COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 59, No. 1 2010

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

Correction and Supplement to the Paper "Generators of the Néron-Severi Group of a Fermat Surface"

by

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(Received March 25, 2010)

Abstract. In the article of the title, we have constructed some explicitly defined curves on a complex Fermat surface, which together with the lines generate a major part of its Néron-Severi group. In this note, we make a correction of a formula in the article and prove it by a simplified argument.

1. Introduction

1.1. In the article [3], we have constructed some explicitly defined curves on a complex Fermat surface, which generate a major part of its Néron-Severi group over the rational numbers, together with the lines on the surface [6].

The aim of this note is to make a correction in the statement of a formula [3, (2.8) in Theorem 3] and to prove it (in §3) by applying a simplified argument which holds in a more general situation (§2). In §1.2 below we review the background and the notation, where the main result of [3] is restated as Theorem 1.2.

It should be remarked that the following problems were posed in [3]: (i) to find curves which generate over $\mathbb{Q}$ the "exceptional" part of the Néron-Severi group not spanned by the lines and curves given by [3, 6], and (ii) to find generators of the Néron-Severi group over $\mathbb{Z}$. As to the problem (ii), see the recent paper [4] which solves the question when the degree of the Fermat surface is relatively prime to 6 and less than 100. As to the problem (i), we have partial results based on the idea to use the Mordell-Weil lattice of some elliptic Delsarte surfaces (in preparation).

1.2. Let X_m^2 be the Fermat surface of degree m defined by the equation

$$x^m + y^m + z^m + w^m = 0$$

in the projective space $\mathbb{P}^3$ over the complex number field $\mathbb{C}$. If we denote by μ_m the group of m-th roots of unity in $\mathbb{C}$, then the abelian group

$$G := (\mu_m \times \mu_m \times \mu_m \times \mu_m)/\text{diagonal}$$

acts on X_m^2 in an obvious manner. The character group $\hat{G}$ of G can be naturally identified with the additive group

$$\{(a_0, a_1, a_2, a_3) \in (\mathbb{Z}/m\mathbb{Z})^4 \mid a_0 + a_1 + a_2 + a_3 = 0\}\,.$$

In order to describe the action of G on the cohomology group $H^2(X^2_m, \mathbb{C})$, for each character $\alpha \in \hat{G}$, let

$$V(\alpha) = \{\xi \in H^2(X^2_m, \mathbb{C}) \mid g^*\xi = \alpha(g)\xi \ (\forall g \in G)\}.$$

It is then well known that

$$H^2(X^2_m, \mathbb{C}) = V(0) \oplus \bigoplus_{\alpha \in \mathfrak{A}^2_m} V(\alpha)$$

and $\dim V(\alpha) = 1$ for any $\alpha \in \{0\} \cup \mathfrak{A}^2_m$, where 0 denotes the trivial character (0, 0, 0, 0) and where

$$\mathfrak{A}^2_m = \{\alpha = (a_0, a_1, a_2, a_3) \in \hat{G} \mid a_i \neq 0 \ (\forall i \in \{0, 1, 2, 3\})\}$$

(see [5]). Define a subset $\mathfrak{B}^2_m$ of $\mathfrak{A}^2_m$ by

$$\mathfrak{B}^2_m = \{(a_0, a_1, a_2, a_3) \in \mathfrak{A}^2_m \mid \langle ta_0 \rangle + \langle ta_1 \rangle + \langle ta_2 \rangle + \langle ta_3 \rangle = 2m \ (\forall t \in (\mathbb{Z}/m\mathbb{Z})^\times\},$$

where, for any $a \in \mathbb{Z}/m\mathbb{Z}$, $\langle a \rangle$ denotes the unique integer such that $0 \leq \langle a \rangle < m$ and $\langle a \rangle \equiv a \pmod{m}$. Then the space of the Hodge cycles on X^2_m of codimension 1 is given by

$$H^{1,1}(X^2_m) \cap H^2(X^2_m, \mathbb{Q}) = V(0) \oplus \bigoplus_{\alpha \in \mathfrak{B}^2_m} V(\alpha).$$

In view of this decomposition, an element of the index set $\{0\} \cup \mathfrak{B}^2_m$ will be called a *Hodge class*.

Given a curve C on X^2_m, we put

$$G_C = \{g \in G \mid g(C) = C\}.$$

For any $\alpha \in \mathfrak{B}^2_m$ we define

$$\omega_\alpha(C) = \frac{1}{|G_C|} \sum_{g \in G} \alpha(g)[C_g],$$

where $C_g = g(C)$ and $[C_g] \in H^2(X^2_m, \mathbb{C})$ denotes the cohomology class of C_g. It is clear from the definition that $\omega_\alpha(C) \in V(\alpha)$. If $\omega_\alpha(C) \neq 0$ then we say that C *represents* the Hodge class α. In order that $\omega_\alpha(C) \neq 0$ it is clearly necessary that $\mathrm{Ker}(\alpha) \supset G_C$, in which case we will write as

$$\omega_\alpha(C) = \sum_{g \in G/G_C} \alpha(g)[C_g]$$

to simplify the notation.

An element $\alpha \in \mathfrak{A}^2_m$ which is equal, up to permutation, to the element $(a,\ m - a,\ b,\ m - b) \in \mathfrak{B}^2_m$ for some $a, b \in \mathbb{Z}/m\mathbb{Z} - \{0\}$ will be called a *decomposable element*. In [6] the second author proved the following.

THEOREM 1.1. *Let C be the line on X^2_m defined by*

$$x + \varepsilon y = z + \varepsilon' w = 0,$$

where $\varepsilon, \varepsilon'$ are $2m$-th roots of unity such that $\varepsilon^m = \varepsilon'^m = -1$. Then C represents the Hogde class $\alpha = (a,\ m-a,\ b,\ m-b)$. More precisely, we have

$$\omega_\alpha(C).\overline{\omega_\alpha(C)} = -m^3 .$$

If $(m,6)=1$, then $\mathfrak{B}_m^2$ consists of only decomposable elements. However, if $(m,6)>1$ and $m \neq 4$, besides decomposable elements there are three types of elements in $\mathfrak{B}_m^2$, which are, up to permutaion, equal to one of the following elements ([6], [1]):

$$\begin{array}{ll} (a,\ a+m/2,\ m/2,\ m/2-2a) & (2 \mid m,\ 2a \neq 0)\,, \\ (a,\ a+m/3,\ a+2m/3,\ m-3a) & (3 \mid m,\ 3a \neq 0)\,, \\ (a,\ a+m/2,\ 2a+m/2,\ m-4a) & (2 \mid m,\ 4a \neq 0)\,. \end{array} \tag{1}$$

We call them *standard elements*. In our joint paper [3], we found curves on X_m^2 which represent the standard elements. The main results of [3] can be restated as follows:

THEOREM 1.2. *For each standard element $\alpha \in \mathfrak{B}_m^2$ listed in* (1), *we define an integer $\nu \in \{2,3,4\}$ and two homogeneous polynomials f_1, f_2 as follows*:

(i) *If $\alpha = (a, a+m/2, m, m/2-2a)$, we set $\nu = 2$ and*

$$f_1 = x^d + y^d + \sqrt{-1}z^d\,,$$
$$f_2 = w^2 - \sqrt[d]{2}xy\,,$$

where $d = m/2$.

(ii) *If $\alpha = (a, a+m/3, a+2m/3, m-3a)$, we set $\nu = 3$ and*

$$f_1 = x^d + y^d + z^d\,,$$
$$f_2 = w^3 - \sqrt[d]{-3}xyz\,,$$

where $d = m/3$.

(iii) *If $\alpha = (a, a+m/2, 2a+m/2, m-4a)$, we set $\nu = 4$ and $d = m/2$.*

(iii-a) *If $4 \mid m$, we set*

$$f_1 = x^d + y^d - \sqrt{2}(xy)^{d/2} + \sqrt{-1}z^d\,,$$
$$f_2 = w^4 - \sqrt[d]{-8}xyz^2\,.$$

(iii-b) *If $4 \nmid m$, we set*

$$f_1 = (x^d + y^d + \sqrt{-1}z^d)z - \frac{\sqrt{2}}{\sqrt[m]{-8}}(xy)^{(d-1)/2}w^2\,,$$
$$f_2 = w^4 - \sqrt[d]{-8}xyz^2\,.$$

Then the curve C defined by $f_1 = f_2 = 0$ on X_m^2 represents the Hodge class α. More precisely we have

$$\omega_\alpha(C).\overline{\omega_\alpha(C)} = -\nu m^3.$$

REMARK 1.3. The values of $\sqrt[d]{2}$, $\sqrt[d]{-3}$, etc. in the defining equation of C are fixed once and for all.

However, in [3] we wrongly stated the theorem in the case of (iii-a); the formula for $\omega_\alpha(C).\overline{\omega_\alpha(C)}$, whose proof was skipped there, was not correct. The purpose of this note is to give the proof of the correct formula stated as above.

REMARK 1.4. The curves C defined in Theorem 1.2 are complete intersection curves in $\mathbb{P}^3$ defined by $f_1 = f_2 = 0$ except for the last case (iii-b), where C is defined in $\mathbb{P}^3$ by three equations $f = f_1 = f_2 = 0$. Although we gave a direct proof of (iii-b) in [3], we will prove, in the last of §3, that case (iii-b) is an easy consequence of case (iii-a).

2. Preliminaries

Let S be a non-singular complex projective surface defined by $f = 0$ in $\mathbb{P}^3$, where f is a homogeneous polynomial of degree m. Suppose that f has the following form

$$f = f_1 f_1^* + f_2 f_2^* , \tag{2}$$

where f_1, f_1^*, f_2, f_2^* are non-constant homogeneous polynomials. If we define a curve C in $\mathbb{P}^3$ by

$$f_1 = f_2 = 0 ,$$

then (2) shows that C is contained in S. Let G be an abelian subgroup of $\mathrm{Aut}(C)$ and put

$$G_C = \{g \in G \mid g(C) = C\} .$$

Let $(f_1)_0$ be the zero divisor of f_1 on S. Suppose we have a subgroup Γ of G with the following properties:

$$G_C \subsetneq \Gamma, \tag{3}$$

$$(f_1)_0 = \sum_{\gamma \in \Gamma / G_C} C_\gamma , \tag{4}$$

where $C_\gamma = \gamma(C)$. Let $[C] \in H^2(S, \mathbb{C})$ be the cohomology class of C. For a fixed *non-trivial* character $\chi \in \hat{\Gamma}$, we set

$$\eta = \sum_{\gamma \in \Gamma / G_C} \chi(\gamma)[C_\gamma] = \sum_{\gamma \in \Gamma / G_C} \overline{\chi(\gamma)} \gamma^*[C] .$$

The following proposition shows that η is non-trivial.

THEOREM 2.1. *Notation being as above, we have*

$$\eta.\overline{\eta} = -m \deg(f_1) \deg(f_1^*) .$$

In the proof of Theorem 2.1 the following two lemmas will play an important rôle.

LEMMA 2.2. *Let C' be the curve on S defined by $f_1^* = f_2 = 0$ and L a hyperplane section on S. Then we have*

$$C + C' \sim \deg(f_2) L , \tag{5}$$

$$\sum_{\gamma \in \Gamma / G_C} C_\gamma \sim \deg(f_1) L , \tag{6}$$

where $\sim$ denotes the linear equivalence.

Proof. These relations immediately follows from (2) and (4) respectively. □

LEMMA 2.3. *Let C^* be the curve on S defined by $f_1^* = f_2^* = 0$. Then $C \cap C^* = \emptyset$.*

Proof. If $C \cap C^*$ were not empty, then for any $P \in C \cap C^*$ we would have

$$\frac{\partial f}{\partial x}(P) = \sum_{i=1}^{2} \left\{ \frac{\partial f_i}{\partial x}(P) \cdot f_i^*(P) + f_i(P) \cdot \frac{\partial f_i^*}{\partial x}(P) \right\} = 0$$

and quite similarly

$$\frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = \frac{\partial f}{\partial w}(P) = 0\,.$$

Then P would be a singular point of S, contradicting to the assumption that S is non-singular. Thus $C \cap C^* = \emptyset$. □

PROPOSITION 2.4. *For any $\gamma \in \Gamma$, the intersection number $(C_\gamma . C)$ on S is given by*

$$(C_\gamma . C) = \begin{cases} -\deg(f_1)\deg(C') + \deg(f_2)\deg(C) & (\gamma \in G_C)\,, \\ \deg(f_2)\deg(C) & (\gamma \notin G_C)\,. \end{cases} \tag{7}$$

Proof. It follows from (5) that

$$(C_\gamma . C) + (C_\gamma . C') = \deg(f_2)\deg(C_\gamma) \tag{8}$$

for any $\gamma \in \Gamma$. We first show formula (7) for $\gamma \notin G_C$. In this case, we have $C'_{\gamma^{-1}} \subset C^*$ in the notation of Lemma 2.3. Thus the lemma implies that $C'_{\gamma^{-1}} \cap C = \emptyset$ and so

$$(C_\gamma . C') = (C . C'_{\gamma^{-1}}) = 0\,. \tag{9}$$

Hence (8) shows that

$$(C_\gamma . C) = \deg(f_2)\deg(C_\gamma)\,.$$

The second equality of (7) follows from this since $\deg(C_\gamma) = \deg(C)$.

On the other hand, if $\gamma \in G_C$, then $C_\gamma = C$ and equation (8) reads

$$(C . C) + (C . C') = \deg(f_2)\deg(C)\,. \tag{10}$$

We use (6) and (9) to obtain

$$(C . C') = \sum_{\gamma \in \Gamma / G_C} (C_\gamma . C') = \deg(f_1)\deg(C')\,. \tag{11}$$

Combining (10) with (11), we have

$$(C . C) = -\deg(f_1)\deg(C') + \deg(f_2)\deg(C)\,.$$

This completes the proof. □

Proof of Theorem 2.1. We first note that

$$\eta.\overline{\eta} = |\Gamma| \sum_{\gamma\in\Gamma/G_C} \chi(\gamma)(C_\gamma.C)\,. \tag{12}$$

This immediately follows from the fact that

$$[C_\gamma].[C_{\gamma'}] = (C_\gamma.C_{\gamma'}) = (C_{\gamma\gamma'^{-1}}.C)\,.$$

Applying Proposition 2.4 to (12), we have

$$\begin{aligned}\sum_{\gamma\in\Gamma/G_C} \chi(\gamma)(C_\gamma.C) &= -\deg(f_1)\deg(C') + \sum_{\gamma\in\Gamma/G_C} \chi(\gamma)\deg(f_2)\deg(C)\\ &= -\deg(f_1)\deg(C')\,.\end{aligned}$$

Here the last equality holds since χ is non-trivial on Γ. Since $\deg(C') = \deg(f_1^*)\deg(f_2)$ and $\deg(f_2) = \frac{m}{|\Gamma|}$, it follows that

$$\eta.\overline{\eta} = -m\deg(f_1)\deg(f_1^*)\,.$$

This proves the theorem. □

3. Proof

3.1. Let m be a positive integer such that $m > 4$ and $m \equiv 0 \pmod 4$. We will prove Theorem 1.2 (iii-a) by applying the results in the previous section to the case where S is the Fermat surface X_m^2.

We first define three polynomials f_1, f_1^*, f_2 by

$$\begin{aligned} f_1 &= x^d + y^d - \sqrt{2}(xy)^{d/2} + \sqrt{-1}z^d\,,\\ f_1^* &= x^d + y^d + \sqrt{2}(xy)^{d/2} - \sqrt{-1}z^d\,,\\ f_2 &= w^4 - \sqrt[d]{-8}xyz^2\,.\end{aligned}$$

These polynomials satisfy the following identity

$$x^m + y^m + z^m + w^m = f_1 f_1^* + \prod_{\varepsilon\in\mu_{m/4}} (w^4 - \varepsilon\sqrt[d]{-8}xyz^2)\,. \tag{13}$$

This implies in particular that the algebraic curve C in $\mathbb{P}^3$ defined by $f_1 = f_2 = 0$ is contained in X_m^2. As in the previous section we set

$$G_C = \{g \in G_m^2 \mid C_g = C\}\,.$$

Now, for any $a \in \mathbb{Z}/m\mathbb{Z}$ such that $4a \neq 0$ we put

$$\beta_a = (a, d+a, d+2a, m-4a) \in \mathfrak{B}_m^2\,.$$

For simplicity we write β for $\beta_1 = (1, d+1, d+2, m-4)$.

LEMMA 3.1. *If* $\alpha \in \mathfrak{B}_m^2$ *and* $\mathrm{Ker}(\alpha) \supset G_C$, *then* $\alpha = \beta_a$ *for some* $a \in \mathbb{Z}/m\mathbb{Z}$ *with* $4a \neq 0$.

Proof. Let ζ be a primitive m-th root of unity. Then two elements

$$g_1 = [1 : \zeta^{-4} : \zeta^2 : 1]\,, \quad g_2 = [1 : \zeta^4 : 1 : \zeta] \in G_m^2$$

generate G_C. Therefore we have $\mathrm{Ker}(\alpha) \supset G_C$ if and only if $\alpha(g_1) = \alpha(g_2) = 1$. If we set $\alpha = (a_0, a_1, a_2, a_3) \in \mathfrak{A}_m^2$, then the latter condition gives rise to the congruence relations

$$a_2 \equiv 2a_1 (\mathrm{mod}\ \frac{m}{2})\,, \quad a_3 \equiv -4a_1 \quad (\mathrm{mod}\ m)\,.$$

This shows that α can be written as

$$\alpha = \begin{cases} (a, a, 2a, m-4a)\,, \\ (a, a+d, 2a+d, m-4a) \end{cases}$$

for some $a \in \mathbb{Z}/m\mathbb{Z}\backslash$ with $4a \neq 0$. But the first element cannot belong to $\mathfrak{B}_m^2$, so we have $\alpha = \beta_a$ as desired. □

We next consider a subgroup

$$H = \{(1 : \zeta_1 : \zeta_2 : \zeta_3) \in G_m^2 \mid \zeta_1^{d/2} = \zeta_2^d = 1\}$$

of G_m^2. For any $h \in H$, the curve $C_h := h(C)$ is defined by $f_1 = h^* f_2 = 0$, where

$$h^* f_2 = w^4 - \overline{\beta(h)}\sqrt[d]{-8}xyz^2\,.$$

It follows that $G_C = H \cap \mathrm{Ker}(\beta)$. Since β is non-trivial on H, we have

$$G_C \subsetneq H\,. \tag{14}$$

Let $(f_1)_0$ be the zero divisor of f_1 on X_m^2. Then (13) shows that

$$(f_1)_0 = \sum_{h \in H/G_C} C_h\,. \tag{15}$$

These properties (14) and (15) allow us to apply Theorem 2.1 to the case where $S = X_m^2$ and $\Gamma = H$.

Clearly β is non-trivial on H. Thus, if we denote by $\chi = \beta_{|H} \in \hat{H}$ the restriction of the character β to H, then χ is non-trivial on H. Put

$$\eta = \sum_{h \in H/G_C} \overline{\chi(h)} h^*[C]\,.$$

LEMMA 3.2. *Notation being as above, we have* $\eta.\overline{\eta} = -\frac{m^3}{4}$.

Proof. Since $\deg(f_1) = \deg(f_1^*) = \frac{m}{2}$, Theorem 2.1 shows that

$$\eta.\overline{\eta} = -m \cdot \frac{m}{2} \cdot \frac{m}{2} = -\frac{m^3}{4}\,.$$

This proves the lemma. □

LEMMA 3.3. *Put* $\omega_{\beta_{km/4+1}} = w_{\beta_{km/4+1}}(C)$ *for* $k = 0, 1, 2, 3$. *Then*

$$\eta = \frac{1}{8}\left(\omega_{\beta_1} + \omega_{\beta_{m/4+1}} + \omega_{\beta_{m/2+1}} + \omega_{\beta_{3m/4+1}}\right)\,.$$

Proof. Recall that, for each $\alpha \in \hat{G}$, $p_\alpha : H^2(X_m^2, \mathbb{C}) \to V(\alpha)$ denotes the projector to $V(\alpha)$. The one-dimensional space $V(\alpha)$ is generated by the cohomology classes of algebraic cycles on X_m^2 if and only if $\alpha \in \{0\} \cup \mathfrak{B}_m^2$. Hence, if $\alpha \notin \{0\} \cup \mathfrak{B}_m^2$, then $p_\alpha([C]) = 0$. Moreover, if $G_C \not\subset \mathrm{Ker}(\alpha)$, then $p_\alpha([C]) = 0$. Therefore Lemma 3.1 shows that

$$[C] = \sum_{\alpha \in \hat{G}} p_\alpha([C]) = p_0([C]) + \sum_{\substack{a \in \mathbb{Z}/m\mathbb{Z} \\ 4a \neq 0}} p_{\beta_a}([C]) .$$

Let $p_\chi = \frac{1}{|H|} \sum_{h \in H} \overline{\chi(h)} h^*$ be the projector to the χ-eigenspace. Then η is related to p_χ by the formula

$$\eta = \frac{|G_C|}{|H|} p_\chi([C]) . \tag{16}$$

Note that

$$p_\chi p_\alpha = \begin{cases} \mathrm{id} & (\alpha_{|H} = \chi) , \\ 0 & (\alpha_{|H} \neq \chi) \end{cases}$$

and $\beta_{a|_H} = \chi$ if and only if $a \equiv 1 \pmod{m/4}$. Hence

$$\begin{aligned} p_\chi([C]) &= p_\chi(p_0([C])) + \sum_{\substack{a \in \mathbb{Z}/m\mathbb{Z} \\ 4a \neq 0}} p_\chi(p_{\beta_a}([C])) \\ &= p_{\beta_1}([C]) + p_{\beta_{m/4+1}}([C]) + p_{\beta_{2m/4+1}}([C]) + p_{\beta_{3m/4+1}}([C]) \\ &= \frac{|G_C|}{|G|} \left(\omega_{\beta_1} + \omega_{\beta_{m/4+1}} + \omega_{\beta_{2m/4+1}} + \omega_{\beta_{3m/4+1}} \right) . \end{aligned}$$

Substituting this into (16), we obtain

$$\eta = \frac{|H|}{|G|} \left(\omega_{\beta_1} + \omega_{\beta_{m/4+1}} + \omega_{\beta_{m/2+1}} + \omega_{\beta_{3m/4+1}} \right) .$$

Since $|G/H| = 8$, this proves the lemma. □

3.2. *Proof of Theorem* 1.2 (iii-a)

We first note that $\omega_\alpha . \overline{\omega}_{\alpha'} = 0$ whenever $\alpha \neq \alpha'$. If $8 \mid m$, then $\frac{km}{4} + 1 \in (\mathbb{Z}/m\mathbb{Z})^\times$ for any $k \in \{0, 1, 2, 3\}$. If we denote by $\mathbb{Q}(\zeta_m)$ the m-th cyclotomic field, then $H^2(X_m^2, \mathbb{Q}(\zeta_m))$ admits an action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}))$ and $\omega_{t \cdot \alpha} \in H^2(X_m^2, \mathbb{Q}(\zeta_m))$ ($t \in (\mathbb{Z}/m\mathbb{Z})^\times$) are Galois conjugate. Since the intersection product is Galois equivariant, we have

$$\omega_{t \cdot \alpha} . \overline{\omega}_{t \cdot \alpha'} = \omega_\alpha . \overline{\omega}_{\alpha'}$$

for any $t \in (\mathbb{Z}/m\mathbb{Z})^\times$. Thus Lemma 3.3 shows that

$$\eta . \overline{\eta} = 4 \times \left(\frac{1}{8} \right)^2 \omega_\beta . \overline{\omega_\beta} = \frac{1}{16} \omega_\beta . \overline{\omega_\beta} .$$

Therefore from Lemma 3.2 we obtain

$$\omega_\beta.\overline{\omega_\beta} = 16\eta.\overline{\eta} = 16 \cdot \left(-\frac{m^3}{4}\right) = -4m^3 .$$

On the other hand, if $4\|m$, then $\frac{m}{4}+1$, $\frac{3m}{4}+1 \notin (\mathbb{Z}/m\mathbb{Z})^\times$, and so the Galois conjugate method above does not work. To avoid this difficulty, we consider a group

$$G_0 = \{(1:\zeta:1:1) \in G_m^2 \mid \zeta \in \mu_4\}$$

of order 4 and define an operator π on $H^2(X_m^2, \mathbb{C})$ to be

$$\pi = \frac{1}{4}\sum_{g\in G_0} \overline{\beta(g)} g^* .$$

It is then clear from the definition that

$$\pi(\omega_{\beta_{\frac{km}{4}+1}}) = \begin{cases} \omega_\beta & (k=0)\,, \\ 0 & (k=1,2,3)\,. \end{cases}$$

Therefore from Lemma 3.3 we have $\pi(\eta) = \frac{1}{8}\omega_\beta$. On the other hand, a simple calculation shows that

$$\pi(\eta).\overline{\pi(\eta)} = \frac{1}{4}\sum_{g\in G_0} \overline{\beta(g)} g^*\eta.\overline{\eta} .$$

As we shall see below, we have

$$g^*\eta.\overline{\eta} = 0 \tag{17}$$

for any $g \in G_0 \setminus \{1\}$. Then the theorem immediately follows from this since

$$\omega_\beta.\overline{\omega_\beta} = 64\pi(\eta).\overline{\pi(\eta)} = 16\eta.\overline{\eta} = -4m^3 .$$

In order to prove (17), note that

$$g^*\eta.\overline{\eta} = |H| \sum_{h\in H} \beta(h)(C_{gh}.C) . \tag{18}$$

If $g \in G_0 \setminus \{1\}$, then C_{gh} is defined by

$$\begin{cases} x^d + \varepsilon^2 y^d - \zeta\sqrt{2}(xy)^{d/2} + \sqrt{-1}z^d = 0\,, \\ w^4 - \overline{\beta(gh)}\sqrt[d]{-8}xyz^2 = 0 \end{cases}$$

for some $\zeta \in \mu_4 \setminus \{1\}$. Since $\beta(h) \in \mu_{m/4}$ and $\beta(g) \in \mu_4 \setminus \{1\}$, we have $\beta(gh) \neq 1$. From this it is easily seen that $(C_{gh}.C) = 4m^2$ for any $g \in G_0 \setminus \{1\}$. Then (17) immediately follows from (18) since β is a non-trivial character on H. This completes the proof of Theorem 1.2, (iii-a). □

3.3. As we have mentioned in Remark 1.4, we will prove that (iii-b) follows from (iii-a). To this end we prepare a lemma. The proof is an easy exercise and we omit it.

LEMMA 3.4. *Let V, V' be $\mathbb{C}$-vector spaces admitting actions of abelian groups G, G' respectively. Suppose that there exists a surjective homomorphism $\psi : G \to G'$ and a $\mathbb{C}$-linear map $f : V \to V'$ such that*

$$f(gv) = \psi(g)f(v)$$

for any $g \in G$, $v \in V$. Let α, α' be characters of G, G' respectively such that $\alpha = \alpha' \circ \pi$, and let

$$p_\alpha = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}g\,, \quad p_{\alpha'} = \frac{1}{|G'|}\sum_{g\in G'}\overline{\alpha'(g)}g$$

be the corresponding projector. Then we have

$$f \circ p_\alpha = p_{\alpha'} \circ f\,.$$

Proof of Theorem 1.2, (iii-b). Let $m > 4$ be an even integer which is not necessarily divisible by 4. Let $\beta = (1, 1 + d, 2 + d, m - 4) \in \mathfrak{A}_m^2$ as in the proof of Theorem 1.2, (iii-a) above and set

$$\tilde{\beta} = (2, 2 + m, 4 + m, 2m - 8) \in \mathfrak{B}_{2m}^2\,.$$

We consider the curve $\tilde{C}$ on X_{2m}^2 defined by

$$\tilde{C} : \begin{cases} x^m + y^m - \sqrt{2}(xy)^{m/2} + \sqrt{-1}z^m = 0\,, \\ w^4 - \sqrt[m]{-8}xyz^2 = 0\,. \end{cases}$$

Then by Theorem 1.2, (iii-a) proved just above we know that

$$\omega_{\tilde{\beta}}(\tilde{C}).\overline{\omega_{\tilde{\beta}}(\tilde{C})} = -4(2m)^3 = -32m^3\,. \tag{19}$$

If we denote by $\varphi : X_{2m}^2 \to X_m^2$ the finite morphism of degree 8 defined by

$$\varphi(x : y : z : w) = (x^2 : y^2 : z^2 : w^2)\,,$$

then we have $C = \varphi(\tilde{C})$ and $[\mathbb{C}(\tilde{C}) : \mathbb{C}(C)] = 4$. This implies that

$$\varphi_*([\tilde{C}]) = 4[C]\,, \tag{20}$$

where $[C] \in H^2(X_m^2)$ and $[\tilde{C}] \in H^2(X_{2m}^2)$ denote the cohomology classes of C and $\tilde{C}$ respectively. For each $\tilde{g} \in G_{2m}^2$, letting $g = \tilde{g}^2$, we have $\varphi_*\tilde{g}^* = g^*$ and $\tilde{\beta}(\tilde{g}) = \beta(g)$. Then Lemma 3.4 together with (20) shows that

$$\varphi_*(p_{\tilde{\beta}}([\tilde{C}]) = p_\beta(\varphi_*[\tilde{C}]) = 4p_\beta([C])\,.$$

Since $|G_{2m}^2| = 8|G_m^2|$ and $|G_{\tilde{C}}| = 4|G_C|$, this implies that

$$\varphi_*(\omega_{\tilde{\beta}}(\tilde{C})) = \frac{|G_{2m}^2|}{|G_{\tilde{C}}|}\varphi_*(p_{\tilde{\beta}}([\tilde{C}])) = 8\cdot\frac{|G_m^2|}{|G_C|}p_\beta([C]) = 8\omega_\beta(C)\,.$$

Thus, $\varphi_*(\omega_{\tilde{\beta}}(\tilde{C})) = 8\omega_\beta(C)$. It then follows from the projection formula that

$$\omega_{\tilde{\beta}}(\tilde{C}).\overline{\omega_{\tilde{\beta}}(\tilde{C})} = 8\omega_\beta(C).\overline{\omega_\beta(C)}\,.$$

Therefore (19) yields

$$\omega_\beta(C).\overline{\omega_\beta(C)} = \frac{1}{8}\omega_{\tilde{\beta}}(\tilde{C}).\overline{\omega_{\tilde{\beta}}(\tilde{C})} = \frac{1}{8}\cdot(-32m^3) = -4m^3\,.$$

This proves Theorem 1.2, (iii-b). □

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