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Conformal Manifolds Admitting Ricci-flat Weyl Structures

by

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Introduction

Weyl geometry is not merely a generalization of Riemannian geometry but also a synthesis of Riemannian geometry and conformal geometry (cf. [6]). From this point of view, we would like to pay attention to conformal manifolds on which Weyl geometry is developed.

A Weyl structure on a conformal manifold (M, C) is regarded as a torsion-free affine connection compatible with the conformal class C. The Weyl conformal curvature tensor W, among others, is a primary invariant of the conformal class C ($\dim M \geqq 4$). Then it is natural to ask ourselves the following question: Is there a Weyl structure on (M, C) whose curvature tensor coincides with the Weyl tensor W of C? It turns out that such a structure is nothing but a Ricci-flat Weyl structure (see [10]). We should remark that not every conformal manifold can admit a Ricci-flat Weyl structure. The existence of such a structure may be rather a strong condition.

Ricci-flat Weyl manifolds form a special class of Einstein-Weyl manifolds which satisfy an analogue of the Einstein equations in Riemannian geometry. In Weyl geometry, it is an interesting and important problem to construct Einstein-Weyl manifolds. Recently there have appeared many research papers on this subject (see [9], [12], [13] and the references therein). Almost all examples, however, are obtained by using known techniques for constructing Einstein metrics. Of course, examples of Ricci-flat Weyl manifolds have been constructed. Through such examples, we wondered that there might be at most two Ricci-flat Weyl structures on a given conformal manifold.

The main purpose of this paper is first to study the number of Ricci-flat Weyl structures on a conformal manifold and then to clarify the structure of compact conformal manifolds which admit Ricci-flat Weyl structures. Since every flat Weyl structure is necessarily Ricci-flat, these studies enable us to obtain some results on conformal manifolds which admit flat Weyl structures.

Let D be a Weyl struture on a conformal manifold (M, C). For each metric g in C, there is a unique 1-form ω_g on M such that $Dg = -2\omega_g \otimes g$. D is said to be closed if the distance curvature θ of D vanishes identically on M, i.e., $d\omega_g = 0$ (cf. [7], [17]). In this case, we can define an characteristic class $ch(D)$ as an element of the first de Rham cohomology group (see [9]). As every Ricci-flat Weyl structure D

is closed, we can exploit the characteristic class $ch(D)$.

Now we shall give a survey of our results. Let (M, C) be a connected conformal manifold with $\dim M \geqq 3$. We start with the number of Ricci-flat Weyl structures on (M, C). To study it, we need the following fact. If there are two distinct Ricci-flat Weyl structures D_1 and D_2 on (M, C), then $ch(D_1)+ch(D_2)=0$. Using this formula, we obtain a general finiteness theorem: *If there is a Ricci-flat Weyl structure D on (M, C) with $ch(D) \neq 0$ then (M, C) admits at most two Ricci-flat Weyl structures* (see Theorem 3.2). Note that, in this general theorem, M is not necessarily assumed to be compact. In the compact case, we can improve the theorem as follows:

A compact conformal manifold (M, C) admits at most two Ricci-flat Weyl structures. Furthermore, (M, C) admits just two Ricci-flat Weyl structures if and only if there is a Ricci-flat Weyl structure D on (M, C) such that $ch(D) \neq 0$ (see Corollary 4.4).

To prove these facts, we shall use the result, due to P. Gauduchon [5], which asserts that for any Weyl structure D on a compact conformal manifold (M, C) there exists a metric $g \in C$, unique up to a constant factor, such that the 1-form ω_g is co-closed with respect to g. The pair (g, ω_g) is often called the Gauduchon gauge.

Our task is then to determine 1) all compact Ricci-flat Riemannian manifolds and 2) all compact conformal manifolds which admit just two Ricci-flat Weyl structures. We shall investigate the second case 2) because the first case falls within Riemannian geometry. Consider a Riemannian manifold (N, h) and a homomorphism ρ of $\boldsymbol{Z}$ into the group of isometries $I(N, h)$. Then we can define a manifold $N(\rho)$ so that it forms a fibre bundle $N(\rho) \rightarrow S^1$ with standard fibre N. For each non-zero real number a, the metric h induces a natural conformal class $C_{h,a}$ on $N(\rho)$. We shall say that $(N(\rho), C_{h,a})$ is a *standard conformal manifold* if (N, h) is an Einstein manifold with positive constant scalar curvature. We can see that $(N(\rho), C_{h,a})$ admits two natural Ricci-flat Weyl structures D_a and D_{-a}. We shall call D_a the *standard Ricci-flat Weyl structure* on $(N(\rho), C_{h,a})$. Then our result can be stated as follows.

Let (M, C) be a compact, connected conformal manifold with $\dim M \geqq 3$. *Assume that there is a Ricci-flat Weyl structure D on (M, C) with $ch(D) \neq 0$. Then there is a compact, standard conformal manifold $(N(\rho), C_{h,a})$ such that D is isomorphic to the standard Ricci-flat Weyl structure D_a on $(N(\rho), C_{h,a})$* (see Theorem 4.5).

In [8], Gauduchon studied the same problem and obtained a weaker result. We can see from Theorem 4.5 that a compact conformal manifold admitting two Ricci-flat Weyl structures is identified with a certain compact standard conformal manifold. However, there are many standard Ricci-flat Weyl structures which are mutually isomorphic. This requires us to deal with the equivalence problem for standard Ricci-flat Weyl structures. Fortunately we can accomplish it for the compact case (see Theorem 6.1).

As for flat Weyl structures, we can prove that a non-Riemannian flat Weyl structure D on a compact conformal manifold is isomorphic to a certain standard flat Weyl structure D_a on $(N(\rho), C_{h,a})$, where (N, h) is a Riemannian manifold of positive constant curvature (see Theorem 5.4 and [8]).

After recalling the rudiments of Weyl geometry, we shall first construct in Section 2 standard Ricci-flat Weyl manifolds. The general finiteness theorem is proved in Section 3. Section 4 is devoted to studying compact conformal manifolds. In Section 5, we deal with flat Weyl structures. In the last section, we study the equivalence problem for standard Ricci-flat Weyl structures.

§1. Weyl structures on conformal manifolds and isomorphisms

Throughout this paper, M will denote a connected, paracompact C^∞-manifold of dimension $n \geqq 3$. All metrics under consideration are assumed to be positive definite.

Now consider a conformal class C on M. C determines a $CO(n)$-structure B on M, where $CO(n)$ denotes the conformal group of degree n. A torsion-free connection on B is called a Weyl connection. When we fix a Weyl connection Γ on B, we call the system $D=(C, \Gamma)$ a Weyl structure on (M, C). As usual, we identify Γ with the induced torsion-free affine connection on M, which we also denote by D. Then, for each metric g in C, there is a unique 1-form ω_g on M such that

$$Dg = -2\omega_g \otimes g \,. \tag{1.1}$$

The pair (g, ω_g) is called a gauge of D. We shall denote the 1-form by $\omega_g(D)$ if the Weyl structure D needs to be specified. Note that the form of the equation (1.1) is invariant under a "Weyl transformation"

$$g \mapsto \bar{g} = e^{2f} g \,, \qquad \omega_g \mapsto \omega_{\bar{g}} = \omega_g - df \,, \tag{1.2}$$

where f is a smooth function on M.

Conversely, for a metric g in C and a 1-form ω on M, there is a unique torsion-free affine connection D such that $Dg = -2\omega \otimes g$. In fact, denoting by ∇ the Levi-Civita connection of g and by ξ the vector field dual to ω with respect to g, we have only to set

$$D_X Y = \nabla_X Y + \omega(X) Y + \omega(Y) X - g(X, Y)\xi \,, \tag{1.3}$$

where X and Y are arbitrary vector fields on M. The affine connection D induces a Weyl connection Γ on B, so that D is a Weyl structure on (M, C). The pair (g, ω) turns out to be a gauge of D. We shall also say that D is determined by (g, ω). It should be remarked that for any smooth function f on M the pair $(e^{2f}g, \omega - df)$ determines the same Weyl structure D.

Let D be a Weyl structure on (M, C). The distance curvature θ of D is defined by $\theta = d\omega_g$, $g \in C$ (see [17] p. 124). Following Gauduchon [7], we shall say that D is closed if θ vanishes identically on M. In this case, the de Rham cohomology class $[\omega_g] \in H^1(M)$ of the closed 1-form ω_g does not depend on the metric $g \in C$. For simplicity, we write $ch(D) = [\omega_g]$, which can be considered as a Chern-Simons characteristic class arising from the Weyl connection Γ on B (see [9]). It is obvious that a closed Weyl structure D on (M, C) is identified with the Levi-Civita connection of a certain Riemannian metric g in C if and only if $ch(D)=0$.

Let D be a Weyl structure on (M, C) and let R be the Ricci tensor of D. Denoting by S the symmetric part of R, we get $R=S-(n/2)\theta$. Then we can see that D is closed if and only if $R=S$ on M. Let us fix a gauge (g, ω) of D and consider the Levi-Civita connection ∇ of g. We denote by $\delta_g\omega$ the codifferential of ω and by $|\omega|_g$ the pointwise norm of ω with respect to g. Then, in terms of the Ricci tensor r_g of g, S is given by

$$S=r_g-\frac{n-2}{2}(H_g(\omega)-2\omega\cdot\omega)+(\delta_g\omega-(n-2)|\omega|^2)\cdot g\,, \tag{1.4}$$

where we put $H_g(\omega)(X, Y)=(\nabla_X\omega)(Y)+(\nabla_Y\omega)(X)$.

For each metric g in C, we define a function Λ_g on M to be the trace of R with respct to g. Λ_g is often called the conformal scalar curvature of D (cf. [13]). Under the Weyl transformation (1.2), the conformal scalar curvature changes as $\Lambda_{\bar{g}}=\Lambda_g\cdot e^{-2f}$. If we set $K=\Lambda_g\cdot g$, then K does not depend on the metric $g\in C$. The traceless Ricci tensor S_0 of D is defined by $S_0=S-(1/n)K$. Then the Ricci tensor R is decomposed as follows

$$R=-\frac{n}{2}\theta+\frac{1}{n}K+S_0\,, \tag{1.5}$$

which corresponds to the irreducible decomposition of the $CO(n)$-module of algebraic 2-tensors (see [10]). Let $\mathscr{T}_s^r(M)$ denote the $C^\infty(M)$-module of all tensor fields of type (r, s) on M, where s denotes the covariant degree. Then we can construct $C^\infty(M)$-module homomorphisms $\rho, \sigma\colon \mathscr{T}_2^0(M)\to\mathscr{T}_3^1(M)$ in such a way that the curvature tensor $\mathscr{R}$ of D is decomposed as follows

$$\mathscr{R}=\rho(\theta)+\frac{1}{2n(n-1)}\sigma(K)+\frac{1}{n-2}\sigma(S_0)+W\,, \tag{1.6}$$

where W is the Weyl conformal curvature tensor of C. Note that if $\dim M=3$ then W vanishes identically on M. The decomposition (1.6) corresponds to the irreducible and orthogonal decomposition of the $CO(n)$-module of algebraic curvature tensors (see [10] and also [6]).

A Weyl structure D on (M, C) is called an Einstein-Weyl structure if the symmetric part S of R is proportional to a metric in C. This is equivalent to saying that the traceless Ricci tensor S_0 of D vanishes identically on M. If D is Ricci-flat, i.e., $R=0$, then D is a closed Einstein-Weyl structure. We have proved in [10] that if D is a closed Einstein-Weyl structure for which $ch(D)\neq 0$, then D is Ricci-flat (see also [8]). A Ricci-flat Weyl structure has the following characteristic property.

PROPOSITION 1.1. *Let D be a Weyl structure on a conformal manifold (M, C) with* $\dim M\geqq 3$. *Then D is Ricci-flat if and only if the curvature tensor of D coincides with the Weyl conformal curvature tensor W of C at eavery point of M. In particular, if* $\dim M=3$, *every Ricci-flat Weyl structure is flat.*

This follows from the decompositions (1.5) and (1.6) (see [10]).

Next we consider isomorphisms. Let (M', C') be another conformal manifold. A diffeomorphism F: $M \to M'$ is called an isomorphism of (M, C) onto (M', C') if, for a metric $g' \in C'$, the metric F^*g' on M belongs to C. This definition does not depend on the choice of metric in C'. In this case, (M, C) and (M', C') are said to be isomorphic. Let D (resp. D') be a Weyl structure on (M, C) (resp. (M', C')). An isomorphism F of (M, C) onto (M', C') is called an isomorphism of D onto D' if F is an affine isomorphism of D onto D'. More precisely, F satisfies

$$F_*(D_X Y) = D'_{F_*X} F_* Y \tag{1.7}$$

for all vector fields X and Y on M. In this case, we shall say that D and D' are isomorphic or equivalent. We shall frequently use the following

PROPOSITION 1.2. *Let (M, C), (M', C'), D and D' be as above. Take an arbitrary gauge (g', ω') of D'. Then an isomorphism F of (M, C) onto (M', C') is an isomorphism of D onto D' if and only if D is determined by the pair $(F^*g', F^*\omega')$.*

Proof. We shall denote by X, Y and Z arbitrary vector fields on M. Define a $(1, 2)$-tensor field β on M by

$$\beta(X, Y) = D_X Y - F_*^{-1}(D'_{F_*X} F_* Y)\,. \tag{1.8}$$

Since D and D' are torsion-free, β is symmetric, i.e., $\beta(X, Y) = \beta(Y, X)$. Let us set $g = F^*g'$ and $\omega = F^*\omega'$. Then it is not hard to verify

$$(D_X g)(Y, Z) = -2\omega(X)g(Y, Z) - (g(\beta(X, Y), Z) + g(Y, \beta(X, Z)))\,. \tag{1.9}$$

Now suppose first F is an affine isomorphism of D onto D'. From (1.7) and (1.8), we have $\beta = 0$, and hence $(D_X g)(Y, Z) = -2\omega(X)g(Y, Z)$. This implies that D is determined by (g, ω).

Conversely, suppose $Dg = -2\omega \otimes g$. The formula (1.9) yields

$$g(\beta(X, Y), Z) + g(Y, \beta(X, Z)) = 0\,.$$

Since β is symmetric, we have $g(\beta(X, Y), Z) = 0$, and hence $\beta = 0$. Then it follows from (1.8) that F is an affine isomorphism of D onto D'.

§2. Standard conformal manifolds

Before going to our general theory, we construct examples of Ricci-flat Weyl manifolds, which will play an important role in the sequel.

We consider the circle $S^1 = \boldsymbol{R}/\boldsymbol{Z}$ and the universal covering space of S^1: $\pi_0: \boldsymbol{R} \to S^1$. We regard it as a principal fibre bundle with structure group $\boldsymbol{Z}$. Let $n \geqq 3$ and let (N, h) be an $(n-1)$-dimensional connected Riemannian manifold. Consider a homomorphism ρ of $\boldsymbol{Z}$ into the full group of isometries $I(N, h)$. It is clear that ρ is completely determined by the isometry $\rho(1)$. Since ρ defines in a natural way a left action of $\boldsymbol{Z}$ on N, the principal fibre bundle $\pi_0: \boldsymbol{R} \to S^1$ determines a fibre bundle $p: N(\rho) \to S^1$ with standard fibre N. More precisely, define a right action of $\boldsymbol{Z}$ on $\boldsymbol{R} \times N$ by

$$(2.1) \qquad (t, x) \cdot m = (t+m, \rho(-m)(x)),$$

where $(t, x) \in \boldsymbol{R} \times N$ and $m \in \boldsymbol{Z}$. Then $N(\rho)$ is the orbit space of $\boldsymbol{R} \times N$ by the action (see [8] and [11]). We denote the action of $m \in \boldsymbol{Z}$ by R_m. The natural surjective map $\pi: \boldsymbol{R} \times N \to N(\rho)$ is a covering projection. Then the universal covering space of $N(\rho)$ is given by $\boldsymbol{R} \times N^*$, where N^* is the universal covering space of N. Moreover, we can see from (2.1) that if $\rho(1)$ coincides with the identity transformation id_N of N then $N(\rho) = S^1 \times N$.

For a non-zero real number a, we define a metric on $\boldsymbol{R} \times N$ by $g_a^* = (adt)^2 + h$. Since g_a^* is invariant under the action of $\boldsymbol{Z}$, there is a unique metric g_a on $N(\rho)$ such that $\pi^* g_a = g_a^*$. By the same reason, the 1-form adt on $\boldsymbol{R} \times N$ induces a unique 1-form ω_a on $N(\rho)$ such that $\pi^* \omega_a = adt$. It should be remarked that ω_a is a parallel 1-form with respect to the metric g_a. For later use, we mention here a special property of g_a. For every point z of $N(\rho)$ and every tangent vector v to $N(\rho)$ at z, the metric g_a satisfies the following inequality:

$$(2.2) \qquad g_a(v, v) \geqq \omega_a(v)^2 .$$

This follows immediately from the definiton of g_a^*.

Now we denote by $C_{h,a}$ the conformal class on $N(\rho)$ determined by g_a and by D_a the Weyl structure determined by the pair (g_a, ω_a). Since $g_{-a} = g_a$, we have $C_{h,-a} = C_{h,a}$. Then both D_a and D_{-a} are Weyl structures on the same conformal manifold $(N(\rho), C_{h,a})$. Since ω_a is closed, D_a and D_{-a} are closed Weyl structures. By definition, we have $ch(D_a) = [\omega_a]$, and hence $ch(D_a) \neq 0$.

PROPOSITION 2.1. *Let the notation be as above. Then the following two conditions are mutually equivalent*:

1) *The Weyl structure D_a on $(N(\rho), C_{h,a})$ is Ricci-flat*;

2) *(N, h) is an Einstein manifold with constant scalar curvature $(n-1)(n-2)$. Moreover, if the condition 1) is satisfied, then D_{-a} is also Ricci-flat.*

Proof. Since the 1-form ω_a is parallel with respect to g_a, it follows from (1.4) that the Ricci tensor S_a of D_a is given by

$$(2.3) \qquad S_a = r_a + (n-2)(\omega_a \cdot \omega_a - |\omega_a|^2 g_a),$$

where r_a is the Ricci tensor of g_a and $|\omega_a|$ is the pointwise norm of ω_a with respect to g_a. Let r be the Ricci tensor of h. Then we get $\pi^* r_a = r$. Since $|\omega_a| = 1$, we have

$$\pi^* S_a = r + (n-2)((adt)^2 - g_a^*) = r - (n-2)h .$$

Therefore the vanishing of S_a is equivalent to the Einstein equation $r = (n-2)h$. The last assertion also follows from (2.3).

We shall say that $(N(\rho), C_{h,a})$ is a *standard conformal manifold* (with fibre (N, h)) if (N, h) satisfies the condition 2) in Proposition 2.1. In this case, D_a will be called a *standard Ricci-flat Weyl structure.* Then Proposition 2.1 says that a standard conformal manifold $(N(\rho), C_{h,a})$ admits at least two Ricci-flat Weyl structures. The relation between D_a and D_{-a} is realized by

$$ch(D_a)+ch(D_{-a})=0\ . \tag{2.4}$$

Now we consider the equivalence between standard Ricci-flat Weyl structures. Let (N, h) and (N', h') be $(n-1)$-dimensional connected Einstein manifolds satisfying the condition 2) in Proposition 2.1. Let $\rho: \boldsymbol{Z}\to I(N, h)$ and $\rho': \boldsymbol{Z}\to I(N', h')$ be homomorphisms. Then we can consider the standard Ricci-flat Weyl structure D_a on $(N(\rho), C_{h,a})$ and the standard Ricci-flat Weyl structure D_b on $(N'(\rho'), C_{h',b})$, where a and b are non-zero real numbers.

PROPOSITION 2.2. *Let the notation be as above. Then D_a and D_b are isomorphic if the following two conditions are satisfied*:

1) $a=\varepsilon b$, *where ε is either $+1$ or -1*;
2) *There is an isometry φ of (N, h) onto (N', h') such that*

$$\varphi\circ\rho(m)=\rho'(\varepsilon m)\circ\varphi\ .$$

for any $m\in\boldsymbol{Z}$.

Proof. Define a diffeomorphism $\Phi: \boldsymbol{R}\times N\to\boldsymbol{R}\times N'$ by $\Phi(t, x)=(\varepsilon t, \varphi(x))$, $(t, x)\in\boldsymbol{R}\times N$. Then we have $\Phi^*(bdt)=adt$ and $\Phi^*g_b^*=g_a^*$. It follows from 2) and (2.1) that $\Phi\circ R_m(t, x)=R_{\varepsilon m}\circ\Phi(t, x)$ for any $(t, x)\in\boldsymbol{R}\times N$ and any $m\in\boldsymbol{Z}$. This means that Φ induces a unique diffeomorphism F of $N(\rho)$ onto $N'(\rho')$ satisfying $\pi\circ\Phi=F\circ\pi$. Then it is easy to verify $F^*g_b=g_a$ and $F^*\omega_b=\omega_a$. Proposition 2.2 follows now from Proposition 1.2.

COROLLARY 2.3. *Let (N, h), ρ and a be as before. If the homomorphism ρ satisfies $\rho(2m)=id_N$ for all $m\in\boldsymbol{Z}$, then the standard Ricci-flat Weyl structures D_a and D_{-a} on $(N(\rho), C_{h,a})$ are isomorphic.*

In fact, we can rewrite the condition as follows: $\rho(m)=\rho(-m)$. Then the conditions 1) and 2) in Proposition 2.2 are satisfied by setting $\varepsilon=-1$ and $\varphi=id_N$. In particular, consider the case where $\rho(m)=id_N$ for all $m\in\boldsymbol{Z}$. Then we get $N(\rho)=S^1\times N$, which we call the trivial standard conformal manifold. We have immediately

COROLLARY 2.4. *Let (N, h) and a be as before. Then the standard Ricci-flat Weyl structures D_a and D_{-a} on the trivial standard conformal manifold $(S^1\times N, C_{h,a})$ are isomorphic.*

It should be remarked that Proposition 2.2 is still valid without the assumption that (N, h) and (N', h') are Einstein manifolds. Here we have restricted ourselves only to Ricci-flat Weyl structures.

§3. A finiteness theorem

In the previous section, we have constructed examples of conformal manifolds which admit at least two Ricci-flat Weyl structures. Then a question arises whether a conformal manifold can admit only finite number of Ricci-flat Weyl structures. To answer this question, we need the following generalization of the crucial relation (2.4).

PROPOSITION 3.1. *Let (M, C) be a connected conformal manifold with* $\dim M \geqq 3$. *If there are two distinct Ricci-flat Weyl structures D_1 and D_2 on (M, C), then*

$$ch(D_1)+ch(D_2)=0 .$$

Proof. Let $\pi: M^* \to M$ be the universal covering space of M. Fix a metric g in C and set $\omega_i = \omega_g(D_i)$ $(i=1, 2)$. Since ω_i is closed, there is a smooth function f_i on M^* such that $\pi^*\omega_i = df_i$ $(i=1, 2)$. Define a metric g_i on M^* by

$$g_i = \exp(2f_i) \cdot \pi^* g \qquad (i=1, 2) \tag{3.1}$$

and smooth functions λ and μ on M^* by $\lambda = f_2 - f_1$ and $\mu = f_1 + f_2$, respectively. Moreover, we set $\omega = \omega_2 - \omega_1$ and $g_0 = e^{\lambda} g_1$. Then we have easily $g_2 = e^{2\lambda} g_1$, $g_0 = e^{\mu}\pi^* g$ and $\pi^*\omega = d\lambda$. Let ξ denote the vector field dual to ω with respect to g and let ξ^* denote the lift of ξ to M^*. It is easy to see that the vector field dual to $d\lambda$ with respect to g_0 is given by $e^{-\mu}\xi^*$. Let $|d\lambda|_0$ denote the pointwise norm of $d\lambda$ with respect to g_0. Then we have

$$|d\lambda|_0^2 = e^{-2\mu} g_0(\xi^*, \xi^*) = e^{-\mu}(\pi^* g)(\xi^*, \xi^*) ,$$

and hence

$$|d\lambda|_0^2 \cdot e^{\mu} = \pi^* f , \tag{3.2}$$

where we put $f = g(\xi, \xi)$.

Let r_i be the Ricci tensor of g_i $(i=1, 2)$. Since $g_2 = e^{2\lambda} g_1$, it follows from the well-known formula that

$$r_2 = r_1 - (n-2)(\nabla d\lambda - d\lambda \cdot d\lambda) + (\Delta\lambda - (n-2)|d\lambda|_1^2) g_1 , \tag{3.3}$$

where ∇, Δ and $|d\lambda|_1$ denote, respectively, the Levi-Civita connection, the Laplacian and the pointwise norm of $d\lambda$ with respect to g_1 (cf. (1.4)). Let D_i^* denote the lift of the connection D_i to M^* $(i=1, 2)$. Then, from (3.1), we can see that D_i^* coincides with the Levi-Civita connection of g_i $(i=1, 2)$. Since D_1^* and D_2^* are Ricci-flat, λ satisfies the following differential equation:

$$(n-2)(\nabla d\lambda - d\lambda \cdot d\lambda) = (\Delta\lambda - (n-2)|d\lambda|_1^2) g_1 .$$

Taking the trace of this equation with respect to g_1, we get

$$2\Delta\lambda = (n-2)|d\lambda|_1^2 , \tag{3.4}$$

and hence

$$(3.5)\qquad (n-2)(\nabla d\lambda - d\lambda\cdot d\lambda) = -(\Delta\lambda)g_1\,.$$

Let ∇^0 be the Levi-Civita connection of g_0. Since $g_0 = e^{\lambda}g_1 \cdot \nabla^0$ is related to ∇ as follows:

$$(3.6)\qquad \nabla^0_X Y = \nabla_X Y + \frac{1}{2}(d\lambda(X)\cdot Y + d\lambda(Y)\cdot X - g_1(X, Y)\operatorname{grad}\lambda)\,,$$

where $\operatorname{grad}\lambda$ is the vector field dual to $d\lambda$ with respect to g_1 (cf. (1.3)). Let us calculate $\nabla^0 d\lambda$. The formula (3.6) yields immediately

$$(\nabla^0_X d\lambda)(Y) = (\nabla_X d\lambda)(Y) - d\lambda(X)\cdot d\lambda(Y) + \frac{1}{2}|d\lambda|_1^2\cdot g_1(X, Y)\,.$$

Substituting (3.5) into this expression, we get

$$\begin{aligned} 2(n-2)(\nabla^0_X d\lambda)(Y) &= -2(\Delta\lambda)g_1(X, Y) + (n-2)|d\lambda|_1^2\cdot g_1(X, Y) \\ &= -(2\Delta\lambda - (n-2)|d\lambda|_1^2)\cdot g_1(X, Y)\,, \end{aligned}$$

and hence $(\nabla^0_X d\lambda)(Y) = 0$ by (3.4). This implies that $|d\lambda|_0$ is a constant. Putting $c = |d\lambda|_0^2$, we see from (3.2) that $c\cdot e^{\mu} = \pi^* f$. If $c = 0$, we get $\omega_1 = \omega_2$. But this cannot occur since $D_1 \neq D_2$. Thus we must have $c > 0$ and $\mu = \pi^*(\log f - \log c)$. Then, by the definition of μ, we finally obtain $\omega_1 + \omega_2 = d(\log f)$, and hence $ch(D_1) + ch(D_2) = 0$. We have thereby proved Proposition 3.1.

Now assume that (M, C) admits a Ricci-flat Weyl structure D with $ch(D) \neq 0$. Suppose there were three distinct Ricci-flat Weyl structures $D_1 = D$, D_2 and D_3 on (M, C). Since $D_1 \neq D_2$, it follows from Proposition 3.1 that $ch(D_1) = -ch(D_2)$. By the same reason, we also have $ch(D_2) = -ch(D_3)$. Hence, $ch(D_1) = ch(D_3)$. But, since $D_1 \neq D_3$, we must have $ch(D_1) = 0$. This contradicts the assumption. Thus we arrive at the following finiteness theorem.

THEOREM 3.2. *Let (M, C) be a connected conformal manifold with* $\dim M \geqq 3$. *If there is a Ricci-flat Weyl structure D on (M, C) such that $ch(D) \neq 0$, then (M, C) admits at most two Ricci-flat Weyl structures.*

We consider again an n-dimensional standard conformal manifold $(N(\rho), C_{h,a})$, where (N, h) is an $(n-1)$-dimensional connected Einstein manifold with constant scalar curvature $(n-1)(n-2)$ and a is a non-zero real number. Then the standard Ricci-flat Weyl structure D_a on $(N(\rho), C_{h,a})$ satisfies $ch(D_a) \neq 0$. Thus we have

COROLLARY 3.3. *Every standard conformal manifold $(N(\rho), C_{h,a})$ admits just two Ricci-flat Weyl structutes, that is, D_a and D_{-a}.*

If we restrict ourselves to (geodesically) complete Ricci-flat Weyl structures, we can prove that a connected conformal manifold can admit at most *one* complete Ricci-flat Weyl structure (cf. [14]).

§4. Compact conformal manifolds and Ricci-flat Weyl structures

In this section, we deal with compact conformal manifolds. Then we can improve Theorem 3.2 and determine the structure of compact conformal manifolds which admit Ricci-flat Weyl structures. To carry out our study, we need the following

LEMMA 4.1. *Let (M, C) be a compact conformal manifold and let D be a Weyl structure on (M, C). Then there exists a Riemannian metric g in C, unique up to a constant factor, such that the 1-form $\omega=\omega_g(D)$ satisfies $\delta_g\omega=0$, where δ_g denotes the codifferential with respect to g.*

This important fact, due to Gauduchon [5], shows how Weyl geometry is properly connected with Riemannian geometry. Lemma 4.1 is proved by using some analysis on compact manifold (see [5] and also [15]). A gauge (g, ω) of D is called the Gauduchon gauge if the 1-form ω is co-closed with respect to g.

LEMMA 4.2 (cf. [8]). *Let (M, C) be a compact conformal manifold with $\dim M \geqq 3$. Assume that there is a Ricci-flat Weyl structure D on (M, C). Let (g, ω) be the Gauduchon gauge of D. Then ω is a parallel 1-form with respect to the Levi-Civita connection of g.*

Proof. Since D is Ricci-flat, we have, from (1.4).

$$r-(n-2)\nabla\omega+(n-2)\omega\cdot\omega-(n-2)|\omega|^2 g=0\,,$$

where r is the Ricci tensor of g and $|\omega|$ is the pointwise norm of ω with respect to g. Let us calculate the pointwise norm $|\nabla\omega|$. Using the usual tensor notation, we get

$$\begin{aligned}(n-2)|\nabla\omega|^2&=(n-2)(\nabla_i\omega_j)(\nabla^i\omega^j)\\&=r_{ij}\nabla^i\omega^j+(n-2)\omega_i\omega_j\nabla^i\omega^j-(n-2)|\omega|^2 g_{ij}\nabla^i\omega^j\,.\end{aligned}\tag{4.1}$$

Define a 1-form α on M by $\alpha_i=r_{ij}\omega^j$. Then

$$\begin{aligned}r_{ij}\nabla^i\omega^j&=\nabla^i(r_{ij}\omega^j)-(\nabla^i r_{ij})\omega^j\\&=-\delta_g\alpha+(\delta_g r)(\omega^\#)\,,\end{aligned}\tag{4.2}$$

where $\omega^\#$ denotes the vector field dual to ω with respect to g. By the differential Bianchi identity, the scalar curvature s of g satisfies $ds=-2\delta_g r$. Then (4.2) can be written as

$$r_{ij}\nabla^i\omega^j=-\delta_g\alpha-\frac{1}{2}\langle ds, \omega\rangle\,,$$

where $\langle\ ,\ \rangle$ denotes the pointwise inner product with respect to g. On the other hand, we have

$$\omega_i\omega_j\nabla^i\omega^j=\frac{1}{2}(\delta_g\omega)|\omega|^2-\frac{1}{2}\delta_g(|\omega|^2\omega)=-\frac{1}{2}\delta_g(|\omega|^2\omega)$$

and $g_{ij}\nabla^i\omega^j=0$. Substituting these results into (4.1), we obtain

$$2(n-2)|\nabla\omega|^2=-(n-2)\delta_g(|\omega|^2\omega)-2\delta_g\alpha-\langle ds,\omega\rangle .$$

Integrating this equation over M with respect to the Riemannian volume element, we get

$$2(n-2)\|\nabla\omega\|^2=-(ds,\omega)=-(s,\delta_g\omega)=0 ,$$

and hence $\nabla\omega=0$, where $(\ ,\ )$ (resp. $\|\cdot\|$) denotes the L^2-inner produdt (resp. L^2-norm) with respect to g. We have thereby proved Lemma 4.2.

THEOREM 4.3. *Let (M, C) be a compact, connected conformal manifold with* $\dim M\geqq 3$. *Assume that there is a Ricci-flat Weyl structure D on (M, C).*

1) *If $ch(D)=0$, there exists a Ricci-flat Riemannian metric g in C such that $Dg=0$. In this case, (M, C) admits only one Ricci-flat Weyl structure D.*

2) *If $ch(D)\neq 0$, there exists another Ricci-flat Weyl structure D' on (M, C) such that $ch(D')=-ch(D)$. In this case, (M, C) admits just two Ricci-flat Weyl structures D and D'.*

Proof. 1) The first assertion may be evident. Let D' be a Ricci-flat Weyl structure on (M, C). Suppose $D'\neq D$. Then it follows from Proposition 3.1 that $ch(D')=0$. Hence there is another Ricci-flat Riemannian metric g' in C such that $D'g'=0$. This implies that there is a non-constant function λ on M satisfying $g'=e^{2\lambda}g$. Since g and $g'$are Ricci-flat metrics, we get

$$2\Delta\lambda=(n-2)|d\lambda|_g^2 , \tag{4.3}$$

where Δ is the Laplacian of g (see (3.3) and (3.4)). Then, by the Hopf maximum principle, we can deduce from (4.3) that λ is a constant. This is a contradiction. Hence we must have $D'=D$.

2) Consider the Gauduchon gauge (g, ω) of D and the Weyl structure D' on (M, C) determined by $(g, -\omega)$. Since by Lemma 4.2 ω is parallel with respect to g, the Ricci tensor S' of D' coincides with that of D (see (1.4)). Hence we have $S'=0$. Suppose $D'=D$. Then we get $\omega=0$, which contradicts the assumption $ch(D)\neq 0$. Hence $D'\neq D$. The last assertion follows immediately from Theorem 3.2.

COROLLARY 4.4. *Let (M, C) be a compact, connected conformal manifold with* $\dim M\geqq 3$. *Then*

1) *(M, C) admits at most two Ricci-flat Weyl structures*;

2) *(M, C) admits just two Ricci-flat Weyl structures if and only if there is a Ricci-flat Weyl structure D on (M, C) such that $ch(D)\neq 0$.*

In this way, we have arrived at the final result on the number of Ricci-flat Weyl structures on a conformal manifold. Then we should clarify the structure of compact conformal manifolds which admit just two Ricci-flat Weyl structures.

THEOREM 4.5. *Let (M, C) be a compact, connected conformal manifold of dimension $n \geqq 3$. Assume that there is a Ricci-flat Weyl structure D on (M, C) satisfying $ch(D) \neq 0$. Then there exist an $(n-1)$-dimensional, compact, connected Einstein manifold (N, h) with constant scalar curvature $(n-1)(n-2)$, a homomorphism ρ of $\boldsymbol{Z}$ into the full group of isometries $I(N, h)$ and a positive number a such that D is isomorphic to the standard Ricci-flat Weyl structure D_a on $(N(\rho), C_{h,a})$.*

Proof. Let us consider the Gauduchon gauge (g, ω) of D. By Lemma 4.2, ω is parallel with respect to g. Since ω never vanishes, effecting a homothetic change of g if necessary, we can assume that the pointwise norm $|\omega|$ with respect to g satisfies $|\omega|=1$ at each point of M. Let $x \in M$ and let $\mathscr{D}_x$ denote the linear space consisting of all tangent vectors v to M at x satisfying $\omega(v)=0$. Since ω is parallel, $\mathscr{D}_X$ is invariant under the action of the holonomy group H_x of (M, g). We can see that the $(n-1)$-dimensional distribution $\mathscr{D}$ is completely integrable. Let us fix a point x_0 of M and consider the leaf N of $\mathscr{D}$ passing through the point x_0. Let $\iota: N \to M$ be the inclusion map and set $h=\iota^* g$. Then (N, h) is a totally geodesic submanifold of (M, g) (see [11]). Since (M, g) is complete, so is (N, h). It is easy to see that the Ricci tensor r_h of (N, h) satisfies $r_h=\iota^* r_g$, where r_g is the Ricci tensor of (M, g). Since D is Ricci-flat and ω is parallel, it follows from (1.4) that

$$r_g=(n-2)(g-\omega \cdot \omega)\,. \tag{4.4}$$

Hence we have $r_h=(n-2)h$ at each point of N. Thus we can conclude that (N, h) is an $(n-1)$-dimensional connected Einstein manifold with constant scalar curvature $(n-1)(n-2)$. Moreover, by Myers' theorem, we know that N is compact.

Let ξ denote the vector field dual to ω. Then ξ is also parallel and satisfies $|\xi|=1$ at each point of M. Let $\{\varphi_t\}$ be the 1-parameter group of diffeomorphisms generated by ξ. Each φ_t is an isometry of (M, g). For each positive number a, we define a differentiable map $\Phi_a: \boldsymbol{R} \times N \to M$ by $\Phi_a(t, x)=\varphi_{at}(x)$, $(t, x) \in \boldsymbol{R} \times N$. Let us describe the differential $(\Phi_a)_*$ of Φ_a. Set $y=\Phi_a(t, x)$. As usual, we identify the tangent space $T_t(\boldsymbol{R})$ with $\boldsymbol{R}$. Take any $u \in \boldsymbol{R}$ and any $v \in T_x(N)$. Then, by the Leibniz formula, we have

$$(\Phi_a)_*(u+v)=au\xi_y+(\varphi_{at})_*(v)\,. \tag{4.5}$$

We define a Riemannian metric on $\boldsymbol{R} \times N$ by $g_a^*=(adt)^2+h$. For all $u, u' \in \boldsymbol{R}$ and all $v, v' \in T_x(N)$, we obtain

$$\begin{aligned}(\Phi_a^* g)(u+v, u'+v') &= g(au\xi_y+(\varphi_{at})_*(v)\,, \quad au'\xi_y+(\varphi_{at})_*(v')) \\ &= a^2uu'+(\varphi_{at}^* g)(v, v') \\ &= a^2uu'+h(v, v')\,,\end{aligned}$$

and hence $\Phi_a^* g=g_a^*$. Then $\Phi_a: (\boldsymbol{R} \times N, g_a^*) \to (M, g)$ is an isometric immersion. Since $(\boldsymbol{R} \times N, g_a^*)$ is complete, it follows that Φ_a is a covering projection.

In order to clarify the structure of the covering space, we consider again the distribution $\mathscr{D}$. We remark that φ_t leaves $\mathscr{D}$ invariant. This implies that φ_t maps each leaf of $\mathscr{D}$ onto a leaf of $\mathscr{D}$. Taking account of this fact, we define a subset Γ of $\boldsymbol{R}$ by

$$\Gamma=\{t\in\boldsymbol{R}\mid\varphi_t(x_0)\in N\}\ .$$

This is equivalent to saying that Γ consists of all $t\in\boldsymbol{R}$ satisfying $\varphi_t(N)=N$. Since N is compact, Γ is a closed subgroup of $\boldsymbol{R}$. We should first prove that Γ contains non-zero numbers. Suppose $\Gamma=\{0\}$. Putting $a=1$, we consider the covering map Φ_1. Assume that $\Phi_1(t,x)=\Phi_1(s,y)$ for (t,x), $(s,y)\in\boldsymbol{R}\times N$. Then we have $\varphi_{t-s}(x)=y$. Since $x, y\in N$, it follows that $\varphi_{t-s}(N)=N$. By the remark above, we get $t-s\in\Gamma$, and hence $t=s$. This means that Φ_1 is injective. Thus we can see that Φ_1 is a diffeomorphism of $\boldsymbol{R}\times N$ onto M. But this is a contradiction since M is compact. Therefore Γ contains non-zero numbers.

Now we shall prove that Γ is discrete in $\boldsymbol{R}$. There are an open neighborhood U of x_0 and a function f on U satisfying $\omega=df$ and $f(x_0)=0$. Shrinking U if necessary, we can assume that $N\cap U$ is given by

$$N\cap U=\{x\in U\mid f(x)=0\}\ .$$

Suppose, for any $\varepsilon>0$, there were a number $t(\varepsilon)\in\Gamma$ such that $0<|t(\varepsilon)|<\varepsilon$. It may be sufficient to consider positive numbers ε such that $\varphi_{t(\varepsilon)}(x_0)\in U$. Then the condition $t(\varepsilon)\in\Gamma$ implies that $\varphi_{t(\varepsilon)}(x_0)$ lies in $N\cap U$. Hence we get

$$\xi_{x_0}(f)=\lim_{\varepsilon\to 0}\frac{1}{t(\varepsilon)}\{f(\varphi_{t(\varepsilon)}(x_0))-f(x_0)\}=0\ .$$

But this is a contradiction since $|\xi|=1$ at x_0. Then there is a positive ε such that $(-\varepsilon,\varepsilon)\cap\Gamma=\{0\}$. Thus Γ is a discrete subgroup of $\boldsymbol{R}$. Putting

$$a=\inf\{|t|\mid t\in\Gamma,\ t\neq 0\}\ , \tag{4.6}$$

we finally obtain $\Gamma=a\cdot\boldsymbol{Z}$.

For each $m\in\boldsymbol{Z}$, we define a diffeomorphism $\rho(m)$ of N by $\rho(m)(x)=\varphi_{am}(x)$, $x\in N$. It is easy to see that each $\rho(m)$ is an isometry of (N,h) and that the map $\rho:\boldsymbol{Z}\to I(N,h)$ is a homomorphism. The group $\boldsymbol{Z}$ acts on $\boldsymbol{R}\times N$ as follows:

$$(t,x)\cdot m=(t+m,\ \rho(-m)(x))\ ,$$

where $(t,x)\in\boldsymbol{R}\times N$ and $m\in\boldsymbol{Z}$. It is clear that the action is free. We denote the right action by R_m. Here we reconsider the covering map $\Phi_a:\boldsymbol{R}\times N\to M$, where a denotes the positive number given by (4.6). For simplicity, we write $\Phi=\Phi_a$. Then it is easy to prove the following properties.

P-1) $\Phi(R_m(t,x))=\Phi(t,x)$ for all $(t,x)\in\boldsymbol{R}\times N$ and all $m\in\boldsymbol{Z}$.

P-2) If two points (t,x) and (s,y) of $\boldsymbol{R}\times N$ satisfy $\Phi(t,x)=\Phi(s,y)$, then there is an integer $m\in\boldsymbol{Z}$ such that $(s,y)=R_m(t,x)$.

By P-1) and P-2), we can conclude that $\Phi:\boldsymbol{R}\times N\to M$ is a principal fibre bundle with structure group $\boldsymbol{Z}$. The orbit space of $\boldsymbol{R}\times N$ by the $\boldsymbol{Z}$-action is nothing but the space $N(\rho)$ (see §2). It follows that Φ induces a unique diffeomorphism F: $N(\rho)\to M$ satisfying $F\circ\pi=\Phi$. Let g_a denote the metric on $N(\rho)$ determined by g_a^*, i.e., $\pi^*g_a=g_a^*$.

Then we have easily $F^*g = g_a$. The 1-form adt on $\boldsymbol{R} \times N$ induces a unique 1-form ω_a such that $\pi^*\omega_a = adt$. Then, since $\Phi^*\omega = adt$ by (4.5), we have $F^*\omega = \omega_a$. Therefore, by Proposition 1.2, we can conclude that F is an isomorphism of the standard Ricci-flat Weyl structure D_a on $(N(\rho), C_{h,a})$ onto D. This completes the proof of Theorem 4.5.

COROLLARY 4.6. *Let (M, C) be a compact, connected conformal manifold with* $\dim M \geqq 3$. *If (M, C) admits two distinct Ricci-flat Weyl structures, then (M, C) is isomorphic to a compact standard conformal manifold.*

COROLLARY 4.7. *Let (M, C) be as above. Assume that (M, C) admits two distinct Ricci-flat Weyl structures. Then $\chi(M) = 0$ and $b_1(M) = 1$, where $\chi(M)$ is the Euler characteristic of M and $b_1(M)$ is the first Betti number of M.*

In fact, considering again the smooth vector field ξ on M, we have immediately $\chi(M) = 0$. Since M is diffeomorphic to $N(\rho)$, we can consider the fibre bundle $p: N(\rho) \to S^1$ with standard fibre N. Then the conclusion $b_1(M) = 1$ follows from the fact that the fundamental group $\pi_1(N)$ of N is finite (see [1]).

By the formula (4.4), we can prove that (M, g) is a compact connected Riemannian manifold of non-negative Ricci curvature. Then the famous work of J. Cheeger and D. Gromoll can be well applied to the space (M, g) (see [2] and [3]). However, Theorem 4.5 cannot be deduced from their general results.

§5. Flat Weyl structures

In this section, we deal with flat Weyl structures. Since every flat Weyl structure is Ricci-flat, almost all results in the previous sections hold good for flat Weyl structures. First of all, we deduce from Proposition 1.1 the following

PROPOSITION 5.1. *Let D be a Weyl structure on a conformal manifold (M, C)* with $\dim M \geqq 3$. *Then the following two conditions are mutually equivalent*:

1) *D is flat*;

2) *D is Ricci-flat, and the Weyl conformal curvature tensor W of C vanishes identically on M.*

Moreover, if the condition 1) *is satisfied, then every Ricci-flat Weyl structure on (M, C) is necessarily flat.*

By virtue of Proposition 5.1, Theorems 3.2 and 4.3 are still valid if we replace the term "Ricci-flat" by "flat". Let us restate Corollary 4.4.

THEOREM 5.2. *Let (M, C) be a compact, connected conformal manifold with* $\dim M \geqq 3$. *Then*

1) *(M, C) admits at most two flat Weyl structures*;

2) *(M, C) admits just two flat Weyl structures if and only if there is a flat Weyl structure D on (M, C) such that $ch(D) \neq 0$.*

To clarify the structure of compact conformal manifolds which admit two flat Weyl structures, we first consider a standard conformal manifold. Let $n \geqq 3$ and let (N, h) be an $(n-1)$-dimensional, connected Einstein manifold with constant scalar curvature $(n-1)(n-2)$. Let ρ be a homomorphism of $\boldsymbol{Z}$ into the group of isometries $I(N, h)$ and let a be a non-zero real number. Then we can consider the standard conformal manifold $(N(\rho), C_{h,a})$ (see §2).

PROPOSITION 5.3. *Let (N, h), ρ and a be as above. Then the following two conditions are equivalent*:

1) *The standard Ricci-flat Weyl structures D_a and D_{-a} on $(N(\rho), C_{h,a})$ are flat*;

2) *(N, h) is a space of constant curvature with sectional curvature $+1$.*

In particular, if $\dim N(\rho)=3$ *or* 4, *then D_a and D_{-a} are flat.*

Proof. By Proposition 5.1, 1) is equivalent to the vanishing of the Weyl conformal curvature tensor W of $C_{h,a}$. If $\dim N=2$, it is obvious that both 1) and 2) are true. If $\dim N \geqq 3$, the vanishing of W is equivalent to saying that the product metric $g_a^* = (adt)^2 + h$ on $\boldsymbol{R} \times N$ is conformally flat. But this is equivalent to the condition 2) (see [1] p. 61). On the other hand, every 3-dimensional Einstein manifold is a space of constant curvature. Therefore, if $\dim N(\rho)=4$, D_a is flat.

We call D_a the *standard flat Weyl structure* on $(N(\rho), C_{h,a})$ if (N, h) satisfies the condition 2) in Proposition 5.3. Then we have the following

THEOREM 5.4. *Let (M, C) be a compact, connected conformal manifold with* $\dim M \geqq 3$. *Assume that there is a flat Weyl structure D on (M, C) satisfying $ch(D) \neq 0$. Then there exist a compact, connected Riemannian space (N, h) of constant curvature with sectional curvature $+1$, a homomorphism $\rho: \boldsymbol{Z} \to I(N, h)$ and a positive number a such that D is isomorphic to the standard flat Weyl structure D_a on $(N(\rho), C_{h,a})$.*

This follows immediately from Theorem 4.5 and Proposition 5.3. It should be remarked that the universal covering space of (N, h) is given by $(S^{n-1}$, can), where $n = \dim M$. Then we can see that M itself is covered by $\boldsymbol{R} \times S^{n-1}$. In the 4-dimensional case, Gauduchon called $N(\rho)$'s "variétés de type $S^1 \times S^3$" (see [8]). The situation as in Theorem 5.4 has been also studied from various points of view (cf. [4], [16]). Finally, from Proposition 5.3 and Theorem 5.4 we have

PROPOSITION 5.5 (cf. [8]). *Let (M, C) be a 3 or 4-dimensional, compact, connected conformal manifold. If (M, C) admits a Ricci-flat Weyl structure D with $ch(D) \neq 0$, then D is flat.*

§6. Equivalence problem for standard Ricci-flat Weyl structures

We know now that every compact conformal manifold which admits non-Riemannian Ricci-flat Weyl structures is isomorphic to a certain standard conformal manifold. Our task is then to study the equivalence problem for standard Ricci-flat Weyl structures. We have already found in Proposition 2.2 a sufficient condition. Here we shall prove the converse of Proposition 2.2 for the compact case.

Let $n \geqq 3$. Let (N, h) and (N', h') be $(n-1)$-dimensional, *compact*, connected Einstein manifolds with positive constant scalar curvature $(n-1)(n-2)$. Let $\rho: \boldsymbol{Z} \to I(N, h)$ and $\rho': \boldsymbol{Z} \to I(N', h')$ be a homomorphisms. Let a and b be non-zero real numbers.

THEOREM 6.1. *Let the notation be as above. Consider the standard Ricci-flat Weyl structure D_a on $(N(\rho), C_{h,a})$ and the standard Ricci-flat Weyl structure D_b on $(N'(\rho'), C_{h',b})$. Then D_a and D_b are isomorphic if and only if the following two conditions are satisfied*:

1) $a = \varepsilon b$, *where* ε *is either* $+1$ *or* -1;
2) *There is an isometry* $\varphi: (N, h) \to (N', h')$ *such that*

$$\varphi \circ \rho(m) = \rho'(\varepsilon m) \circ \varphi$$

for any $m \in \boldsymbol{Z}$.

Proof. Suppose F: $N(\rho) \to N'(\rho')$ is an isomorphism of D_a onto D_b. Consider the gauge (g_a, ω_a) of D_a and the gauge (g_b, ω_b) of D_b constructed in §2. Since ω_a (resp. ω_b) is parallel with respect to g_a (resp. g_b), (g_a, ω_a) (resp. (g_b, ω_b)) is the Gauduchon gauge of D_a (resp. D_b) (see Lemma 4.1). Since F is an isomorphism of D_a onto D_b, it follows from Proposition 1.2 that $(F^*g_b, F^*\omega_b)$ is also the Gauduchon gauge of D_a. Thus there is a positive constant λ such that

$$F^*g_b = \lambda g_a\,. \tag{6.1}$$

Furthermore, we get

$$F^*\omega_b = \omega_a\,. \tag{6.2}$$

We shall first prove $\lambda = 1$. Define a vector field ξ_a^* on $\boldsymbol{R} \times N$ (resp. ξ_b^* on $\boldsymbol{R} \times N'$) by

$$\xi_a^* = \frac{1}{a}\left(\frac{\partial}{\partial t}\right) \qquad \left(\text{resp. } \xi_b^* = \frac{1}{b}\left(\frac{\partial}{\partial t}\right)\right). \tag{6.3}$$

Since the vector field ξ_a^* (resp. ξ_b^*) is invariant under the action of $\boldsymbol{Z}$, there is a unique vector filed ξ_a on $N(\rho)$ (resp. ξ_b on $N'(\rho')$) such that $\pi_*\xi_a^* = \xi_a$ (resp. $\pi_*\xi_b^* = \xi_b$). The equation (6.2) yields $\omega_b(F_*\xi_a) = 1$. Then it follows from (2.2) that

$$g_b(F_*\xi_a, F_*\xi_a) \geqq (\omega_b(F_*\xi_a))^2 = 1\,.$$

On the other hand, we have

$$g_a(\xi_a, \xi_a) = (\pi^*g_a)(\xi_a^*, \xi_a^*) = g_a^*(\xi_a^*, \xi_a^*) = 1\,.$$

These results together with (6.1) yields $\lambda \geqq 1$. Next let us consider the inverse map $F^{-1}: N'(\rho') \to N(\rho)$. Then we get $(F^{-1})^*g_a = \lambda^{-1}g_b$ and $\omega_a((F^{-1})_*\xi_b) = 1$. By the same argument, we can verify $\lambda^{-1} \geqq 1$. Therefore we finally obtain $\lambda = 1$.

The equation (6.2) means that F maps each fibre of $\rho: N(\rho) \to S^1$ onto a fibre of $p': N'(\rho') \to S^1$. It follows that F induces a diffeomorphism f_0 of S^1 onto S^1 satisfying $p' \circ F = f_0 \circ p$. Then there is a diffeomorphism $f: \boldsymbol{R} \to \boldsymbol{R}$ such that $\pi_0 \circ f = f_0 \circ \pi_0$, where $\pi_0: \boldsymbol{R} \to S^1$ is the covering projection. Then (6.2) can be rewritten as $f^*(bdt) = adt$. Hence we get $f(t) = \varepsilon t + c$, where we put $\varepsilon = a/b$ and c is a constant. But, since f induces f_0, we must have $\varepsilon = \pm 1$. This proves 1).

Now we can see that F maps the fibre $N_0 = p^{-1}(\pi_0(0))$ onto the fibre $N'_c = (p')^{-1}(\pi_0(c))$. We define a diffeomorphism $\beta: N \to N_0 \subset N(\rho)$ by $\beta(x) = \pi(0, x)$, $x \in N$, and a diffeomorphism $\beta': N' \to N'_c \subset N'(\rho')$ by $\beta'(y) = \pi(c, y)$, $y \in N'$. Since $F(N_0) = N'_c$, we can define a diffeomorphism φ of N onto N' by

$$\beta' \circ \varphi = F \circ \beta . \tag{6.4}$$

We have easily $\beta^* g_a = h$ and $(\beta')^* g_b = h'$. Then, from (6.4), we see that φ is an isometry of (N, h) onto (N', h').

Here we consider again vector fields ξ_a on $N(\rho)$ and ξ_b on $N'(\rho')$. Let $\{\Phi_s\}$ (resp. $\{\Phi'_s\}$) denote the 1-parameter group of diffeomorphisms generated by ξ_a^* (resp. ξ_b^*). Then, from (6.3), we have the following explicit expression:

$$\Phi_s(t, x) = \left(\frac{s}{a} + t, x\right) \quad \left(\text{resp. } \Phi'_s(t, y) = \left(\frac{s}{b} + t, y\right)\right),$$

where $(t, x) \in \boldsymbol{R} \times N$ (resp. $(t, y) \in \boldsymbol{R} \times N'$). Let $\{\varphi_s\}$ (resp. $\{\varphi'_s\}$) denote the 1-parameter group of diffeomorphisms generated by ξ_a (resp. ξ_b). Since $F_* \xi_a = \xi_b$, we have

$$F \circ \varphi_s = \varphi'_s \circ F , \tag{6.5}$$

for all $s \in \boldsymbol{R}$. The relation $\pi_* \xi_a^* = \xi_a$ yields $\pi \circ \Phi_s = \varphi_s \circ \pi$. Therefore, for all $m \in \boldsymbol{Z}$ and all $x \in N$, we get

$$\varphi_{am}(\pi(0, x)) = \pi(\Phi_{am}(0, x)) = \pi(m, x) ,$$

and hence

$$\varphi_{am} \circ \beta = \beta \circ \rho(m) . \tag{6.6}$$

In a similar way, we have

$$\varphi'_{bm} \circ \beta' = \beta' \circ \rho'(m) . \tag{6.7}$$

From (6.4), (6.5) and (6.6), we have immediately

$$\beta'(\varphi \circ \rho(m)(x)) = \varphi'_{am} \circ \beta' \circ \varphi(x) ,$$

for all $m \in \boldsymbol{Z}$ and all $x \in N$. Using the relation $a = \varepsilon b$ and (6.7), we get

$$\beta'(\varphi \circ \rho(m)(x)) = \beta'(\rho'(\varepsilon m) \circ \varphi(x)) ,$$

and hence $\varphi \circ \rho(m) = \rho'(\varepsilon m) \circ \varphi$. This proves 2).

The converse is proved in Proposition 2.2. We have thereby proved Theorem 6.1.

Corollary 6.2. *Let (N, h), ρ and a be as before. Then the standard Ricci-flat Weyl structures D_a and D_{-a} on $(N(\rho), C_{h,a})$ are isomorphic if and only if there is an* *isometry φ of (N, h) such that*

$$\varphi \circ \rho(m) = \rho(-m) \circ \varphi$$

for any $m \in \boldsymbol{Z}$.

This follows immediately from Theorem 6.1. Now we set $\rho^*(m) = \rho(-m)$ for any $m \in \boldsymbol{Z}$. Then ρ^* is also a homomorphism of $\boldsymbol{Z}$ into the group of isometries $I(N, h)$. The condition in Corollary 6.2 is equivalent to saying that $\rho(m)$ and $\rho^*(m)$ are conjugate in $I(N, h)$ for all $m \in \boldsymbol{Z}$. Moreover, we have

COROLLARY 6.3. *Let (N, h), ρ, ρ^* and a be as above. Then the standard Ricci-flat Weyl structure D_{-a} on $(N(\rho), C_{h,-a})$ is isomorphic to the standard Ricci-flat Weyl structure D_a on $(N(\rho^*), C_{h,a})$.*

In view of Corollary 6.3, it is sufficient to consider only positive numbers a. Note that the homomorphism ρ is determined by the isometry $\rho(1)$ of (N, h).

Let us now fix an $(n-1)$-dimensional, compact, connected Einstein manifold (N, h) with constant scalar curvature $(n-1)(n-2)$. Then we can see from Theorem 6.1 and Corollary 3.3 that an n-dimensional, compact, standard conformal manifold with fibre (N, h) is completely determined by the following data:

1) a positive number a;
2) a representative $\rho(1)$ of a conjugate class in $I(N, h)$.

If we consider (S^{n-1}, can), then the determination of all conjugate classes in $I(S^{n-1}, can)$ is reduced to the well-known result of linear algebra.

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