COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 51, No. 1 2002

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

# On the Zeros of the Riemann Zeta Function

by

Akio FUJII

(Received November 9, 2001)

## §1. Introduction

We shall give a study of the high moments of the remainder term in the Riemann-von Mangoldt formula for the number of the zeros of the Riemann zeta function $\zeta(s)$. Littlewood [3] and Selberg [4] have studied the same problem under the assumption of the Riemann Hypothesis (that all complex zeros of $\zeta(s)$ have real part 1/2). Here we shall study it without assuming any unproved hypothesis. A new criterion for the Riemann Hypothesis will be presented in this context.

We denote the non-trivial zeros of $\zeta(s)$ by $\rho = \beta + i\gamma$ with real numbers $\beta$ and $\gamma$. We suppose in this article that $T > T_o$. Let $N(T)$ denote the number of the zeros $\beta + i\gamma$ of $\zeta(s)$ in $0 < \gamma < T$, $0 < \beta < 1$, when $T \neq \gamma$ for any $\gamma$. When $T = \gamma$ for some $\gamma$, then we put

$$N(T) = \frac{1}{2}(N(T+0) + N(T-0)) .$$

Let

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right) \quad \text{for} \quad T \neq \gamma ,$$

where the argument is obtained by the continuous variation along the straight lines joining 2, $2 + iT$, and $\frac{1}{2} + iT$, starting with the value zero. When $T = \gamma$, then we put

$$S(T) = \frac{1}{2}(S(T+0) + S(T-0)) .$$

Then the well known Riemann-von Mangoldt formula (cf. p. 212 of Titchmarsh [5] ) states that

$$N(T) = \frac{1}{\pi}\vartheta(T) + 1 + S(T) ,$$

where $\vartheta(T)$ is the continuous function defined by

$$\vartheta(T) = \Im\left(\log\Gamma\left(\frac{1}{4} + \frac{iT}{2}\right)\right) - \frac{1}{2}T\log\pi$$

with $\vartheta(0) = 0$, $\Gamma(s)$ being the gamma-function. It is well known that

$$\vartheta(T) = \frac{T}{2}\log\frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + \frac{1}{48T} + \frac{7}{5760T^3} + \cdots$$

and that, for $T > T_o$, we have

$$S(T) \ll \log T\,.$$

The last estimate was refined under the Riemann Hypothesis (R.H.) by Littlewood [3] and later by Selberg [4] in different ways as follows:

$$S(T) = O\left(\frac{\log T}{\log\log T}\right)\,.$$

Here we are concerned with the average behavior of $S(T)$. The first average is well known and classical. Littlewood [3] and Selberg [4] have shown that

$$\int_0^T S(t)\,dt = O(\log T)\,.$$

It seems to be difficult to go beyond this bound without assuming any unproved hypothesis. In fact, it is noticed on p. 335 of Titchmarsh [5] that

*the Lindelöf hypothesis is equivalent to the statement that*

$$\int_0^T S(t)\,dt = o(\log T) \quad (T \to \infty)\,.$$

If we assume the Riemann Hypothesis, then we have, due to Littlewood [3] and Selberg [4],

$$\int_0^T S(t)\,dt = O\left(\frac{\log T}{\log\log T}\right)\,.$$

The purpose of the present article is to give a study of the higher moments

$$\tilde{S}_m(t)$$

for $m \geq 2$ of $S(t)$, which will be defined below. When $T \neq \gamma$, then we put

$$\tilde{S}_0(T) = S(T)$$

and

$$\tilde{S}_m(T) = \int_0^T \tilde{S}_{m-1}(t)\,dt + C_m$$

for any integer $m \geq 1$, where $C_m$'s are the constants defined by

$$C_{2k-1} = (-1)^{k-1}\frac{1}{\pi}\underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k-1}\log|\zeta(\sigma)|(d\sigma)^{2k-1}$$

for $m = 2k-1$ and

$$C_{2k} = (-1)^{k-1}\underbrace{\int_{\frac{1}{2}}^{1}\int_{\sigma}^{1}\cdots\int_{\sigma}^{1}}_{2k}(d\sigma)^{2k} = (-1)^{k-1}\frac{1}{(2k)!\cdot 2^{2k}}$$

for $m = 2k$, respectively. When $T = \gamma$, then we put

$$\tilde{S}_m(T) = \frac{1}{2}(\tilde{S}_m(T+0) + \tilde{S}_m(T-0)).$$

Concerning $\tilde{S}_m(T)$ for $m \geq 2$, Littlewood [3] and Selberg [4] have shown under the Riemann Hypothesis that

$$\tilde{S}_m(T) \ll_m \frac{\log T}{(\log\log T)^{m+1}}.$$

In this article we shall study $\tilde{S}_m(T)$ for $m \geq 2$ without assuming any unproved hypothesis. We shall first describe the relation between $\tilde{S}_m(T)$ and the integral $I_m(T)$ which will be defined as follows. When $T \neq \gamma$, then we put for $k \geq 1$

$$I_{2k-1}(T) = \frac{1}{\pi}(-1)^{k-1}\Re\Bigg\{\underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k-1} \log\zeta(\sigma+iT)\,(d\sigma)^{2k-1}\Bigg\}$$

and

$$I_{2k}(T) = \frac{1}{\pi}(-1)^{k}\Im\Bigg\{\underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k} \log\zeta(\sigma+iT)\,(d\sigma)^{2k}\Bigg\}.$$

When $T = \gamma$, then we put for $m \geq 1$

$$I_m(T) = \frac{1}{2}(I_m(T+0) + I_m(T-0)).$$

For $\sigma \geq \frac{1}{2}$ and $T > T_o$, let $N(\sigma, T)$ be the number of the zeros $\beta + i\gamma$ of $\zeta(s)$ such that $\beta > \sigma$ and $0 < \gamma < T$ when $T \neq \gamma$. When $T = \gamma$, then we put

$$N(\sigma, T) = \frac{1}{2}((N(\sigma, T+0) + N(\sigma, T-0)).$$

We now describe a relation between $\tilde{S}_m(T)$ and $I_m(T)$ as follows.

LEMMA 1.

$$\tilde{S}_1(T) = I_1(T)$$

*and for any integer* $m \geq 2$

$$\tilde{S}_m(T) = I_m(T) + 2\sum_{\substack{h+2r=m\\ r\geq 1, h\geq 0}} (-1)^{r-1}\tilde{N}_{h,2r}(T),$$

*where we put for* $h \geq 1$ *and* $r \geq 1$

$$\tilde{N}_{h,2r}(T) = \underbrace{\int_0^T\cdots\int_0^t}_{h}\ \underbrace{\int_{\frac{1}{2}}^1\int_{\sigma}^1\cdots\int_{\sigma}^1}_{2r} N(\sigma, t)(d\sigma)^{2r}(dt)^h$$

*and for* $h = 0$ *and* $r \geq 1$

$$\tilde{N}_{0,2r}(T) = \underbrace{\int_{\frac{1}{2}}^{1}\int_{\sigma}^{1}\cdots\int_{\sigma}^{1}}_{2r} N(\sigma, T)(d\sigma)^{2r} .$$

The multiple integral in the definition of $I_m(T)$ can be simplified into a single integral as follows.

LEMMA 2. *For any integer $m \geq 1$, we have*

$$I_m(T) = -\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\frac{1}{2}}^{\infty}\left(\sigma - \frac{1}{2}\right)^m \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma\right\}.$$

Thus we are led to evaluate first the integral $I_m(T)$. Concerning this, we shall prove the following

THEOREM 1. *For any integer $m \geq 1$, we have*

$$I_m(T) \ll_m \log T ,$$

*where the constant involved in the upper bound may depend on $m$.*

The case $m = 1$ was obtained by Littlewood [3] and Selberg [4]. Theorem 1 implies immediately the following.

THEOREM 1′. *For any integer $m \geq 2$, we have*

$$\tilde{S}_m(T) = 2\sum_{\substack{h+2r=m\\ r\geq 1, h\geq 0}} (-1)^{r-1}\tilde{N}_{h,2r}(T) + O(\log T) .$$

Now we are led to evaluate the sum

$$2\sum_{\substack{h+2r=m\\ r\geq 1, h\geq 0}} (-1)^{r-1}\tilde{N}_{h,2r}(T) .$$

We can obtain an upper bound for it as an application of Selberg's density theorem (cf. p. 232 of Selberg [4]): for $T > T_o$ and some positive constant $C$,

$$N(\sigma, T) \ll T\log T \cdot e^{-C(\sigma - \frac{1}{2})\log T}$$

uniformly for $\sigma \geq \frac{1}{2}$. Namely, we have first

$$\underbrace{\int_{\frac{1}{2}}^{1}\int_{\sigma}^{1}\cdots\int_{\sigma}^{1}}_{2r} N(\sigma, T)(d\sigma)^{2r} \ll \frac{T}{(\log T)^{2r-1}} .$$

Consequently, we have

$$\tilde{N}_{h,2r}(T) \ll \frac{T^{h+1}}{(\log T)^{2r-1}}$$

and

$$2\sum_{\substack{h+2r=m\\ r\geq 1,h\geq 0}}(-1)^{r-1}\tilde{N}_{h,2r}(T)\ll\frac{T^{m-1}}{\log T}.$$

Combining this with Theorem 1′, we get the following theorem concerning $\tilde{S}_m(T)$.

THEOREM 2. *For any integer $m\geq 2$, we have*

$$\tilde{S}_m(T)\ll\frac{T^{m-1}}{\log T}.$$

Obviously, there is a big gap between our result and the conditional result mentioned above. As we have seen above, this gap comes from the possible zeros off the critical line. We can see it in such a form that does not involve the integrals. Concretely, we describe the sum

$$2\sum_{\substack{h+2r=m\\ r\geq 1,h\geq 0}}(-1)^{r-1}\tilde{N}_{h,2r}(T)$$

as follows. By the definition of $N(\sigma,T)$, we have for $h\geq 1$

$$\begin{aligned}\tilde{N}_{h,2r}(T)&=\underbrace{\int_0^T\cdots\int_0^t}_{h}\underbrace{\int_{\frac{1}{2}}^1\int_\sigma^1\cdots\int_\sigma^1}_{2r}\sideset{}{'}\sum_{\substack{\beta+i\gamma\\ \beta>\sigma,0<\gamma<t}}1(d\sigma)^{2r}(dt)^h\\ &=\frac{1}{(2r)!}\underbrace{\int_0^T\cdots\int_0^t}_{h}\sideset{}{'}\sum_{\substack{\beta+i\gamma\\ \beta>\frac{1}{2},0<\gamma<t}}\left(\beta-\frac{1}{2}\right)^{2r}(dt)^h\\ &=\frac{1}{(2r)!\cdot h!}\sum_{\substack{\beta+i\gamma\\ \beta>\frac{1}{2},0<\gamma<T}}\left(\beta-\frac{1}{2}\right)^{2r}(T-\gamma)^h,\end{aligned}$$

where the dash indicates that when $t=\gamma$, then we use the halving convention as above. When $h=0$, then the above formula is seen still to be valid. Consequently, we may restate lemma 1 and Theorem 1′ as follows, respectively.

LEMMA 1′. *For any integer $m\geq 2$, we have*

$$\tilde{S}_m(T)=I_m(T)+2\sum_{\substack{h+2r=m\\ r\geq 1,h\geq 0}}(-1)^{r-1}\frac{1}{(2r)!\cdot h!}\sideset{}{'}\sum_{\substack{\beta+i\gamma\\ \beta>\frac{1}{2},0<\gamma<T}}\left(\beta-\frac{1}{2}\right)^{2r}(T-\gamma)^h,$$

*where the dash denotes the halving convention as above when $h=0$.*

THEOREM 3. *For any integer $m\geq 2$, we have*

$$\tilde{S}_m(T) = 2 \sum_{\substack{h+2r=m \\ r\geq 1, h\geq 0}} (-1)^{r-1} \frac{1}{(2r)! \cdot h!} \sum_{\substack{\beta+i\gamma \\ \beta>\frac{1}{2}, 0<\gamma<T}} \left(\beta - \frac{1}{2}\right)^{2r} (T-\gamma)^h + O(\log T)\,.$$

From Theorem 1′ or from Theorem 3, we see immediately that if there exists a zero $\beta_o + i\gamma_o$ satisfying $\beta_o > \frac{1}{2}$ and $\gamma_o > 0$, then we have for any integer $m \geq 3$ and for all $T > 2\gamma_o$

$$\tilde{S}_m(T) \geq A_m \left(\beta_o - \frac{1}{2}\right)^2 T^{m-2}\,,$$

where $A_m$ is an absolute positive constant which may depend on $m$. Consequently, we get the following criterion for the Riemann Hypothesis.

THEOREM 4. *The following statement is equivalent to the Riemann Hypothesis. For any integer* $m \geq 3$, *we have*

$$\tilde{S}_m(T) = o(T^{m-2}) \quad (T \to \infty)\,.$$

We shall give the proofs of these results as precise as possible. In the next section, we shall give the proofs of Lemmas 1 and 2. In section 3, we shall justify the arguments used in section 2. In section 4, we shall prove Theorem 1.

We notice here that $\tilde{S}_2(T)$ is treated in Fujii [1] (cf. Theorem VIII on p. 171 and also p. 186 of Fujii [1]). However, as noticed in Fujii [2], there is a gap in the argument and should be corrected as in the present article.

Finally, the author wishes to express his gratitude to Professor Aleksander Ivić. The constant communications with him has stimulated the author to write down this article.

## §2. Proof of Lemmas 1 and 2

We shall use the following two relations. Suppose first that $T \neq \gamma$. Then, by p. 300 of Littlewood [3] ( and also pp. 220–221 of Titchmarsh [5]), we have for $-1 \leq \sigma \leq 1$

$$\Im\left\{\int_0^T \log\zeta(\sigma+it)\,dt\right\} = \int_\sigma^\infty \log|\zeta(\sigma+iT)|d\sigma - \int_\sigma^\infty \log|\zeta(\sigma)|d\sigma$$

and

$$2\pi \int_\sigma^1 \nu(\sigma, T)\,d\sigma = \Im\left\{\int_\sigma^\infty \log\zeta(\sigma+iT)\,d\sigma\right\} + \int_0^T \log|\zeta(\sigma+it)|dt\,,$$

where we put

$$\nu(\sigma, T) = \begin{cases} N(\sigma, T) - \frac{1}{2} & \text{if } \sigma < 1 \\ 0 & \text{if } \sigma \geq 1\,. \end{cases}$$

When $T = \gamma$, then the relations hold by halving conventions as above.

We shall prove Lemma 1 by induction.

First we have, using the above relation and the definition of $C_1$,

$$\begin{aligned}\tilde{S}_1(T) &= \frac{1}{\pi}\Im\left\{\int_0^T \log\zeta\left(\frac{1}{2}+it\right)dt\right\} + C_1\\ &= \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma+iT)|d\sigma - \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma)|d\sigma + C_1\\ &= I_1(T)\,.\end{aligned}$$

Next we have

$$\begin{aligned}\tilde{S}_2(T) &= \int_0^T \tilde{S}_1(t)\,dt + C_2\\ &= \int_0^T I_1(t)\,dt + C_2\\ &= \frac{1}{\pi}\int_0^T\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma+it)|d\sigma dt + C_2\,.\end{aligned}$$

Here we can interchange the order of the integrations, which will be justified in the next section. Thus we get

$$\begin{aligned}\tilde{S}_2(T) &= \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\int_0^T \log|\zeta(\sigma+it)|dtd\sigma + C_2\\ &= \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\left\{-\Im\left(\int_{\sigma}^{\infty}\log\zeta(\sigma'+iT)d\sigma'\right) + 2\pi\int_{\sigma}^{1}\nu(\sigma',T)\,d\sigma'\right\}d\sigma + C_2\\ &= I_2(T) + 2\int_{\frac{1}{2}}^{1}\int_{\sigma}^{1}N(\sigma',T)d\sigma'd\sigma - \int_{\frac{1}{2}}^{1}\int_{\sigma}^{1}d\sigma'd\sigma + C_2\\ &= I_2(T) + 2\int_{\frac{1}{2}}^{1}\int_{\sigma}^{1}N(\sigma',T)d\sigma'd\sigma = I_2(T) + 2\tilde{N}_{0,2}(T)\,,\end{aligned}$$

where the definition of $\tilde{N}_{h,2r}(T)$ is described in the introduction. For $m = 3$, we have

$$\begin{aligned}\tilde{S}_3(T) &= \int_0^T \tilde{S}_2(t)\,dt + C_3\\ &= \int_0^T I_2(t)\,dt + 2\int_0^T\int_{\frac{1}{2}}^{1}\int_{\sigma}^{1}N(\sigma',t)d\sigma'd\sigma dt + C_3\\ &= -\frac{1}{\pi}\Im\left\{\int_0^T\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\log\zeta(\sigma'+it)d\sigma'd\sigma dt\right\} + 2\tilde{N}_{1,2}(T) + C_3\,.\end{aligned}$$

Here we can interchange the order of the integrations to obtain

$$\begin{aligned}\tilde{S}_3(T) =& -\frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\left\{\int_{\sigma'}^{\infty}\log|\zeta(\sigma''+iT)|d\sigma''\right.\\ &\left.-\int_{\sigma'}^{\infty}\log|\zeta(\sigma'')|d\sigma''\right\}d\sigma' d\sigma + 2\tilde{N}_{1,2}(T) + C_3\\ =& I_3(T) + 2\tilde{N}_{1,2}(T)\,.\end{aligned}$$

Now for $k \geq 2$, we get, by the induction hypothesis,

$$\begin{aligned}\tilde{S}_{2k}(T) =& \int_0^T \tilde{S}_{2k-1}(t)\,dt + C_{2k}\\ =& \int_0^T I_{2k-1}(t)\,dt + 2\int_0^T \sum_{\substack{h+2r=2k-1\\ r\geq 1, h\geq 0}} (-1)^{r-1}\tilde{N}_{h,2r}(t)\,dt + C_{2k}\\ =& \frac{1}{\pi}(-1)^{k-1}\int_0^T \Re\Bigg\{\underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k-1}\log\zeta(\sigma+iT)\,(d\sigma)^{2k-1}\Bigg\}\,dt\\ &+ 2\int_0^T \sum_{\substack{h+2r=2k-1\\ r\geq 1, h\geq 0}} (-1)^{r-1}\tilde{N}_{h,2r}(t)\,dt + C_{2k}\\ =& \frac{1}{\pi}(-1)^{k-1}\Re\Bigg\{\underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k-1}\int_0^T\log\zeta(\sigma+iT)\,dt\,(d\sigma)^{2k-1}\Bigg\}\\ &+ 2\int_0^T \sum_{\substack{h+2r=2k-1\\ r\geq 1, h\geq 0}} (-1)^{r-1}\tilde{N}_{h,2r}(t)\,dt + C_{2k}\\ =& \frac{1}{\pi}(-1)^{k-1}\Bigg\{\underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k-1}\Bigg\{-\Im\Bigg(\int_{\sigma}^{\infty}\log\zeta(\sigma+iT)\,d\sigma\Bigg)\\ &+ 2\pi\int_{\sigma}^{1}\nu(\sigma,T)\,d\sigma\Bigg\}\,(d\sigma)^{2k-1}\Bigg\}\\ &+ 2\int_0^T \sum_{\substack{h+2r=2k-1\\ r\geq 1, h\geq 0}} (-1)^{r-1}\tilde{N}_{h,2r}(t)\,dt + C_{2k}\end{aligned}$$

$$= \frac{1}{\pi}(-1)^k \Im\Bigg\{ \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k} \log \zeta(\sigma + iT)\,(d\sigma)^{2k} \Bigg\}$$

$$+ 2 \cdot (-1)^{k-1} \underbrace{\int_{\frac{1}{2}}^{1} \int_{\sigma}^{1} \cdots \int_{\sigma}^{1}}_{2k} N(\sigma, T)\,(d\sigma)^{2k}$$

$$+ (-1)^k \underbrace{\int_{\frac{1}{2}}^{1} \int_{\sigma}^{1} \cdots \int_{\sigma}^{1}}_{2k} (d\sigma)^{2k}$$

$$+ 2 \int_0^T \sum_{\substack{h+2r=2k-1 \\ r\geq 1, h\geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(t)\,dt + C_{2k}$$

$$= I_{2k}(T) + 2 \sum_{\substack{h+2r=2k \\ r\geq 1, h\geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(T)\,.$$

In the same manner, we get for $k \geq 2$,

$$\tilde{S}_{2k+1}(T) = I_{2k+1}(T) + 2 \sum_{\substack{h+2r=2k+1 \\ r\geq 1, h\geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(T)\,.$$

This proves Lemma 1.

Lemma 2 can be proved by the integration by parts. We shall indicate only the first two steps. We may suppose that $T \neq \gamma$. Then

$$\frac{i^m}{m!} \int_{\frac{1}{2}}^{\infty} \left(\sigma - \frac{1}{2}\right)^m \frac{\zeta^{'}}{\zeta}(\sigma + iT)\,d\sigma$$

$$= \frac{i^m}{m!} \Bigg\{ \Bigg[ \left(\sigma - \frac{1}{2}\right)^m \log \zeta(\sigma + iT) \Bigg]_{\frac{1}{2}}^{\infty} - m \int_{\frac{1}{2}}^{\infty} \left(\sigma - \frac{1}{2}\right)^{m-1} \log \zeta(\sigma + iT)\,d\sigma \Bigg\}$$

$$= -m \frac{i^m}{m!} \int_{\frac{1}{2}}^{\infty} \left(\sigma - \frac{1}{2}\right)^{m-1} \log \zeta(\sigma + iT)\,d\sigma\,.$$

If $m \geq 2$, then the last integral is

$$= m\frac{i^m}{m!}\left[\left(\sigma-\frac{1}{2}\right)^{m-1}\int_{\sigma}^{\infty}\log\zeta(\sigma+iT)\right)d\sigma\Bigg]_{\frac{1}{2}}^{\infty}$$

$$- m(m-1)\frac{i^m}{m!}\int_{\frac{1}{2}}^{\infty}\left(\sigma-\frac{1}{2}\right)^{m-2}\int_{\sigma}^{\infty}\log\zeta(\sigma+iT)(d\sigma)^2$$

$$= -m(m-1)\frac{i^m}{m!}\int_{\frac{1}{2}}^{\infty}\left(\sigma-\frac{1}{2}\right)^{m-2}\int_{\sigma}^{\infty}\log\zeta(\sigma+iT)(d\sigma)^2 .$$

We continue this procedure and get Lemma 2.

## §3. A supplemental remark to the proof of Lemma 1

Here we shall justify the interchanges of integration in the previous section. For this purpose we shall prove Lemmas 3 and 5 which will be given below.

LEMMA 3. *For any* $\sigma \geq \frac{1}{2}$, *we have*

$$\int_0^T\int_{\sigma}^3 \log|\zeta(\sigma'+it)|d\sigma'\,dt = \int_{\sigma}^3\int_0^T \log|\zeta(\sigma'+it)|dtd\sigma' .$$

We may suppose that $\sigma = \frac{1}{2}$. To justify the interchange of integration, it is enough to prove the convergence of the integral

$$\int_{\frac{1}{2}}^3\int_0^T |\log|\zeta(\sigma+it)||dtd\sigma .$$

Let $\gamma_n^*$ run over the positive different imaginary parts of the zeros of $\zeta(s)$ satisfying

$$\gamma_n^* < \gamma_{n+1}^* \quad \text{for } n = 1, 2, 3, \cdots .$$

Let $\gamma_N^*$ be the first $\gamma_n^*$ satisfying $T \leq \gamma_n^*$. We take positive $\varepsilon < 1$ such that

$$0 < \varepsilon < \frac{1}{2}\min_{n\leq N}(\gamma_{n+1}^* - \gamma_n^*) .$$

The last integral above is bounded by

$$\sum_{n=1}^N\int_{\frac{1}{2}}^3\left(\int_{\gamma_{n-1}^*+\varepsilon}^{\gamma_n^*-\varepsilon} + \int_{\gamma_n^*-\varepsilon}^{\gamma_n^*} + \int_{\gamma_n^*}^{\gamma_n^*+\varepsilon}\right)|\log|\zeta(\sigma+it)||dtd\sigma ,$$

where we put $\gamma_0^* + \varepsilon = 0$. We denote the possible zeros with an imaginary part $\gamma_n^*$ and a real part $> \frac{1}{2}$ by

$$\beta_n^{**}(1) + i\gamma_n^*, \cdots, \beta_n^{**}(m) + i\gamma_n^*, \cdots, \beta_n^{**}(M) + i\gamma_n^* ,$$

where we suppose that when $M \geq 2$, we have

$$\beta_n^{**}(m) < \beta_n^{**}(m+1) \quad \text{for any} \quad m \geq 1.$$

If $M \geq 1$, then we take positive $\delta = \delta_n$ such that

$$0 < \delta \leq \frac{1}{2}\min_m(\beta_n^{**}(m+1) - \beta_n^{**}(m),\, \beta_n^{**}(1) - \frac{1}{2},\, 1 - \beta_n^{**}(M))\,,$$

where we put $\beta_n^{**}(m+1) - \beta_n^{**}(m) = \infty$ when $M = 1$. Then the last sum is further decomposed as follows.

$$\sum_{n=1}^{N}\left(\int_{\frac{1}{2}}^{\beta_n^{**}(1)-\delta} + \int_{\beta_n^{**}(1)-\delta}^{\beta_n^{**}(1)+\delta} + \cdots + \int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m)+\delta} + \int_{\beta_n^{**}(m)+\delta}^{\beta_n^{**}(m+1)-\delta}\right.$$
$$\left. + \cdots + \int_{\beta_n^{**}(M)+\delta}^{3}\right)\cdot\left(\int_{\gamma_{n-1}^{*}+\varepsilon}^{\gamma_n^{*}-\varepsilon} + \int_{\gamma_n^{*}-\varepsilon}^{\gamma_n^{*}} + \int_{\gamma_n^{*}}^{\gamma_n^{*}+\varepsilon}\right)\cdot|\log|\zeta(\sigma+it)||dtd\sigma\,.$$

If there are no zeros with the imaginary part $\gamma_n^{*}$ and a real part $> \frac{1}{2}$, then we do not have to decompose the integral over the interval $[\frac{1}{2},\ 3]$ as we have done above.

Now typical terms are

$$I_1(n,m) = \int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m)+\delta}\int_{\gamma_n^{*}-\varepsilon}^{\gamma_n^{*}}|\log|\zeta(\sigma+it)||dtd\sigma,$$

$$I_2(n,m) = \int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m+1)-\delta}\int_{\gamma_{n-1}^{*}+\varepsilon}^{\gamma_n^{*}-\varepsilon}|\log|\zeta(\sigma+it)||dtd\sigma$$

and

$$I_3(n,m) = \int_{\beta_n^{**}(m)+\delta}^{\beta_n^{**}(m+1)-\delta}\int_{\gamma_n^{*}-\varepsilon}^{\gamma_n^{*}}|\log|\zeta(\sigma+it)||\,dtd\sigma$$

Here we shall use the following expression of $\log\zeta(s)$ (cf. Theorem 9.6 B of Titchmarsh [5]).

LEMMA 4. *For $s \neq \rho$, we have*

$$\log\zeta(s) = \sum_{|t-\gamma|\leq 1}\log(s-\rho) + O(\log t)\,,$$

*where $s = \sigma + it$, $-1 \leq \sigma \leq 2$, $t \geq 1$ and $-\pi < \Im\log(s-\rho) \leq \pi$.*

We denote the multiplicity of $\rho$ by $\mu(\rho)$. Then we get first

$$\begin{aligned} &I_1(n,m) \\ &\quad\ll \sum_{\substack{\rho=\beta+i\gamma \\ \gamma_n^{*}-\varepsilon-1<\gamma<\gamma_n^{*}+1}}\int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m)+\delta}\int_{\max(\gamma_n^{*}-\varepsilon,\gamma-1)}^{\min(\gamma_n^{*},\gamma+1)}|\log|\sigma+it-\beta-i\gamma||dtd\sigma \\ &\qquad + \varepsilon\delta\log T \end{aligned}$$

$$\ll \mu\left(\frac{1}{2}+i\gamma_n^*\right)\int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m)+\delta}\int_{\gamma_n^*-\varepsilon}^{\gamma_n^*}\left|\log\left|\sigma+it-\frac{1}{2}-i\gamma_n^*\right|\right|dtd\sigma$$

$$+\sum_{j=1}^{M}\mu(\beta_n^{**}(j)+i\gamma_n^*)\int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m)+\delta}\int_{\gamma_n^*-\varepsilon}^{\gamma_n^*}\left|\log\left|\sigma+it-\beta_n^{**}(j)-i\gamma_n^*\right|\right|dtd\sigma$$

$$+\sum_{\substack{\gamma\neq\gamma_n^*\\ \gamma_n^*-\varepsilon-1<\gamma<\gamma_n^*+1}}\int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m)+\delta}\int_{\gamma_n^*-\varepsilon}^{\gamma_n^*}|\log|\sigma+it-\beta-i\gamma||dtd\sigma+\varepsilon\delta\log T$$

$$=I_1'+I_1''+I_1'''+\varepsilon\delta\log T,\quad \text{say.}$$

Since for $|u|\leq\varepsilon$ and for $\frac{1}{2}\leq\sigma\leq 1$

$$\max((\sigma-\beta_n^{**}(j))^2,u^2)<(\sigma-\beta_n^{**}(j))^2+u^2<\frac{1}{4}+\varepsilon^2<1$$

and

$$|\log|\sigma-\beta_n^{**}(j)+iu||\ll\min(|\log|\sigma-\beta_n^{**}(j)||,|\log|u||),$$

we get

$$I_1''\ll\sum_{j=1}^{M}\mu(\beta_n^{**}(j)+i\gamma_n^*)\int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m)+\delta}\int_{-\varepsilon}^{0}|\log|u||dud\sigma$$

$$\ll\varepsilon|\log\varepsilon|\delta\cdot\sum_{\gamma=\gamma_n^*}1\ll\log T.$$

In the same manner, we get

$$I_1'\ll\delta\varepsilon|\log\varepsilon|\cdot\log T\ll\log T.$$

For $I_1'''$, since

$$\varepsilon\leq|t-\gamma|\leq 1+\varepsilon,$$

we have

$$\varepsilon^2\leq(\sigma-\beta)^2+\varepsilon^2\leq(\sigma-\beta)^2+(t-\gamma)^2\leq\frac{1}{4}+(1+\varepsilon)^2$$

and

$$|\log|\sigma+it-\beta-i\gamma||\ll|\log\varepsilon|.$$

Hence, we get

$$I_1'''\ll\delta\varepsilon|\log\varepsilon|\log T.$$

Thus we get

$$I_1(n,m)\ll\log T.$$

In a similar manner, we have

$$I_2(n,m)$$

$$\ll \sum_{\gamma_{n-1}^*-1+\varepsilon<\gamma<\gamma_n^*+1-\varepsilon} \int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m+1)-\delta} \int_{\gamma_{n-1}^*+\varepsilon}^{\gamma_n^*-\varepsilon} |\log((\sigma-\beta)^2+(t-\gamma)^2)|dtd\sigma$$

$$+\log T\ .$$

Since

$$(\sigma-\beta)^2<(\sigma-\beta)^2+\varepsilon^2\le(\sigma-\beta)^2+(t-\gamma)^2\le 2\,,$$

the last sum is

$$\ll \sum_{\gamma_{n-1}^*-1+\varepsilon<\gamma<\gamma_n^*+1-\varepsilon} \int_{\beta_n^{**}(m)-\delta}^{\beta_n^{**}(m+1)-\delta} |\log|\sigma-\beta||d\sigma$$

$$\ll \sum_{\gamma_{n-1}^*-1+\varepsilon<\gamma<\gamma_n^*+1-\varepsilon} \int_{\beta_n^{**}(m)-\delta}^{1} |\log|\sigma-\beta||d\sigma$$

$$\ll \sum_{\gamma_{n-1}^*-1+\varepsilon<\gamma<\gamma_n^*+1-\varepsilon} \left|\int_{-1}^{1}\log|x|dx\right|$$

$$\ll \log T.$$

Hence, we get

$$I_2(n,m)\ll\log T\ ,$$

and similarly the same bound for $I_3(n,m)$. Thus for each $\beta_n^{**}(m)+i\gamma_n^*$, each double integral is convergent. In the same manner, we have the same conclusion when $M\not\geq 1$. Consequently, the integral

$$\int_{\frac{1}{2}}^{3}\int_0^T |\log|\zeta(\sigma+it)||dtd\sigma$$

is convergent. In a similar manner, we can prove the following

LEMMA 5. *For any $\sigma\ge\frac{1}{2}$, we have*

$$\int_0^T\int_\sigma^3 \Im\log\zeta(\sigma'+it)d\sigma'\,dt=\int_\sigma^3\int_0^T \Im\log\zeta(\sigma'+it)dtd\sigma'.$$

We may suppose that $\sigma=\frac{1}{2}$. We use the same notations and the argument as above. We shall treat only one typical singular part, namely,

$$\int_{\beta_n^{**}(m)}^{\beta_n^{**}(m)+\delta}\int_{\gamma_n^*}^{\gamma_n^*+\varepsilon} |\Im(\log(\sigma+it-\beta_n^{**}(m)-i\gamma_n^*))|dtd\sigma\ .$$

This is

$$
\begin{aligned}
&= \int_0^\delta \int_0^\varepsilon |\Im(\log(x+iy))| dy dx \\
&\ll \int_0^\delta \int_0^\varepsilon \left| \frac{\pi}{2} - \mathrm{Arctan}\left(\frac{x}{y}\right) \right| dy dx \\
&\ll \delta\varepsilon + \int_0^\delta \int_0^\varepsilon \mathrm{Arctan}\left(\frac{x}{y}\right) dy dx \,.
\end{aligned}
$$

The last integral is

$$
\begin{aligned}
&= \int_0^\delta \left[ y \cdot \mathrm{Arctan}\left(\frac{x}{y}\right) \right]_0^\varepsilon dx - \int_0^\delta \int_0^\varepsilon y \cdot \left( \mathrm{Arctan}\left(\frac{x}{y}\right) \right)' dy dx \\
&= \int_0^\delta \varepsilon \cdot \mathrm{Arctan}\left(\frac{x}{\varepsilon}\right) dx + \int_0^\delta \int_0^\varepsilon yx \frac{1}{y^2+x^2} dy dx \,.
\end{aligned}
$$

The former integral is bounded. The latter integral is

$$
= \int_0^\delta x \frac{1}{2} (\log(x^2+\varepsilon^2) - \log(x^2))\, dx \,.
$$

This is also bounded.

## §4. Proof of Theorem 1

Let $s = \sigma + it$. We suppose that $\sigma \geq \frac{1}{2}$ and $t \geq 2$. Let $X$ be a positive number satisfying $4 \leq X \leq t^2$. We put

$$
\sigma_{X,t} = \frac{1}{2} + 2 \max_{\rho} \left( \beta - \frac{1}{2}, \frac{2}{\log X} \right),
$$

where $\rho$ runs through all zeros $\beta + i\gamma$ of $\zeta(s)$ for which

$$
|t - \gamma| \leq \frac{X^{3|\beta - \frac{1}{2}|}}{\log X} \,.
$$

We put

$$
\Lambda_X(n) = \begin{cases} \Lambda(n) & \text{for} \quad 1 \leq n \leq X \\ \Lambda(n) \dfrac{\log^2 \frac{X^3}{n} - 2\log^2 \frac{X^2}{n}}{2\log^2 X} & \text{for} \quad X \leq n \leq X^2 \\ \Lambda(n) \dfrac{\log^2 \frac{X^3}{n}}{2\log^2 X} & \text{for} \quad X^2 \leq n \leq X^3 \end{cases}
$$

and

$$
\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^k \quad \text{with a prime } p \text{ and an integer } k \geq 1 \\ 0 & \text{otherwise}\,. \end{cases}
$$

Then for $\sigma \ge \sigma_{X,t}$, $2 \le X \le t^2$, $t \ge 2$, we have, by p.239 of Selberg[4],

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n<X^3} \frac{\Lambda_X(n)}{n^s} + O\left(X^{\frac{1}{4}-\frac{\sigma}{2}} \left| \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}} \right|\right) + O(X^{\frac{1}{4}-\frac{\sigma}{2}} \log t)\,.$$

Now we have

$$\begin{aligned}
&-\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\frac{1}{2}}^{\infty}\left(\sigma-\frac{1}{2}\right)^m \frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma\right\} \\
&\quad= -\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\sigma_{X,t}}^{\infty}\left(\sigma-\frac{1}{2}\right)^m \frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma\right\} \\
&\qquad -\frac{1}{\pi}\Im\left\{\frac{i^m}{m!(m+1)}\left(\sigma_{X,t}-\frac{1}{2}\right)^{m+1}\frac{\zeta'}{\zeta}(\sigma_{X,t}+it)\right\} \\
&\qquad +\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\frac{1}{2}}^{\sigma_{X,t}}\left(\sigma-\frac{1}{2}\right)^m\left(\frac{\zeta'}{\zeta}(\sigma_{X,t}+it)-\frac{\zeta'}{\zeta}(\sigma+it)\right)d\sigma\right\} \\
&\quad= J_1+J_2+J_3, \quad \text{say}\,.
\end{aligned}$$

$$\begin{aligned}
J_1 &= -\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\sigma_{X,t}}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\left(-\sum_{n<X^3}\frac{\Lambda_X(n)}{n^s}\right)d\sigma\right\} \\
&\quad +\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\sigma_{X,t}}^{\infty}\left(\sigma-\frac{1}{2}\right)^m O\left(X^{\frac{1}{4}-\frac{\sigma}{2}}\left|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\right|\right)d\sigma\right\} \\
&\quad +\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\sigma_{X,t}}^{\infty}\left(\sigma-\frac{1}{2}\right)^m O(X^{\frac{1}{4}-\frac{\sigma}{2}}\log t)\,d\sigma\right\} \\
&= {J_1}' + J_1'' + J_1''', \quad \text{say}\,.
\end{aligned}$$

$$\begin{aligned}
{J_1}' &= \frac{1}{\pi}\Im\left\{\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{it}}\frac{i^m}{m!}\int_{\sigma_{X,t}}^{\infty}\left(\sigma-\frac{1}{2}\right)^m e^{-\sigma\log n}\,d\sigma\right\} \\
&= \frac{1}{\pi}\Im\left\{\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{it}}\frac{i^m}{m!}\frac{1}{\sqrt{n}(\log n)^{m+1}}\Gamma\left(m+1,\left(\sigma_{X,t}-\frac{1}{2}\right)\log n\right)\right\} \\
&= \frac{1}{\pi}\Im\left\{\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{it}}\frac{i^m}{m!}\frac{m!}{\sqrt{n}(\log n)^{m+1}}e^{-(\sigma_{X,t}-\frac{1}{2})\log n}\sum_{\nu=0}^{m}\frac{((\sigma_{X,t}-\frac{1}{2})\log n)^\nu}{\nu!}\right\} \\
&= \frac{1}{\pi}\Im\left\{i^m\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}(\log n)^{m+1}}\sum_{\nu=0}^{m}\frac{((\sigma_{X,t}-\frac{1}{2})\log n)^\nu}{\nu!}\right\},
\end{aligned}$$

where we put for any $Y > 0$ and for $\Re(s) > 0$,

$$\Gamma(s,Y)=\int_Y^\infty x^{s-1}e^{-x}dx\,.$$

$$\begin{aligned}J_1''&\ll\Big|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\Big|\int_{\sigma_{X,t}}^\infty\Big(\sigma-\frac12\Big)^m X^{\frac14-\frac{\sigma}{2}})\,d\sigma\\&\ll\Big|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\Big|\Gamma\Big(m+1,\frac12\Big(\sigma_{X,t}-\frac12\Big)\log X\Big)\frac{1}{(\log X)^{m+1}}\\&\ll\Big(\sigma_{X,t}-\frac12\Big)^{m+1}\Big|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\Big|.\end{aligned}$$

$$J_1'''\ll\log t\int_{\sigma_{X,t}}^\infty\Big(\sigma-\frac12\Big)^m X^{\frac14-\frac{\sigma}{2}}\,d\sigma\ll\Big(\sigma_{X,t}-\frac12\Big)^{m+1}\log t\,.$$

Thus we get

$$\begin{aligned}J_1=&\frac1\pi\Im\Big\{i^m\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}(\log n)^{m+1}}\sum_{\nu=0}^m\frac{((\sigma_{X,t}-\frac12)\log n)^\nu}{\nu!}\Big\}\\&+O\Big(\Big(\sigma_{X,t}-\frac12\Big)^{m+1}\Big|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\Big|\Big)+O\Big(\Big(\sigma_{X,t}-\frac12\Big)^{m+1}\log t\Big).\end{aligned}$$

Similarly, we get

$$J_2=O\Big(\Big(\sigma_{X,t}-\frac12\Big)^{m+1}\Big|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\Big|\Big)+O\Big(\Big(\sigma_{X,t}-\frac12\Big)^{m+1}\log t\Big).$$

To treat $J_3$, we can apply the arguments on pp. 243–246 of Selberg [4]. We notice first that

$$J_3=\begin{cases}\dfrac{(-1)^k}{(2k+1)!\pi}\displaystyle\int_{\frac12}^{\sigma_{X,t}}\Big(\sigma-\frac12\Big)^{2k+1}\Re\Big(\frac{\zeta'}{\zeta}(\sigma_{X,t}+it)-\frac{\zeta'}{\zeta}(\sigma+it)\Big)d\sigma\\ \qquad\qquad\text{for } m=2k+1\\ \dfrac{(-1)^k}{(2k)!\pi}\displaystyle\int_{\frac12}^{\sigma_{X,t}}\Big(\sigma-\frac12\Big)^{2k}\Im\Big(\frac{\zeta'}{\zeta}(\sigma_{X,t}+it)-\frac{\zeta'}{\zeta}(\sigma+it)\Big)d\sigma\\ \qquad\qquad\text{for } m=2k\,.\end{cases}$$

When $m=2k+1$,

$$\int_{\frac{1}{2}}^{\sigma_{X,t}} \left(\sigma - \frac{1}{2}\right)^{2k+1} \Re\left(\frac{\zeta'}{\zeta}(\sigma_{X,t} + it) - \frac{\zeta'}{\zeta}(\sigma + it)\right) d\sigma$$

$$= \sum_{\rho} \int_{\frac{1}{2}}^{\sigma_{X,t}} \left(\sigma - \frac{1}{2}\right)^{2k+1} \left\{ \frac{\sigma_{X,t} - \beta}{(\sigma_{X,t} - \beta)^2 + (t-\gamma)^2} - \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t-\gamma)^2} \right\} d\sigma$$

$$+ O\left(\left(\sigma_{X,t} - \frac{1}{2}\right)^{2k+2} \log t\right)$$

$$\ll \left(\sigma_{X,t} - \frac{1}{2}\right)^{2k+3} \sum_{\rho} \frac{1}{(\sigma_{X,t} - \beta)^2 + (t-\gamma)^2}$$

$$\ll \left(\sigma_{X,t} - \frac{1}{2}\right)^{2k+2} \left( \left| \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}} \right| + \log t \right)$$

$$\ll \left(\sigma_{X,t} - \frac{1}{2}\right)^{m+1} \left( \left| \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}} \right| + \log t \right),$$

and the same bound is seen to hold also when $m = 2k$. Consequently, we get

$$-\frac{1}{\pi} \Im \left\{ \frac{i^m}{m!} \int_{\frac{1}{2}}^{\infty} \left(\sigma - \frac{1}{2}\right)^m \frac{\zeta'}{\zeta}(\sigma + it)\, d\sigma \right\}$$

$$= \frac{1}{\pi} \Im \left\{ i^m \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it} (\log n)^{m+1}} \sum_{\nu=0}^{m} \frac{((\sigma_{X,t} - \frac{1}{2}) \log n)^\nu}{\nu!} \right\}$$

$$+ O\left(\left(\sigma_{X,t} - \frac{1}{2}\right)^{m+1} \left| \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}} \right| \right) + O\left(\left(\sigma_{X,t} - \frac{1}{2}\right)^{m+1} \log t\right).$$

We choose $X = (\log t)^{\frac{2}{3}}$. Then we see that the right hand side is

$$\ll \log t\,.$$

This proves our Theorem 1.

## References

[ 1 ] A. Fujii, Some problems of Diophantine Approximation in the theory of the Riemann zeta function (III), Comment. Math. Univ. Sancti Pauli, **42** (1993), 161–187.

[ 2 ] A. Fujii, On the discrepancy estimates of the zeros of the Riemann zeta function, Comment. Math. Univ. Sancti Pauli, **51** (2002), 19–51.

[ 3 ] J. E. Littlewood, On the zeros of Riemann's zeta function, Proc. of Camb. Phil. Soc., **22** (1924), 295–318.

[ 4 ] A. Selberg, Collected Papers, Springer Verlag, Vol. 1, (1989).

[ 5 ] E. C. Titchmarsh, The theory of the Riemann zeta function (2nd ed. rev. by D. R. Heath-Brown), Oxford Univ. Press, (1988), 1951.

Department of mathematics, Rikkyo University
Nishi-ikebukuro,Toshimaku, Tokyo 171–8501, Japan