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Research for γ_5-hermiticity in the minimal doubling fermion

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Research for γ_5-hermiticity in the minimal doubling fermion

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Abstract

The Lattice gauge theory is a useful tool for investigating strong coupling physics. In this formulation, space-time coordinate is discretized and physical degrees of freedom are defined at sites and links. Regarding a lattice spacing as a cut-off regulator, we can naturally regularize UV divergence and calculate observables using some techniques, e.g., strong coupling expansion, Monte Carlo simulations, and so on.

As is well known, lattice fermion breaking γ_5-hermiticity causes the sign problem, expectation values in theories broken γ_5-hermiticity have complex phase and are obtained as complex values. Especially, in high density region of the finite temperature and density QCD, we hardly obtain expectation values because of this problem, thus it is considered as one of the problems which should be solved.

In this thesis, we investigate γ_5-hermiticity using the minimal doubling fermions. We analyze kinetic term of lattice fermion, which is assumed translation invariant, continuum and periodic function, using γ_5-hermiticity, R-hermiticity and PT symmetry. We show that the properties are related to each other. And we show that a PT symmetric kinetic term can not reduce doublers.

We also formulate two dimensional fermions without γ_5-hermiticity based on the minimal doubling fermion. From discussions of the eigenvalue distribution and the number of poles for our fermions, we find out an appropriate fermion for application to practical analyses. This fermion has the same symmetries as the usual minimal doubling fermions, but γ_5-hermiticity does not preserve. As simple tests for application of the fermion, we

apply the non-γ_5-hermiticity fermion to the 2D Gross-Neveu model. As a result, we obtain similar phase diagrams to ones using the naive fermion.

Contents

1 Introduction

Lattice gauge theory is a powerful tool for revealing nonperturbative quark dynamics [1]. In this formulation, the space-time coordinate is discretized and physical variables are defined at sites and links. An inverse lattice spacing is identify with a ultraviolet cut-off regulator, therefor, ultraviolet divergence is naturally regularized in this theory. In addition, we can calculate observables using some proper techniques; especially, Monte Carlo simulations are applied for investigating nonperturbative physics, e.g., quantum chromo dynamics (QCD), nuclear physics, and other theories. Despite the very simple structure, numerical simulations in lattice gauge theory feed us great results and development of nonperturbative physics, with progression of computers. However, not only those advantages but also some difficulties remain or are appeared: fine-tuning for broken symmetry, finite volume effect, space-time continuum limit, broken symmetries by lattice fermions, and so on.

As is well known, the naive lattice fermion has redundant physical degrees of freedom, doublers; this is called the doubling problem. We cannot remove doublers without breaking some symmetries or properties, because of the no-go theorem of Nilsen and Ninomiya [2]-[4]. To overcome this problem, many lattice fermions have been formulated, e.g., the Wilson fermion [1] and the KS fermion [5]. In particular, chiral symmetry is one of the important symmetry in QCD and nuclear physics. Though quarks satisfy chiral symmetry classically, spontaneously chiral symmetry breaking and quark condensation are caused by quantum correction, which is called axial or chiral anomaly. The symmetry breaking is trigger for massless boson, called Goldstone boson, and the boson is interpreted as lightest pseud-scalar meson, that is pion. In lattice gauge theory with the naive fermions, the quark condensation is not caused because contributions to chiral anomaly by fermions are canceled by each doublers. In this context, preserving exact chiral symmetry is a significant problem in analyzing non-perturbative QCD, but it is one of the symmetry in the no-go theorem and is incompatible with the removal of doublers.

To overcome this problem, many physicists have formulated various lattice fermions, e.g., Wilson fermion, which breaks chiral symmetry [1], Kogut-Suskind fermion, which regards doublers as flavors [5], SLAC fermion, which breaks locality[6] and so on. In analyzes using Wilson fermion, chiral symmetry is broken for single pole fermions, thus, fine-tuning for mass parameter is needed to restore chiral symmetry. On the other hand, Ginsparg and Wilson found out a relationship for Dirac operator which automatically restore chiral symmetry without fine-tuning in the continuum limit [11]. The relationship is called the Ginsparg-Wilson relation. Few years later, Neuburger found out a solution for the relation [12], and Luscher formulated chiral exact lattice fermion based on the relationship [13]. Through the Ginsparg-Wilson relation, the discussion for chiral symmetry in the lattice gauge theory was over for the present. However, we have to pay much numerical costs to analyze theories using the Ginsparg-Wilson fermion.

In recent years, Creutz formulated an exact chiral symmetric lattice fermion [16] in hexagonal lattice space-time, and Borici applied it to orthogonal lattice [17]. One the other hand, a few decades ago, Karsten constructed a fermion formulation with the same structure, but with a different action to the Creutz one [14]. These fermions are called the minimal doubling fermions [14]-[16]. The minimal doubling fermions break (hyper-)cubic symmetry and some discrete symmetries, such as charge conjugation (C), parity transformation (P), time reflection (T), and so on. Many properties of the fermions have been analyzed in the orthogonal lattice [14]-[28] and hyperdiamond lattice [30]. In quantum theory, we must fine-tune some parameters to preserve these broken symmetries, however, it is difficult to adjust them generally.

By the way, finite temperature and density physics are main subjects for the lattice gauge theory. In a finite density theory, it is well known that a fermion bilinear term in the action is broken γ_5-hermiticity by a chemical potential term. The γ_5-hermiticity guarantees the hermiticity of the Hamiltonian and is also a reality condition for fermion determinant appearing when fermions are integrated out from the partition function. In general, the fermion determinant in a finite density theory is not a real number but rather a complex number. For the estimation of observables, we need to use an appropriate reweighting method. However, in the high density region, the complex phase of fermion determinant fluctuates in a wide range, thus, expectation values approach to zero and can be hardly estimated. No one knows general resolutions for this problem, and this problem is still an open problem, called the sign problem.

In this thesis, we focus on γ_5-hermiticity. As mentioned above, γ_5-hermiticity is needed for avoiding the sign problem, and also needed for solutions of the Ginsparg-Wilson relation. Additionally, the γ_5-hermiticity is one of the symmetries in the Nielsen-Ninomiya theorem. As a strategy to solve the sign problem or obtain lessons of resolutions, we formulate lattice fermion for calculation of observables, leaving γ_5-hermiticity broken. In this approach, we use lattice artifact of this fermion as a procedure for avoiding or reducing complex phases in expectation values. And we also conceive that we should understand lattice fermions more deeply because (1) fermion is one of the most important object in the field theory without regard to lattice theory, (2) we need to systematically control doublers or their properties and (3) in the numerical context very superior lattice fermions which are unknown to us yet might exist. This thesis a start pint for these motivations.

This thesis is constructed as follows. In Sect.2, we firstly present reviews of the lattice gauge theory and the minimal doubling fermions. In Sect.3, we analyze the translation-invariant, continuum, and periodic function lattice fermion kinetic term using γ_5-hermiticity, R-hermiticity, and PT symmetry. These symmetries and hermiticities are related to each other. For example, assuming that a translation-invariant kinetic term with continuum and periodic function does not have PT symmetry, it can have R-hermiticity or γ_5-hermiticity. R-hermiticity is a reality or Hermite condition for renormalized coupling constants perturbatively. We show that a PT-symmetric kinetic term cannot reduce doublers. As a simple example, we apply minimal doubling fermions that do not have PT symmetry or R-hermiticity to the 2D N-flavor GrossNeveu model and calculate renormalization group flows. In this flow, complex or non-Hermite coupling constants are caused by quantum correction. In Sect.4.1, we formulate 2D fermions without γ_5-hermiticity (non-γ_5-hermiticity fermions) based on the minimal doubling fermion. As with the minimal doubling fermion, the non-γ_5-hermiticity fermion breaks some discrete symmetries. We obtain the eigenvalue distribution and the number of poles for the fermions and discuss the selection rule for an optimum fermion to apply to a practical analysis. For simple application tests, the 2D Gross-Neveu model is studied using the non-γ_5-hermiticity fermion. We draw two sorts of phase diagrams, parity broken phase diagrams, called Aoki phase and chiral broken phase diagrams in massless and an imaginary chemical potential system. By the analyzing these models with the fermion, we expect that we can more deeply understand the structure of lattice fermions, and thus the sign problem. In Subsect. 4.1, we construct non-γ_5-hermiticity fermions based on the minimal doubling fermion and investigate their symmetries and properties. In Subsect. 4.2, we study parity broken phase diagrams for the 2D Gross-Neveu model using the non-γ_5-hermiticity fermion. In Subsect. 4.3, we also draw chiral broken phase diagrams for the Gross-Neveu model adding a imaginary chemical potential in two dimensions. In Subsect. 4.4, we discuss a reality condition for observables from the eigenvalue distribution of the fermions with an imaginary chemical potential. Final section is devoted to the summary and discussion.

2 Basic review

In this section, we write basic reviews needed to understand this thesis.

2.1 Lattice theory

In this subsection, we review the lattice gauge theory [1]. The lattice gauge theory is one of the techniques for investigating nonperturbative physics. Roughly speaking, the lattice gauge theory is a field theory defined on discrete space-time. We usually define scalar fields and fermions on sites, and gauge fields on links, which are connections between two neighboring sites. To obtain results of target continuum theories by way of the lattice gauge theory, we need to take three steps: (1) we discretize space-time of target theory, (2) calculate observables using some technique, and (3) take the continuum limit. Now, we introduce a method to obtain a classical lattice theory from a classical continuum theory. For simplicity, we firstly treat four dimensional ϕ^4 theory, then a free fermion system, and finally SU(N) gauge theory.

We define the Euclidean continuum 4D ϕ^4 theory as follows:

$$S_{\phi^4}^{\text{cont.}} = \int d^4x \left[\frac{1}{2} \sum_{\mu} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]. \tag{2.1}$$

where μ is space-time index which runs over from 1 to 4. The action has translation symmetry, SO(4) Euclid symmetry and $\mathbf{Z}_2$ symmetry. To define lattice action, we discretize space-time. We replace some parts of the continuum action as follows:

$$\begin{aligned} x &\rightarrow na, \\ \int d^4x &\rightarrow a^4 \sum_n, \\ \partial_\mu \phi(x) &\rightarrow \frac{\phi(na + \hat{\mu}a) - \phi(na)}{a}, \end{aligned}$$

where n, a and $\hat{\mu}$ are a lattice site, which is 4D vector, a lattice spacing, and unit vectors, which are directed to μ-directions, respectively. And, we define lattice action of the ϕ^4 theory so that the action satisfies the following lattice principle:

- in the classical continuum limit, the lattice theory must realize the continuum theory.

- we must preserve symmetries as many as possible.

From the principle, we can get the ϕ^4 lattice theory as follows:

$$S^{\text{lat.}}_{\phi^4} = a^4 \sum_n \left[\frac{1}{2} \sum_\mu (\partial^f_\mu \phi(na))^2 + \frac{1}{2} m^2 \phi^2(na) + \frac{\lambda}{4!} \phi^4(na) \right], \tag{2.2}$$

In lattice theory, we can define some difference operators, e.g. forward, backward, and symmetric difference operators [1]:

$$\partial^f_\mu \phi(na) \equiv \frac{\phi(na+\hat{\mu}a) - \phi(na)}{a}, \tag{2.3}$$

$$\partial^b_\mu \phi(na) \equiv \frac{\phi(na) - \phi(na-\hat{\mu}a)}{a}, \tag{2.4}$$

$$\partial^s_\mu \phi(na) \equiv \frac{1}{2} \left(\partial^f_\mu + \partial^b_\mu \right) \phi(na). \tag{2.5}$$

The lattice action preserves discrete translation symmetry, discrete Euclid symmetry, called hypercubic symmetry, and $\mathbf{Z}_2$ symmetry. However, continuum translation symmetry and continuum Euclid symmetry are broken. In the continuum limit, we can easily obtain the continuum action Eq.(2.1) from the lattice action Eq.(2.2) using Taylor expansion.

Note that the lattice action Eq.(2.2) is one of the lattice actions which we can define. For example, we can add $a^8 \sum_n g^{(6)} \phi^6(na)$, with a dimensionless parameter $g^{(6)}$, to the lattice action Eq.(2.2) because this term vanishes in the continuum limit. This action does not interfere with the lattice principle.

Next, we construct a lattice action from the continuum free fermion theory. We define the continuum free fermion action as follows:

$$S^{\text{cont.}}_{\text{free }\psi} = \int d^4x \bar{\psi} (\sum_\mu \partial_\mu \gamma_\mu + m) \psi, \tag{2.6}$$

In the similar way to the ϕ^4 theory, we define lattice free fermion action,

$$S^{\text{lat.}}_{\text{free }\psi} = a^4 \sum_n \bar{\psi}(na) (\sum_\mu \partial^s_\mu \gamma_\mu + m) \psi(na), \tag{2.7}$$

where γ_μ is γ-matrix. Now, we apply symmetric difference operator, ∂^s_μ, to the action

[1] In the lattice principle we can use not only ∂^f_μ but also ∂^b_μ or ∂^s_μ. However, the scalar lattice action using ∂^s_μ generates 2^4 doublers in quantum theory.

rather than ∂^f_μ or ∂^b_μ because the lattice action using ∂^s_μ does not preserve hermiticity. We will discuss this issue in the next subsection.

Finally, we construct QCD action on lattice space-time from the continuum QCD action. The continuum action is defined as follows:

$$S^{\text{cont.}}_{\text{QCD}} = \int d^4x \,\text{tr}\left[\frac{1}{4}F^2_{\mu\nu} + \bar{\psi}(\nabla_\mu\gamma_\mu + m)\psi\right], \tag{2.8}$$

where $F_{\mu\nu}$ is a field strength defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and ∇_μ is a covariant derivative, which is defined as $\nabla_\mu = \partial_\mu + igA_\mu$. The A_μ and g are SU(N) gauge fields and a gauge coupling constant, respectively. The action preserves gauge symmetry whose transformation is defined as follows:

$$\begin{aligned} \psi(x) &\rightarrow \Omega(x)\psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x)\Omega^\dagger(x), \\ A_\mu &\rightarrow \Omega(x)A_\mu(x)\Omega^\dagger(x) + \frac{i}{g}\Omega^\dagger(x)\partial_\mu\Omega(x), \end{aligned} \tag{2.9}$$

where $\Omega(x)$ is an arbitrary SU(N) function.

To construct a lattice action from the continuum action Eq.(2.8), we need to define the gauge field in lattice space-time. However, we cannot simply define the gauge field on sites for the two reasons : (1) the gauge field has Euclid indices, namely directions, and (2) we cannot define a gauge invariant action because the derivative operator in the kinetic terms is replaced with the difference operator; hence, the gauge symmetry is broken at $\mathcal{O}(a)$ in the case of defining gauge fields on sites. For these problems, we define a link variable lying from na to $na + \hat{\mu}a$, instead of the gauge field,

$$U(na, na + \hat{\mu}a) = \exp\left[ig\int_{na}^{na+\hat{\mu}a} dx_\mu\, A_\mu(x)\right] \equiv U_\mu(na), \tag{2.10}$$

and its gauge transformation,

$$U(na, na + \hat{\mu}a) \rightarrow \Omega(na)U(na, na + \hat{\mu}a)\Omega^\dagger(na + \hat{\mu}a). \tag{2.11}$$

Hermite conjugate of the link variable is defined as an opposite directed link variable,

$$U^\dagger(na, na + \hat{\mu}a) = U(na + \hat{\mu}a, na) \equiv U^\dagger_\mu(na). \tag{2.12}$$

Note that the gauge field in the continuum theory is represented as an element of Lie algebra, on the other hand, the link variable is defined as an element of Lie group. The domain of the SU(N) Lie group is compact, therefore, we do not need to perform gauge fixing because we do not over-count link variable spectrum in the path integral formulation.

Here, we discretize all terms in the continuum action. The kinetic term of the gauge field is represented as *plaquette*, which is closed Wilson loop defined as

$$S_U^{\text{lat.}} = \frac{1}{8} \sum_{n,\mu,\nu\neq\mu} \mathrm{tr}\left[U_\mu(na)U_\nu(na+\hat{\mu}a)U_\mu^\dagger(na+\hat{\nu}a)U_\nu^\dagger(na) + (\text{h.c.})\right], \tag{2.13}$$

where (h.c.) denotes hermite conjugate of 1st term of r.h.s. in this action [2]. For the fermion part, we can define a gauge invariant lattice action, replacing the difference operator with the following covariant difference operator,

$$\nabla_\mu^f \psi(na) \equiv \frac{U_\mu(na)\psi(na+\hat{\mu}a)U_\mu^\dagger(na) - \psi(na)}{a}, \tag{2.15}$$

$$\nabla_\mu^b \psi(na) \equiv \frac{\psi(na) - U_\mu^\dagger(na-\hat{\mu}a)\psi(na-\hat{\mu}a)U_\mu(na-\hat{\mu}a)}{a}, \tag{2.16}$$

$$\begin{aligned}\nabla_\mu^s \psi(na) &\equiv \frac{U_\mu(na)\psi(na+\hat{\mu}a)U_\mu^\dagger(na) - U_\mu^\dagger(na-\hat{\mu}a)\psi(na-\hat{\mu}a)U_\mu(na-\hat{\mu}a)}{2a} \\ &= \frac{\nabla_\mu^f + \nabla_\mu^b}{2}\psi(na).\end{aligned} \tag{2.17}$$

Using the operator, we can define the fermi-bilinear term on lattice from the continuum QCD action,

$$S_\psi^{\text{lat.}} = a^4 \sum_n \bar{\psi}(na)(\sum_\mu \nabla_\mu^s \gamma_\mu + m)\psi(na). \tag{2.18}$$

From the Eq.(2.13) and (2.18), the lattice QCD action is defined as follows:

$$S_{\text{QCD}}^{\text{lat.}} = S_U^{\text{lat.}} + S_\psi^{\text{lat.}}. \tag{2.19}$$

[2]Path of the plaquette is not unique. For example, we can define as,

$$\begin{aligned} S_U^{\text{lat.}} = \sum_{n,\mu,\nu\neq\mu} & c \cdot \mathrm{tr}\left[U_\mu(na)U\mu(na+\hat{\mu}a)U_\nu(na+2\hat{\mu}a)\right. \\ & \left.\cdot U_\mu^\dagger(na+\hat{\mu}a+\hat{\nu}a)U_\mu^\dagger(na+\hat{\nu}a)U_\nu^\dagger(na) + (\text{h.c.})\right]. \end{aligned} \tag{2.14}$$

where c is a constant, which is adjusted in the continuum limit.

2.2 Doubling problem and Nielsen-Ninomiya theorem

In the lattice theory, a lattice fermion has a serious problem, called "doubling problem". The lattice fermion Eq.(2.7) generates 2^4 degenerate spectra, called "doublers", in quantum theory. The degenerate degrees of freedom affect observables or physics, which we are interested in. Unfortunately, a no-go theorem was discovered by Nielsen and Ninomiya [2]-[4]. The theorem states that we need to break important symmetry or property for reducing the doublers. In this subsection, we briefly present the doubling problem and the Nielsen-Ninomiya theorem.

As a start point, we rewrite the lattice fermion action Eq.(2.7) in momentum space,

$$\psi(na) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\psi(p)\exp(ian\cdot p),$$
$$\bar{\psi}(na) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\bar{\psi}(p)\exp(ian\cdot p). \tag{2.20}$$

and the identity,

$$\int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\exp(ia(n-m)\cdot p) = \frac{1}{a^4}\delta_{n,m}. \tag{2.21}$$

From the transformation, we obtain the action in momentum space,

$$\begin{aligned} S^{\text{lat.}}_{\text{free }\psi} &= \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\bar{\psi}(-p)\left[\sum_\mu \frac{i}{a}\sin(p_\mu a)\gamma_\mu + m\right]\psi(p) \\ &\equiv \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\bar{\psi}(-p)D_{\text{n}}(p)\psi(p). \end{aligned} \tag{2.22}$$

The Dirac operator $D_{\text{n}}(p)$ is called naive fermion. In quantum theory, a propagator of the fermion is an inverse of the Dirac operator. The physical degrees of freedom of fermions appear on the poles preserving the the following dispersion relation,

$$\begin{aligned} |aD_{\text{n}}(p)|^2 &= \sum_\mu \sin^2(ap_\mu) + (am)^2 \\ &= 0. \end{aligned} \tag{2.23}$$

In massless case, we can find the following 16 solutions for the dispersion relation,

$$\begin{aligned}(p_1,p_2,p_3,p_4) &= (0,0,0,0),(0,0,0,\pi/a),(0,0,\pi/a,0),\cdots,\\ &\quad(\pi/a,\pi/a,0,\pi/a),(\pi/a,\pi/a,\pi/a,0),(\pi/a,\pi/a,\pi/a,\pi/a).\end{aligned} \tag{2.24}$$

Therefore, the naive fermion generates 2^4 times spectra in quantum theory even if we define only a fermion in classical theory.

The doublers affect physics of chiral symmetry, e.g. chiral anomaly. To see that simply, we see chirality of the doublers. We define the chiral matrix as follows:

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \tag{2.25}$$

Now, we expand the Dirac operator around momentum $(0,0,0,\pi/a)$ and take continuum limit,

$$S^{\text{cont.}}_{(0,0,0,\pi/a)} = \int_{-\infty}^{\infty}\frac{d^4p}{(2\pi)^4}\bar{\psi}(-p)\left[i\sum_{i=1}^{3}p_i\gamma_i + ip_4(-\gamma_4)+m\right]\psi(p). \tag{2.26}$$

Similarly to Eq.(2.25), we can define the chiral matrix of the continuum fermion Eq.(2.26),

$$\tilde{\gamma}_5 = \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_4,$$

where $\tilde{\gamma}_i = \gamma_i$ for $i=1,2,3$ and $\tilde{\gamma}_4 = -\gamma_4$. Hence,

$$\tilde{\gamma}_5 = -\gamma_5. \tag{2.27}$$

The Eq.(2.27) shows that the doubler which appears on the pole $(0,0,0,\pi)$ has an opposite chiral charge to the fermion appearing on the pole $(0,0,0,0)$. From the same argument, half numbers of doubler have the same chiral charge and the others have an opposite chiral charge. This fact shows that the naive fermion does not cause chiral anomaly in the U(1) gauge theory because the anomaly is canceled by the doublers.

From here, we shortly refer to Nielsen-Ninomiya theorem [2]-[4]. The statement of the theorem, which is proven in free theory, is that a single pole fermion cannot simultaneously preserve all of the symmetries or properties :

- Translation invariance

- Chiral symmetry
- Locality
- (γ_5-)hermiticity

where the symmetries and properties are defined as follows:

$$S^{\mathrm{cont.}} = \int d^4x \bar{\psi}(x) D(x) \psi(x) \quad \rightarrow \quad S^{\mathrm{lat.}} = a^4 \sum_{n,m} \bar{\psi}(na) D(n,m) \psi(ma),$$

Translation invariance:

$$D(n,m) \;\; = \;\; D(n-m), \tag{2.28}$$

Chiral symmetry:

$$\{D(n,m), \gamma_5\} \;\; = \;\; 0, \tag{2.29}$$

Locality:

$$n - m \;\; < \;\; \infty, \tag{2.30}$$

γ_5-hermiticity:

$$\gamma_5 D^\dagger(n,m) \gamma_5 \;\; = \;\; D(n,m), \tag{2.31}$$

There are some proof of the theorem, but we do not mention detail here. Some single pole fermions have been formulated to reduce doublers. For example, lattice fermion using the forward difference operator breaks γ_5-hermiticity,

$$aD_{\mathrm{f}}(p) = \sum_\mu \left[\exp(ip_\mu a) - 1\right] \gamma_\mu, \tag{2.32}$$

the Wilson fermion breaks chiral symmetry,

$$aD_{\mathrm{W}}(p) = \sum_\mu \left[i \sin(p_\mu a) \gamma_\mu + \kappa(1 - \cos(p_\mu a))\right], \tag{2.33}$$

where κ is a dimensionless hopping parameter, and the SLAC fermion breaks locality,

$$aD_{\mathrm{SLAC}}(p) = i\sum_{\mu} ap_\mu\gamma_\mu, \tag{2.34}$$

and so on.

In many numerical simulations, the Wilson or SLAC fermion is often used because the other symmetries often seem to be more important than chiral symmetry. Though the Wilson term, which is the second term of the Eq.(2.33), vanishes in the classical continuum limit, the chiral symmetry is broken in quantum theory even if in the massless theory. To preserve chiral symmetry, we must perform fine-tuning of chiral broken relevant and marginal operators, which have canonical dimensions less than and equal to space-time dimensions, respectively. In general, there are the more numbers of relevant and marginal operators in higher dimensions. Thus, we need to perform many simulations to determine fine-tuning parameters numerically as functions depend on lattice spacing. On the other hand, Ginsparg and Wilson discovered a relationship called the Ginsparg-Wilson relation [11]. And some Dirac operators, which is called as the overlap fermions, formulated based on the relationship. The overlap fermions automatically restore chiral symmetry without fine-tuning in the quantum continuum limit. However, we have to pay a great cost for numerical simulations to analyze theories using the fermion. Because of that, for a long time, exact chiral symmetric fermions have been looked for.

2.3 Minimal doubling fermion

In this subsection, we briefly review minimal doubling fermions [14]-[17]. We firstly define minimal doubling fermion in two-dimensions, then in four-dimensions.

We define the kinetic terms of naive action(NA) and two minimal doubling actions(MDAs) in two-dimensional momentum space as follows [3] :

$$S_{\mathrm{kin}} = \int \frac{d^2p}{(2\pi)^2}\bar{\psi}(-p)D(p)\psi(p), \tag{2.35}$$

where the subscript "kin" means a kinetic term, and

[3] We fix the lattice spacing as $a = 1$.

$$D(p) = \begin{cases} \sum_{\mu=1,4} i \sin p_\mu \gamma_\mu & \equiv D_{\mathrm{n}}(p) \\ i(\sin p_1 + \cos p_4 - 1)\gamma_1 + i(\sin p_4 + \cos p_1 - 1)\gamma_4 & \equiv D_{\mathrm{md1}}(p) \\ i(\sin p_1 + \cos p_4 - 1)\gamma_1 + i \sin p_4 \gamma_4 & \equiv D_{\mathrm{md2}}(p) \end{cases} . \tag{2.36}$$

The subscripts 1 and 4 mean the space and time components respectively. D_{md1} and D_{md2} are called the "twisted ordering action" and the "dropped twisted ordering action" respectively [4].

In two dimensions, the NA has four zero-modes and the MDAs have two, which appear in the following momenta:

$$\begin{aligned} D_{\mathrm{n}} &: \ \tilde{p} = (0,0),\ (0,\pi),\ (\pi,0)\,\mathrm{and}\,(\pi,\pi), \\ D_{\mathrm{md1}} &: \ \tilde{p} = (0,0)\,\mathrm{and}\,(\pi/2,\pi/2), \\ D_{\mathrm{md2}} &: \ \tilde{p} = (0,0)\,\mathrm{and}\,(0,\pi). \end{aligned} \tag{2.37}$$

The doublers appear around each zero-mode as $D(p) = D(\tilde{p} + q)$ with $D(\tilde{p}) = 0$, and the spectra are overcounting in observables. In the case of the NA, the half number of doublers have the same chirality and the others have opposite.

In cases of the MDAs, they have opposite chirality to each other. The NA and MDAs have γ_5-hermiticity:

$$\gamma_5 D(p) \gamma_5 = D^\dagger(p). \tag{2.38}$$

For massless fermions, they also have chiral symmetry:

$$\gamma_5 D(p) + D(p)\gamma_5 = 0. \tag{2.39}$$

The MDAs break (hyper-)cubic symmetry and some discrete symmetries. We define charge conjugation(C), parity transformation(P) and time reflection(T) acting on a

[4] We have another choice of D_{md2} action, $D_{\mathrm{md2}}(p) = i(\sin p_4 + \cos p_1 - 1)\gamma_4 + i \sin p_1 \gamma_1$. This action does not have CP and T symmetry but has CT and P symmetries. In addition, this action cannot be proved reflection symmetry, or reflection positivity. We can apply the same argument in this thesis to another D_{md2} [21]

Table 1: The discrete symmetries for the NA and the MDAs

	C	P	T	CP	CT	PT	CPT
naive	○	○	○	○	○	○	○
md1	×	×	×	×	×	×	○
md2	×	×	○	○	×	×	○

fermion kinetic term [5]:

$$\begin{aligned} \mathrm{C} &: \quad D(p_1,p_4) = -CD^{\top}(-p_1,-p_4)C^{-1}, \\ \mathrm{P} &: \quad D(p_1,p_4) = \gamma_4 D(-p_1,p_4)\gamma_4, \\ \mathrm{T} &: \quad D(p_1,p_4) = \gamma_1 D(p_1,-p_4)\gamma_1, \end{aligned} \tag{2.40}$$

where C is a charge conjugation matrix.

We present these symmetric properties of the NA and MDAs in Tab 1 [6].

Finally, we extend the minimal doubling fermions to four dimensional ones.

$$D^{(4)}_{\mathrm{md1}}(p) = i\sum_{\mu}\sin p_\mu\gamma_\mu + i\sum_{\mu}(\cos p_{\mu\,(\mathrm{mod}.4)+1} - 1)\gamma_{\mu\,(\mathrm{mod}.4)+1}, \tag{2.41}$$

$$D^{(4)}_{\mathrm{md2}}(p) = i\sum_{\mu}\sin p_\mu\gamma_\mu + i\sum_{i=1}^{3}(\cos p_i - 1)\gamma_4 \tag{2.42}$$

These fermions have only two poles at the following momenta,

$$\begin{aligned} D^{(4)}_{\mathrm{md1}} &: \quad \tilde{p} = (0,0,0,0)\,\mathrm{and}\,(\pi/2,\pi/2,\pi/2,\pi/2), \\ D^{(4)}_{\mathrm{md2}} &: \quad \tilde{p} = (0,0,0,0)\,\mathrm{and}\,(0,0,0,\pi). \end{aligned} \tag{2.43}$$

The four dimensional minimal doubling fermions have the same symmetries as the two dimensional ones.

The minimal doubling fermions break discrete symmetries and (hyper)cubic symmetry. In quantum theory, we must perform fine-tuning parameters to restore the broken symmetries. In gauge theory with the twisted ordering action, there are two relevant operators, whose canonical dimension is three, which are clearly needed fine-tuning (same

[5] We can apply the same laws to four-dimensional theory, using $p_1 \to \mathbf{p}$ instead.

[6] "T" represents site and link reflection in lattice space. In the case of D_{md2}, it has link reflection positivity [18].

indices are summed),

$$
\begin{aligned}
\mathcal{O}_3^{(1)} &= \bar{\psi} i \gamma_4 \psi, \\
\mathcal{O}_3^{(2)} &= \bar{\psi} \gamma_4 \gamma_5 \psi,
\end{aligned}
$$

and eight marginal operators, whose canonical dimension is four,

$$
\begin{aligned}
\mathcal{O}_4^{(1)} &= \bar{\psi} D_\mu \gamma_4 \psi, \\
\mathcal{O}_4^{(2)} &= \bar{\psi} D_4 \gamma_4 \psi, \\
\mathcal{O}_4^{(3)} &= \bar{\psi} i D_\mu \gamma_\mu \gamma_5 \psi, \\
\mathcal{O}_4^{(4)} &= \bar{\psi} i D_4 \gamma_4 \gamma_5 \psi, \\
\mathcal{O}_4^{(5)} &= F_{\mu\nu} F_{\mu\nu}, \\
\mathcal{O}_4^{(6)} &= F_{\mu\nu} \tilde{F}_{\mu\nu}, \\
\mathcal{O}_4^{(7)} &= F_{\mu 4} F_{\mu 4}, \\
\mathcal{O}_4^{(8)} &= F_{\mu 4} \tilde{F}_{\mu 4},
\end{aligned}
$$

where the $\tilde{F}_{\mu\nu}$ is a magnetic field, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. On the other hand, there are a relevant operator in the case of the dropped twisted ordering action,

$$
\mathcal{O}_3^{(1)} = \bar{\psi} i \gamma_4 \psi,
$$

and four marginal operators,

$$
\begin{aligned}
\mathcal{O}_4^{(1)} &= \bar{\psi} i D_\mu \gamma_\mu \psi, \\
\mathcal{O}_4^{(2)} &= \bar{\psi} i D_4 \gamma_4 \psi, \\
\mathcal{O}_4^{(3)} &= F_{\mu\nu} F_{\mu\nu}, \\
\mathcal{O}_4^{(4)} &= F_{\mu 4} F_{\mu 4}.
\end{aligned}
$$

We can easily see that there are many operators, which are needed to fine-tune, because several symmetries in the continuum action are broken by the doubler suppressing terms of the minimal doubling fermions. Actually, reducing the numerical time of fine-tuning for these operators is a practical problem in the numerical simulations.

3 γ_5-hermiticity, R-hermiticity and PT symmetry

In this section, we define γ_5-hermiticity, R-hermiticity and PT symmetry in lattice fermion formulation and show how they restrict a kinetic term. γ_5-hermiticity is closely related to sign problem, i.e., the fermion determinant is not a positive value, and it is sometimes used as an hermite condition. R-hermiticity is a classical hermite condition, which is used in e.g., Ref. [7]. In quantum theory, we will show that this condition restricts effective coupling constants to real values in perturbation. PT symmetry is important for a fermion kinetic term and doublers. We will discuss these issues in detail below.

For a concrete discussion, we will focus on only kinetic terms in four dimensions. We can easily extend the following discussion to even dimensions. We define a translation-invariant kinetic term in momentum space as follows (the lattice space $a = 1$):

$$S = \int \frac{d^4k}{(2\pi)^4} \bar{\psi}(-k) D(k) \psi(k), \tag{3.1}$$

with

$$D(k) = \sum_{\mu=1}^{4} f_\mu(k)\gamma_\mu, \tag{3.2}$$

where $f_\mu(k)$ are complex numbers in general and $f(k)_\mu \to ik_\mu$ in the classical continuum limit.

We define γ_5-hermiticity, R-hermiticity and PT symmetry as follows [7]:

$$\gamma_5\text{-hermiticity} \;:\; D(k) = \gamma_5 D^\dagger(k)\gamma_5, \tag{3.3}$$

$$\text{R-hermiticity} \;:\; D(k) = D^\dagger(-k), \tag{3.4}$$

$$\text{PT symmetry} \;:\; D(k) = \gamma_5 D(-k)\gamma_5. \tag{3.5}$$

These conditions are not independent from each other. We can easily derive that, a kinetic term satisfies two of the three conditions is a sufficient condition for that the other condition is automatically satisfied. However, this is not a necessary condition. If $f_\mu(k)$ is pure imaginary, γ_5-hermiticity assures an anti-hermite condition for the kinetic term and a real positive fermion determinant. R-hermiticity is also used as an hermite

[7] In following discussion, we can use C symmetry instead of PT symmetry because of CPT symmetry is satisfied.

condition, e.g. in Ref.[7]; however, it is not suitable because the forward-difference kinetic term, $D_{\text{fd}}(k) = \sum_\mu \left(e^{ik_\mu} - 1\right)\gamma_\mu$, satisfies this condition.

Hence, we will show that R-hermiticity is a condition for real effective coupling constants in perturbation. We assume that a fermion kinetic term has R-hermiticity and that the effective coupling constants have the following form:

$$g_{eff} \;=\; g_0 + \sum_{n=1}^{\infty} I^{(n)}, \tag{3.6}$$

with

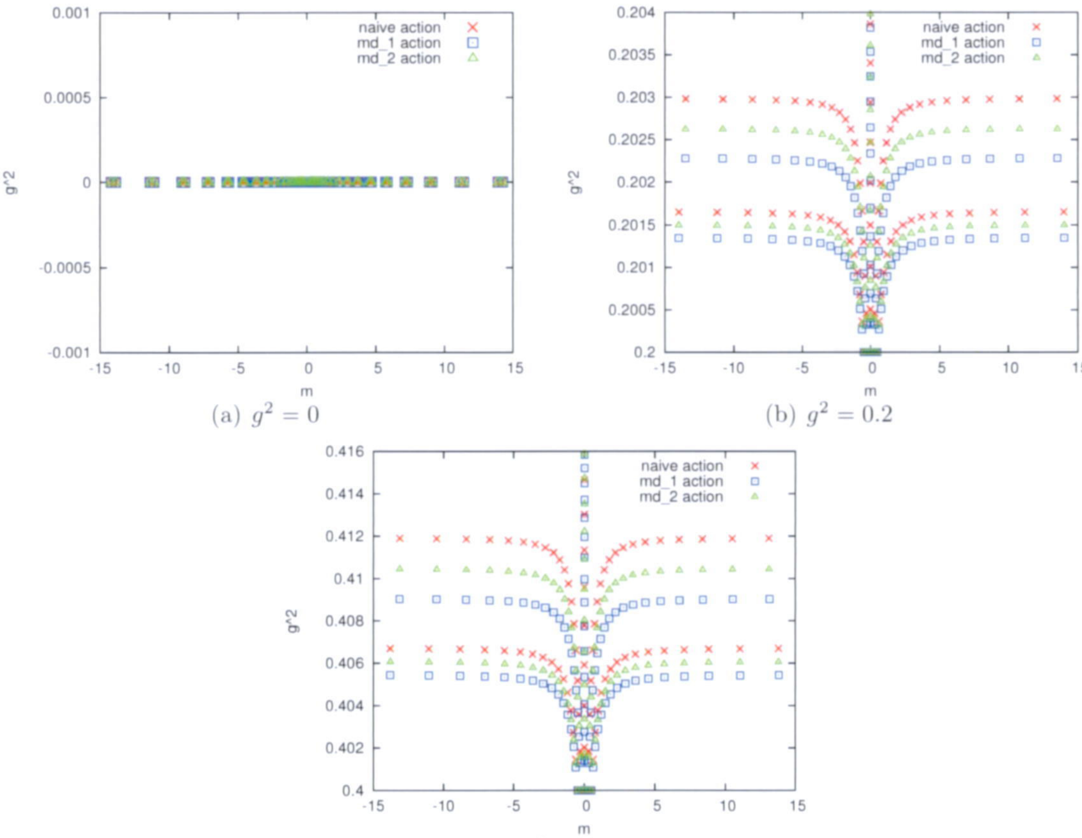
$$I^{(n)} \;=\; \int_{-\pi}^{\pi} \prod_{i=1}^{r} \frac{d^4 k_i}{(2\pi)^4} \cdot I^{(n)}_{\alpha_1\beta_1\cdots\alpha_r\beta_r}(-k_1,\cdots,-k_r) \cdot \prod_{j=1}^{r} S_{\alpha_j\beta_j}(k_j), \tag{3.7}$$

where g_0 is a real bare coupling constant whereas g_{eff} is an effective coupling constant. $S_{\alpha\beta}(k)$ is a fermion propagator and $I^{(n)}$ is the n-loop quantum effect, which is constructed from r-fermion propagators. If hermite conjugate acts on the second term on the r.h.s. of Eq.(3.7), the effective parameter is real if the following condition is satisfied;

$$I^{(n)}_{\alpha_1\beta_1\cdots\alpha_r\beta_r}(-k_1,\cdots,-k_r) = I^{(n)\dagger}_{\beta_1\alpha_1\cdots\beta_r\alpha_r}(k_1,\cdots,k_r), \tag{3.8}$$

where this equation is derived by replacing $S_{\alpha_j\beta_j}(k_j) \to S^\dagger_{\beta_j\alpha_j}(-k_j)$, which is R-hermiticity. If the action is constructed from hermite terms except a fermion kinetic term, this equation is satisfied; hence, Eq.(3.8) is the hermite condition for $I^{(n)}$. Therefore R-hermiticity is a reality or hermite condition for coupling constants as long as Eq.(3.8) is satisfied.

Figures.1-3 show the renormalization group flows(RGFs) of the Gross-Neveu model in two dimensions using the naive action(NA) and the minimal doubling actions(MDAs). This is the simplest model for visualizing complex or non-hermite coupling constants caused by quantum correction. We define MDAs and Gross-Neveu model in subsection 2.3 and appendix A.2 respectively. And we explain how to calculate the Wilsonian RGFs in appendix A.3. Because the MDAs have only γ_5-hermiticity, the mass term has an off-diagonal or complex quantum correction, which is proportional to a γ matrix. A more complicate example is given in Ref.[22]. Using the Wilsonian method [10], we calculate numerically the RGFs for the mass and coupling constant starting from the trivial fixed point, $m = g^2 = 0$. In the case of the MDAs, we use doublers as the different flavor fermions, and, in the case of the NA, we use only two poles, $\tilde{p} = (0,0)$ and (π,π). We



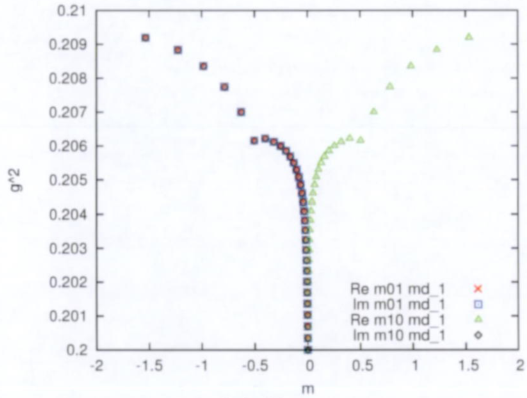
(a) $g^2 = 0$

(b) $g^2 = 0.2$

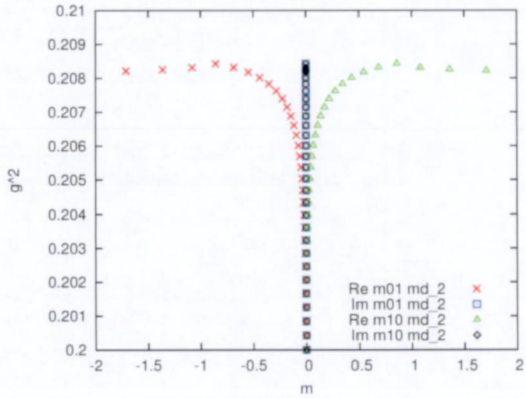
(c) $g^2 = 0.4$

Figure 1: The RGFs of the NA and MDAs. The initial parameters are $m_{11} = m_{22} = m_{33} = m_{44} = 0, \pm 0.25, \pm 0.5, g^2 = (a)0, (b)0.2, (c)0.4$. The RGFs run from the initial conditions toward infrared, which the g^2 are increasing.

represent the spinor indices explicitly, and we distinguish 0, 1 from 2, 3 as different flavors. We assume that high-frequency modes of fields $\psi(1 < |k|)$, $\bar{\psi}(1 < |k|)$, and $\sigma(1 < |k|)$ are not effective, and we neglect their contributions. We choose the initial conditions for the mass to be $m_{00} = m_{11} = m_{22} = m_{33} = 0, \pm 0.25, \pm 0.5$, and for the coupling constant to be $g^2 = 0, 0.2, 0.4$. We set the off-diagonal mass components equal to zero in all cases. We will calculate numerically the one-loop quantum effects and RGFs, which run from



(a) D_{md1}



(b) D_{md2}

Figure 2: The off-diagonal mass and coupling constant of the MDAs, (a)D_{md1}, (b)D_{md2}. The initial parameters are $m_{11} = m_{22} = m_{33} = m_{44} = 0, g^2 = 0.2$. The RGFs run from the initial conditions toward infrared which the g^2 are increasing. The RGFs have irregular forms and the off-diagonal mass components show non-hermiticity.

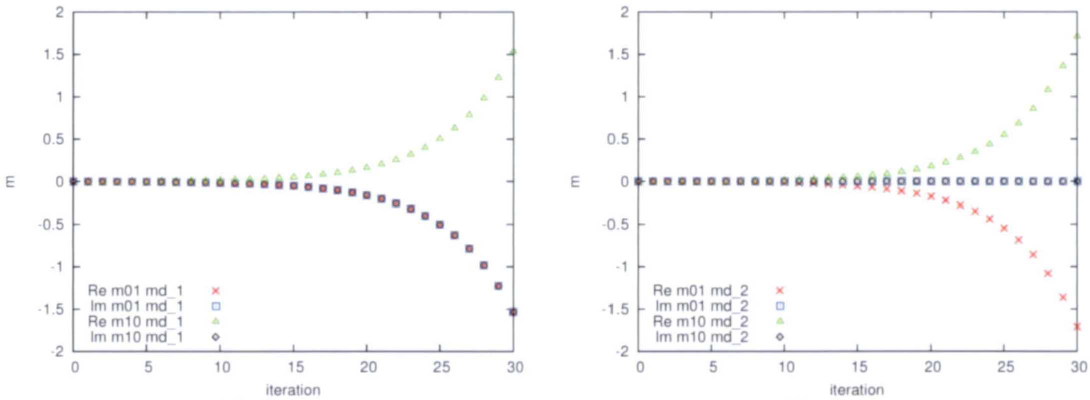
the initial conditions [8]. In our calculation we define γ matrices as follows:

$$\gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.9}$$

The results are shown in Fig.1. The RGFs of the NA and MDAs have similar forms and the difference among each value is $O(10^{-3})$. In the MDA cases, however, off-diagonal mass components (or complex eigenvalues in a diagonal matrix) are generated by the RGFs, except the initial value, which is a trivial fixed point, $m = 0, g^2 = 0$. We show this fact in Fig.2 and 3 with an initial condition of $m = 0, g^2 = 0.2$. Figures.2(a) and (b) show the RGFs of $D_{\mathrm{md1}}(p)$ and $D_{\mathrm{md2}}(p)$, respectively. Figures.3(a) and (b) show the relationship between the off-diagonal mass components and iterations using D_{md1} and D_{md2}, respectively. Though the off-diagonal mass components amplify as the flows approach IR because of the scaling effect, they do not break chiral $\mathbf{Z}_4$ symmetry [9]. These Δm, which is defined as $\Delta m \propto i\gamma_\mu$, are not always complex but off-diagonal. We can choose a γ matrix representation, which diagonalizes one matrix, e.g., $\gamma_1 =$

[8]To estimate integrating part of the one-loop calculation, we used the sectional measurement method. We chose the division length to be $\Delta p_\mu = 0.01$. The error is $O(0.01^2)$ by one iteration.

[9]These generated couplings are essentially different from mass. They do not break chiral symmetry because $\Delta m \propto i\gamma_\mu$. These terms couple $\bar{\psi}_L$ to ψ_L and $\bar{\psi}_R$ to ψ_R.



(a) D_{md1} (b) D_{md2}

Figure 3: The off-diagonal mass and iteration of the MDAs, (a)D_{md1}, (b)D_{md2}. The initial parameters are $m_{11} = m_{22} = m_{33} = m_{44} = 0, g^2 = 0.2$. The off-diagonal mass components are generated and have non-hermiticity.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. In this representation, shifted mass is complex and diagonal, $\Delta m \propto i\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

In principle, these terms can be canceled by counterterms; therefore, we can fine-tune the perturbation [22]. However, nonperturbative analysis is difficult and this problem must be solved in future work.

Next, we will show that PT symmetry is always broken if we add extra kinetic terms to a NA to reduce it to doublers [10].

Statement.

In even dimensions, a PT-symmetric kinetic term with assumed periodicity and continuity always has 2^D or more than 2^D poles.

Proof.

For simplicity, we also assume translation invariance [11]. A general 2π periodic and

[10] This argument has been discussed in Ref.[23], although not mathematically.

[11] We can similarly arrive at same statement without translation invariance. In the case of non-translation invariance, a kinetic term has two-momenta dependence $D(k,p)$, and at least 4^d doublers appear.

continuum $D(k)$ has the following form:

$$D(k) = \sum_{\mu,\nu=1}^{d} \sum_{n \in \mathbf{N}^d}^{\infty} [(A_{\mu\nu}(n) + iB_{\mu\nu}(n)) \cos(n_\nu k_\nu) + (C_{\mu\nu}(n) + iD_{\mu\nu}(n)) \sin(n_\nu k_\nu) + E_{\mu\nu}] \gamma_\mu \quad (3.10)$$

where $A_{\mu\nu}(n), B_{\mu\nu}(n), C_{\mu\nu}(n), D_{\mu\nu}(n)$ are real constants and $E_{\mu\nu}$ are complex constants. From PT symmetry,

$$A_{\mu\nu}(n) = B_{\mu\nu}(n) = E_{\mu\nu} = 0, \quad \text{for all} \quad \mu, \nu, n. \quad (3.11)$$

The $D(k)$ always has two poles at $k = 0$ and π for each dimension. Therefore $D(k)$ has 2^D or more than 2^D poles.

This statement means that we cannot reduce the number of doublers using PT-symmetric kinetic terms [12]. In a numerical simulation context, γ_5-hermiticity is a very important condition to avoid the sign problem. Assuming translation invariance, R-hermiticity is not satisfied if $D(k)$ satisfies γ_5-hermiticity but not PT symmetry. Therefore, the effective parameters have explicit non-hermiticity.

In the process of rewriting from Minkowskian to Euclidean, hermite fermion kinetic terms transmute to anti-hermite ones. In the Minkowski formulation, we forbid non-hermite or complex couplings using the hermite condition. In contrast, the definition of "hermite" in Euclidean space is ambiguous. Though some MDAs have reflection symmetry or reflection positivity, which are equal to the hermite conjugate or unitarity in Minkowski space, these conditions do not properly have hermiticity in quantum theory. Similarly, γ_5-hermiticity is commonly used as an hermite condition, but we cannot forbid non-hermite or complex couplings directly. A kinetic term that reduces the number of doublers allows the possibility of generating these anti-hermite effective coupling constants. R-hermiticity is a criterion to remove non-hermiticity, at least, for 2D Gross Neveu model or models coupling scalar fields.

[12] We cannot apply this statement to the non-γ_μ linear case, e.g., the Wilson fermion.

4 Non γ_5-hermiticity minimal doubling fermions

4.1 2D non-γ_5-hermiticity fermions

In this subsection, we define 2D fermions without γ_5-hermiticity (non-γ_5-hermiticity fermions) based on the minimal doubling fermion and investigate the symmetries and properties of the fermions. The minimal doubling fermions were formulated by Karsten et al. [14]-[17] and do not interfere with the no-go theorem of Nielsen-Ninomiya because two doublers, which are a $\pm$-chiral charge pair, appear. The fermions preserve translation invariance, chiral symmetry, locality and γ_5-hermiticity, but break (hyper)cubic symmetry and some discrete symmetries, e.g. charge conjugation, parity symmetry and so on. We refer the reader to Refs. [14]-[29] for more details on the minimal doubling fermion in detail.

Now, we construct non-γ_5-hermiticity fermions by adding PT symmetry breaking terms because lattice fermions with PT-symmetric doubler-suppressing kinetic terms always generate 2^d doublers [29]. We define five free massless non-γ_5-hermiticity fermions in coordinate space as follows (lattice spacing $a = 1$) [13]:

$$S = \sum_{n,m} \bar{\psi}_n D^{(2)}_{nm} \psi_m,$$

$$D^{(2)}_{1\ nm} = \sum_{\mu=1,4} \frac{1}{2}(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\mu + \frac{\kappa}{2}\left(2\delta_{n,m} - \delta_{n+\hat{1},m} - \delta_{n-\hat{1},m}\right) \cdot \gamma_1, \tag{4.1}$$

$$D^{(2)}_{2\ nm} = \sum_{\mu=1,4} \frac{1}{2}(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\mu + \frac{\kappa}{2}\left(2\delta_{n,m} - \delta_{n+\hat{4},m} - \delta_{n-\hat{4},m}\right) \cdot \gamma_1, \tag{4.2}$$

$$D^{(2)}_{3\ nm} = \sum_{\mu=1,4} \frac{1}{2}(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\mu + \frac{\kappa}{2}\sum_{\mu=1,4}(2\delta_{n,m} - \delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\mu, \tag{4.3}$$

$$D^{(2)}_{4\ nm} = \sum_{\mu=1,4} \frac{1}{2}(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\mu + \frac{\kappa}{2}\sum_{\mu,\nu=1,4\ \mu\neq\nu}(2\delta_{n,m} - \delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\nu, \tag{4.4}$$

$$D^{(2)}_{5\ nm} = \sum_{\mu=1,4} \frac{1}{2}(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\mu + \frac{\kappa}{2}\sum_{\mu=1,4}(2\delta_{n,m} - \delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_1, \tag{4.5}$$

[13] We define the non-γ_5-hermiticity fermions up to exchanging a temporal index and a spatial one: In the $D^{(2)}_1$ case, $\bar{D}^{(2)}_1(p_1, p_4) = \sum_{\mu=1,4} i \sin p_\mu \cdot \gamma_\mu + \kappa(1 - \cos p_1) \cdot \gamma_1$. The $D^{(2)}_1$ preserves P symmetry but breaks T symmetry defined below.

and in momentum space,

$$\begin{aligned}
S &= \int \frac{d^2p}{(2\pi)^2}\, \bar{\psi}(-p)D^{(2)}(p)\psi(p), \\
D_1^{(2)}(p) &= \sum_{\mu=1,4} i\ \sin p_\mu \cdot \gamma_\mu + \kappa(1-\cos p_1)\cdot\gamma_1, && (4.6) \\
D_2^{(2)}(p) &= \sum_{\mu=1,4} i\ \sin p_\mu \cdot \gamma_\mu + \kappa(1-\cos p_4)\cdot\gamma_1, && (4.7) \\
D_3^{(2)}(p) &= \sum_{\mu=1,4} i\ \sin p_\mu \cdot \gamma_\mu + \kappa \sum_{\mu=1,4}(1-\cos p_\mu)\cdot\gamma_\mu, && (4.8) \\
D_4^{(2)}(p) &= \sum_{\mu=1,4} i\ \sin p_\mu \cdot \gamma_\mu + \kappa \sum_{\mu,\nu=1,4\ \mu\neq\nu}(1-\cos p_\mu)\cdot\gamma_\nu, && (4.9) \\
D_5^{(2)}(p) &= \sum_{\mu=1,4} i\ \sin p_\mu \cdot \gamma_\mu + \kappa \sum_{\mu=1,4}(1-\cos p_\mu)\cdot\gamma_1, && (4.10)
\end{aligned}$$

where κ is a hopping parameter which is a real number [14]. Note that $D_3^{(2)}$ at $\kappa=-1(+1)$ is just a Dirac operator with a forward(backward) difference operator. Next, we define charge conjugation(C), parity transformation(P) and time reversal(T) as follows:

$$\begin{aligned}
\mathrm{C} &: \quad D(p_1,p_4) \to -CD^\top(-p_1,-p_4)C^{-1}, \\
\mathrm{P} &: \quad D(p_1,p_4) \to \gamma_4 D(-p_1,p_4)\gamma_4, \qquad (4.11) \\
\mathrm{T} &: \quad D(p_1,p_4) \to \gamma_1 D(p_1,-p_4)\gamma_1,
\end{aligned}$$

where C is a charge conjugation matrix. In two dimensions, we can define $C=i\gamma_1$ as a charge conjugation matrix where $\gamma_1=\sigma^2$ and $\gamma_4=\sigma^1$. We also define a Dirac operator preserving chiral symmetry and γ_5-hermiticity as follows:

$$\begin{aligned}
\text{chiral} &: \quad D(p_1,p_4) = -\gamma_5 D(p_1,p_4)\gamma_5, \\
\gamma_5\text{-hermiticity} &: \quad D(p_1,p_4) = \gamma_5 D^\dagger(p_1,p_4)\gamma_5,
\end{aligned}$$

where γ_5 is a chiral matrix defined as $\gamma_5 = i\gamma_1\gamma_4$. All the Dirac operators have preserved chiral symmetry and CPT symmetry but broken C, P, CT and PT symmetries. $D_1^{(2)}$, $D_2^{(2)}$, and $D_5^{(2)}$ also preserve T and CP symmetry, but $D_3^{(2)}$ and $D_4^{(2)}$ do not. $D_1^{(2)}$, $D_2^{(2)}$ and $D_5^{(2)}$ have the discrete symmetries as the Karsten-Wilczek minimal doubling fermions [14, 15], and $D_3^{(2)}$ and $D_4^{(2)}$ have the same ones as the Borici-Creutz ones [16, 17]. We

[14]We regard indices 1 and 4 as temporal and spacial directions, respectively.

	C	P	T	CP	CT	PT	CPT	chi	γ_5-h
$D_1^{(2)}$	×	×	○	○	×	×	○	○	×
$D_2^{(2)}$	×	×	○	○	×	×	○	○	×
$D_3^{(2)}$	×	×	×	×	×	×	○	○	×
$D_4^{(2)}$	×	×	×	×	×	×	○	○	×
$D_5^{(2)}$	×	×	○	○	×	×	○	○	×

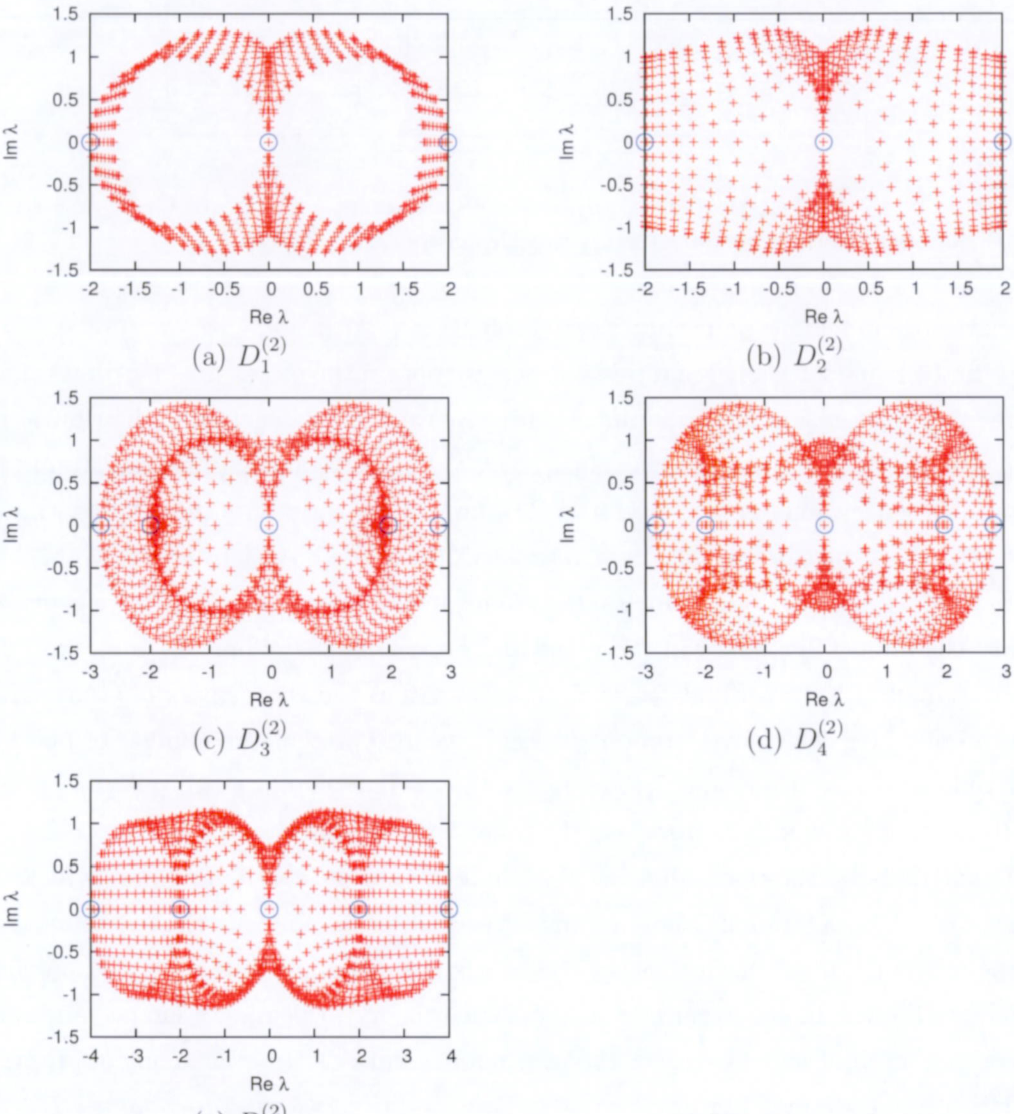
Table 2: The symmetric properties of discrete symmetries, which are C, P, and so on, chiral symmetry and γ_5-hermiticity for the Dirac operators $D_{1-5}^{(2)}$.

summarize their symmetric properties in Table 2.

To find the model application possibilities, we obtain the eigenvalue distribution and the number of doublers of the fermions. Firstly, we present eigenvalue distribution in Fig.4. We clearly see that the eigenvalues of $D_2^{(2)}$, $D_3^{(2)}$, $D_4^{(2)}$ and $D_5^{(2)}$ spread around the origin. The eigenvalues spread over the Reλ-Imλ plane entirely within the continuum limit. This is a typical feature of the sign problem. In the $D_1^{(2)}$ case, by contrast, there are spaces that are enclosed eigenvalues in the plane. In the continuum limit, the eigenvalues tend to the infinity boundary in the plane and are distributed along the Imλ axis. The distribution of $D_1^{(2)}$ in the limit has the same features as the Dirac operators that satisfy γ_5-hermiticity, e.g. the naive fermion. In Fig.5, we also present the number of poles for the fermions at $\kappa = 0.5, 1$, and 2. The figure shows that there are two poles, $p = (0, 0)$ and $(0, \pi)$, in $D_1^{(2)}$ at $\kappa \leq 1$. However, the other fermions have more than two poles.

As noted above, the eigenvalue distribution of $D_1^{(2)}$ in the continuum limit is identical with the continuum fermion, whose eigenvalues are distributed along the imaginary axis. On the other hand, the eigenvalues of the others are distributed on the Reλ-Imλ plane entirely within the limit. According to the lattice theory principles, a lattice fermion in the continuum limit should recover the continuum fermion, $\bar{\psi} ip \cdot \gamma\psi$. Using appropriate regularization, e.g. small mass or anti-boundary condition, we can probably use $D_{2-5}^{(2)}$ to analyze a model. By the lattice principle, however, the Dirac operators $D_{2-5}^{(2)}$ should be excluded from the candidates for application to a practical calculation.

According to the analysis of the number of poles, there are generally an odd number or more than four poles. The intuitive reason for odd or many doublers is that the dispersion relations of the fermions are complex. For example, in the $D_2^{(2)}$ case, the



(a) $D_1^{(2)}$ (b) $D_2^{(2)}$ (c) $D_3^{(2)}$ (d) $D_4^{(2)}$ (e) $D_5^{(2)}$

Figure 4: The eigenvalue distribution of Dirac operators (a) $D_1^{(2)}$, (b) $D_2^{(2)}$, (c) $D_3^{(2)}$, (d) $D_4^{(2)}$ and (e) $D_5^{(2)}$. The horizontal and vertical axes denote the real and the imaginary parts of eigenvalues, respectively. The hopping parameter, the fermion mass and lattice size are fixed at $\kappa = 1$, $m = 0$, and $L^2 = 36 \times 36$. The blue circle points denote eigenvalues for momenta $p = (0, 0)$, $(0, \pi)$, $(\pi, 0)$, and (π, π).

dispersion relation is derived as follows:

$$\sum_{\mu=1,4} \sin^2 p_\mu - \kappa^2(1-\cos p_4)^2 - 2i\kappa \sin p_1(1-\cos p_4) = 0. \tag{4.12}$$

The first term on the l.h.s. has an opposite signature to the second term. This can easily lead to six real solutions for Eq.(4.12),

$$\begin{aligned}(p_1,p_4) &= (0,0),(\pi,0),\\ &\quad (0,4\tan^{-1}[\kappa+\sqrt{1+\kappa^2}]),(\pi,4\tan^{-1}[\kappa+\sqrt{1+\kappa^2}]),\\ &\quad (0,-4\tan^{-1}[\kappa-\sqrt{1+\kappa^2}]),(\pi,-4\tan^{-1}[\kappa-\sqrt{1+\kappa^2}]).\end{aligned} \tag{4.13}$$

In a similar way, the solutions for $D_3^{(2)}$ at $\kappa=-1$, which is the forward difference fermion, are obtained as

$$(p_1,p_4) = (0,0),(\pi/2,-\pi/2),(-\pi/2,\pi/2). \tag{4.14}$$

Now, we study the relationship between doublers and γ_5-hermiticity in detail. We define a Dirac operator $D(p)$ which is assumed by continuity, periodicity, translation invariance, locality, and γ_μ-linear as follows:

$$\begin{aligned} D(p) &= \sum_{\mu,\nu,n_{\mu\nu}} \left[A_{\mu\nu n_{\mu\nu}} \sin(n_{\mu\nu}p_\mu) + B_{\mu\nu n_{\mu\nu}} \cos(n_{\mu\nu}p_\mu)\right]\cdot\gamma_\nu \\ &\equiv \sum_\nu \mathcal{D}_\nu(p)\cdot\gamma_\nu, \end{aligned} \tag{4.15}$$

where $A_{\mu\nu n_{\mu\nu}}, B_{\mu\nu n_{\mu\nu}}$ are complex constant numbers and $n_{\mu\nu}\in\mathbb{N}+\{0\}$. We assume that $D(p)=0$ at $p=\tilde{p}$. From the Taylor expansion around $\tilde{p}$,

$$\begin{aligned} D(\tilde{p}+\delta p) &= \sum_{\mu,\nu,n_{\mu\nu}} \left[A_{\mu\nu n_{\mu\nu}} \cos(n_{\mu\nu}\tilde{p}_\mu) - B_{\mu\nu n_{\mu\nu}} \sin(n_{\mu\nu}\tilde{p}_\mu)\right] n_{\mu\nu}\delta p_\mu\cdot\gamma_\nu + O(\delta p^2) \\ &= \sum_{\mu,\nu} \left.\frac{\partial\mathcal{D}_\nu(p)}{\partial p_\mu}\right|_{p=\tilde{p}} \delta p_\mu\cdot\gamma_\nu + O(\delta p^2). \end{aligned} \tag{4.16}$$

If we take the continuum limit, only the δp-linear terms survive and are imposed by the

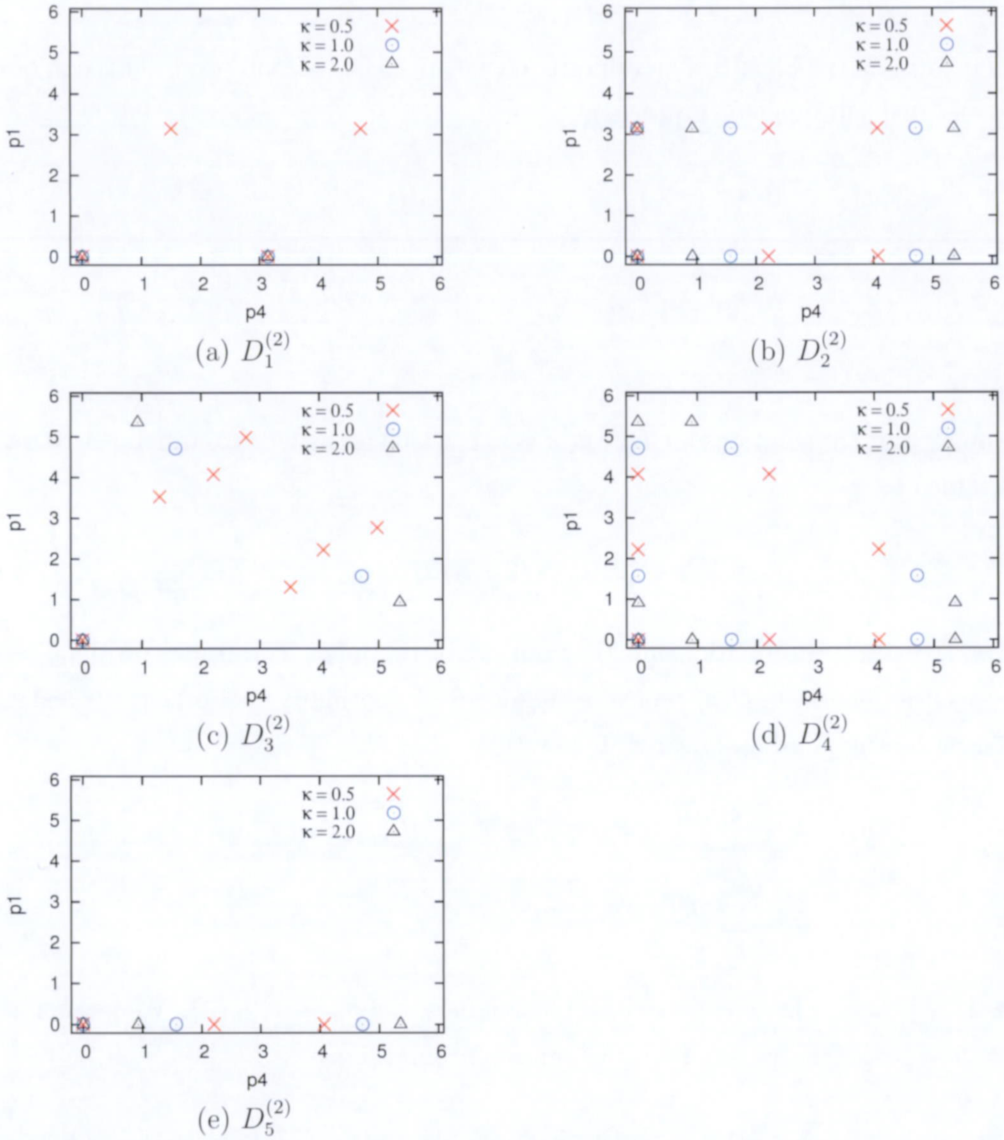


Figure 5: The pole distribution of five Dirac operators (a) $D_1^{(2)}$, (b) $D_2^{(2)}$, (c) $D_3^{(2)}$, (d) $D_4^{(2)}$, and (e) $D_5^{(2)}$. The horizontal and vertical axes denote p_4 and p_1 respectively. The cross points, circle points and triangle points represent poles at $\kappa = 0.5, 1$, and 2, respectively.

following condition,

$$\sum_{\mu} \frac{\partial \mathcal{D}_\nu(p)}{\partial p_\mu}\bigg|_{p=\tilde{p}} \delta p_\mu \neq 0 \quad \text{for any } \nu, \tag{4.17}$$

because if this condition is not satisfied, $D(p)$ has no excitation modes or flat directions around $p = \tilde{p}$. From Eq.(4.17) and the intermediate-value theorem, we can derive that $D(p)$ generates even doublers which are pairs of $\pm$-chirality. However, this derivation is insufficient where $D(p)$ is not satisfied with γ_5-hermiticity because its poles appear at not only $D(\tilde{p}) = 0$. If we assume γ_5-hermiticity to $D(p)$, the complex constants $A_{\mu\nu n_{\mu\nu}}, B_{\mu\nu n_{\mu\nu}}$ are pure imaginary. Hence $D^2(p) = 0$ is a necessary and sufficient condition for $D(p) = 0$; namely, $\mathcal{D}_\nu(p)$ at $p = \tilde{p}$ satisfies a dispersion relation if and only if $\mathcal{D}_\nu(\tilde{p}) = 0$ for any ν. Therefore, $D(p)$ always generates even doublers which are $\pm$-chiral pairs, i.e. in the framework of Nielsen-Ninomiya theorem [2]-[4]. On the other hand, we can obtain the Taylor expansion of $D_3^{(2)}(\tilde{p} + \delta p)$, which does not satisfy with γ_5-hermiticity, around $\tilde{p} = (\pi/2, -\pi/2)$ as

$$D_3^{(2)}(\tilde{p} + \delta p) = (-\delta p_1 + i - 1) \cdot \gamma_1 + (\delta p_4 - i - 1) \cdot \gamma_4 + O(\delta p^2). \tag{4.18}$$

We can see that $D_3^{(2)}(\tilde{p})$ is not equal to zero but the squared $D_3^{(2)}(\tilde{p})$ is. In addition, the Eq.(4.18) does not approach the continuum fermion, $D_c(p) = i\sum_\mu p_\mu \gamma_\mu$, in the continuum limit. From this discussion, we can state that non-γ_5-hermiticity affects not only odd doublers but also non-trivial doublers, which appear at $D(\tilde{p}) \neq 0$ and $D^2(\tilde{p}) = 0$. Additionally the non-trivial doublers do not approach an appropriate continuum limit.

Finally, we prove the reflection positivity for Dirac operators defined as follows:

$$\bar{D}^{(2)}_{1\ nm} = \sum_{\mu=1,4} \frac{1}{2}\left(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}\right) \cdot \gamma_\mu + \frac{\kappa}{2}\left(2\delta_{n,m} - \delta_{n+\hat{4},m} - \delta_{n-\hat{4},m}\right) \cdot \gamma_4, \tag{4.19}$$

$$\bar{D}^{(2)}_{2\ nm} = \sum_{\mu=1,4} \frac{1}{2}\left(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}\right) \cdot \gamma_\mu + \frac{\kappa}{2}\left(2\delta_{n,m} - \delta_{n+\hat{1},m} - \delta_{n-\hat{1},m}\right) \cdot \gamma_4, \tag{4.20}$$

$$\bar{D}^{(2)}_{5\ nm} = \sum_{\mu=1,4} \frac{1}{2}\left(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}\right) \cdot \gamma_\mu + \frac{\kappa}{2} \sum_{\mu=1,4} \left(2\delta_{n,m} - \delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}\right) \cdot \gamma_4, \tag{4.21}$$

The reflection positivity is the unitarity condition in Euclidean space. Although there are two kinds of reflection positivity, site-reflection and link-reflection, we prove only the link-reflection positivity from now on [15]. Now, we prove it in the only $\bar{D}^{(2)}_1$ case. We can also prove the link-reflection positivity of $\bar{D}^{(2)}_2$ and $\bar{D}^{(2)}_5$ in way similar way, described below. Here, we define an anti-linear mapping Θ acting on the fermions as follows:

$$\Theta(\psi_{n_1,n_4}) = \bar{\psi}_{n_1,1-n_4}\gamma_4, \quad \Theta(\bar{\psi}_{n_1,n_4}) = \gamma_4\psi_{n_1,1-n_4}, \tag{4.22}$$

and on fermion bilinear,

$$\begin{aligned} \Theta(\bar{\psi}_{n_1,n_4}\Gamma\psi_{m_1,m_4}) &= \Theta(\psi_{m_1,m_4})\Gamma^\dagger\Theta(\bar{\psi}_{n_1,n_4}) \\ &= \bar{\psi}_{m_1,1-m_4}\gamma_4\Gamma^\dagger\gamma_4\psi_{n_1,1-n_4}, \end{aligned} \tag{4.23}$$

where Γ is an arbitrary function depending on the γ-matrix. Let us denote the fermions in the half-space with $n_4 \geq 1$ by $\psi^{(+)}$ and $\bar{\psi}^{(+)}$, and in the other half-space $n_4 \leq 0$ by $\psi^{(-)}$ and $\bar{\psi}^{(-)}$. According to the above notation, the action Eq.(4.1) can be written as

$$S = S_+[\psi^{(+)}, \bar{\psi}^{(+)}] + S_-[\psi^{(-)}, \bar{\psi}^{(-)}] + S_c[\psi^{(+)}, \bar{\psi}^{(+)}, \psi^{(-)}, \bar{\psi}^{(-)}], \tag{4.24}$$

where

$$\begin{aligned} S_c &= \frac{1-\kappa}{2}\bar{\psi}^{(-)}_{n_1,0}\gamma_4\psi^{(+)}_{n_1,1} - \frac{1+\kappa}{2}\bar{\psi}^{(+)}_{n_1,1}\gamma_4\psi^{(-)}_{n_1,0} \\ &= \frac{1-\kappa}{2}\bar{\psi}^{(-)}_{n_1,0}\Theta(\bar{\psi}^{(-)}_{n_1,0}) + \frac{1+\kappa}{2}\psi^{(-)\top}_{n_1,0}\Theta(\psi^{(-)\top}_{n_1,0}), \end{aligned} \tag{4.25}$$

S_+ depends only on the fermions in positive time, and related with S_- as

$$\begin{aligned} \Theta(S_+[\psi^{(+)}, \bar{\psi}^{(+)}]) &= S^\dagger_+[\Theta(\psi^{(+)}), \Theta(\bar{\psi}^{(+)})]) \\ &= S_-[\psi^{(-)}, \bar{\psi}^{(-)}]. \end{aligned} \tag{4.26}$$

For proof of the reflection positivity, we must show $\langle\Theta(F)F\rangle \geq 0$, where F is an arbitrary

[15] The procedure for proof of the site-reflection positivity is similar to the case of link-reflection, but we cannot prove the site-reflection positivity.

function depending on positive time fermions, $\psi^{(-)}$ and $\bar{\psi}^{(-)}$. $\langle\Theta(F)F\rangle$ can be written as follows:

$$\begin{aligned}\langle\Theta(F)F\rangle &= Z^{-1}\int D\bar{\psi}^{(-)}D\psi^{(-)}F[\psi^{(-)},\bar{\psi}^{(-)}]e^{-S_{-}[\psi^{(-)},\bar{\psi}^{(-)}]}\\ &\quad\cdot\int D\bar{\psi}^{(+)}D\psi^{(+)}F^{\dagger}[\psi^{(+)},\bar{\psi}^{(+)}]e^{-S_{+}[\psi^{(+)},\bar{\psi}^{(+)}]}\cdot e^{-S_c}. \end{aligned}\tag{4.27}$$

From Eq.(4.25) and demanding that S_c is positive, the expectation value $\langle\Theta(F)F\rangle$ is positive and converge for every $-1\le\kappa\le 1$.

We can prove the link-reflection positivity for $\bar{D}_2^{(2)}$ at any κ and $\bar{D}_5^{(2)}$ at $-1\le\kappa\le 1$ from the same discussion.

4.2 Gross-Neveu model in two dimensions

In this section, we apply the non-γ_5-hermiticity fermion defined in Subsect.4.1 to the 2D Gross-Neveu model and draw phase diagrams [32]. The model is investigated as a toy model for QCD and can be solved exactly in the large-N limit using the saddle point approximation. In this limit, we can also obtain the parity broken phase diagram, called Aoki phase [33]-[43].

Now, we apply the Dirac operator $D_1^{(2)}$ in Eq.(4.1) and study the phase diagrams. At first sight, the fermion seems to be unsuitable for a practical calculation because it does not preserve γ_5-hermiticity. These analyses are simple tests for application of the fermion to a concrete model. In this paper, we analyze two cases, (1) parity symmetry breaking and (2) chiral symmetry breaking, which is caused in the model with an imaginary chemical potential. We also compare the phase diagrams using the fermion with those using the naive fermion. The chiral broken phase diagrams are studied in the next section.

Here, we define the continuum Gross-Neveu model in two dimensions as follows,

$$S_{\mathrm{c}}^{\mathrm{GN}}=\int d^2x\left[\bar{\psi}\left(\partial\cdot\gamma+m\right)\psi-\frac{g_\sigma^2}{2N}\left(\bar{\psi}\psi\right)^2-\frac{g_\pi^2}{2N}\left(i\bar{\psi}\gamma_5\psi\right)^2\right],\tag{4.28}$$

where we omit the flavor indices, $\bar{\psi}\psi\equiv\sum_{i=1}^{N}\bar{\psi}^i\psi^i$. In this action, the fermions have imposed $\mathrm{U_V}(1)$ symmetry:

$$\begin{aligned}\psi &\rightarrow e^{i\theta}\,\psi,\\ \bar{\psi} &\rightarrow \bar{\psi}\,e^{-i\theta}.\end{aligned} \tag{4.29}$$

In massless case, the fermions preserve continuum chiral symmetry for $g_\sigma^2 = g_\pi^2$ and chiral $\mathbf{Z}_4$ symmetry for $g_\sigma^2 \neq g_\pi^2$. Then, we introduce an auxiliary scalar field σ and an auxiliary pseudo-scalar field π. The partition function in the continuum theory is defined as follows,

$$Z_{\mathrm{c}} = \int D\psi D\bar{\psi} D\sigma D\pi\; e^{-S^{\mathrm{GN}}_{\mathrm{c,aux}}}, \tag{4.30}$$

where

$$S^{\mathrm{GN}}_{\mathrm{c,aux}} = \int d^2x \left[\bar{\psi}\,(\,\partial\cdot\gamma + m + \sigma + \pi i\gamma_5\,)\,\psi + \frac{N}{2g_\sigma^2}\sigma^2 + \frac{N}{2g_\pi^2}\pi^2\right]. \tag{4.31}$$

Integrating out the auxiliary fields σ and π, the action (4.31) recovers the former action, Eq.(4.28), from the following relations:

$$\begin{aligned}\frac{N}{g_\sigma^2}\sigma &= -\bar{\psi}\psi,\\ \frac{N}{g_\pi^2}\pi &= -i\bar{\psi}\gamma_5\psi.\end{aligned} \tag{4.32}$$

Next, we define a lattice action from the continuum action. We choose Dirac operator as $\bar{D}_1^{(2)}$, defined in Eq.(4.19). Note that the Dirac operator $D_1^{(2)}$ is not suitable for our purpose because we study the parity broken phase diagrams. One difference between $\bar{D}_1^{(2)}$ and $D_1^{(2)}$ is the components in the doubler suppressing terms, temporal or spatial. $\bar{D}_1^{(2)}$ preserves parity symmetry but not time-reversal symmetry. Hence, $D_1^{(2)}$, which has broken parity symmetry, is not fit for our purpose. We define the lattice action of the Gross-Neveu model as follows:

$$S^{\mathrm{GN}}_{\mathrm{lat,aux}} = \sum_{n,m}\bar{\psi}_n\left[\,\bar{D}^{(2)}_{1\;nm} + (m + \sigma_n + \pi_n i\gamma_5\,)\,\delta_{n,m}\right]\psi_m + \frac{N}{2}\sum_n\left[\frac{\sigma_n^2}{g_\sigma^2} + \frac{\pi_n^2}{g_\pi^2}\right]. \tag{4.33}$$

where the lower indices denote a coordinate in lattice space. Integrating out the fermion, we can obtain an effective action $S_{\rm eff}$:

$$Z = \int D\sigma_n D\pi_n e^{-NS_{\rm eff}(\sigma,\pi)}, \tag{4.34}$$

$$S_{\rm eff}(\sigma,\pi) = \frac{1}{2}\sum_n \left[\frac{\sigma_n^2}{g_\sigma^2}+\frac{\pi_n^2}{g_\pi^2}\right] - {\rm Tr}\ \log D, \tag{4.35}$$

where D is defined as follows:

$$D_{nm} = \bar{D}^{(2)}_{1\ nm} + (m+\sigma_n+\pi_n i\gamma_5\ )\,\delta_{n,m}. \tag{4.36}$$

In the large-N limit, we can integrate out the auxiliary field σ and π from the partition function using the saddle point approximation. The solutions $\bar{\sigma}_n, \bar{\pi}_n$ are given by the saddle point condition,

$$\frac{\delta S_{\rm eff}(\sigma,\pi)}{\delta\sigma_n} = \frac{\delta S_{\rm eff}(\sigma,\pi)}{\delta\pi_n} = 0. \tag{4.37}$$

We impose translation invariance, $\bar{\sigma}_n = \sigma_0$ and $\bar{\pi}_n = \pi_0$ for any n; the partition function is written as

$$Z = e^{-V\cdot V_{\rm eff}(\sigma_0,\pi_0)}, \tag{4.38}$$

$$V_{\rm eff} = \frac{1}{2}\left[\frac{\sigma_0^2}{g_\sigma^2}+\frac{\pi_0^2}{g_\pi^2}\right] - \frac{1}{V}{\rm Tr}\ \log D, \tag{4.39}$$

where V is the volume of the system. The last term in Eq.(4.39) is obtained using Fourier transformation:

$$\begin{aligned} {\rm Tr}\ \log D &= V\cdot\int\frac{d^2k}{(2\pi)^2}\ \log\left[(m+\sigma_0)^2+\pi_0^2+H(k)\right], \\ &\equiv V\cdot\int\frac{d^2k}{(2\pi)^2}\ \mathcal{E}(k), \end{aligned} \tag{4.40}$$

where

$$H(k) = \sin^2 k_1 + \left[\sin k_4 - i\kappa(1-\cos k_4)\right]^2. \tag{4.41}$$

We can easily see that Eq.(4.40) is real because $\mathcal{E}(k)$ is a k_1-even function and the sum

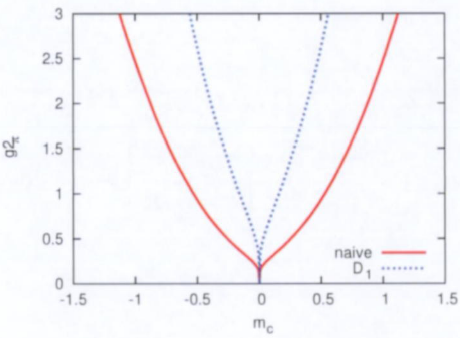


Figure 6: The phase diagram of Aoki phase using the naive fermion (red line) and the non-γ_5-hermiticity fermion $\bar{D}_1^{(2)}$ (blue line) with $g_\sigma^2 = g_\pi^2/2$. We fix the hopping parameter of the non-γ_5-hermiticity fermion at $\kappa = 1$. The horizontal and vertical axes denote the critical mass m_c and the squared four-fermi coupling constant g_π^2, respectively. The center and both sides in this diagram are parity broken and symmetric phase respectively.

of $\mathcal{E}(k_1, k_4)$ with $\mathcal{E}(k_1, -k_4)$ is real:

$$
\begin{aligned}
&\mathcal{E}(k_1, k_4) + \mathcal{E}(k_1, -k_4) \\
= \ &\log\left[\left\{(m+\sigma_0)^2 + \pi_0^2 + \sum_{\mu=1,4} \sin^2 k_\mu - \kappa^2(1-\cos k_4)^2\right\}^2 \right. \\
&\left. + \left\{2\kappa \sin k_4(1-\cos k_4)\right\}^2\right]. \qquad (4.42)
\end{aligned}
$$

According to the saddle point condition (4.37), we can obtain the following equation:

$$
\frac{\sigma_0}{g_\sigma^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2\sigma_m}{\sigma_m^2 + \pi_0^2 + H(k)}, \qquad (4.43)
$$

$$
\frac{\pi_0}{g_\pi^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2\pi_0}{\sigma_m^2 + \pi_0^2 + H(k)}, \qquad (4.44)
$$

where $\sigma_m = m + \sigma_0$. π_0 is an order parameter for parity symmetry breaking; hence, π_0 approaches zero near the critical line in the parameter space. Near the line ,we can derive the following gap equations:

$$
\frac{\sigma_0}{g_\sigma^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2\sigma_{m_c}}{\sigma_{m_c}^2 + H(k)}, \qquad (4.45)
$$

$$
\frac{1}{g_\pi^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2}{\sigma_{m_c}^2 + H(k)}, \qquad (4.46)
$$

where $\sigma_{m_c} = m_c + \sigma_0$. m_c represents a critical mass depending on g_σ^2 and g_π^2. At

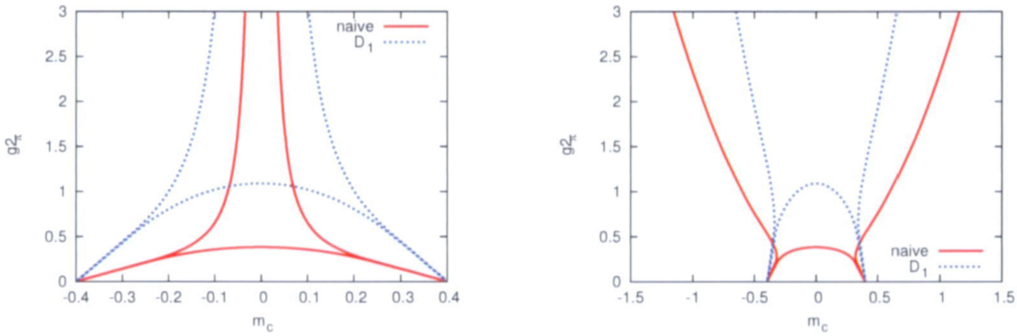


Figure 7: The phase diagrams for the Aoki phase using the naive fermion (red line) and the non-γ_5-hermiticity fermion $\bar{D}_1^{(2)}$ (blue line) adding the flavored mass term with (left) $g_\sigma^2 = g_\pi^2$ and (right) $g_\sigma^2 = g_\pi^2/2$. We fix the flavored mass factor and the hopping parameter of the non-γ_5-hermiticity fermion at $m_f = 0.4$ and $\kappa = 1$, respectively. The horizontal and vertical axes denote the critical mass m_c and the squared four-fermi coupling constant g_π^2 respectively. The center and both sides in this diagram are parity broken and symmetric phase respectively.

$g_\sigma^2 = g_\pi^2 \equiv g^2$, we can obtain the critical mass by dividing Eq.(4.45) by Eq.(4.46):

$$\sigma_0 = \sigma_{m_c} \quad \Rightarrow \quad m_c = 0 \ \text{ for any } g^2, \tag{4.47}$$

and a pion mass m_π^2 on the critical line as follows:

$$\begin{aligned} m_\pi^2 &\propto \left\langle \frac{\delta^2 S_{\text{eff}}}{\delta\pi_0 \delta\pi_0} \right\rangle \\ &= V \cdot \left[\frac{1}{g^2} - \int \frac{d^2k}{(2\pi)^2} \frac{2}{\sigma_{m_c}^2 + \pi_0^2 + H(k)} + \int \frac{d^2k}{(2\pi)^2} \frac{4\pi_0^2}{\left(\sigma_{m_c}^2 + \pi_0^2 + H(k)\right)^2} \right]. \end{aligned} \tag{4.48}$$

From (4.46) and the fact that π_0 approaches to zero near the critical line, the pion mass is obtained as

$$m_\pi^2 = 0. \tag{4.49}$$

We also study phase diagrams with fermions adding a flavored mass term [44, 45]. A flavored mass is defined as follows:

$$m_f(p) \;=\; \begin{cases} m_f \cdot \cos p_1 \cos p_4 & \text{for the naive fermion} \\ m_f \cdot \cos p_1 & \text{for the non- } \gamma_5\text{-hermiticity fermion} \end{cases}, \tag{4.50}$$

where m_f on r.h.s. is constant [16]. We add the flavored mass to the fermion mass term:

$$m \;\rightarrow\; m + m_f(p).$$

From this modification, the gap equations change as follows,

$$\frac{\sigma_0}{g_\sigma^2} \;=\; \int \frac{d^2k}{(2\pi)^2} \frac{2(\sigma_{m_c} + m_f(k))}{(\sigma_{m_c} + m_f(k))^2 + H(k)}, \tag{4.51}$$

$$\frac{1}{g_\pi^2} \;=\; \int \frac{d^2k}{(2\pi)^2} \frac{2}{(\sigma_{m_c} + m_f(k))^2 + H(k)}. \tag{4.52}$$

We present the phase diagram without the flavored mass in Fig. 6 and with the flavored mass in Fig. 7 [17]. We fix the hopping parameter at $\kappa = 1$. The four-fermi couplings in the action are related as $g_\sigma^2 = g_\pi^2/2$ in Fig. 6. In Fig. 7, we fix the hopping parameter and the flavored mass factor at $\kappa = 1$, and $m_f = 0.4$ respectively, and the four-fermi couplings are related as (left) $g_\sigma^2 = g_\pi^2$ and (right) $g_\sigma^2 = g_\pi^2/2$. We clearly see that the phase diagrams using the non-γ_5-hermiticity fermion have a very similar structure to those using the naive fermion. The critical mass and the four-fermi couplings are real numbers, despite using a fermion without γ_5-hermiticity.

4.3 Gross-Neveu model with imaginary chemical potential

In this section, we focus on the 2D Gross-Neveu model with an imaginary chemical potential and study the chiral broken phase diagrams [46, 47]. The outline for obtaining the chiral broken phase diagram is the same as in Subsect.4.2.

[16]We can choose some variations as a flavored mass [45]. In our definition, the doublers of non-γ_5-hermiticity fermions defined in Eq.(4.19) at momenta $(0,0)$ and $(0,\pi)$ have $m + m_f$ and $m - m_f$ masses respectively. On the other hand, the half doublers of the naive fermion at $(0,0)$ and (π,π) also have $m + m_f$ and those at $(0,\pi)$ and $(\pi,0)$ have $m - m_f$. By the definition, we can compare phase diagrams using fermions that have the same mass spectra but different numbers of doublers.

[17]In the phase diagrams using the fermions plus the flavored mass, there is 1st-order phase transition at the bottom of the diagrams enclosing the critical line. Because of that, we cannot use the gap equation, Eqs.(4.51) and (4.52), in this area. However, we ask leave not to correct this, to emphasize that we can obtain solutions for the gap equations.

We define the 2D Gross-Neveu model adding an imaginary chemical potential term as follows:

$$\begin{aligned} S_{\mathrm{c}}^{\mathrm{GN}} &= \int d^2x \left[\bar{\psi}\left(\partial\cdot\gamma+m\right)\psi \right. \\ & \left. -\frac{g_\sigma^2}{2N}\left(\bar{\psi}\psi\right)^2-\frac{g_\pi^2}{2N}\left(i\bar{\psi}\gamma_4\psi\right)^2+i\mu\,\bar{\psi}\gamma_4\psi\right], \end{aligned} \tag{4.53}$$

where μ is a chemical potential. Note that the third term in the action is different from Eq.(4.28), replacing γ_5 with γ_4. This action imposes $\mathrm{U_V}(1)$ symmetry. In addition, the action preserves chiral $\mathbf{Z}_4$ symmetry for $m=0$ and any g_π^2.

Introducing an auxiliary scalar field σ and an auxiliary vector field π_4, we rewrite the action as follows:

$$\begin{aligned} S_{\mathrm{c}}^{\mathrm{GN}} &= \int d^2x \left[\bar{\psi}\left(\partial\cdot\gamma+\sigma+\pi_4 i\gamma_4\right)\psi \right. \\ & \left. +\frac{N}{2g_\sigma^2}\left(\sigma-m\right)^2+\frac{N}{2g_\pi^2}\left(\pi_4-\mu\right)^2\right]. \end{aligned} \tag{4.54}$$

where

$$\frac{N}{g_\sigma^2}(\sigma-m) = -\bar{\psi}\psi, \tag{4.55}$$

$$\frac{N}{g_\pi^2}(\pi_4-\mu) = -i\bar{\psi}\gamma_4\psi. \tag{4.56}$$

In this analysis, we adopt $D_1^{(2)}$, defined in Eq.(4.1), as a lattice fermion. We cannot apply $\bar{D}_1^{(2)}$, defined in Eq.(4.19), because its determinant is always a complex number. In other words, the coupling constants in the chiral phase diagrams are always complex numbers. We discuss this issue in the next section.

We discretize space-time and write down a lattice action as follows:

$$\begin{aligned} S_{\mathrm{lat,aux}}^{\mathrm{GN}} &= \sum_{n,m}\bar{\psi}_n\left[\,D_{1\,nm}^{(2)}+\left(\sigma_n+\pi_{4n}i\gamma_4\,\right)\delta_{n,m}\right]\psi_m \\ & +\frac{N}{2}\sum_n\left[\frac{1}{g_\sigma^2}(\sigma_n-m)^2+\frac{1}{g_\pi^2}(\pi_{4n}-\mu)^2\right]. \end{aligned} \tag{4.57}$$

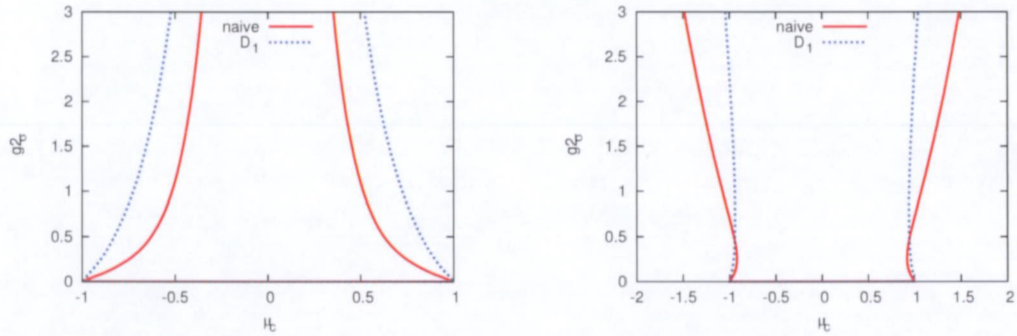


Figure 8: The chiral broken phase diagrams using the naive fermion (red line) and the non-γ_5-hermiticity fermion $D_1^{(2)}$ (blue line) with (left) $g_\sigma^2 = g_\pi^2$ and (right) $g_\sigma^2 = 2g_\pi^2$. We choose the hopping parameter in the non-γ_5-hermiticity fermion to be $\kappa = 2$. The horizontal and vertical axes denote the critical chemical potential and the squared four-fermi coupling constant g_σ^2. The center and both sides in this diagram are chiral broken and symmetric phase respectively.

where $D_{1\,nm}^{(2)}$ is defined in Eq.(4.1). Integrating fermions in the action, an effective action is obtained as follows:

$$Z = \int D\sigma_n D\pi_{4n} e^{-NS_{\mathrm{eff}}(\sigma,\pi_4)}, \tag{4.58}$$

$$S_{\mathrm{eff}}(\sigma,\pi_4) = \frac{1}{2}\sum_n \left[\frac{1}{g_\sigma^2}(\sigma_n - m)^2 + \frac{1}{g_\pi^2}(\pi_{4n} - \mu)^2\right] - \mathrm{Tr}\ \log D, \tag{4.59}$$

where

$$\mathrm{Tr}\log D = V \cdot \int \frac{d^2k}{(2\pi)^2}\ \log\left[\sigma_n^2 + \tilde{H}(k)\right], \tag{4.60}$$

and

$$\tilde{H}(k) = [\sin k_1 - i\kappa(1-\cos k_1)]^2 + (\sin k_4 + \pi_{4n})^2. \tag{4.61}$$

We can integrate out the auxiliary fields σ and π_4 in the large N limit, and solutions are obtained from the saddle point approximation:

$$\frac{\delta S_{\mathrm{eff}}(\sigma,\pi_4)}{\delta\sigma_n} = \frac{\delta S_{\mathrm{eff}}(\sigma,\pi_4)}{\delta\pi_{4n}} = 0. \tag{4.62}$$

Imposing translation invariance on the solutions $\bar{\sigma}_n = \sigma_0$ and $\bar{\pi}_{4n} = \pi_{40}$, we can derive the following gap equations,

$$\frac{\sigma_0 - m}{g_\sigma^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2\sigma_0}{\sigma_0^2 + \tilde{H}(k)}, \tag{4.63}$$

$$\frac{\pi_{40} - \mu}{g_\pi^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2(\pi_{40} + \sin k_4)}{\sigma_0^2 + \tilde{H}(k)}, \tag{4.64}$$

Now we fix the fermion mass at $m = 0$ and draw the chiral broken phase diagrams. The auxiliary field σ_0 approaches zero near the critical line because σ_0 is an order parameter for chiral symmetry breaking. Hence, gap equations for the chiral broken phase diagrams are derived as follows:

$$\frac{1}{g_\sigma^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2}{\tilde{H}(k)}, \tag{4.65}$$

$$\frac{\pi_{40} - \mu_c}{g_\pi^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2(\pi_{40} + \sin k_4)}{\tilde{H}(k)}, \tag{4.66}$$

where μ_c is a critical chemical potential. At $g_\sigma^2 = g_\pi^2 \equiv g^2$, we can also obtain a meson mass m_σ^2 on the critical line from the following equation:

$$\begin{aligned} m_\sigma^2 &\propto \left\langle \frac{\delta^2 S_{\text{eff}}}{\delta\sigma_0 \delta\sigma_0} \right\rangle \\ &= V \cdot \left[\frac{1}{g^2} - \int \frac{d^2k}{(2\pi)^2} \frac{2}{\sigma_0^2 + \tilde{H}(k)} + \int \frac{d^2k}{(2\pi)^2} \frac{4\sigma_0^2}{(\sigma_0^2 + \tilde{H}(k))^2} \right]. \end{aligned} \tag{4.67}$$

From Eq.(4.65) and the fact that σ_0 approaches zero on the critical line, we can obtain the meson mass on the critical line,

$$m_\sigma^2 = 0. \tag{4.68}$$

The chiral broken phase diagrams with the naive fermion and the non-γ_5-hermiticity fermion $\bar{D}_1^{(2)}$ are presented in Fig.8. In this figure, we set the hopping parameter at $\kappa = 2$, and the four-fermi coupling constants g_σ^2 and g_π^2 are related as (left) $g_\sigma^2 = g_\pi^2$ and (right) $g_\sigma^2 = 2g_\pi^2$. We can see that the phase diagram using the naive fermion has a qualitatively very similar structure to that using the non-γ_5-hermiticity fermion. As with the non-chemical potential case in Subsect.4.2, all of the couping constants in the phase diagrams are real numbers.

4.4 Reality condition

In Sects. 4.2 and 4.3, we investigated the parity and chiral phase diagrams for the Gross-Neveu model using the non-γ_5-hermiticity fermion. The results showed not only that the phase structure using the non-γ_5-hermiticity fermion has a qualitatively similar structure to the naive fermion, but also that the coupling constants in the phase diagram are real numbers despite breaking γ_5-hermiticity. In this section, we discuss why the coupling constants in the phase diagrams drawn in Sects. 4.2 and 4.3 are real numbers:

To see the reason, we compare two fermions adding an imaginary chemical potential:

$$D_c^{(2)}(p) = \sum_{\mu=1,4} i\ \sin p_\mu \cdot \gamma_\mu + \kappa(1-\cos p_1)\cdot\gamma_1 + i\mu\cdot\gamma_4 \equiv \sum_{\mu=1,4} f_\mu(p)\cdot\gamma_\mu. \tag{4.69}$$

$$\bar{D}_c^{(2)}(p) = \sum_{\mu=1,4} i\ \sin p_\mu \cdot \gamma_\mu + \kappa(1-\cos p_4)\cdot\gamma_4 + i\mu\cdot\gamma_4 \equiv \sum_{\mu=1,4} \bar{f}_\mu(p)\cdot\gamma_\mu, \tag{4.70}$$

The former has a γ_5-hermiticity breaking term with a temporal index, and the latter has one with a spatial index. Firstly, we present the eigenvalue distribution of these fermions in Fig.9. Because a determinant of the Dirac operator is obtained by the product of all of the eigenvalues, the eigenvalues of the Dirac operator must be complex conjugate pairs for a real determinant. Figure 9 shows that all the eigenvalues obtained from $D_c^{(2)}$ have complex conjugate pairs, but the eigenvalues of $\bar{D}_c^{(2)}$ do not. The Dirac operators seem to have very similar forms, but only $D_c^{(2)}$ preserves hermiticity, which indicates a real determinant.

To investigate in more detail, we obtain the products of the eigenvalues of the Dirac operators, namely determinants. The determinants are obtained as follows:

$$\det D_c^{(2)} = \prod_p \sum_{\mu=1,4} (-1)\cdot f_\mu^2(p), \tag{4.71}$$

$$\det \bar{D}_c^{(2)} = \prod_p \sum_{\mu=1,4} (-1)\cdot \bar{f}_\mu^2(p), \tag{4.72}$$

where $f_\mu(p)$ and $\bar{f}_\mu(p)$ are defined in Eqs.(4.69) and (4.70). A determinant of a Dirac operator that preserves γ_5-hermiticity is a real number because $\sum_{\mu=1,4} f_\mu^2(p)$ for any momentum is always real. By contrast, a determinant of fermion without γ_5-hermiticity is, in general, a complex number. However, if an arbitrary Dirac operator $D(p) =$

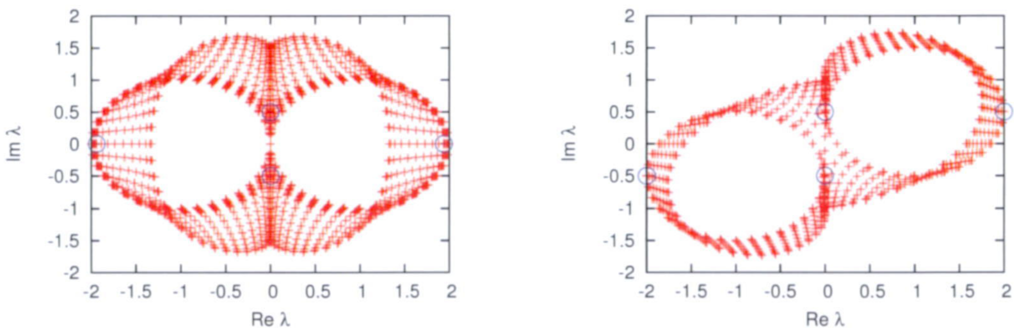


Figure 9: The eigenvalue distribution of the Dirac operators with an imaginary chemical potential: (left) $D_c^{(2)}$ and (right) $\bar{D}_c^{(2)}$. We set the hopping parameter and the chemical potential at $\kappa = 1.0$ and $\mu = 0.5$, respectively. The lattice size is 36×36. The blue circle points denote eigenvalues of momenta $p = (0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π).

$\sum_\mu \mathcal{F}_\mu(p) \cdot \gamma_\mu$ satisfies the following condition, we can obtain its Dirac determinant, which is real:

$$\sum_{\mu=1,4} \mathcal{F}_\mu^2(p) = \sum_{\mu=1,4} \mathcal{F}_\mu^{2\,*}(\tilde{p}) \quad \text{for any } p, \tag{4.73}$$

where $*$ denotes complex conjugate and $\tilde{p}$ is a momentum which satisfies this condition. The fermion $D_c^{(2)}$ defined in Eq.(4.69) satisfies the condition at $\tilde{p} = (-p_1, p_4)$; hence, the determinant of $D_c^{(2)}$ is a real number. On the other hand, we cannot obtain the determinant of $\bar{D}_c^{(2)}$ as a real number because $\bar{D}_c^{(2)}$ cannot satisfy the condition for any $\tilde{p}$. This fact suggests that although $\det D_c^{(2)}(p_1, p_4)$ is a complex number, it is real-valued by the product with $\det D_c^{(2)}(-p_1, p_4)$ because $\det D_c^{(2)}(p_1, p_4)$ and $\det D_c^{(2)}(p_1, p_4)$ are related with the following condition,

$$\det D_c^{(2)}(p_1, p_4) = \left[\det D_c^{(2)}(-p_1, p_4)\right]^* . \tag{4.74}$$

In the $\bar{D}_c^{(2)}$ case, however, $\det \bar{D}_c^{(2)}(p)$ cannot be real-valued by any momentum mode. Therefore, Eq.(4.73) guarantees reality for the fermions, at least, in free theory and Yukawa theory.

5 Conclusion and discussion

Conclusion

We have analyzed the translation-invariant, continuum and periodic function lattice fermion kinetic term using γ_5-hermiticity, R-hermiticity and PT symmetry. These conditions are not independent, because satisfying two of the three conditions is a sufficient condition for the other condition. However, it is not a necessary condition. Additionally we have suggested that R-hermiticity is a condition for removing non-hermiticity or complex couplings. In principle, these terms can be canceled by counterterms; therefore, we can fine-tune the perturbation [22]. However, nonperturbative analysis is difficult and this problem must be solved in future work.

We have proved that the PT-symmetric kinetic term does not reduce doublers. Because minimal doubling fermions have only γ_5-hermiticity it generates a renormalized non-Hermite or complex mass by quantum correction. As a simple example of non-R-hermiticity, we visualize the complex coupling constant using one-loop Wilsonian renormalization group flows of the two-flavor Gross-Neveu model in two dimensions.

We have constructed fermions without γ_5-hermiticity (non-γ_5-hermiticity fermions) based on the minimal doubling fermion in two dimensions and investigated symmetries and properties of the fermions. The fermions preserve translation invariance, chiral symmetry and locality but break cubic symmetry and some discrete symmetries. To investigate the model application possibilities, we have studied the eigenvalue distribution and the number of poles for the fermions. The eigenvalues of $D_1^{(2)}$, defined in Eq. (4.1), are distributed along the imaginary axis in the continuum limit. However, the eigenvalues of the other operators, defined in Eqs. (4.2)-(4.5), are distributed in the entire plane in the limit. We can also see that the fermions have more than four or odd poles in general. In the $D_1^{(2)}$ case, only two poles appear at $\kappa \geq 1$. We have also stated that non-γ_5-hermiticity affects not only odd number of doublers but also non-trivial doublers which appear at $D(\tilde{p}) \neq 0$ and $D^2(\tilde{p}) = 0$ and do not have appropriate kinetic term in the continuum limit. Also, the non-trivial doublers do not approach an appropriate continuum limit. In addition, we have proved the link-reflection positivity in the $\bar{D}_1^{(2)}$ case. From the proof, the hopping parameter is restricted to $|\kappa| \leq 1$.

As simple tests for application to a concrete model, we studied the parity broken phase diagram, called Aoki phase, for the 2D Gross-Neveu model using the non-γ_5-hermiticity fermion. We also studied the chiral broken phase diagram for the Gross-Neveu model, adding an imaginary chemical potential in the massless case. Both phase diagrams are qualitatively very similar to those using the naive fermion. All of the coupling constants in the analyses were obtained as real numbers, despite using a fermion without γ_5-hermiticity.

We have discussed the reason for the reality for the Gross-Neveu model using the non-γ_5-hermiticity fermion, $D_1^{(2)}$. To understand this, we investigated the eigenvalue distribution of $D_c^{(2)}$ and $\bar{D}_c^{(2)}$, defined in Eqs. (4.69) and (4.70). We can see that the eigenvalues of $D_c^{(2)}$ are complex conjugate pairs, but the eigenvalues of $\bar{D}_c^{(2)}$ are not. Hence, the determinant of $D_c^{(2)}$ is a real number but $\bar{D}_c^{(2)}$ is not. Although a determinant of $D_c^{(2)}(p1, p2)$ is obtained as a complex number, it is real-valued by the product with a determinant of $D_c^{(2)}(-p1, p4)$. Therefore all the coupling constants in the theory using the non-γ_5-hermiticity fermion $D_1^{(2)}$ are real numbers, and the theory preserves the hermiticity, despite broken the γ_5-hermiticity.

Forward to resolution of the sign problem

In the high density region, we can not obtain observable even if we use Monte Carlo simulation, because Dirac determinant has complex phase. As is well known, general continuum Dirac operator with a chemical potential has eigenvalues distributed entirely, such as $D_2^{(2)}$, in the Reλ-Imλ plane. However, eigenvalues of the $\bar{D}_1^{(2)}$ do not have the aspect. On the other hand, the non-γ_5-hermiticity fermions $\bar{D}_1^{(2)}$ defined in Eq. (4.19) seem to be fermions with a momentum-dependent chemical potential, replacing the hopping parameter κ with a chemical potential μ. In this procedure, we can interpret this fermion as that half of doublers, which appear at momenta $(0,0)$ and $(\pi,0)$, have zero-chemical potential, the others have 2μ chemical potential at $(0,\pi)$ and (π,π). Thanks to the lattice artifact, we might be able to obtain observables without regard to the mass regularization or boundary condition at finite lattice spacing, at least, in free theory or Yukawa theory.

We will investigate gauge theory using a non-γ_5-hermiticity fermion and higher-dimensional extension in future work.

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A Appendix

A.1 Notation

In this subsection we summarize the notation, which we use in this thesis, in arbitrary even dimensions, $D = 2n$ for $n \in \mathbb{N}$.

A.1.1 Minkowski space

Metric $g^{\mu\nu}$:

$$\begin{aligned} g_{\mu\nu} &= g^{\mu\nu} = \mathrm{diag}(+,-,-,\cdots,-), \\ & \qquad\qquad \text{for } \mu,\nu = 0,1,2,3,5,\cdots,D, \qquad (\mathrm{A.1.1}) \\ A_\mu &= g_{\mu\nu}A^\nu, \qquad A^\mu = g^{\mu\nu}A_\nu, \qquad (\mathrm{A.1.2}) \\ A\cdot B &\equiv g_{\mu\nu}A^\mu B^\nu = A^\mu B_\mu = A_\mu B^\mu = A_0 B_0 - \mathbf{A}\cdot\mathbf{B}, \qquad (\mathrm{A.1.3}) \end{aligned}$$

Dirac matrix γ^μ :

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}, \qquad (\mathrm{A.1.4}) \\ (\gamma^\mu)^\dagger &= \gamma^0\gamma^\mu\gamma^0 = \gamma_\mu, \qquad (\mathrm{A.1.5}) \\ \mathrm{Tr}\ \gamma^\mu &= 0, \qquad (\mathrm{A.1.6}) \end{aligned}$$

Chiral matrix γ^{D+1} :

$$\begin{aligned} \gamma^{D+1} &= (i)^{{}_{D-1}\mathrm{C}_2}\ \gamma^0\gamma^1\cdots\gamma^D, \qquad (\mathrm{A.1.7}) \\ (\gamma^{D+1})^\dagger &= \gamma^{D+1}, \qquad (\mathrm{A.1.8}) \\ \mathrm{Tr}\ \gamma^5 &= 0, \qquad (\mathrm{A.1.9}) \end{aligned}$$

Charge conjugation matrix C :

$$
\begin{aligned}
C^{-1} &= C^{\dagger} = C^{\top} = -C, && \text{(A.1.10)} \\
C\gamma^{\mu}C^{-1} &= -\gamma^{\mu\top}, && \text{(A.1.11)} \\
C\gamma^{D+1}C^{-1} &= \gamma^{D+1\top}, && \text{(A.1.12)}
\end{aligned}
$$

A.1.2 Euclid space

To formulate field theory in Euclid space from Minkowski space, we carry out Wick rotation and redefine time,

$$
t \;\rightarrow\; \tau = it, \tag{A.1.13}
$$

and Dirac matrix,

$$
\begin{aligned}
\gamma^{0} \rightarrow \gamma^{0} = \gamma^{E}_{4}, \qquad \gamma^{i} \rightarrow \gamma^{j} = i\,\gamma^{E}_{j}, \quad & && \text{(A.1.14)} \\
\text{for } j = 1, 2, 3, 5, \cdots D, &
\end{aligned}
$$

where τ and the upper index "E" denote time and Dirac matrix in Euclid space, respectively. From the manipulation, a metric and Dirac matrix in Euclid space have the following properties,

Metric $\eta_{\mu\nu}$:

$$
\begin{aligned}
\eta_{\mu\nu} &= \mathrm{diag}(+,+,+,\cdots,+) = \delta_{\mu\nu}, \\
&\qquad \text{for } \mu,\nu = 1, 2, \cdots, D, && \text{(A.1.15)} \\
A_{\mu} &= \eta_{\mu\nu}A^{\nu} = \eta^{\mu\nu}A_{\nu} = A^{\mu}, && \text{(A.1.16)} \\
A\cdot B &\equiv \eta_{\mu\nu}A_{\mu}B_{\nu} = A_{\mu}B_{\mu}, && \text{(A.1.17)} \\
& && \text{(A.1.18)}
\end{aligned}
$$

Dirac matrix γ_μ :

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \tag{A.1.19}$$
$$(\gamma_\mu)^\dagger = \gamma_\mu, \tag{A.1.20}$$
$$\mathrm{Tr}\ \gamma_\mu = 0, \tag{A.1.21}$$

Chiral matrix γ_{D+1} :

$$\gamma_{D+1} = (i)^{{}_D\mathrm{C}_2}\ \gamma_1\gamma_2\cdots\gamma_D, \tag{A.1.22}$$
$$(\gamma_{D+1})^\dagger = \gamma_{D+1}, \tag{A.1.23}$$
$$\mathrm{Tr}\ \gamma_{D+1} = 0, \tag{A.1.24}$$

Charge conjugation matrix C :

$$C^{-1} = C^\dagger = C^\top = -C, \tag{A.1.25}$$
$$C\gamma^\mu C^{-1} = -\gamma_\mu^\top, \tag{A.1.26}$$
$$C\gamma_{D+1}C^{-1} = \gamma_{D+1}^\top, \tag{A.1.27}$$

A.1.3 Other notes

Condition for fermion

$$\text{Anti-fermion} \ : \ \bar{\psi} = \psi^\dagger\gamma^0, \quad \text{only for Minkowski} \tag{A.1.28}$$
$$\text{Charge conjugation} \ : \ \psi^c = C\bar{\psi}^\top, \ \ \bar{\psi}^c = -\psi^\top C^{-1}, \tag{A.1.29}$$
$$\text{Time reversal} \ : \ \psi(x_0, \mathbf{x}) \to i\gamma_5\gamma_4\psi(-x_0, \mathbf{x}), \tag{A.1.30}$$
$$\text{Parity transformation} \ : \ \psi(x_0, \mathbf{x}) \to i\gamma_4\psi(x_0, -\mathbf{x}), \tag{A.1.31}$$
$$\text{Chiral projection} \ : \ \psi_\mp = \frac{1 \mp \gamma^5}{2}\psi, \tag{A.1.32}$$
$$\text{Majorana condition} \ : \ \psi^c = \psi, \tag{A.1.33}$$
$$\text{Weyl condition} \ : \ \psi_\mp = \psi, \tag{A.1.34}$$

Identity

Fierz identity :

$$\frac{1}{2^{d/2}}\,\mathrm{Tr}\left[\Gamma^A\,(\Gamma^B)^\dagger\right] = \delta^{AB}, \tag{A.1.35}$$

$$\frac{1}{2^{d/2}}\sum_{A=1}^{2^{d/2}}(\Gamma^A)_{\alpha\beta}(\Gamma^A)^\dagger_{\rho\sigma} = \delta_{\alpha\sigma}\delta_{\rho\beta}, \tag{A.1.36}$$

where

$$\Gamma^A = \left\{\mathbf{1},\gamma^{\mu_1},\gamma^{\mu_1\mu_2},\cdots,\gamma^{\mu_1\mu_2\cdots\mu_d}\right\}, \tag{A.1.37}$$

$$\gamma^{\mu_1\mu_2\cdots\mu_i} = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_i}, \quad \text{for } \mu_1<\mu_2<\cdots<\mu_d \tag{A.1.38}$$

A.2 N-flavors Gross-Neveu model and Renormalization Group Flow in Two Dimensions

In this section, we describe the N-flavors Gross-Neveu(GN) model [32] and calculate Wilsionian renormalization group flows(RGFs) using the NA and MDAs numerically. Firstly we will review the N-flavors GN model and then we will calculate the RGFs.

We define the continuum Euclidean Lagrangian of N-flavors GN model as follows:

$$\mathcal{L}_{\mathrm{GN}} = \bar\psi\left(\partial\cdot\gamma+m\right)\psi-\frac{g^2}{2N}\left(\bar\psi\psi\right)^2, \tag{A.2.39}$$

where m is a fermion mass and g is a coupling constant of four fermi interaction. We omit flavor indices if we do not have to write explicitly, $\bar\psi\psi\equiv\sum_{i=1}^N\bar\psi_i\psi_i$, where "$i$" means flavor degrees of freedom.

This Lagrangian has $U(1)$ symmetry:

$$\begin{aligned}\psi &\rightarrow e^{i\theta}\psi,\\ \bar\psi &\rightarrow \bar\psi e^{-i\theta}.\end{aligned} \tag{A.2.40}$$

In the case of massless fermions, this Lagrangian has chiral $\mathbf{Z}_4$ symmetry:

$$\begin{aligned}\psi &\rightarrow (i\gamma_5)^n\,\psi,\\ \bar\psi &\rightarrow \bar\psi\,(i\gamma_5)^n. \quad (n=0,1,2,3)\end{aligned} \tag{A.2.41}$$

In the case of massive fermions, chiral $\mathbf{Z}_4$ symmetry reduces to chiral $\mathbf{Z}_2$ symmetry($n = 0, 2$). In addition, if all flavors have same masses it has the $SU(N)_F$ symmetry:

$$\begin{aligned} \psi_i &\rightarrow U_{ij}\psi_j, \\ \bar{\psi}_i &\rightarrow \bar{\psi}_j U_{ji}^\dagger, \quad (U \in SU(N)) \end{aligned} \tag{A.2.42}$$

It is convenient to redefine the GN action using an auxiliary scalar field σ instead of $(\bar{\psi}\psi)$:

$$\mathcal{L}_{\mathrm{GN}} = \bar{\psi}\left(\partial \cdot \gamma + m\right)\psi + \frac{N}{2}\sigma^2 + g\sigma\bar{\psi}\psi. \tag{A.2.43}$$

According to this manipulation, we can obtain the action which involves Yukawa interaction instead of four fermi interaction. According to perturbative calculation, the NG model has asymptotic freedom property [8, 9].

A.3 Wilsonian Renormalization Group

In this appendix, we review a method to calculate the Wilsonian renormalization group flow in the case of GN model in two dimension [10].

We define the partition function of the GN model in momentum space [18]:

$$Z = \int D\sigma D\psi D\bar{\psi} \exp(-S_{\mathrm{GN}}), \tag{A.3.44}$$

with

$$S_{\mathrm{GN}} = \int_{0<|p|<1} \frac{d^2p}{(2\pi)^2}\mathcal{L}_{\mathrm{GN}}, \tag{A.3.45}$$

where

$$D\sigma = \prod_{0<|k|<1} d\sigma(k), \quad D\psi = \prod_{0<|k|<1} d\psi(k), \quad D\bar{\psi} = \prod_{0<|k|<1} d\bar{\psi}(k), \tag{A.3.46}$$

and $\mathcal{L}_{\mathrm{GN}}$ is given in (A.2.43). We can treat of N as a mass parameter of auxiliary field σ. Here we assumed that the high frequency modes had already integrated and they do

[18] We omit the subscript which means flavors.

not contribute effectively. Then we split the field configurations as follows:

$$\sigma(p) = \sigma_l(p) + \sigma_h(p), \tag{A.3.47}$$

where

$$\begin{aligned} \sigma_l(p) &= \sigma(p) \quad \text{if } 0 < |p| < \frac{4}{5}, \qquad \text{zero otherwise,} & \text{(A.3.48)} \\ \sigma_h(p) &= \sigma(p) \quad \text{if } \frac{4}{5} < |p| < 1, \qquad \text{zero otherwise,} & \text{(A.3.49)} \end{aligned}$$

and the other fields are also split similarly [19]. We choice renormalization conditions as follows:

$$\begin{aligned} \Gamma^{(2)}_{\psi}(0,0) &= -m_R, & \text{(A.3.50)} \\ \Gamma^{(2)}_{\sigma}(0,0) &= -N_R, & \text{(A.3.51)} \\ \Gamma^{(3)}(0,0,0) &= -g_R, & \text{(A.3.52)} \end{aligned}$$

where $\Gamma^{(i)}$ are renormalized i-point functions, m_R, N_R, g_R are renormalized parameters and arguments of $\Gamma^{(i)}$ are external momenta. In order to obtain effective parameters, we calculate one-loop effect and integrate out only high frequency modes:

$$\begin{aligned} m_{R\alpha\beta} &= \left(\frac{5}{4}\right)^{-2} \eta_{\psi}^2 \left[m - g^2 \int_{\frac{4}{5}<|k|<1} \frac{d^2k}{(2\pi)^2} S_{\alpha\beta}(k) D(k) \right], & \text{(A.3.53)} \\ \frac{N_R}{2} &= \left(\frac{5}{4}\right)^{-2} \eta_{\sigma}^2 \left[\frac{N}{2} + \frac{g^2}{2} \int_{\frac{4}{5}<|k|<1} \frac{d^2k}{(2\pi)^2} \mathrm{tr}\left[S(k)S(k)\right] \right], & \text{(A.3.54)} \\ g_R &= \left(\frac{5}{4}\right)^{-4} \eta_{\psi}^2 \eta_{\sigma} \left[g + g^3 \int_{\frac{4}{5}<|k|<1} \frac{d^2k}{(2\pi)^2} (S(k)S(k))_{\alpha\beta} D(k) \right] \cdot \delta_{\alpha\beta}, & \\ & & \text{(A.3.55)} \end{aligned}$$

where $S(k)$ and $D(k)$ are propagators of each fields presented below, "tr" is a trace operation of the fermionic indices and η_{ψ} and η_{σ} are rescaling parameters of fermion and auxiliary field respectively. We can define these parameters with dimensional analysis in

[19] On account of numerical efficiency, we choose a division which split between σ_l and σ_h as $p = \frac{4}{5}$.

the following values:

$$\eta_\psi = \left(\frac{5}{4}\right)^{3/2}, \tag{A.3.56}$$

$$\eta_\sigma = \frac{5}{4}. \tag{A.3.57}$$

We can obtain propagators from the GN action:

$$S(k) = \left[D_f(\tilde{k}+k)+m\right]^{-1}, \tag{A.3.58}$$

$$D(k) = \frac{1}{N}, \tag{A.3.59}$$

where $D_f(k)$ is one of the lattice fermion kinetic terms in (2.36) and $\tilde{k}$ is a zero-mode momentum in (2.38). Substituting (A.3.58) and (A.3.59) to (A.3.54)-(A.3.55), we can obtain effective mass and coupling constant after integrating out over fields $\sigma(\frac{4}{5} < |k| < 1), \psi(\frac{4}{5} < |k| < 1)$ and $\bar{\psi}(\frac{4}{5} < |k| < 1)$.

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