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IKEBUKURO TOKYO
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Modules of Vector Valued Siegel Modular Forms of Half Integral Weight

by

Tomoyoshi IBUKIYAMA

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Abstract. In this paper, we determine explicitly the modules of certain spaces of vector valued Siegel modular forms of half integral weight of degree two with character. Here the discrete group is the Hecke type congruence subgroup $\Gamma_0^{(2)}(4)$ of $Sp(2,\mathbb{R})$ and the weights of the forms are roughly $\det^{k-1/2} Sym(2)$ and $\det^{k-1/2} Sym(4)$ for any natural number k, where $\det^{k-1/2}$ is a certain automorphy factor of weight $k-1/2$ and $Sym(j)$ is the symmetric tensor representation of $GL(2)$ of degree j. The character is a certain standard non-trivial character ψ of $\Gamma_0^{(2)}(4)$ defined later. Our main theorems are given in §3 Theorem 3.2 and §3 Theorem 3.3. Precise definitions will be given in §1. We note that there is no modular forms of one variable of half-integral weight with character ψ belonging to $\Gamma_0^{(1)}(4)$, but in the case of degree two, both forms with character and without character appear. The dimensions of both of these vector valued Siegel cusp forms of degree two of weight $\det^{k-1/2} Sym(j)$ are obtained by Tsushima [8] when $k \geq 5$. Also for the case without character, Tsushima [9] obtained an explicit description of the module for $\det^{k-1/2} Sym(2)$ already. But as for the case with character, the situation is fairly different and more difficult, since it is like a difference between modular forms of even weight and odd weight, and usually in the theory of Siegel modular forms, the latter is more difficult to construct. We overcome this difficulty by using a theory of differential operators acting on a triple of Siegel modular forms. We have already developed the theory of such differential operators in [4], [6] in the case of integral weight, but here we apply it to the case of half integral weight. The necessary modification of the proof will be indicated in §2 Theorem 2.1. The explicit results in this paper were used for the experimental calculation in [5].

1. Preliminaries

We denote by H_n the Siegel upper half space of degree n and by $Sp(n,R)$ the usual symplectic group over any commutative ring R:

$$Sp(n,R) = \left\{ g \in GL(2n,R);\ {}^t g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\},$$

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where 1_n is the $n \times n$ unit matrix. Now we define a vector valued Siegel modular form of half integral weight. Let $\Gamma_0^{(n)}(4)$ be the subgroup of $Sp(n, \mathbb{Z})$ define by

$$\Gamma_0^{(n)}(4) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z}); C \equiv 0 \bmod 4 \right\} .$$

We denote by ψ the character of $\Gamma_0^{(n)}(4)$ defined by $\psi(\gamma) = \left(\frac{-1}{\det D}\right)$, where the right hand side is the Kronecker symbol. We define the theta function $\theta(Z)$ of $Z \in H_n$ by

$$\theta(Z) = \sum_{p \in \mathbb{Z}^n} e({}^t pZp) ,$$

where we write $e(x) = \exp(2\pi i x)$. Then for any $\gamma \in \Gamma_0^{(n)}(4)$, we have $(\theta(\gamma Z)/\theta(Z))^2 = \det(CZ + D)\psi(\gamma)$. So we can regard $\theta(\gamma Z)/\theta(Z)$ as the automorphy factor of $\Gamma_0^{(n)}(4)$ of weight 1/2. In order to define more general action beyond $\Gamma_0^{(n)}(4)$, we introduce the metaplectic group $\widetilde{G}_n$. As a set, $\widetilde{G}_n$ consists of elements $(g, \phi(Z))$ where $g \in Sp(n, \mathbb{R})$ and $\phi(Z)$ is a holomorphic function on H_n such that $|\phi(Z)| = |\det(CZ + D)|^{1/2}$. The group operation is defined by $(g_1, \phi_1(Z))(g_2, \phi_2(Z)) = (g_1 g_2, \phi_1(g_2 Z)\phi_2(Z))$. This group $\widetilde{G_n}$ is a covering group of $Sp(n, \mathbb{R})$. For any integer k and a rational representation (ρ, V) of $GL(n, \mathbb{C})$, we define an action of $\widetilde{G_n}$ on V-valued functions $F(Z)$ on H_n by

$$F|_{k-1/2,\rho}(g, \phi(Z)) = \phi(Z)^{-2k+1}\rho(CZ + D)^{-1}F(gZ) .$$

When ρ is the trivial representation, we write $F|_{k-1/2,\rho} = F|_{k-1/2}$. The group $\Gamma_0^{(n)}(4)$ can be embedded into $\widetilde{G}_n$ by the following group homomorphism:

$$\iota : \Gamma_0^{(n)}(4) \ni \gamma \to (\gamma, \theta(\gamma Z)/\theta(Z)) \in \widetilde{G}_n .$$

We denote the image of this map by $\widetilde{\Gamma}_0^{(n)}(4)$ and we often identify these two groups.

Now we assume that $n = 2$. We denote by $\rho_j = Sym(j)$ the symmetric tensor representation of $GL(2, \mathbb{C})$ of degree j. We denote by V_j the space of homogeneous polynomials $P(u_1, u_2)$ of degree j in u_1, u_2. We fix a realization of $Sym(j)$ by the representation $\rho_j(U)P(u_1, u_2) = P((u_1, u_2)U)$ for any $U \in GL(2, \mathbb{C})$. We write $\mu_j(g, Z) = \rho_j(CZ + D)$ for any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{R})$. Let χ be any character of $\Gamma_0^{(2)}(4)$. For any non-negative integers k and j, a Siegel modular form of $\Gamma_0^{(2)}(4)$ of weight $\det^{k-1/2} Sym(j)$ with character χ is defined to be a V_j-valued holomorphic function F on H_2 such that

$$F(\gamma Z) = \chi(\gamma)\,(\theta(\gamma Z)/\theta(Z))^{2k-1}\,\mu_j(\gamma, Z)F(Z)$$

for any $\gamma \in \Gamma_0^{(2)}(4)$. We denote by $A_{k-1/2,j}(\Gamma_0^{(2)}(4), \chi)$ the space of these functions. Also for any integer k and a character χ, we denote by $A_k(\Gamma_0^{(2)}(4), \chi)$ the space of holomorphic functions on H_2 such that

$$F(\gamma Z) = \chi(\gamma)\det(CZ + D)^k F(Z)$$

for all $\gamma \in \Gamma_0^{(2)}(4)$. When χ is the trivial character, we simply write $A_{k-1/2,j}(\Gamma_0^{(2)}(4)) = A_{k-1/2,j}(\Gamma_0^{(2)}(4), \chi)$ and $A_k(\Gamma_0^{(2)}(4)) = A_k(\Gamma_0^{(2)}(4), \chi)$. If $j = 0$ we also write $A_{k-1/2}(\Gamma_0^{(2)}(4), \chi) = A_{k-1/2,0}(\Gamma_0^{(2)}(4), \chi)$.

For any V_j-valued function F on H_2, we define the Siegel Φ-operator as usual by

$$\Phi(F) = \lim_{\lambda\to\infty} F\begin{pmatrix}\tau & 0\\ 0 & i\lambda\end{pmatrix}.$$

We say that $F \in A_{k-1/2,j}(\Gamma_0^{(2)}(4), \chi)$ is a cusp form if $\Phi(F|_{k-1/2,j}[\widetilde{g}]) = 0$ for any $\widetilde{g} = (g, \phi(Z)) \in \widetilde{G}_2$ with $g \in Sp(2,\mathbb{Z})$. We denote by $S_{k-1/2,j}(\Gamma_0^{(2)}(4), \chi)$ the space of cusp forms.

Since $\det(CZ + D)^l \psi(\gamma)^l = (\theta(\gamma Z)/\theta(Z))^{2l}$ for any $\gamma \in \Gamma_0^{(2)}(4)$, we see that, if $F \in A_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi)$ and $G \in A_l(\Gamma_0^{(2)}(4), \psi^l)$, then we have $FG \in A_{k+l-1/2,j}(\Gamma_0^{(2)}(4), \psi)$. Hence, if we define a ring A of Siegel modular forms of integral weight by $A = \bigoplus_{k=0}^{\infty} A_k(\Gamma_0(4), \psi^k)$, then for a fixed j, the module $\oplus_{k=1}^{\infty} A_{k-1/2,j}(\Gamma_0(4), \psi)$ is an A-module. For $j = 2$ and 4, we shall give the generators of these A-modules in §3. Finally, we quote the following theorem given in Tsushima[8] as Theorem 4.6.

THEOREM 1.1. *We have*

$$A_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi) = S_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi).$$

Proof. Tsushima's argument in [8] seems insufficient for the non scalar-valued case, since he ignored the automorphy factor of the symmetric tensor representation in his argument there and some part of his claim in the proof is not valid for non scalar-valued case. So we give here a correct proof essentially based on his idea. Tsushima has shown in [8] that a complete set of representatives of one-dimensional cusps of $\Gamma_0^{(2)}(4)$ are given by P_i $(1 \le i \le 4)$ where

$$P_1 = 1_4, \qquad P_2 = \begin{pmatrix}0 & 1_2\\ -1_2 & 0\end{pmatrix}, \qquad P_3 = \begin{pmatrix}1_2 & 0\\ S & 1_2\end{pmatrix}, \qquad P_4 = \begin{pmatrix}1_2 & 0\\ T & 1_2\end{pmatrix},$$

for $S = \begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}$, $T = \begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}$, and the $n \times n$ unit matrix 1_n. We fix any $\phi_i(Z)$ such that $\widetilde{P_i} = (P_i, \phi_i(Z)) \in \widetilde{G}_2$. So we must show that for any $F \in A_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi)$ and i with $1 \le i \le 4$, we have $\Phi(F|_{k-1/2,j}\widetilde{P_i}) = 0$, where we write $F|_{k-1/2,\rho_j}\widetilde{P_i} = F|_{k-1/2,j}\widetilde{P_i}$ for short. For $\gamma \in \Gamma_0^{(2)}(4)$, we write $J(\gamma, Z) = \theta(\gamma Z)/\theta(Z)$ and we put

$$\phi_2 = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}.$$

We can easily show by direct calculation that $P_i\phi_2P_i^{-1} \in \Gamma_0^{(2)}(4)$ and $\psi(P_i\phi_2P_i^{-1}) = -1$ for all $i = 1, \ldots, 4$. If we denote by $Z_0 = \begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}$ the restriction of Z to the diagonal, then obviously $\phi_2 Z_0 = Z_0$. So we have $J(P_i\phi_2 P_i^{-1}, P_i Z_0) = \theta(P_i\phi_2 Z_0)/\theta(P_i Z_0) = 1$. So we have $F(P_i Z_0) = F(P_i\phi_2 P_i^{-1} P_i Z_0) = -\mu_j(P_i\phi_2 P_i^{-1}, P_i Z_0)F(P_i Z_0)$. This means that

$$(F|_{k-1/2,j}[\widetilde{P_i}])(Z_0) = \\ -\mu_j(P_i, Z_0)^{-1}\mu_j(P_i\phi_2 P_i^{-1}, P_i Z_0)\mu_j(P_i, Z_0)(F|_{k-1/2,j}[\widetilde{P_i}])(Z_0)\,.$$

By the cocycle condition of the automorphy factor and the fact that $\phi_2 Z_0 = Z_0$, we have

$$\begin{aligned}\mu_j(P_i\phi_2 P_i^{-1}, P_i Z_0) &= \mu_j(P_i\phi_2, Z_0)\mu_j(P_i^{-1}, P_i Z_0)\\ &= \mu_j(P_i, Z_0)\mu_i(\phi_2, Z_0)\mu(P_i, Z_0)^{-1}\,,\end{aligned}$$

so we have

$$\begin{aligned}(1) \qquad (F|_{k-1/2,j}[\widetilde{P_i}])(Z_0) &= -\mu_j(\phi_2, Z_0)(F|_{k-1/2,j}[\widetilde{P_i}])(Z_0).\\ &= -\rho_j\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(F|_{k-1/2,j}[\widetilde{P_i}])(Z_0)\,.\end{aligned}$$

So if we write $(F|_{k-1/2,j}[\widetilde{P_i}])(Z_0) = \sum_{\nu=0}^{j} f_{i,\nu}(Z_0)u_1^{j-\nu}u_2^{\nu}$, then the action of $\rho_j\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ on this function is given by $\sum_{\nu=0}^{j}(-1)^{\nu} f_{i,\nu}(Z_0)u_1^{j-\nu}u_2^{\nu}$, and we have $f_{i,\nu}(Z_0) = 0$ for all even ν. (Tsushima claimed in his proof in [8] that we have always $(f|_{k-1/2,j}[P_i])(Z_0) = 0$ but this is not true as we can see in the examples in section 3, i.e. $f_{i,\nu}(Z_0)$ is not identically zero in general when ν is odd.) The proof of the theorem is completed by using Arakawa's argument in [1] as follows. We have the Fourier expansion:

$$F|_{k-1/2,j}[\widetilde{P_i}](Z_0) = \sum_{T\in L} C_i(T)e(\mathrm{Tr}(TZ))$$

where L is a lattice in the space of 2×2 symmetric rational matrices and $C_i(T)$ is a homogeneous polynomial of degree j in u_1 and u_2 with constant coefficients. Then we have

$$\Phi(F|_{k-1/2,j}[\widetilde{P_i}](Z)) = \lim_{\omega\to i\infty}(F|_{k-1/2,j}[\widetilde{P_i}])(Z_0) = \sum_m C_i\begin{pmatrix}m & 0\\ 0 & 0\end{pmatrix}e(m\tau)\,,$$

where m runs over all rational numbers m such that $\begin{pmatrix}m & 0\\ 0 & 0\end{pmatrix}\in L$. So we must show that $C_i\begin{pmatrix}m & 0\\ 0 & 0\end{pmatrix} = 0$ for all such m. Now we define an element of $\Gamma_0^{(2)}(4)$ by

$$\gamma_l = \begin{pmatrix}U_l & 0\\ 0 & {}^tU_l^{-1}\end{pmatrix}$$

where $U_l = \begin{pmatrix}1 & 0\\ l & 1\end{pmatrix}$. We have $\psi(\gamma_l) = 1$. For each i with $1\le i\le 4$, we put $\delta_{i,l} = P_i\gamma_l P_i^{-1}$. Then we see $\delta_{i,l}\in\Gamma_0^{(2)}(4)$ and $\psi(\delta_{i,l}) = 1$, so we have $F|_{k-1/2,j}[\iota(\delta_{i,l})] = F$.

As is known in [10] Lemma 5.1 and 5.2, for $\widetilde{g} = (g, \phi(Z)) \in \widetilde{G}_2$ with $g \in Sp(2, \mathbb{Z})$ and $\gamma \in \Gamma_0^{(2)}(4)$ such that $g^{-1}\gamma g \in \Gamma_0^{(2)}(4)$, we have

$$\widetilde{g}^{-1}\iota(\gamma)\widetilde{g} = \iota(g^{-1}\gamma g)\,. \tag{2}$$

So we have

$$\begin{aligned} F|_{k-1/2,j}[\widetilde{P}_i]|_{k-1/2,j}[\iota(\gamma_l)] &= F|_{k-1/2,j}[\widetilde{P}_i\iota(\gamma_l)\widetilde{P}_i^{-1}\widetilde{P}_i] \\ &= F|_{k-1/2,j}[\iota(\delta_{i,l})\widetilde{P}_i] = F|_{k-1/2,j}\widetilde{P}_i\,. \end{aligned}$$

We see $\theta(\gamma_l Z)/\theta(Z) = 1$, so we have

$$(F|_{k-1/2,j}[\widetilde{P}_i])(U_l Z_0\,{}^tU_l) = \rho_j({}^tU_l)F|_{k-1/2,j}[\widetilde{P}_i](Z_0)\,.$$

Since ${}^tU_l\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}U_l = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$, this means that

$$C_i\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} = \rho_j\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}C_i\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}.$$

By taking $l > 0$, it is easy to see that the coefficient of $C_i\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$ at $u_1^{j-\nu}u_2^{\nu}$ are all zero except for the coefficient of u_1^j. But we have already shown before that $f_{i,\nu}(Z_0) = 0$ for all even ν. So we see $C_i\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} = 0$ and hence $\Phi(F|_{k-1/2,j}[\widetilde{P}_i]) = 0$ for all i with $1 \leq i \leq 4$. □

2. Differential operators

We construct the generators of $\oplus_{k=1}^{\infty}A_{k-1/2,j}(\Gamma_0(4), \psi)$ as an A-module by means of differential operators. So, we first explain about differential operators. In [4], we gave a certain general theory on differential operators on Siegel modular forms, which contains a theory of vector valued holomorphic differential operators with constant coefficients which give a new Siegel modular form from several given scalar valued Siegel modular forms of various integral weights for general degree. It is easy to see that this theory is valid also for the case of half integral weight. We explain the necessary modification below. We fix a natural number n and r. For each i with $1 \leq i \leq r$, we denote an $n \times n$ symmetric matrix with variable components by R_i. Let (ρ, V) be a finite dimensional rational representation of $GL(n, \mathbb{C})$. We denote by $Q(R_1, R_2, \ldots, R_r)$ a V-valued polynomial in the $n(n+1)r/2$ components of $R_1, \ldots, R_r$. Let d_i be an integer such that $d_i \geq n$. We denote by $\mathcal{H}(\rho, d_i, r)$ the space of V-valued polynomials $Q(R_1, \ldots, R_r)$ which satisfy the following conditions (1), (2).

(1) Let X_i be a $n \times d_i$ matrix of variable coefficients. If we put

$$P(X_1, X_2, \ldots, X_r) = Q(X_1^tX_1, X_2^tX_2, \ldots, X_r^tX_r)\,,$$

then P is pluriharmonic with respect to $X = (X_1, \ldots, X_r)$. Namely if we put $X = (x_{i\nu})$ $(1 \le i \le n, 1 \le \nu \le d_1 + dr)$, then for any i, j with $1 \le i, j \le n$, we have

$$\sum_{\nu=1}^{rd} \frac{\partial^2 P}{\partial x_{i\nu} x_{j\nu}} = 0 .$$

(2) For any $A \in GL(n, \mathbb{C})$, we have

$$Q(AR_1 {}^tA, AR_2 {}^tA, \ldots, AR_r {}^tA) = \rho(A) Q(R_1, \ldots, R_r) .$$

For $Z_l \in H_n$ we denote by $z_{l,ij}$ the (i, j) component of Z_l and define a matrix of differential operators by

$$\frac{\partial}{\partial Z_l} = \frac{1}{2\pi i} \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial z_{l,ij}} \right)_{1 \le i, j \le n} .$$

For any vector valued polynomial $Q = Q(R_1, \ldots, R_r)$ in $n(n+1)r/2$ variables, we define a vector valued operator $\mathcal{D}_Q$ by

$$\mathcal{D}_Q = Q \left(\frac{\partial}{\partial Z_1}, \ldots, \frac{\partial}{\partial Z_r} \right) .$$

We denote by Δ the image of diagonal embedding of $H_n \ni Z$ to $(Z, \ldots, Z) \in H_n^r = \overbrace{H_n \times \cdots \times H_n}^{r \text{ times}}$, and we denote by Res_Δ the restriction of functions of $(H_n)^r$ to Δ.

THEOREM 2.1. *Let κ_i $(1 \le i \le r)$ be even or odd integers. For any holomorphic function F on the r product $(H_n)^r$ and any element $(g, \phi(Z)) \in \widetilde{G}_n$, we have*

$$\begin{aligned} &\mathrm{Res}_\Delta(\mathcal{D}_Q(\phi(Z_1)^{-\kappa_1} \cdots \phi(Z_r)^{-\kappa_r} F(gZ_1, \ldots, gZ_r)) \\ &\quad = \phi(Z)^{-(\kappa_1 + \cdots + \kappa_r)} \rho(CZ + D)^{-1} (\mathrm{Res}_\Delta(\mathcal{D}_Q F))(gZ) , \end{aligned}$$

if and only if $Q \in \mathcal{H}(\rho, d_i, r)$.

Proof. We can regard that both sides of the above commutation relation are actions of $\widetilde{G}_n$. We can show that the following elements

$$\left(\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}, 1 \right), \qquad \left(\begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix}, 1 \right), (\text{all } {}^tS = S \in M_n(\mathbb{R}))$$

$$\left(\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \det(Z/i)^{1/2} \right), \qquad (1_4, c) \quad (\text{all } c \in \mathbb{C} \text{ with } |c| = 1)$$

of $\widetilde{G}_n$ give the same action on both sides almost in the same way as in the proof of Theorem 2 in [4], taking $d_i = \kappa_i$ and $d = \sum_{i=1}^r \kappa_i$. Here we fix any branch of $\det(Z/i)^{1/2}$. The rest is easy since $\widetilde{G}_n$ is generated by these elements and since this is an action of $\widetilde{G}_n$. □

COROLLARY 2.2. *We fix a rational representation (ρ, V) of $GL(n, \mathbb{C})$. If $F_1 \in A_{k_1 - 1/2}(\Gamma_0^{(n)}(4))$ and $F_l \in A_{k_i}(\Gamma_0^{(n)}(4), \psi^{k_i})$ for $2 \le i \le r$, then for any $Q \in \mathcal{H}(\det \otimes \rho, 2k_1 - 1, 2k_2, \ldots, 2k_r, r)$, the V-valued function $\mathrm{Res}_\Delta(\mathcal{D}_Q F)$ belongs to $A_{k_1 + \cdots + k_r + 1/2, \rho}(\Gamma_0^{(n)}(4), \psi)$.*

Proof. Since $\det(CZ+D) = (\theta(\gamma Z)/\theta(Z))^2\psi(\gamma)$ for $\gamma \in \Gamma_0^{(n)}(4)$, the automorphy factor of $\mathrm{Res}_\Delta(\mathcal{D}_Q F)$ is

$$\begin{aligned}&\det(CZ+D)^{k_2+\cdots+k_r+1}\psi(\gamma)^{k_2+\cdots+k_r}(\theta(\gamma Z)/\theta(Z))^{2k_1-1}\\&\quad = (\theta(\gamma Z)/\theta(Z))^{2k_1+\cdots+2k_r+1}\psi(\gamma)\end{aligned}$$

□

Examples of the above Q in case of $n=2$, $r=3$ and $j=2$ or 4 have been given in [6]. We reproduce them here for convenience of the readers.

We denote by $R=(r_{lm})$, $S=(s_{lm})$, $T=(t_{lm})$, three 2×2 symmetric matrices of independent variable coefficients. Let k_1, k_2, k_3 be positive rational numbers such that $2k_i \in \mathbb{Z}$ $(1\le i\le 3)$. We define

$$\{F,G,H\}_{\det Sym(2)} = \mathrm{Res}_\Delta(\mathcal{D}_Q(F(Z_1)G(Z_2)H(Z_3)))$$

for the following polynomial

$$Q(R,S,T) = \begin{vmatrix} r_{11} & s_{11} & t_{11}\\ r_{12} & s_{12} & t_{12}\\ k_1 & k_2 & k_3\end{vmatrix} u_1^2 - \begin{vmatrix} r_{11} & s_{11} & t_{11}\\ k_1 & k_2 & k_3\\ r_{22} & s_{22} & t_{22}\end{vmatrix} u_1u_2 + \begin{vmatrix} k_1 & k_2 & k_3\\ r_{12} & s_{12} & t_{12}\\ r_{22} & s_{22} & t_{22}\end{vmatrix} u_2^2 .$$

We also define

$$\{F,G,H\}_{\det Sym(4)} = \mathrm{Res}_\Delta(\mathcal{D}_Q(F(Z_1)G(Z_2)H(Z_3))$$

for the polynomial $Q(R,S,T) = \sum_{\nu=0}^4 Q_\nu(R,S,T)u_1^{4-\nu}u_2^\nu$, where we put

$$Q_0(R,S,T) = (k_2+1)\begin{vmatrix} (k_1+1)r_{11} & k_2 & k_3\\ r_{11}^2 & s_{11} & t_{11}\\ r_{11}r_{12} & s_{12} & t_{12}\end{vmatrix} - (k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{11} & k_3\\ r_{11} & s_{11}^2 & t_{11}\\ r_{12} & s_{11}s_{12} & t_{12}\end{vmatrix},$$

$$\begin{aligned}Q_1(R,S,T) = {}&2(k_2+1)\begin{vmatrix} (k_1+1)r_{12} & k_2 & k_3\\ r_{11}r_{12} & s_{11} & t_{11}\\ r_{12}^2 & s_{12} & t_{12}\end{vmatrix} - 2(k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{12} & k_3\\ r_{11} & s_{11}s_{12} & t_{11}\\ r_{12} & s_{12}^2 & t_{12}\end{vmatrix}\\ &+(k_2+1)\begin{vmatrix} (k_1+1)r_{11} & k_2 & k_3\\ r_{11}^2 & s_{11} & t_{11}\\ r_{11}r_{22} & s_{22} & t_{22}\end{vmatrix} - (k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{11} & k_3\\ r_{11} & s_{11}^2 & t_{11}\\ r_{22} & s_{11}s_{22} & t_{22}\end{vmatrix},\end{aligned}$$

$$Q_2(R,S,T) = 3(k_2+1)\begin{vmatrix} (k_1+1)r_{12} & k_2 & k_3\\ r_{11}r_{12} & s_{11} & t_{11}\\ r_{22}r_{12} & s_{22} & t_{22}\end{vmatrix} - 3(k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{12} & k_3\\ r_{11} & s_{11}s_{12} & t_{11}\\ r_{22} & s_{22}s_{12} & t_{22}\end{vmatrix},$$

$$\begin{aligned}Q_3(R,S,T) = {}&2(k_2+1)\begin{vmatrix} (k_1+1)r_{12} & k_2 & k_3\\ r_{12}^2 & s_{12} & t_{12}\\ r_{12}r_{22} & s_{22} & t_{22}\end{vmatrix} - 2(k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{12} & k_3\\ r_{12} & s_{12}^2 & t_{12}\\ r_{22} & s_{12}s_{22} & t_{22}\end{vmatrix}\\ &+(k_2+1)\begin{vmatrix} (k_1+1)r_{22} & k_2 & k_3\\ r_{11}r_{22} & s_{11} & t_{11}\\ r_{22}^2 & s_{22} & t_{22}\end{vmatrix} - (k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{22} & k_3\\ r_{11} & s_{11}s_{22} & t_{11}\\ r_{22} & s_{22}^2 & t_{22}\end{vmatrix},\end{aligned}$$

$$Q_4(R,S,T) = (k_2+1)\begin{vmatrix} (k_1+1)r_{22} & k_2 & k_3 \\ r_{22}r_{12} & s_{12} & t_{12} \\ r_{22}^2 & s_{22} & t_{22} \end{vmatrix} - (k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{22} & k_3 \\ r_{12} & s_{22}s_{12} & t_{12} \\ r_{22} & s_{22}^2 & t_{22} \end{vmatrix}.$$

Now we consider three numbers k_1, k_2, k_3 allowing duplication and we assume that two among k_1, k_2, k_3 are integers and the other one is a non-integral half-integer. In this case, if $F \in A_{k_1}(\Gamma_0^{(2)}(4), \psi^{l_1})$, $G \in A_{k_2}(\Gamma_0^{(2)}(4), \psi^{l_2})$ and $H \in A_{k_3}(\Gamma_0^{(2)}(4), \psi^{l_3})$, then we have

$$\{F,G,H\}_{\det Sym(2)} \in A_{k_1+k_2+k_3+1}(\Gamma_0^{(2)}(4), \psi^{\lambda_1+\lambda_2+\lambda_3+1})$$
$$\{F,G,H\}_{\det Sym(4)} \in A_{k_1+k_2+k_3+1}(\Gamma_0^{(2)}(4), \psi^{\lambda_1+\lambda_2+\lambda_3+1}),$$

where $\lambda_i = k_i + l_i$ if k_i is an integer and $\lambda_i = l_i$ if k_i is not an integer. The above change of the power of ψ is easily explained by using the fact $(\theta(\gamma Z)/\theta(Z))^{2k} = \det(CZ+D)^k\psi(\gamma)^k$ for any integral k.

3. Theorems and Proofs

For $Z \in H_2$ and any $m = {}^t(m', m'') \in \mathbb{Z}^4$, we define theta constants of characteristic m by

$$\theta_m(Z) = \sum_{p\in\mathbb{Z}^2} e\left(\frac{1}{2}{}^t\big(p+\frac{m'}{2}\big)Z\big(p+\frac{m'}{2}\big) + {}^t\big(p+\frac{m'}{2}\big)\frac{m''}{2}\right).$$

We write

$$f_1 = (\theta_{0000}(2Z))^2 = \theta^2,$$
$$x_2 = (\theta_{0000}(2Z)^4 + \theta_{0001}(2Z)^4 + \theta_{0010}(2Z)^4 + \theta_{0011}(2Z)^4)/4,$$
$$g_2 = \theta_{0000}(2Z)^4 + \theta_{0100}(2Z)^4 + \theta_{1000}(2Z)^4 + \theta_{1100}(2Z)^4,$$
$$f_3 = (\theta_{0001}(2Z)\theta_{0010}(2Z)\theta_{0011}(2Z))^2.$$

We know that $f_1 \in A_1(\Gamma_0^{(2)}(4), \psi)$, x_2, $g_2 \in A_2(\Gamma_0^{(2)}(4))$ and $f_3 \in A_3(\Gamma_0^{(2)}(4), \psi)$ (See [7]). It has been shown there also that f_1, g_2, x_2, f_3 are algebraically independent and

$$\oplus_{k=0}^{\infty} A_k(\Gamma_0(4), \psi^k) = \mathbb{C}[f_1, g_2, x_2, f_3].$$

As explained before, the module $\oplus_{k=1}^{\infty} A_{k-1/2,j}(\Gamma_0(4), \psi)$ is an A-module for each non-negative integer j, where we put $A = \mathbb{C}[f_1, g_2, x_2, f_3]$. As was shown in Theorem 1.1, we have $A_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi) = S_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi)$ for any natural number j and in particular, this space is zero if j is odd.

PROPOSITION 3.1. *We have*

$$\sum_{k=1}^{\infty} \dim A_{k-1/2,2}(\Gamma_0(4), \psi)t^k = \frac{t^6+2t^7}{(1-t)(1-t^2)^2(1-t^3)}$$

and

$$\sum_{k=1}^{\infty} \dim A_{k-1/2,4}(\Gamma_0(4), \psi)t^k = \frac{3t^5 + 3t^6 - t^8}{(1-t)(1-t^2)^2(1-t^3)} .$$

Proof. The first result is due to Tsushima [9]. The second results for $k \geq 5$ is due to Tsushima [8]. The proof for $k \leq 4$ will be given in the proof of Theorem 3.3. □

3.1. Case of Sym(2)

We put

$$F_{11/2,2}(Z) = \{\theta, g_2, x_2\}_{\det Sym(2)} ,$$
$$G_{13/2,2}(Z) = \{\theta, g_2, f_3\}_{\det Sym(2)} ,$$
$$H_{13/2,2}(Z) = \{\theta, x_2, f_3\}_{\det Sym(2)} .$$

Then we have $F_{11/2,2} \in A_{11/2,2}(\Gamma_0(4), \psi)$ and $G_{13/2,2}, H_{13/2,2} \in A_{13/2,2}(\Gamma_0(4), \psi)$.

THEOREM 3.2. *We have*

$$\oplus_{k=1}^{\infty} A_{k-1/2,2}(\Gamma_0(4), \psi) = AF_{11/2,2} \oplus AG_{13/2,2} \oplus AH_{13/2,2} .$$

Here the notation $\oplus$ means the direct sum as modules.

Proof. Comparing with the dimension formula, all we need to do is to show that three vectors $F_{11/2,2}$, $G_{13/2,2}$, $H_{13/2,2}$ are linearly independent over the ring A. We identify any V_j valued function $F = \sum_{i=0}^{j} F_i u_1^{j-i} u_2^i$ as a column vector ${}^t(F_0, F_2, \ldots, F_j)$. Then if we put

$$F = \begin{pmatrix} F_{11/2,2} & G_{13/2,2} & H_{13/2,2} \end{pmatrix} ,$$

we can regard this as a 3×3 matrix. The linear independence is proved if we show that $\det(F)$ is not identically zero as a holomorphic function of Z. In general, if $B = (b_{lm})$ is a 4×4 matrix and $C = (\widetilde{b}_{lm})_{2 \leq l,m \leq 4}$ is the 3×3 matrix where $\widetilde{b}_{lm}$ is the (l, m) cofactor of B, then it is well known in the theory of linear algebra that we have

$$\det(C) = b_{11} \det(B)^2 . \tag{3}$$

For $Z \in H_2$, we write $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}$ and put $\partial_1 = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$, $\partial_2 = \frac{1}{4\pi i}\frac{\partial}{\partial z}$, $\partial_3 = \frac{1}{2\pi i}\frac{\partial}{\partial \omega}$. Applying (3) to

$$B = \begin{pmatrix} \theta_{0000}/2 & 2x_2 & 2g_2 & 3f_3 \\ \partial_1\theta_{0000} & \partial_1 x_2 & \partial_1 g_2 & \partial_1 g_3 \\ \partial_2\theta_{0000} & \partial_2 x_2 & \partial_2 g_2 & \partial_2 g_3 \\ \partial_3\theta_{0000} & \partial_3 x_2 & \partial_3 g_2 & \partial_3 g_3 \end{pmatrix} ,$$

we have $\det(F) = \det(C) = \theta_{0000} \det(B)^2/2$. Since θ_{0000}, x_2, g_2 and f_3 are algebraically independent, the functional determinant of x_2/θ_{0000}^4, g_2/θ_{0000}^4 and f_3/θ_{0000}^6 is not identically zero, and since this is equal to a non-zero constant times $\det(B)/\theta_{0000}^{15}$, we see that $\det(B)$ is not identically zero either. So we have $\det(F) \neq 0$ as a function. This means that three vectors $F_{11/2,2}$, $G_{13/2,2}$, $H_{13/2,2}$ are linearly independent over the ring of holomorphic functions on H_2, a fortioli over A. □

Actually we have $\det(F) \in M_{43/2}(\Gamma_0(43))$.

For the sake of completeness, we give a more direct proof too. We denote by f_{lm} the (l,m)-component of the above 3×3 matrix F. For $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}$, we put $q = e(\tau)$, $\zeta = e(z)$, $u = e(\omega)$. Then by calculating the Fourier coefficients, we have

$$\begin{aligned}
f_{11} &= u(384q^2(\zeta - \zeta^{-1}) - 2304q^3(\zeta - \zeta^{-1}) + O(q^4)) + O(u^2)\,,\\
f_{21} &= u(768q^2(\zeta + \zeta^{-1}) - 4608q^3(\zeta + \zeta^{-1}) + O(q^4)) + O(u^2)\,,\\
f_{31} &= u^2(-384q(\zeta - \zeta^{-1}) + 1152q^2(\zeta - \zeta^{-1}) + 384q^2(\zeta^3 - \zeta^{-3})\\
&\quad - 10368q^3(\zeta - \zeta^{-1}) - 1152q^3(\zeta^3 - \zeta^{-3}) + O(q^4)) + O(u^3)\,,\\
f_{12} &= u(768q^2(\zeta - \zeta^{-1}) - 1536q^3(\zeta - \zeta^{-1})\\
&\quad + 1024q^3(\zeta^2 - \zeta^{-2}) + O(q^4)) + O(u^2)\,,\\
f_{22} &= u(1024q^2 + 1536q^2(\zeta + \zeta^{-1}) - 4096q^3\\
&\quad - 3072q^3(\zeta + \zeta^{-1}) + 1024q^3(\zeta^2 + \zeta^{-2}) + O(q^4)) + O(u^2)\,,\\
f_{32} &= u^2(-768q(\zeta - \zeta^{-1}) + 5376q^2(\zeta - \zeta^{-1}) + 768q^2(\zeta^3 - \zeta^{-3})\,,\\
&\quad - 8448q^3(\zeta - \zeta^{-1}) - 5376q^3(\zeta^3 - \zeta^{-3}) + O(q^4)) + O(u^3)\,,\\
f_{13} &= u(-512q^3(\zeta^2 - \zeta^{-2}) + O(q^4)) + O(u^2)\,,\\
f_{23} &= u(-512q^2 + 2048q^3 - 512q^3(\zeta^2 + \zeta^{-2}) + O(q^4)) + O(u^2)\,,\\
f_{33} &= u^3(512q(\zeta^2 - \zeta^{-2}) - 3072q^2(\zeta^2 - \zeta^{-2})\\
&\quad - 512q^2(\zeta^4 - \zeta^{-4}) + 8192q^3(\zeta^2 - \zeta^{-2})\\
&\quad + 3072q^3(\zeta^4 - \zeta^{-4}) + O(q^4)) + O(u^4)\,.
\end{aligned}$$

By direct calculation, we see

$$\det(F) = 1207959552u^4(q^6(\zeta - \zeta^{-1})^2 + O(q^7)) + O(u^5) \neq 0\,.$$

3.2. Case of Sym(4)

Before stating next theorem, we review the Hecke operator $U(4)$ acting on $A_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi)$. We put

$$g_2 = \begin{pmatrix} 2^{-1}1_2 & 0 \\ 0 & 2 \cdot 1_2 \end{pmatrix}$$

and $\widetilde{g_2} = (g_2, 2) \in \widetilde{G}_2$. By using the explicit formula of the multiplier system of $\theta(Z)$ in Proposition 4.1 in [10], we can show that for any $\gamma \in \Gamma_0^{(2)}(4) \cap g_2^{-1}\Gamma_0^{(2)}(4)g_2$, we have $\widetilde{g_2}\iota(\gamma)\widetilde{g_2}^{-1} = \iota(g_2\gamma g_2^{-1})$. So for the double coset

$$U(4) = \widetilde{\Gamma}_0^{(2)}(4)\,(g_2, 2)\,\widetilde{\Gamma_0}^{(2)}(4) = \bigcup_{i=1}^{r} \widetilde{\Gamma_0}^{(2)}(4)\widetilde{g_2}\widetilde{h}_r\,, \quad (\widetilde{h}_r \in \widetilde{\Gamma}_0^{(2)}(4))\,,$$

we can define the action on any $F(z) \in A_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi)$ by

$$F|_{k-1/2,j}U(4) = 2^{2k+m-7}\sum_r \psi(h_r)F|_{k-1/2,j}[\tilde{g}_2\tilde{h}_r]\,.$$

If we write $\widetilde{h}_r = (h_r, \theta(h_r Z)/\theta(Z))$, then $h_r \in \Gamma_0^{(2)}(4)$ can be chosen as representatives of $\left(\Gamma_0^{(2)}(4) \cap g_2^{-1}\Gamma_0^{(2)}(4)g_2\right)\backslash \Gamma_0^{(2)}(4)$, so a complete set of representatives of $\widetilde{\Gamma}_0^{(2)}(4)\backslash U(4)$ is given by

$$\widetilde{g}\left(\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} 2^{-1}1_2 & 2^{-1}S \\ 0 & 2\cdot 1_2 \end{pmatrix}, 2\right) \in \widetilde{g}\,\widetilde{\Gamma}_0^{(2)}(4)\,,$$

where S runs over 2×2 integral symmetric matrices which are representatives mod 4. Hence if we write the Fourier expansion of $F(Z)$ by $F(Z) = \sum_T a(T)e(\mathrm{Tr}(TZ))$ where T runs over positive definite half-integral symmetric matrices and $a(T) \in V_j$, we have

$$F|_{k-1/2,j}U(4) = \frac{1}{64}\sum_{S={}^tS\in M_2(\mathbb{Z}/4\mathbb{Z})} F\left(\frac{Z+S}{4}\right) = \sum_T a(4T)e(\mathrm{Tr}(TZ))\,.$$

Since the last expression does not contain k and j and simple, we may regard this as a definition of $U(4)$ and we write $F|_{k-1/2,j}U(4) = F|U(4)$. We put

$$\begin{aligned}
F_{9/2,4a} &= \{f_1, g_2, \theta\}_{Sym(4)}/96\,,\\
F_{9/2,4b} &= \{f_1, x_2, \theta\}_{Sym(4)}/48\,,\\
F_{9/2,4c} &= \{f_1, g_2, \theta\}_{Sym(4)}|U(4)/12288\,,\\
F_{11/2,4a} &= \{g_2, g_2, \theta\}_{Sym(4)}/1152\,,\\
F_{11/2,4b} &= \{g_2, \theta, x_2\}_{Sym(4)}/144\,,\\
F_{11/2,4c} &= \{x_2, x_2, \theta\}_{Sym(4)}/2304\,.
\end{aligned}$$

THEOREM 3.3. *We have*

$$\oplus_{k=1}^{\infty} A_{k-1/2,4}(\Gamma_0^{(2)}(4)) = AF_{9/2,4a} \oplus AF_{9/2,4b} \oplus AF_{9/2,4c} \oplus AF_{11/2,4a} \oplus AF_{11/2,4b} \oplus BF_{11/2,4c}\,,$$

where we put $A = \mathbb{C}[f_1, g_2, x_2, f_3]$ as before and $B = \mathbb{C}[f_1, g_2, f_3]$. Here the notation $\oplus$ means the direct sum as modules.

In order to prove Theorem 3.3, we need the following lemma.

LEMMA 3.4. *Five vectors $F_{9/2,4a}$, $F_{9/2,4b}$, $F_{11/2.4a}$, $F_{11/2,4b}$, $F_{11/2,4c}$ are linearly independent over the ring of holomorphic functions on H_2.*

Proof. Regarding modular forms in $A_{k+1/2,j}(\Gamma_0^{(2)}(4), \psi)$ as five dimensional column vectors as before, we can show that the function

$$\det(F_{9/2,4a}, F_{9/2,4b}, F_{11/2,4a}, F_{11/2,4b}, F_{11.2,4c})$$

is not identically zero as the function on H_2. Indeed, if we put $q = e(\tau)$, $u = e(\omega)$, $\zeta = e(z)$ as before and put

$$H = (h_{ij}) = (F_{9/2,4a}, F_{9/2,4b}, F_{11/2,4a}, F_{11/2,4b}, F_{11/2,4c}),$$

then mainly by computer calculation we have

$$\begin{aligned}
h_{11} &= u(6q^2(\zeta-\zeta^{-1}) - 12q^3(\zeta-\zeta^{-1}) + 6q^3(\zeta^2-\zeta^{-2}) \\
&\quad - 30q^4(\zeta-\zeta^{-1}) - 12q^4(\zeta^2-\zeta^{-2}) + 18q^4(\zeta^3-\zeta^{-3}) + O(q^5)) + O(u^2), \\
h_{21} &= u(q + 2q^2 + 12q^2(\zeta+1/\zeta) - 2q^2(\zeta^2+\zeta^{-2}) - 12q^3 - 24q^3(\zeta+\zeta^{-1}) \\
&\quad + 14q^3(\zeta^2+\zeta^{-2}) - 8q^4 - 60q^4(\zeta+\zeta^{-1}) - 12q^4(\zeta^2+\zeta^{-2}) \\
&\quad + 12q^4(\zeta^3+\zeta^{-3}) + O(q^5)) + O(u^2), \\
h_{31} &= u(-3q^2(\zeta^2-\zeta^{-2}) + 12q^3(\zeta^2-\zeta^{-2}) + O(q^5)) + O(u^2), \\
h_{41} &= u(-q + q^2(4-(\zeta^2+\zeta^{-2})) + q^3(4(\zeta^2+\zeta^{-2})) - 16q^4 + O(q^5)) + O(u^2), \\
h_{51} &= u^2(-6q(\zeta-\zeta^{-1}) - 6q^2(\zeta-\zeta^{-1}) + 6q^2(\zeta^3-\zeta^{-3}) + 30q^3(\zeta-\zeta^{-1}) \\
&\quad + 6q^3(\zeta^3-\zeta^{-3}) + 24q^4(\zeta-\zeta^{-1}) - 30q^4(\zeta^3-\zeta^{-3}) \\
&\quad - 6q^4(\zeta^5-\zeta^{-5}) + O(q^5)) + O(u^3), \\
h_{12} &= u(-6q^3(\zeta^2-\zeta^{-2}) + 12q^4(\zeta^2-\zeta^{-2}) + O(q^5)) + O(u^2), \\
h_{22} &= u(-q - 2q^2 + 2q^2(\zeta^2+\zeta^{-2}) + 12q^3 - 14q^3(\zeta^2+\zeta^{-2}) \\
&\quad + 8q^4 + 12q^4(\zeta^2+\zeta^{-2}) + O(q^5)) + O(u^2), \\
h_{32} &= u(3q^2(\zeta^2-\zeta^{-2}) - 12q^3(\zeta^2-\zeta^{-2}) + O(q^5)) + O(u^2), \\
h_{42} &= u(q - 4q^2 + q^2(\zeta^2+\zeta^{-2}) - 4q^3(\zeta^2+\zeta^{-2}) + 16q^4 + O(q^4)) + O(u^2), \\
h_{52} &= u^3(6q(\zeta^2-\zeta^{-2}) - 6q^2(\zeta^4-\zeta^{-4}) - 48q^3(\zeta^2-\zeta^{-2}) \\
&\quad + 48q^4(\zeta^4-\zeta^{-4}) + O(q^5)) + O(u^4), \\
h_{13} &= u(2q^2(\zeta-\zeta^{-1}) - 12q^3(\zeta-\zeta^{-1}) - 2q^3(\zeta^2-\zeta^{-2}) \\
&\quad + 54q^4(\zeta-\zeta^{-1}) + 12q^4(\zeta^2-\zeta^{-2}) - 18q^4(\zeta^3-\zeta^{-3}) + O(q^5)) + O(u^2), \\
h_{23} &= u(q - 2q^2 - 2q^2(\zeta^2+\zeta^{-2}) - 2q^3(\zeta^2+\zeta^{-2}) + 16q^4 \\
&\quad + 48q^4(\zeta+\zeta^{-1}) + 36q^4(\zeta^2+\zeta^{-2}) - 48q^4(\zeta^3+\zeta^{-3}) + O(q^5)) + O(u^2), \\
h_{33} &= u(-12q^2(\zeta-\zeta^{-1}) - 3q^2(\zeta^2-\zeta^{-2}) + 72q^3(\zeta-\zeta^{-1}) \\
&\quad - 180q^4(\zeta-\zeta^{-1}) + 36q^4(\zeta^2-\zeta^{-2}) - 36q^4(\zeta^3-\zeta^{-3}) + O(q^5)) + O(u^2), \\
h_{43} &= u(-q - 8q^2(\zeta+\zeta^{-1}) - q^2(\zeta^2+\zeta^{-2}) + 12q^3 + 48q^3(\zeta+\zeta^{-1}) \\
&\quad - 120q^4(\zeta+\zeta^{-1}) + 12q^4(\zeta^2+\zeta^{-2}) - 8q^4(\zeta^3+\zeta^{-3}) + O(q^5)) + O(u^2), \\
h_{53} &= u^2(-2q(\zeta-\zeta^{-1}) - 42q^2(\zeta-\zeta^{-1}) + 2q^2(\zeta^3-\zeta^{-3}) \\
&\quad + 90q^3(\zeta-\zeta^{-1}) + 42q^3(\zeta^3-\zeta^{-3}) - 184q^4(\zeta-\zeta^{-1})
\end{aligned}$$

$$
\begin{aligned}
&\quad - 90q^4(\zeta^3 - \zeta^{-3}) - 2q^4(\zeta^5 - \zeta^{-5}) + O(q^5)) + O(u^3)\,,\\
h_{14} &= u(4q^2(\zeta - \zeta^{-1}) + 24q^3(\zeta - \zeta^{-1}) + 10q^3(\zeta^2 - \zeta^{-2})\\
&\quad - 156q^4(\zeta - \zeta^{-1}) + 36q^4(\zeta^2 - \zeta^{-2}) - 12q^4(\zeta^3 - \zeta^{-3}) + O(q^5)) + O(u^2)\,,\\
h_{24} &= u(3q + 10q^2 + 4q^2(\zeta + \zeta^{-1}) - 6q^2(\zeta^2 + \zeta^{-2})\\
&\quad + 72q^3(\zeta + \zeta^{-1}) + 10q^3(\zeta^2 + \zeta^{-2}) - 80q^4 - 372q^4(\zeta + \zeta^{-1})\\
&\quad + 108q^4(\zeta^2 + \zeta^{-2}) - 44q^4(\zeta^3 + \zeta^{-3}) + O(q^5)) + O(u^2)\,,\\
h_{34} &= u(-12q^2(\zeta - \zeta^{-1}) - 9q^2(\zeta^2 - \zeta^{-2}) + 72q^3(\zeta - \zeta^{-1})\\
&\quad - 180q^4(\zeta - \zeta^{-1}) + 108q^4(\zeta^2 - \zeta^{-2}) - 36q^4(\zeta^3 - \zeta^{-3})) + O(u^2),\\
h_{44} &= u(-3q - 8q^2(\zeta + \zeta^{-1}) - 3q^2(\zeta^2 + \zeta^{-2}) + 36q^3 + 48q^3(\zeta + \zeta^{-1})\\
&\quad - 120q^4(\zeta + \zeta^{-1}) + 36q^4(\zeta^2 + \zeta^{-2}) - 8q^4(\zeta^3 + \zeta^{-3}) + O(q^5)) + O(u^2)\,,\\
h_{54} &= u^2(-4q(\zeta - \zeta^{-1}) + 12q^2(\zeta - \zeta^{-1}) + 4q^2(\zeta^3 - \zeta^{-3})\\
&\quad - 108q^3(\zeta - \zeta^{-1}) - 12q^3(\zeta^3 - \zeta^{-3}) + 304q^4(\zeta - \zeta^{-1})\\
&\quad + 108q^4(\zeta^3 - \zeta^{-3}) - 4q^4(\zeta^5 - \zeta^{-5}) + O(q^5)) + O(u^3)\,,\\
h_{15} &= u(-q^3(\zeta^2 - \zeta^{-2}) + 8q^5(\zeta^2 - \zeta^{-2}) + O(q^6)) + O(u^2)\,,\\
h_{25} &= u(-q^2 - q^3(\zeta^2 + \zeta^{-2}) + 8q^4 + O(q^5)) + O(u^2)\,,\\
h_{35} &= u^2(12q^3(\zeta^2 - \zeta^{-2}) + 3q^3(\zeta^4 - \zeta^{-4})\\
&\quad - 72q^4(\zeta^2 - \zeta^{-2}) - 18q^4(\zeta^4 - \zeta^{-4}) + O(q^5)) + O(u^3)\,,\\
h_{45} &= u^2(q - 6q^2 + 18q^3 + 8q^3(\zeta^2 + \zeta^{-2}) + q^3(\zeta^4 + \zeta^{-4})\\
&\quad - 44q^4 - 48q^4(\zeta^2 + \zeta^{-2}) - 6q^4(\zeta^4 + \zeta^{-4}) + O(q^5)) + O(u^3)\,,\\
h_{55} &= u^3(q(\zeta^2 - \zeta^{-2}) - 4q^2(\zeta^2 - \zeta^{-2}) - q^2(\zeta^4 - \zeta^{-4})\\
&\quad + 28q^3(\zeta^2 - \zeta^{-2}) + 4q^3(\zeta^4 - \zeta^{-4}) - 96q^4(\zeta^2 - \zeta^{-2})\\
&\quad - 28q^4(\zeta^4 - \zeta^{-4}) + O(q^5)) + O(u^4)\,.
\end{aligned}
$$

Hence we have

$$\det(H) = (6912q^9(\zeta - \zeta^{-1})^3 + O(q^{10}))u^6 + O(u^7)\,.$$

So $\det(H)$ is not identically zero and Lemma is proved.

Proof of Theorem 3.3. First of all we can show the following relation.

$$
\begin{aligned}
(4)\quad &(-30f_1^3 + 27f_3 + 13f_1g_2 + 8f_1x_2)F_{9/2,4a}\\
&+ (-24f_1^3 + 4f_1g_2 + 32f_1x_2)F_{9/2,4b} + (-6f_1^3 + 9f_3 + 3f_1g_2)F_{9/2,4c}\\
&+ (18f_1^2 - 6g_2 - 12x_2)F_{11/2,4a} + (576f_1^2 - 96g_2 - 768x_2)F_{11/2,4c} = 0
\end{aligned}
$$

This is proved as follows. Since $\dim S_{15/2}(\Gamma_0^{(2)}(4),\psi)=20$, there should exist a non-trivial linear relation between 21 forms obtained by multiplying f_1^3, f_1x_2, f_1g_2, f_3 to $F_{9/2,4a}$, $F_{9/2,4b}$, $F_{9.2,4c}$, or f_1^2, x_2, g_2 to $F_{11/2,4a}$, $F_{11/2,4b}$, $F_{11/2,4c}$. The explicit linear relation is obtained by using explicit Fourier coefficients of these forms. The details are omitted here, since it is a routine work. Now assume that

$$G_1F_{9/2,4a}+G_2F_{9/2,4b}+G_3F_{9/2,4c}+G_4F_{11/2,4a}+G_5F_{11/2,4b}+G_6F_{11/2,4c}=0$$

for some $G_1,G_2,G_3\in M_{k-1}(\Gamma_0^{(2)}(4),\psi^{k-1})$ and $G_4,G_5,G_6\in M_k(\Gamma_0^{(2)}(4),\psi^k)$. We multiply $6f_1-9f_3-3f_1g_2$ to both sides of this equality. Then by the above relation (4) and Lemma 3.4, we have

$$\begin{aligned}
(6f_1^3-9f_3-3f_1g_2)G_1&=(-30f_1^3+27f_3+13f_1g_2+8f_1x_2)G_3\\
(6f_1^3-9f_3-3f_1g_2)G_2&=(-24f_1^3+4f_1g_2+32f_1x_2)G_3\\
(6f_1^3-9f_3-3f_1g_2)G_4&=(18f_1^2-6g_2-12x_2)G_3\\
(6f_1^3-9f_3-3f_1g_2)G_5&=0\\
(6f_1^3-9f_3-3f_1g_2)G_6&=(576f_1^2-96g_2-768x_2)G_3
\end{aligned}$$

Since f_1,g_2,x_2,f_3 are algebraically independent and $G_1,G_3\in A=\mathbb{C}[f_1,g_2,x_2,f_3]$, we see from the first relation that

$$G_3=(6f_1^3-9f_3-3f_1g_2)G_0$$

for some $G_0\in M_{k-4}(\Gamma_0(4),\psi^k)$. So we have

$$\begin{aligned}
G_3&=(6f_1^3-9f_3-3f_1g_2)G_0\\
G_1&=(-30f_1^3+27f_3+13f_1g_2+8f_1x_2)G_0\\
G_2&=(-24f_1^3+4f_1g_2+32f_1x_2)G_0\\
G_4&=(18f_1^2-6g_2-12x_2)G_0\\
G_5&=0\\
G_6&=(576f_1^2-96g_2-768x_2)G_0
\end{aligned}$$

So if we assume that $G_6\in B=\mathbb{C}[f_1,g_2,f_3]$, then $G_0=0$. So if we put

$$M=AF_{9/2,4a}+AF_{9/2,4b}+AF_{9/2,4c}+AF_{11/2,4a}+AF_{11/2,4b}+BF_{11/2,4c},$$

then this is a direct sum of modules. If we put $M_{k-1/2}=M\cap A_{k-1/2,4}(\Gamma_0(4),\psi)$, then we have

$$\begin{aligned}
\sum_{k=1}^{\infty}(\dim M_{k-1/2})t^k&=\frac{3t^5+2t^6}{(1-t)(1-t^2)^2(1-t^3)}+\frac{t^6}{(1-t)(1-t^2)(1-t^3)}\\
&=\frac{3t^5+3t^6-t^8}{(1-t)(1-t^2)^2(1-t^3)}.
\end{aligned}$$

By the dimension formula of $A_{k-1/2,4}(\Gamma_0^{(2)}(4), \psi)$ due to Tsushima [8], we see that the above generating function is the true generating function of $\dim A_{k-1/2,4}(\Gamma_0^{(2)}(4), \psi)$ for $k \geq 5$, so we see that $M_{k-1/2} = A_{k-1/2,4}(\Gamma_0^{(2)}(4), \psi)$ for $k \geq 5$. Finally we show that for $k \leq 4$, we have $\dim A_{k-1/2,4}(\Gamma_0^{(2)}(4), \psi) = 0$. Since

$$\theta^2 A_{k-1/2,4}(\Gamma_0^{(2)}(4), \psi) = f_1 A_{k-1/2,4}(\Gamma_0^{(2)}(4), \psi) \subset A_{k+1/2,4}(\Gamma_0^{(2)}(4), \psi)\,,$$

it is sufficient to prove that $A_{7/2,4}(\Gamma_0^{(2)}(4), \psi) = 0$. We already know the basis of $A_{k-1/2,4}(\Gamma_0^{(2)}(4), \psi)$ for $k \geq 5$, so for any $F \in A_{7/2,4}(\Gamma_0^{(2)}(4), \psi)$, the forms $f_1 F$ of weight $9/2$ and $x_2 F$ of weight $11/2$ are written as follows.

$$f_1 F = \alpha_1 F_{9/2,4a} + \alpha_2 F_{9/2,4b} + \alpha_3 F_{9/2,4c}\,, \tag{5}$$

$$x_2 F = \beta_1 f_1 F_{9/2,4a} + \beta_2 f_1 F_{9/2,4b} + \beta_3 f_1 F_{9/2,4c} + \beta_4 F_{11/2,4a} + \beta_5 F_{11/2,4b} + \beta_6 F_{11/2,4c}\,, \tag{6}$$

where α_i $(1 \leq i \leq 3)$ and β_i $(1 \leq i \leq 6)$ are some constants. Multiplying x_2 to (5) and f_1 to (6) and subtracting the equalities, we have a linear relation over $\mathbb{C}$ of the following 9 forms: $x_2 F_{9/2,4a}$, $x_2 F_{9/2,4b}$, $x_2 F_{9/2,4c}$, $f_1^2 F_{9/2,4a}$, $f_1^2 F_{9/2,4b}$, $f_1^2 F_{9/2,4c}$, $f_1 F_{11/2,4a}$, $f_1 F_{11/2,4b}$ and $f_1 F_{11/2,4c}$. Since the weight of these forms is $13/2 > 5 - 1/2$, we have already shown that these are linearly independent over $\mathbb{C}$. So comparing the coefficients, we see that $\alpha_i = \beta_i = 0$ for all i. So we have $f_1 F = 0$. Since f_1 and F are holomorphic on H_2 and f_1 is not identically zero, we have $F = 0$. Thus we prove Theorem 3.3 completely. □

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Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1–1
Toyonaka, Osaka 560–0043
Japan
and
Max Planck Institut für Mathematik
Vivatsgasse 7, Bonn, 53111 Germany
E-mail: ibukiyam@math.sci.osaka-u.ac.jp