilde{F}_q\big(M_2;\ X_q\big).$$

Therefore, by substituting $X_q = \alpha_q$ in the above identity and by using the relations $pf_{M_0} = f_{M_1} = p^{-1} f_{M_2}$ and $D_{M_0} = D_{M_1} = D_{M_2}$, we obtain

$$p^{-(2n-1)(k-\frac{1}{2})} C\left(|D_{M_1}|\right) f_{M_1}^{k-\frac{1}{2}} \sum_i \det D_i^{-n-\frac{1}{2}}$$

$$\times \prod_{q|f_{M_1[diag(p^{-1t}D_i,1)]}} \widetilde{F}_q\left(M_1[\mathrm{diag}(p^{-1t}D_i, 1)];\ \alpha_q\right)$$

$$= p^{-(2n-1)(k-\frac{1}{2})} \Bigg\{ \Phi_0(\alpha_p)\, p^{k-\frac{1}{2}} C\left(|D_{M_0}|\right) f_{M_0}^{k-\frac{1}{2}} \prod_{q|f_{M_0}} \widetilde{F}_q\left(M_0;\ \alpha_q\right)$$

$$+ \Phi_1(\alpha_p)\, C\left(|D_{M_1}|\right) f_{M_1}^{k-\frac{1}{2}} \prod_{q|f_{M_1}} \widetilde{F}_q\left(M_1;\ \alpha_q\right)$$

$$+ \Phi_2(\alpha_p)\, p^{-k+\frac{1}{2}} C\left(|D_{M_2}|\right) f_{M_2}^{k-\frac{1}{2}} \prod_{q|f_{M_2}} \widetilde{F}_q\left(M_2;\ \alpha_q\right) \Bigg\}.$$

Thus

$$\phi_m | V(\Gamma_{2n-1} g \Gamma_{2n-1})$$

$$= p^{-(2n-1)(k-\frac{1}{2})}\Big(p^{k-\frac{1}{2}}\phi_{\frac{m}{p^2}}|U(p^2),\, \phi_m|U(p),\, p^{-k+\frac{1}{2}}\phi_{mp^2}\Big)\begin{pmatrix}\varPhi_0(\alpha_p)\\ \varPhi_1(\alpha_p)\\ \varPhi_2(\alpha_p)\end{pmatrix}.$$

Hence there exist Laurent polynomials $\varPhi_{j,l}(X_p) \in \mathbb{C}[X_p + X_p^{-1}]$ ($j = 0, 1, 2,\ l = 0, \ldots, 2n-1$) which satisfy

(5.2)
$$\begin{aligned}&\phi_m|(V_{0,2n-1}(p^2), \ldots, V_{2n-1,0}(p^2))\\ &\quad = p^{-(2n-1)(k-\frac{1}{2})}\Big(p^{k-\frac{1}{2}}\phi_{\frac{m}{p^2}}|U(p^2),\, \phi_m|U(p),\, p^{-k+\frac{1}{2}}\phi_{mp^2}\Big)\\ &\quad\times\begin{pmatrix}\varPhi_{0,0}(\alpha_p) & \cdots & \varPhi_{0,2n-1}(\alpha_p)\\ \varPhi_{1,0}(\alpha_p) & \cdots & \varPhi_{1,2n-1}(\alpha_p)\\ \varPhi_{2,0}(\alpha_p) & \cdots & \varPhi_{2,2n-1}(\alpha_p)\end{pmatrix}.\end{aligned}$$

Here the polynomials $\varPhi_{j,l}(X_p)$ depend on the choices of p and m, but not on the choice of f which is the preimage of the Duke-Imamoglu-Ibukiyama-Ikeda lift F. The m-th Fourier-Jacobi coefficient $e_{k+n,m}^{(2n-1)}$ of Siegel-Eisenstein series satisfies also the identity (5.2). Thus, because of Theorem 1.1 and of Lemma 5.1, we obtain

$$\begin{pmatrix}\varPhi_{0,0}(p^{k-\frac{1}{2}}) \cdots \varPhi_{0,2n-1}(p^{k-\frac{1}{2}})\\ \varPhi_{1,0}(p^{k-\frac{1}{2}}) \cdots \varPhi_{1,2n-1}(p^{k-\frac{1}{2}})\\ \varPhi_{2,0}(p^{k-\frac{1}{2}}) \cdots \varPhi_{2,2n-1}(p^{k-\frac{1}{2}})\end{pmatrix} = \begin{pmatrix}0 & 1\\ p^{-n-\frac{1}{2}} & p^{-n-\frac{1}{2}}(-1 + p\,\delta_{p|m})\\ 0 & p^{-2n+1}\end{pmatrix} A'_{2,2n}(p^{-(k-\frac{1}{2})}).$$

Furthermore, this identity holds for infinitely many k. Hence we obtain

(5.3)
$$\begin{pmatrix}\varPhi_{0,0}(X_p) \cdots \varPhi_{0,2n-1}(X_p)\\ \varPhi_{1,0}(X_p) \cdots \varPhi_{1,2n-1}(X_p)\\ \varPhi_{2,0}(X_p) \cdots \varPhi_{2,2n-1}(X_p)\end{pmatrix} = \begin{pmatrix}0 & 1\\ p^{-n-\frac{1}{2}} & p^{-n-\frac{1}{2}}(-1 + p\,\delta_{p|m})\\ 0 & p^{-2n+1}\end{pmatrix} A'_{2,2n}(X_p^{-1}).$$

In particular, we get $A'_{2,2n}(X_p) = A'_{2,2n}(X_p^{-1})$. Due to the identities (5.2) and (5.3), we thus obtain Theorem 1.2.

5.4. Proof of Theorem 1.3.

We remark that the m-th Fourier-Jacobi coefficient ϕ_m of $\mathcal{F}$ belongs to $J_{k+n,m}^{(2n-1)\,cusp}$. From the identity (2.1) in Section 2.4 and from Theorem 1.2 we obtain

$$\begin{aligned}&\phi_m(\tau,0)|(T_{0,2n-1}(p^2), \ldots, T_{2n-1,0}(p^2))\\ &\quad = p^{2nk+n-1}\big(\phi_{\frac{m}{p^2}}(\tau,0), \phi_m(\tau,0), \phi_{mp^2}(\tau,0)\big)\\ &\quad\times\begin{pmatrix}0 & 1\\ p^{-k-n} & p^{-k-n}(-1 + p\,\delta_{p|m})\\ 0 & p^{-2k-2n+2}\end{pmatrix} A'_{2,2n}(\alpha_p).\end{aligned}$$

Due to the identity $\mathcal{F}\left(\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}\right) = \sum_{m>0}\phi_m(\tau,0)e(m\omega)$, we have

$$\sum_{m>0}\big\{\phi_{\frac{m}{p^2}}(\tau,0) + p^{-k-n}(-1 + p\,\delta_{p|m})\phi_m(\tau,0) + p^{-2k-2n+2}\phi_{mp^2}(\tau,0)\big\}e(m\omega)$$

$$= p^{-2k-2n+2}\ F\left(\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}\right)\Big|_{\omega} T_{1,0}(p^2)\,,$$

where in the RHS we regard that the Hecke operator $T_{1,0}(p^2)$ acts on $F\left(\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}\right)$ as a function of $\omega \in \mathfrak{H}_1$ for a fixed $\tau \in \mathfrak{H}_1$. Therefore

$$\sum_{m>0} \phi_m(\tau,0)|(T_{0,2n-1}(p^2),\dots,T_{2n-1,0}(p^2))e(m\omega)$$
$$= p^{2nk+n-1}\left(p^{-k-n}F\left(\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}\right), p^{-2k-2n+2}F\left(\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}\right)\Big|_{\omega} T_{1,0}(p^2)\right)A'_{2,2n}(\alpha_p)\,.$$

We denote by $\langle h_1(\omega), h_2(\omega)\rangle_{\omega}$ the Petersson inner product of two elliptic modular forms h_1, h_2. The symbol $\lambda_g(p^2)$ denotes the eigenvalue of g for $T_{1,0}(p^2)$.

Because $\left\langle F\left(\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}\right),\ g(\omega)\right\rangle_{\omega} = \mathcal{F}_{f,g}(\tau)$ and because

$$\left\langle F\left(\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}\right)\Big|_{\omega} T_{1,0}(p^2),\ g(\omega)\right\rangle_{\omega} = \lambda_g(p^2)\mathcal{F}_{f,g}(\tau)\,,$$

we obtain

$$\mathcal{F}_{f,g}|(T_{0,2n-1}(p^2),\dots,T_{2n-1,0}(p^2))$$
$$= \left\langle \sum_{m>0}\phi_m(\tau,0)|(T_{0,2n-1}(p^2),\dots,T_{2n-1,0}(p^2))e(m\omega),\ g(\omega)\right\rangle_{\omega}$$
$$= p^{2nk+n-1}\left(p^{-k-n}\mathcal{F}_{f,g},\ p^{-2k-2n+2}\lambda_g(p^2)\mathcal{F}_{f,g}\right)A'_{2,2n}(\alpha_p)\,.$$

Therefore we proved Theorem 1.3.

5.5. Proof of Corollary 1.4.

Let $\{\mu_0,\mu_1,\dots,\mu_{2n-1}\}$ be the Satake parameter of $\mathcal{F}_{f,g}$ at a prime p. We recall

$$A'_{2,2n}(X_p) = B'_{2,2n}(p^{\frac{3}{2}-n}X_p, p^{\frac{5}{2}-n}X_p,\dots,p^{-\frac{3}{2}+n}X_p)\,,$$

where the matrices $A'_{2,2n}$ and $B'_{2,2n}$ are defined in Section 2.6. Because of the construction of $A'_{2,2n}(X_p)$, the matrix $A'_{2,2n}(\alpha_p)$ determines a Satake parameter $\{\mu_2,\dots,\mu_{2n-1}\}$ up to the action of the Weyl group W_{2n-1}. Hence we can take

$$\{\mu_2,\dots,\mu_{2n-1}\} = \{p^{\frac{3}{2}-n}\alpha_p, p^{\frac{5}{2}-n}\alpha_p,\dots,p^{-\frac{3}{2}+n}\alpha_p\}\,.$$

Now, from Section 2.6 and Section 2.3, we recall

$$(\varphi(T_{0,2n-1}(p^2)),\varphi(T_{1,2n-2}(p^2)),\dots,\varphi(T_{2n-1,0}(p^2)))$$
$$= \left(\prod_{i=2}^{2n-1} X_i\right)(\varphi(T_{0,1}(p^2)),\varphi(T_{1,0}(p^2)))B'_{2,2n}(X_2,\dots,X_{2n-1})$$

and $\mu_0^2\mu_1\cdots\mu_{2n-1} = p^{(2n-1)k}$, where φ is the Satake isomorphism denoted in Section 2.6, and where $T_{l,2n-l}(p^2)$ $(l = 0,\dots,2n)$ is the Hecke operator denoted in 2.4. Furthermore, from a straightforward calculation we have

$$\varphi(T_{0,1}(p^2)) = p^{-1}X_0^2X_1\,,$$
$$\varphi(T_{1,0}(p^2)) = p^{-1}X_0^2X_1(pX_1^{-1} + (p-1) + pX_1)\,.$$

From Theorem 1.3 and the above relations, we have

$$p^{2nk+n-1}\left(p^{-k-n},\ p^{-2k-2n+2}\lambda_g(p^2)\right)A'_{2,2n}(\alpha_p)$$
$$= \left(\prod_{i=2}^{2n-1}\mu_i\right)(p^{-1}\mu_0^2\mu_1,\ p^{-1}\mu_0^2\mu_1(p\mu_1^{-1} + (p-1) + p\mu_1))B'_{2,2n}(\mu_2,\ldots,\mu_{2n-1})\,.$$

Hence, from the fact that the rank of the matrix $A'_{2,2n}(\alpha_p)$ is two, we obtain

$$p\mu_1^{-1} + (p-1) + p\mu_1 = p^{-k-n+2}\lambda_g(p^2)\,.$$

On the other hand, we have $\lambda_g(p^2) = p^{k+n-2}(p\beta_p^2 + (p-1) + p\beta_p^{-2})$. Thus we can take $\mu_1 = \beta_p^2$. Hence we obtain

$$\{\mu_1, \mu_2, \mu_3, \ldots, \mu_{2n-1}\} = \left\{\beta_p^2,\ p^{\frac{3}{2}-n}\alpha_p,\ p^{\frac{5}{2}-n}\alpha_p, \ldots, p^{-\frac{3}{2}+n}\alpha_p\right\}$$

up to the action of the Weyl group W_{2n-1}.

The standard L-function of $\mathcal{F}_{f,g}$ is

$$L(s, \mathcal{F}_{f,g}, \mathrm{st}) = \prod_p \left\{(1-p^{-s})\prod_{i=1}^{2n-1}\left\{(1-\mu_i p^{-s})(1-\mu_i^{-1}p^{-s})\right\}\right\}^{-1}$$
$$= \prod_p \left\{(1-p^{-s})(1-\beta_p^2 p^{-s})(1-\beta_p^{-2}p^{-s})\right.$$
$$\left.\times \prod_{i=1}^{2n-2}\left\{(1-\alpha_p p^{i+\frac{1}{2}-n-s})(1-\alpha_p^{-1}p^{-i-\frac{1}{2}+n-s})\right\}\right\}^{-1}$$
$$= L(s, g, \mathrm{Ad})\prod_p\prod_{i=1}^{2n-2}\left\{(1-\alpha_p p^{i+\frac{1}{2}-n-s})(1-\alpha_p^{-1}p^{i+\frac{1}{2}-n-s})\right\}^{-1}\,.$$

Since $L(s, f) = \prod_p \left\{(1-\alpha_p p^{k-\frac{1}{2}-s})(1-\alpha_p^{-1}p^{k-\frac{1}{2}-s})\right\}^{-1}$, we obtain Corollary 1.4.

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