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Description of the Associated Varieties for the Discrete Series Representations of a Semisimple Lie Group

—An elementary proof by means of differential operators of gradient type—

by

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Introduction

Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. The corresponding complexified Lie algebras are denoted respectively by $\mathfrak{g}$ and $\mathfrak{k}$. We assume Harish-Chandra's rank condition rank $G=$ rank K, which is necessary and sufficient for G to have a non-empty set of discrete series, consisting of square-integrable irreducible unitary representations of G ([6]).

Concrete geometric realizations of discrete series representations have been obtained in several ways (see e.g., the survey article [4]). Among others, Hotta and Parthasarathy [8] realize such representations on the kernel spaces of certain G-invariant differential operators $\mathscr{D}_\lambda$ of gradient-type (originally due to Schmid [10]), defined on vector bundles over the symmetric space G/K, by using some elementary differential calculus on G/K (see §5). Here λ denotes the highest weight of the lowest K-type of the corresponding discrete series representation of G. As we have shown in [19] and [22], the operators $\mathscr{D}_\lambda$ allow us to determine the embeddings of discrete series into various important induced G-modules.

The purpose of this paper is to give an elementary proof, based on the above work of Hotta-Parthasarathy, for the following theorem, which describes the associated varieties of Harish-Chandra $(\mathfrak{g}, K)$-modules of the discrete series.

THEOREM (Theorem 3.1). *If H_Λ is the $(\mathfrak{g}, K)$-module of discrete series with Harish-Chandra parameter $\Lambda=\lambda+\rho_c-\rho_n$ (see §2), then its associated variety $\mathscr{V}(H_\Lambda)\subset\mathfrak{g}$ (see §1 for the definition) coincides with the nilpotent cone $K_{\mathbf{C}}\mathfrak{p}_-$. Here $K_{\mathbf{C}}$ is the analytic subgroup of adjoint group $G_{\mathbf{C}}:=\mathrm{Int}(\mathfrak{g})$ of $\mathfrak{g}$, with Lie algebra $\mathfrak{k}$, and $\mathfrak{p}_-$ denotes the sum of root subspaces of $\mathfrak{g}$ corresponding to the non-compact roots which are negative with respect to Λ.*

It follows from this theorem that the vareity $\mathscr{V}(U(\mathfrak{g})/I_\Lambda)$ defined by the primitive ideal $I_\Lambda:=\mathrm{Ann}_{U(\mathfrak{g})}(H_\lambda)$ in the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ equals the closure of the

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cone $G_C\mathfrak{p}_-$ (Theorem 3.2). We note that the Gelfand-Kirillov dimensions $d(H_\Lambda) := \dim \mathscr{V}(H_\Lambda)$ of discrete series can be computed explicitly by specifying the unique nilpotent K_C-orbit in $\mathfrak{p}$, or the unique nilpotent G_C-orbit in $\mathfrak{g}$ (see Proposition 3.8), which intersects $\mathfrak{p}_-$ densely. See [5], [17] and [21, Theorem 4] for some explicit computation by combinatorial method.

We know that Theorem 3.1 can be deduced from the results in [3, III] and [11], by passing to D-modules via Beilinson-Bernstein correspondence. In fact, (a) the associated variety of a Harish-Chandra module is gained, through the moment map, as the image of characteristic variety of corresponding D-module over the complexified flag variety X of G, and (b) the characteristic vareity of a discrete series D-module can be specified as the conormal bundle of a closed K_C-orbit on X. However these (a) and (b) rely on several deep results about the classification of irreducible G-representations through D-modules, K_C-orbit structure of the variety X, etc., although the associated variety is a very simple object defined for each finitely generated $U(\mathfrak{g})$-module in a purely algebraic context (see §1). From these reasons, we make here a short-cut and give a direct and an elementary proof for the description of the variety $\mathscr{V}(H_\Lambda)$ only by using some basic facts on realization of discrete series. This method allows us to get new information on the annihilator ideal of the graded module $\mathrm{Gr}\, H_\Lambda$ over the symmetric algebra $S(\mathfrak{g}) \simeq \mathrm{gr}\, U(\mathfrak{g})$ of $\mathfrak{g}$ (Theorem 6.5). Here are placed our motivation and emphasis of this presentation.

In order to prove Theorem 3.1 we first assume that the parameter λ is sufficiently regular, and we pass to a graded space of coefficients of Taylor expansions of analytic sections in $\mathrm{Ker}\, \mathscr{D}_\lambda$. This space of coefficients, say $\mathrm{Gr}(\mathrm{Ker}\, \mathscr{D}_\lambda)$, admits a natural $S(\mathfrak{g})$-module structure. Then it can be shown by using Theorem 1 of [8] that the corresponding annihilator ideal in $S(\mathfrak{g})$ defines the associated variety of dual discrete series module H_Λ^* as the common zero. Another point in proving Theorem 3.1 is that the $S(\mathfrak{g})$-module $\mathrm{Gr}(\mathrm{Ker}\mathscr{D}_\lambda)$ is characterized as the kernel of a differential operator $\mathrm{Gr}[\mathscr{D}_\lambda]$ defined on a space of (vector-valued) polynomial functions on $\mathfrak{p}$, which naturally arises from $\mathscr{D}_\lambda$ by passing to the gradation ([8], see Theorem 5.1).

These two results allow us to establish Theorem 3.1 for sufficiently regular λ, by examining in §6 the annihilator ideal of $S(\mathfrak{g})$-module $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ more closely. We further find that for such a λ the ideal $\mathrm{Ann}_{S(\mathfrak{g})} \mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ coincides with its radical in $S(\mathfrak{g})$ (Corollary 6.6). Finally, our theorem for arbitrary λ follows from the result for the above sufficiently regular case, with in mind Zuckerman's translation principle (see §7).

This paper is organized as follows. We begin with introducing in §§1–2 the associated varieties for $U(\mathfrak{g})$-modules and the discrete series for G. §3 describes the variety $\mathscr{V}(H_\Lambda)$ (Theorem 3.1), and we deduce two important consequences (Theorem 3.2 and Proposition 3.8) from Theorem 3.1. The succeeding four sections, §§4–7, are devoted to proving Theorem 3.1.

The first version of this article was written in April 1993, and the main results have been announced in [21].

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1. Associated varieties for $U(\mathfrak{g})$-modules

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We begin with introducing two important invariants: the associated variety and Gelfand-Kirillov dimension, for finitely generated $U(\mathfrak{g})$-modules.

Denote by $(U_k(\mathfrak{g}))_{k=0,1,\cdots}$ the natural increasing filtration of $U(\mathfrak{g})$, when $U_k(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ generated by elements $X_1 \cdots X_m$ $(m \leq k)$ with $X_j \in \mathfrak{g}$ $(1 \leq j \leq m)$. By the Poincaré-Birkhoff-Witt theorem, we can and do identify the associated graded ring

$$\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{k \geq 0} U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g}) \qquad (U_{-1}(\mathfrak{g}) := (0))$$

with the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k \geq 0} S^k(\mathfrak{g})$ of $\mathfrak{g}$ in the canonical way. Here $S^k(\mathfrak{g})$ denotes the homogeneous component of $S(\mathfrak{g})$ of degree k.

Let H be a finitely generated $U(\mathfrak{g})$-module. Take a finite-dimensional subspace H_0 of H such that $H = U(\mathfrak{g})H_0$. Setting $H_k = U_k(\mathfrak{g})H_0$ $(k = 1, 2, \cdots)$, one gets an increasing filtration $(H_k)_k$ of H and correspondingly a finitely generated, graded $S(\mathfrak{g})$-module

$$M = \operatorname{gr}(H; H_0) := \bigoplus_{k \geq 0} M_k \text{ with } M_k = H_k/H_{k-1}\,. \tag{1.1}$$

The annihilator $\operatorname{Ann}_{S(\mathfrak{g})} M := \{D \in S(\mathfrak{g}) \mid Dv = 0 \ (\forall v \in M)\}$ of M is a graded ideal of $S(\mathfrak{g})$, and it defines an algebraic cone in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$:

$$\mathscr{V}(M) := \{\lambda \in \mathfrak{g}^* \mid f(\lambda) = 0 \ (\forall f \in \operatorname{Ann}_{S(\mathfrak{g})} M)\}\,,$$

as the common zero of elements of $\operatorname{Ann}_{S(\mathfrak{g})} M$. Here $S(\mathfrak{g})$ is looked upon as the polynomial ring over $\mathfrak{g}^*$ in the canonical way. It is then easily seen that the variety $\mathscr{V}(M)$ does not depend on the choice of a generating subspace H_0. So, hereafter we write $\mathscr{V}(H)$ for this invariant $\mathscr{V}(M)$ of H.

DEFINITION. (Cf. [14], [20]) For a finitely generated $U(\mathfrak{g})$-module H, the variety $\mathscr{V}(H) \subset \mathfrak{g}^*$ and its dimension $d(H) := \dim \mathscr{V}(H)$ are called respectively the *associated variety* and the *Gelfand-Kirillov dimension* of H.

It should be noticed that, by the Hilbert-Serre theorem (cf. [20, Th. 1.1]), the map $k \mapsto \dim H_k$ coincides with a polynomial in k of degree $d(H)$, for sufficiently large k.

2. Discrete series for a semisimple Lie group

Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. The corresponding Lie algebras are denoted respectively by $\mathfrak{g}_0$ and $\mathfrak{k}_0$. Then one has a Cartan decomposition $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ of $\mathfrak{g}_0$. We always assume the rank condition $\operatorname{rank} G=\operatorname{rank} K$, which is necessary and sufficient for G to have a non-empty discrete series. In this section we collect some basic facts and fix notation on the discrete series representations of G.

Take a maximal abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$, which is a compact Cartan subalgebra of $\mathfrak{g}_0$ by the above assumption on G. Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$, and we write $\mathfrak{h}\subset\mathfrak{g}$ for the complexification of a real vector subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ by dropping the subscript '0'. If $\alpha\in\mathfrak{t}^*$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{t}$, the corresponding root subspace

$$\mathfrak{g}_\alpha:=\{X\in\mathfrak{g}\,|\,[Z, X]=\alpha(Z)X\ (\forall Z\in\mathfrak{t})\}$$

is contained either in $\mathfrak{k}$ or in $\mathfrak{p}$. A root α is said to be compact or non-compact according as $\mathfrak{g}_\alpha\subset\mathfrak{k}$ or $\mathfrak{g}_\alpha\subset\mathfrak{p}$. We denote the totality of roots (resp. compact roots, non-compact roots) by Δ (resp. Δ_c, Δ_n).

Now fix a positive system Δ_c^+ of Δ_c. Let Ξ be the set of linear forms Λ on $\mathfrak{t}$ satisfying the following three conditions:

(2.1) $(\Lambda, \alpha)\geq 0$ for any $\alpha\in\Delta_c^+$, i.e., Λ is Δ_c^+-dominant,

(2.2) $(\Lambda, \alpha)\neq 0$ for any $\alpha\in\Delta$, i.e., Λ is Δ-regular,

(2.3) the map $Z\mapsto\exp\langle\Lambda+\rho, Z\rangle$ $(Z\in\mathfrak{t}_0)$ defines a well-defined unitary character of the Cartan subgroup $T:=\exp\mathfrak{t}_0$, i.e., $\Lambda+\rho$ is T-integral.

Here $(\cdot,\cdot)$ denotes the bilinear form on $\mathfrak{t}^*$ induced canoniacally from the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{t}$, and ρ is half the sum of positive roots in Δ with respect to any fixed positive system of Δ. Notice that the condition (2.3) does not depend on the choice of a positive system which defines ρ.

By Harish-Chandra, there exists a bijective correspondence, say $\Lambda\mapsto\pi_\Lambda$, from Ξ onto the set of (equivalence classes of) discrete series representations of G (see e.g., [19, I, Prop. 1.1]). We say that the discrete series representation π_Λ has *Harish-Chandra parameter* Λ.

What is more important in this article is however the lowest K-type property which characterizes the discrete series π_Λ. To be precise, for a Δ_c^+-dominant, T-integral linear form $\mu\in\mathfrak{t}^*$, let (τ_μ, V_μ) denote the finite-dimensional irreducible K-module with highest weight μ. We set for a $\Lambda\in\Xi$.

$$\lambda:=\Lambda-\rho_c+\rho_n=(\Lambda-2\rho_c)+\rho=(\Lambda+2\rho_n)-\rho\,, \tag{2.4}$$

where half the sum ρ of positive roots is defined by the positive system $\Delta^+:=\{\alpha\in\Delta\,|\,(\Lambda, \alpha)>0\}$, and $\rho_c:=(1/2)\cdot\sum_{\alpha\in\Delta_c^+}\alpha$, $\rho_n:=\rho-\rho_c$.

PROPOSITION 2.1. (See e.g., [4]) (i) *The discrete series representation π_Λ, looked upon as a K-module, has lowest K-type τ_λ:*

(a) π_Λ *contains* τ_λ *with multiplicity one,*

(b) *the highest weight of any irreducible K-representation occurring in* π_Λ *is of the form*

$$\lambda + \sum_{\alpha\in\Delta^+} n_\alpha \alpha$$

with non-negative integers n_α.

(ii) *Conversely, if an irreducible unitary representation* π *of* G *satisfies* (a) *and* (b), *then* π *is unitarily equivalent to* π_Λ.

Suggested by this proposition, we call $\lambda = \Lambda - \rho_c + \rho_n$ the *lowest highest weight* (or the *Blattner parameter*) of π_Λ.

3. Description of the associated varieties for the discrete series

For a $\Lambda \in \Xi$, let H_Λ be the Harish-Chandra $(\mathfrak{g}, K)$-module corresponding to π_Λ, which is gained by passing to the K-finite part of π_Λ. It follows that H_Λ is irreducible as a $U(\mathfrak{g})$-module because of the irreducibility of the corresponding G-representation π_Λ. See e.g., [18, I, 2.4] for the definition and basic facts on Harish-Chandra $(\mathfrak{g}, K)$-modules.

In this section, we describe the associated varieties $\mathscr{V}(H_\Lambda)$ for the discrete series (Theorem 3.1), and deduce from it two important consequences (Theorem 3.2 and Proposition 3.8).

3.1. Varieties $\mathscr{V}(H_\Lambda)$ and $\mathscr{V}(U(\mathfrak{g})/I_\Lambda)$

Now we put

$$\mathfrak{p}_\pm := \bigoplus_{\alpha\in\Delta_n^+} \mathfrak{g}_{\pm\alpha}\,, \tag{3.1}$$

where $\Delta_n^+ = \{\alpha \in \Delta_n \mid (\Lambda, \alpha) > 0\}$ denotes the set of non-compact positive roots with respect to Λ. Notice that the subspaces $\mathfrak{p}_\pm$ depend only on the chamber in which the Harish-Chandra parameter Λ lives. Let G_C be the adjoint group of $\mathfrak{g}$, and K_C be the analytic subgroup of G_C corresponding to the Lie subalgebra $\mathfrak{k}$.

THEOREM 3.1. *The associated variety* $\mathscr{V}(H_\Lambda)$ *of discrete series Harish-Chandra module* H_Λ *coincides with the nilpotent cone* $K_C\mathfrak{p}_-$. *Here* $\mathscr{V}(H_\Lambda)$ *is regarded as a variety in* $\mathfrak{g}$ *by identifying* $\mathfrak{g}^*$ *with* $\mathfrak{g}$ *through the Killing form of* $\mathfrak{g}$.

We will prove this theorem in the succeeding sections, §§4–7, by using the gradient-type differential operators $\mathscr{D}_\lambda$ on G/K whose kernels realize the discrete series representations of G (cf. [8]).

The above theorem allows us to describe also the variety $\mathscr{V}(U(\mathfrak{g})/I_\Lambda)$ associated to the primitive ideal $I_\Lambda := \mathrm{Ann}_{U(\mathfrak{g})} H_\Lambda$, as follows.

THEOREM 3.2. *One has the equality* $\mathscr{V}(U(\mathfrak{g})/I_\Lambda) = \overline{G_C\mathfrak{p}_-}$, *where* $\bar{A}$ *denotes the Zariski closure of a subset* A *of* $\mathfrak{g}$, *and* $U(\mathfrak{g})$ *acts on* $U(\mathfrak{g})/I_\Lambda$ *by left multiplication.*

This is a direct consequence of Theorem 3.1 together with the following proposition.

PROPOSITION 3.3. ([14]) *Let H be an irreducible* $(\mathfrak{g}, K)$*-module and* $I=\mathrm{Ann}_{U(\mathfrak{g})}H$ *be the corresponding primitive ideal of* $U(\mathfrak{g})$. *Then variety* $\mathscr{V}_I := \mathscr{V}(U(\mathfrak{g})/I)$ *is related to the associated variety* $\mathscr{V}(H)$ *of* H *as*

$$\mathscr{V}_I = \overline{G_C \mathscr{V}(H)}\,. \tag{3.2}$$

3.2. For the sake of completeness we show here how to prove Proposition 3.3. The proof uses four fundamental facts on the nilpotent G_C- or K_C-orbits, associated varieties and primitive ideals, which we are going to list up.

LEMMA 3.4. (Cf. [20, Lemma 3.1]) *Let* $\mathscr{N}$ *be the variety of all nilpotent elements of* $\mathfrak{g}$, *and put* $\mathscr{N}(\mathfrak{p}) := \mathscr{N} \cap \mathfrak{p}$. *If* H *and* $I=\mathrm{Am}_{U(\mathfrak{g})}H$ *are as in Proposition* 3.3, *the variety* $\mathscr{V}_I$ (*resp.* $\mathscr{V}(H)$) *is a* G_C*-stable* (*resp.* K_C*-stable*) *cone contained in* $\mathscr{N}$ (*resp. in* $\mathscr{V}_I \cap \mathfrak{p} \subset \mathscr{N}(\mathfrak{p})$).

LEMMA 3.5. (Joseph, cf. [13, Th. 3.1]) *For the above* H *and* I, *one has the equality* $\dim \mathscr{V}_I = 2\dim \mathscr{V}(H)$.

LEMMA 3.6. (See e.g., [3, III, §4]) *The variety* $\mathscr{V}_I$ *associated to a primitive ideal* $I=\mathrm{Ann}_{U(\mathfrak{g})}H \subset U(\mathfrak{g})$ *is the closure of a single nilpotent* G_C*-orbit* $\mathcal{O}_1$ *in* $\mathfrak{g}$: $\mathscr{V}_I = \overline{\mathcal{O}_1}$.

LEMMA 3.7. (Kostant-Rallis, cf. [9, Prop. 5]) *If* $\mathcal{O}$ *is a nilpotent* K_C*-orbit in* $\mathfrak{p}$, *the dimension of* G_C*-orbit* $\mathcal{O}_1 := G_C\mathcal{O}$ *containing* $\mathcal{O}$, *is equal to* $2\dim\mathcal{O}$.

Proof of Proposition 3.3. The inclusion $\overline{G_C\mathscr{V}(H)} \subset \mathscr{V}_I$ in (3.2) is clear from Lemma 3.4. To show the converse inclusion, take a nilpotent K_C-orbit $\mathcal{O}$ in $\mathfrak{p}$ such that $\dim\mathscr{V}(H) = \dim\mathcal{O}$. Such an $\mathcal{O}$ actually exists since the number of nilpotent K_C-orbits in $\mathfrak{p}$ is finite (see [7, Chap. III, Th. 4.8]). Set $\mathcal{O}_1 = G_C\mathcal{O}(\subset \mathscr{V}_I)$. Then it follows from Lemmas 3.5 and 3.7 that

$$\dim\mathcal{O}_1 = 2\dim\mathcal{O} = 2\dim\mathscr{V}(H) = \dim\mathscr{V}_I\,.$$

Hence $\mathcal{O}_1$ is an open subset of $\mathscr{V}_I$. By virtue of Lemma 3.6, we conclude that $\mathscr{V}_I = \overline{\mathcal{O}_1} \subset \overline{G_C\mathscr{V}(H)}$. Q.E.D.

REMARK. The varieties $\mathscr{V}(H)$, $\mathscr{V}_I$ are closely related to the asymptotic support and wave front set of the distribution character of H ([1]; see also [16]).

3.3. Theorem 3.2, combined with Lemmas 3.5 and 3.6, gives the following proposition, which is useful for computing explicitly the Gelfand-Kirillov dimensions of the discrete series (see [21, §8]).

PROPOSITION 3.8. *For a* $\Lambda \in \Xi$, *define a subspace* $\mathfrak{p}_- \subset \mathscr{N}(\mathfrak{p})$ *as in* (3.1).

(i) *If* $\Omega_{\mathfrak{p}_-}$ *denotes the set of nilpotent* G_C*-orbits* $\mathcal{O}_1$ *in* $\mathfrak{g}$ *such that* $\mathcal{O}_1 \cap \mathfrak{p}_- \neq \varnothing$, *there exists a unique orbit* $\mathcal{O}_{\mathfrak{p}_-} \in \Omega_{\mathfrak{p}_-}$ *for which* $\overline{\mathcal{O}_{\mathfrak{p}_-}} \supset \mathcal{O}_1$ *holds for any* $\mathcal{O}_1 \in \Omega_{\mathfrak{p}_-}$.

(ii) *The Gelfand-Kirillov dimension* $d(H_\Lambda)$ *of discrete series* $U(\mathfrak{g})$*-module* H_Λ

coincides with $(1/2)\dim \mathcal{O}_{\mathfrak{p}_-}$.

4. Associated varieties and realization of Harish-Chandra modules on G/K

For a finite-dimensional representation (τ, V_τ) of K, let $\mathscr{A}(\tau)$ be the space of real analytic functions $f: G \to V_\tau$ satisfying

$$f(gk) = \tau(k)^{-1} f(g) \quad (g \in G, k \in K). \tag{4.1}$$

The group G acts on $\mathscr{A}(\tau)$ by left translation, and $\mathscr{A}(\tau)$ turns to be a $U(\mathfrak{g})$-module through differentiation. We call $\mathscr{A}(\tau)$ the G- and $U(\mathfrak{g})$-module analytically induced from τ.

This section develops a general method for describing the associated variety $\mathscr{V}(H)$ of a Harish-Chandra module H in relation with a realization in $\mathscr{A}(\tau)$ of the K-finite dual module H^*. This is a preliminary step for the proof of Theorem 3.1.

4.1. $(S(\mathfrak{g}), K)$-module $\mathrm{Gr}\,\mathscr{A}(\tau)$

At first, we define subspaces $\mathscr{A}_{(k)}$ $(k \in \boldsymbol{Z})$ of $\mathscr{A}(\tau)$ by

$$\mathscr{A}_{(k)} := \{ f \in \mathscr{A}(\tau) \mid (X^m f)(1) = 0 \ \ (\forall X \in \mathfrak{p}, 0 \leq \forall m \leq k) \} \tag{4.2}$$

for $k \geq 0$, and $\mathscr{A}_{(k)} := \mathscr{A}(\tau)$ for $k < 0$, where 1 denotes the identity element of G. Then $(\mathscr{A}_{(k)})_{k \in \boldsymbol{Z}}$ is a decreasing filtration of $\mathscr{A}(\tau)$ such that

$$\text{each } \mathscr{A}_{(k)} \text{ is a } K\text{-stable subspace of } \mathscr{A}(\tau), \tag{4.3}$$

$$\dim \mathscr{A}(\tau)/\mathscr{A}_{(k)} < \infty \quad \text{and} \quad \bigcap_k \mathscr{A}_{(k)} = (0), \tag{4.4}$$

$$U_m(\mathfrak{g})\mathscr{A}_{(k)} \subset \mathscr{A}_{(k-m)} \quad \text{for all integers } k, m \geq 0. \tag{4.5}$$

Correspondingly, one obtains a graded $S(\mathfrak{g})$-module

$$\mathrm{Gr}\,\mathscr{A}(\tau) := \bigoplus_k \mathscr{A}_{(k)}/\mathscr{A}_{(k+1)}, \tag{4.6}$$

which admits by (4.3) a K-module structure, compatible with the $S(\mathfrak{g})$-action.

It is not difficult to analyze this $(S(\mathfrak{g}), K)$-module. To do this, let $(X_i)_{i=1}^s$ and $(X_i^*)_{i=1}^s$ be two bases of the vector space $\mathfrak{p}$ such that $B(X_i, X_j^*) = \delta_j^i$ (Kronecker's δ) for the Killing form B of $\mathfrak{g}$. We put

$$\iota_k(f) := \sum_{|\nu| = k+1} \frac{1}{\nu!} (X^*)^\nu \otimes (X^\nu f)(1) \in S^{k+1}(\mathfrak{p}) \otimes V_\tau \quad (f \in \mathscr{A}_{(k)}), \tag{4.7}$$

where $X^\nu := X_1^{\nu_1} \cdots X_s^{\nu_s}$, $(X^*)^\nu := (X_1^*)^{\nu_1} \cdots (X_s^*)^{\nu_s}$ and $\nu! = \nu_1 \cdots \nu_s!$ for multi-indices $\nu = (\nu_1, \cdots, \nu_s)$ of length $|\nu| := \nu_1 + \cdots + \nu_s = k+1$. Observe that the assignment $\mathscr{A}_{(k)} \ni f \mapsto \iota_k(f) \in S^{k+1}(\mathfrak{p}) \otimes V_\tau$ is independent of the choice of dual bases $(X_i)_i$ and $(X_i^*)_i$, and ι_k naturally gives rise to a K-isomorphism:

$$\tilde{\iota}_k : \mathscr{A}_{(k)}/\mathscr{A}_{(k+1)} \simeq S^{k+1}(\mathfrak{p}) \otimes V_\tau, \tag{4.8}$$

where $S^{k+1}(\mathfrak{p})$ is looked upon as a K-module by the adjoint action.

Through the Killing form B, we identify the symmetric algebra $S(\mathfrak{p})=\bigoplus_{k\geq 0} S^k(\mathfrak{p})$ of $\mathfrak{p}$ with the ring generated by polynomial functions on $\mathfrak{g}$ which vanish identically on $\mathfrak{k}$. Let $S(\mathfrak{g})$ act on $S(\mathfrak{p})$ canonically as the ring of constant coefficient differential operators on the vector space $\mathfrak{g}$. It should be noticed that the action of a $Y\in\mathfrak{g}$ on $X^m\in S(\mathfrak{p})$ with $X\in\mathfrak{p}$, $m=1, 2, \cdots$, is given by

$$Y\cdot X^m=mB(X, Y)X^{m-1}, \tag{4.9}$$

since one has for $Z\in\mathfrak{g}$,

$$\begin{aligned}(Y\cdot X^m)(Z)&=\frac{d}{dt}X^m(Z+tY)\Big|_{t=0}\\ &=\frac{d}{dt}B(X, Z+tY)^m\Big|_{t=0}\\ &=mB(X, Y)B(X, Z)^{m-1}\\ &=(mB(X, Y)X^{m-1})(Z).\end{aligned}$$

Summing up the isomorphisms $\tilde{\imath}_k (k\in\boldsymbol{Z})$ in (4.8), one obtains the following lemma which visualizes the structure of $\mathrm{Gr}\,\mathscr{A}(\tau)$.

LEMMA 4.1. *The map* $\tilde{\imath}:=\bigoplus_k\tilde{\imath}_k$ *gives a graded* $(S(\mathfrak{g}), K)$*-module isomorphism from* $\mathrm{Gr}\,\mathscr{A}(\tau)$ *onto the tensor product* $S(\mathfrak{p})\otimes V_\tau$, *where* $S(\mathfrak{g})$ *acts on* V_τ *trivially.*

Proof. Since each $\tilde{\imath}_k$ is a K-isomorphism, the map $\tilde{\imath}$ commutes with the K-action. So we only have to prove that

$$\tilde{\imath}_{k-1}(Y\cdot(f+\mathscr{A}_{(k+1)}))=Y\cdot\tilde{\imath}_k(f+\mathscr{A}_{(k+1)}) \tag{4.10}$$

for $f\in\mathscr{A}_{(k)}$ and $Y\in\mathfrak{g}$. Writing Y by means of the basis $(X_i)_i$ of $\mathfrak{p}$ as

$$Y\equiv\sum_{i=1}^{s}B(X_i^*, Y)X_i \mod\mathfrak{k},$$

one sees that

$$Y\cdot(f+\mathscr{A}_{(k+1)})=\left(\sum_{i=0}^{s}B(X_i^*, Y)X_if\right)+\mathscr{A}_{(k)}\quad\text{in }\mathscr{A}_{(k-1)}/\mathscr{A}_{(k)}(\subset\mathrm{Gr}\,\mathscr{A}(\tau)), \tag{4.11}$$

since the subspace $\mathscr{A}_{(k)}$ is stable under $\mathfrak{k}$. With (4.9) in mind, the left hand side of (4.10) is calculated as

$$\begin{aligned}\tilde{\imath}_{k-1}(Y\cdot(f+\mathscr{A}_{(k+1)}))&=\sum_{|\nu|=k}\frac{1}{\nu!}(X^*)^\nu\otimes\left(\sum_i X^\nu B(X_i^*, Y)X_if\right)(1)\\ &=\sum_{|\nu|=k}\sum_i\frac{1}{\nu!}B(X_i^*, Y)(X^*)^\nu\otimes((X^\nu X_i)f)(1)\end{aligned}$$

$$= \sum_{|\nu|=k} \sum_i \frac{1}{(\nu+e_i)!} B(X_i^*, Y) X_i \cdot (X^*)^{\nu+e_i} \otimes (X^{\nu+e_i} f)(1)$$

$$= \sum_i B(X_i^*, Y) \left(\sum_{|\mu|=k+1} \frac{1}{\mu!} X_i \cdot (X^*) \mu \otimes (X^\mu f)(1) \right)$$

$$= \sum_{|\mu|=k+1} \frac{1}{\mu!} Y \cdot (X^*)^\mu \otimes (X^\mu f)(1) = Y \cdot \tilde{\iota}_k (f + \mathscr{A}_{(k+1)}),$$

where $e_i := (\delta_j^i)_{1 \le j \le s}$, and $((X^\nu X_i) f)(1) = (X^{\nu+e_i} f)(1)$ holds since $f \in \mathscr{A}_{(k)}$. We thus get the lemma. Q.E.D.

4.2. Variety $\mathscr{V}(H)$ in relation with $\mathrm{Gr}_\xi(H^*)$

Now let H be an irreducible $(\mathfrak{g}, K)$-module. Then the full dual space H' of H, consisting of all linear forms on H, has a $(\mathfrak{g}, K)$-module structure contragredient to H. The K-finite part of H', say H^*, is an irreducible $(\mathfrak{g}, K)$-submodule of H'.

If (τ, V_τ) is a finite-dimensional K-module occurring in H^*, there exists, by a reciprocity theorem of Frobenius type, a $(\mathfrak{g}, K)$-module embedding ξ from H^* into the analytically induced module $\mathscr{A}(\tau)$. Setting

$$H^*_{(k),\xi} := \xi(H^*) \cap \mathscr{A}_{(k)} \qquad (k \in \boldsymbol{Z}) \tag{4.12}$$

with $\mathscr{A}_{(k)} \subset \mathscr{A}(\tau)$ in (4.2), we get a decreasing filtration $(H^*_{(k),\xi})_k$ of $\xi(H^*) \simeq H^*$ with properties (4.3)–(4.5). Write $\mathrm{Gr}_\xi(H^*)$ for the corresponding $(S(\mathfrak{g}), K)$-module:

$$\bigoplus_k H^*_{(k),\xi} / H^*_{(k+1),\xi} \subset \mathrm{Gr}\, \mathscr{A}(\tau). \tag{4.13}$$

On the other hand, the filtration $(H^*_{(k),\xi})_k$ of H^* gives rise to an increasing filtration $(H_{k,\xi})_k$ of H with

$$H_{k,\xi} := \{ v \in H \mid \langle w^*, v \rangle = 0 \ \ (\forall w^* \in H^*_{(k),\xi}) \}, \tag{4.14}$$

by passing to the orthogonal in H. If

$$\mathrm{gr}_\xi(H) := \bigoplus_k H_{k+1,\xi} / H_{k,\xi} \tag{4.15}$$

denotes the corresponding graded $(S(\mathfrak{g}), K)$-module, the dual pairing $\langle \cdot, \cdot \rangle$ on $H^* \times H$ naturally induces a non-degenerate $(S(\mathfrak{g}), K)$-invariant pairing on $\mathrm{Gr}_\xi(H^*) \times \mathrm{gr}_\xi(H)$. By using the latter pairing, one easily finds that

$$\mathrm{Ann}_{S(\mathfrak{g})} \mathrm{Gr}_\xi(H^*) = \mathrm{Ann}_{S(\mathfrak{g})} \mathrm{gr}_\xi(H), \tag{4.16}$$

and that

$$\mathrm{gr}_\xi(H) = \mathrm{gr}(H; H_{0,\xi}) \qquad \text{(see (1.1))}. \tag{4.17}$$

We have thus obtained the following proposition, which enables us to describe the associated variety $\mathscr{V}(H)$ of Harish-Chandra module H by means of the annihilator ideal of $\mathrm{Gr}_\xi(H^*)$.

PROPOSITION 4.2. *Under the above notation one has the equality*

$$\mathscr{V}(H)=\{X\in\mathfrak{g}\mid f(X)=0 \text{ for all } f\in\operatorname{Ann}_{S(\mathfrak{g})}(\operatorname{Gr}_{\xi}(H^*))\}\,. \tag{4.18}$$

5. Graded modules $\operatorname{Gr} H_\Lambda$ and differential operators $\mathscr{D}_\lambda$ of gradient-type

Let H_Λ be the $(\mathfrak{g}, K)$-module of discrete series π_Λ with Harish-Chandra parameter $\Lambda\in\Xi$. Since the lowest K-type $(\tau_\lambda, V_\lambda)$, $\lambda=\Lambda-\rho_c+\rho_n$, appears in H_Λ with multiplicity one (see Proposition 2.1), there exists a unique, up to scalar multiples, $(\mathfrak{g}, K)$-module embedding ξ_λ from H_Λ into the analytically induced module $\mathscr{A}(\tau_\lambda)$.

This section interprets after Hotta-Parthasarathy [8], the $(S(\mathfrak{g}), K)$-module $\operatorname{Gr} H_\Lambda:=\operatorname{Gr}_{\xi_\lambda}(H_\Lambda)$ defined in 4.2, by means of the gradient-type differential operator $\mathscr{D}_\lambda$ whose kernel realizes π_Λ. Here we treat H_Λ itself instead of its dual $(\mathfrak{g}, K)$-module H_Λ^*, by noting that

$$H_\Lambda^*\simeq H_{-w_0\Lambda} \quad \text{as } (\mathfrak{g}, K)\text{-modules}\,, \tag{5.1}$$

for the longest element w_0 of the Weyl group of Δ_c.

5.1. Operator $\mathscr{D}_\lambda$ and realization of discrete series

Let $(X_i)_{i=1}^s$ and $(X_i^*)_{i=1}^s$ be dual bases of $\mathfrak{p}$ as in 4.1. We set for $f\in\mathscr{A}(\tau_\lambda)$,

$$\nabla_\lambda f(g):=\sum_{i=1}^{s} R_{X_i}f(g)\otimes X_i^* \qquad (g\in G)\,, \tag{5.2}$$

where R_X denotes the left G-invariant vector field on G defined by

$$R_X f(g):=\frac{d}{dt}(f(g\exp tY)+\sqrt{-1}f(g\exp tZ))_{|t=0}$$

for $X=Y+\sqrt{-1}Z$ with $Y, Z\in\mathfrak{g}_0$. It is then easy to see that ∇_λ is independent of the choice of dual bases and that it defines a first order, left G-invariant differential operator from $\mathscr{A}(\tau_\lambda)$ to $\mathscr{A}(\tau_\lambda\otimes\operatorname{Ad}_{\mathfrak{p}})$. Here $\operatorname{Ad}_{\mathfrak{p}}$ denotes the adjoint representation of K on $\mathfrak{p}$.

Notice that the tensor product K-representation $\tau_\lambda\otimes\operatorname{Ad}_{\mathfrak{p}}$ decomposes into irreducibles as

$$\tau_\lambda\otimes\operatorname{Ad}_{\mathfrak{p}}\simeq\bigoplus_{\beta\in\Delta_n}[m_\beta]\cdot\tau_{\lambda+\beta}\,, \tag{5.3}$$

where the multiplicity m_β of $\tau_{\lambda+\beta}$ is either 1 or 0 for every $\beta\in\Delta_n$. Let $(\tau_\lambda^\pm, V_\lambda^\pm)$ be the subrepresentations of $\tau_\lambda\otimes\operatorname{Ad}_{\mathfrak{p}}$ such that $\tau_\lambda^\pm\simeq\bigoplus_{\beta\in\Delta_n^+}[m_{\pm\beta}]\cdot\tau_{\lambda\pm\beta}$, and $P_\lambda: V_\lambda\to V_\lambda^-$ be the projection along the decomposition $V_\lambda=V_\lambda^-\oplus V_\lambda^+$.

We now put

$$\mathscr{D}_\lambda f(g):=P_\lambda(\nabla_\lambda f(g)) \qquad (f\in\mathscr{A}(\tau_\lambda))\,. \tag{5.4}$$

Then $\mathscr{D}_\lambda$ gives a G-invariant differential operator from $\mathscr{A}(\tau_\lambda)$ to $\mathscr{A}(\tau_\lambda^-)$.

It follows immediately from the lowest K-type property of H_Λ that

$$\xi_\lambda(H_\Lambda) \subset \operatorname{Ker} \mathscr{D}_\lambda . \tag{5.5}$$

Moreover, the following result, due to Hotta-Parthasarathy, Schmid and Wallach, says that the L^2-kernel of $\mathscr{D}_\lambda$ realizes the discrete series π_Λ.

PROPOSITION 5.1. (Cf. [19, I, Th. 1.5]) *For any $\Lambda \in \Xi$, the $(\mathfrak{g}, K)$-module $\xi_\lambda(H_\Lambda)$, isomorphic to H_Λ, consists exactly of all functions $f \in \operatorname{Ker} \mathscr{D}_\lambda$ which are left K-finite and square-integrable on G.*

5.2. Polynomialization $\operatorname{Gr}[\mathscr{D}_\lambda]$ **and realization of** $\operatorname{Gr} H_\Lambda$

Let $(\mathscr{A}_{(k)})_{k \in \mathbf{Z}}$ (resp. $(\mathscr{A}^-_{(k)})_{k \in \mathbf{Z}}$) be the decreasing filtration of $\mathscr{A}(\tau_\lambda)$ (resp. $\mathscr{A}(\tau^-_\lambda)$), defined by (4.2). Since $\mathscr{D}_\lambda$ sends $\mathscr{A}_{(k)}$ into $\mathscr{A}^-_{(k-1)}$, the operator $\mathscr{D}_\lambda$ induces an $(S(\mathfrak{g}), K)$-homomorphism, say $\operatorname{Gr}[\mathscr{D}_\lambda]$, from $\operatorname{Gr} \mathscr{A}(\tau_\lambda)$ to $\operatorname{Gr} \mathscr{A}(\tau^-_\lambda)$. Through the isomorphism $\tilde{\iota}$ in Lemma 4.1, we regard this homomorphism as a map

$$\operatorname{Gr}[\mathscr{D}_\lambda] : S(\mathfrak{p}) \otimes V_\lambda \to S(\mathfrak{p}) \otimes V^-_\lambda , \tag{5.6}$$

which is called the *polynomialization* of $\mathscr{D}_\lambda$.

Observe that $\operatorname{Gr}[\mathscr{D}_\lambda]$ is given as

$$(\operatorname{Gr}[\mathscr{D}_\lambda] f)(Y) = P_\lambda\left(\sum_i (X_i f)(Y) \otimes X_i^* \right) \qquad (Y \in \mathfrak{g}) \tag{5.7}$$

for $f \in S(\mathfrak{p}) \otimes V_\lambda$. Here $S(\mathfrak{p}) \otimes V$, $V = V_\lambda$ or V^-_λ, is identified in the canonical way with the space of V-valued polynomial functions on $\mathfrak{g}$ which are identically zero on $\mathfrak{k}$.

By virtue of (5.5), one can easily deduce the inclusion

$$\operatorname{Gr} H_\Lambda = \operatorname{Gr}_{\xi_\lambda}(H_\Lambda) \subset \operatorname{Ker}(\operatorname{Gr}[\mathscr{D}_\lambda]) \tag{5.8}$$

for every Harish-Chandra module H_Λ of discrete series. Furthermore, the discussion in [8, page 160] combined with the Blattner multiplicity formula (cf. [19, I, Prop. 1.2]) immediately gives the following theorem.

THEOREM 5.2. (Hotta-Parthasarathy) *The equality* $\operatorname{Gr} H_\Lambda = \operatorname{Ker}(\operatorname{Gr}[\mathscr{D}_\lambda])$ *holds in* (5.8) *provided that the lowest highest weight $\lambda = \Lambda - \rho_c + \rho_n$ of H_Λ is far from the walls*:

$$\lambda - \sum_{\beta \in Q} \beta \text{ is } \Delta_c^+\text{-dominant for any subset } Q \text{ of } \Delta_n^+ . \tag{5.9}$$

Combining this theorem with Proposition 4.2, we make an essential step toward the proof of Theorem 3.1, as in

THEOREM 5.3. *Let $H_\Lambda (\Lambda \in \Xi)$ be a Harish-Chandra module of discrete series, and $H^*_\Lambda \simeq H_{-w_0 \Lambda}$ (see (5.1)) be its dual $(\mathfrak{g}, K)$-module. If $\lambda = \Lambda - \rho_c + \rho_n$ is far from the walls, the associated variety $\mathscr{V}(H^*_\Lambda)$ of discrete series H^*_Λ is determined by the annihilator of operator* $\operatorname{Gr}[\mathscr{D}_\lambda]$ *in* (5.6):

(5.10) $$\mathscr{V}(H_\lambda^*)=\{X\in\mathfrak{g}\mid f(X)=0,\ \forall f\in \mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])\}\,.$$

REMARK. By (5.8) and Proposition 4.2, the inclusion $\subset$ is always true in (5.10) without any assumption on the regularity of λ.

6. $(S(\mathfrak{g}), K)$-modules $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ and the corresponding annihilator ideals

We now go into more detailed structure of graded $(S(\mathfrak{g}), K)$-modules $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])\subset S(\mathfrak{p})\otimes V_\lambda$ defined in 5.2, and their annihilators $\mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda]))\subset S(\mathfrak{g})$.

6.1. Generating subspaces of $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ as K-modules

Let $f=X^m\otimes v$ be an element of $S(\mathfrak{p})\otimes V_\lambda$ with $X\in\mathfrak{p}$, $v\in V_\lambda$ and an integer $m\geq 0$. In view of (5.7) one can compute $\mathrm{Gr}[\mathscr{D}_\lambda]f\in S(\mathfrak{p})\otimes V_\lambda^-$ as

(6.1) $$\mathrm{Gr}[\mathscr{D}_\lambda]f=mX^{m-1}\otimes P_\lambda(v\otimes X)\,,$$

where P_λ is, as in 5.1, the projection from $V_\lambda=V_\lambda^+\oplus V_\lambda^-$ onto V_λ^-. This implies that f lies in $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ if and only if $v\otimes X\in V_\lambda^+$. Notice that, if v_λ is a non-zero highest weight vector of V_λ, the vector $v_\lambda\otimes X_+$ belongs to V_λ^+ for every $X_+\in\mathfrak{p}_+=\sum_{\alpha\in\Delta_n^+}\mathfrak{g}_\alpha$.

For any subset A of $S(\mathfrak{p})\otimes V_\lambda$, let $\{A\}_K$ denote the K-submodule of $S(\mathfrak{p})\otimes V_\lambda$ generated by A. The above discussion leads us to

PROPOSITION 6.1. *The kernel* $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ *contains the K-submodule* $\{S(\mathfrak{p}_+)\otimes v_\lambda\}_K$.

Conversely, we can prove that $\{S(\mathfrak{p}_+)\otimes v_\lambda\}_K$ exhausts $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ in the following sense.

THEOREM 6.2. *For each fixed integer $m\geq 0$, there exists a constant $c_m>0$ such that*

(6.2) $$\mathrm{Ker}^m(\mathrm{Gr}[\mathscr{D}_\lambda])=\{S^m(\mathfrak{p}_+)\otimes v_\lambda\}_K$$

holds if the lowest highest weight λ satisfies the condition

(6.3) $$(\lambda,\alpha)\geq c_m \qquad \textit{for all } \alpha\in\Delta_c^+\,.$$

Here $\mathrm{Ker}^m(\mathrm{Gr}[\mathscr{D}_\lambda]):=\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])\cap(S^m(\mathfrak{p})\otimes V_\lambda)$ *denotes the homogeneous component of* $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ *of degree m.*

This theorem plays a definitive role in proving Theorem 3.1.

To prove Theorem 6.2, we first specify the K-module structure of $\{S^m(\mathfrak{p}_+)\otimes v_\lambda\}_K$. For a positive integer m, let Φ_m denote the totality of all subsets $\delta=\{\beta_1,\beta_2,\cdots,\beta_m\}$ consisting of m non-compact roots $\beta_l\in\Delta_n^+$ $(1\leq l\leq m)$, and we set

(6.4) $$X(\delta):=X_{\beta_1}\cdots X_{\beta_m}\in S^m(\mathfrak{p}_+)\,,$$

(6.5) $$\gamma(\delta):=\beta_1+\cdots+\beta_m\in\mathfrak{t}^*\,.$$

Here $X_\beta \in \mathfrak{g}_\beta$ ($\beta \in \Delta_n^+$) are fixed non-zero root vectors. Note that $X(\delta)$ ($\delta \in \Phi_m$) form a basis of the linear space $S^m(\mathfrak{p}_+)$.

Let $\{\alpha_1, \cdots, \alpha_u\}$ be the simple system of Δ_c^+. Define a positive number κ_m as the maximum of coefficients $n_k(\delta, \delta')$ $(1 \le k \le u)$ of expansions:

$$\gamma(\delta) - \gamma(\delta') = \sum_{k=1}^{u} n_k(\delta, \delta')\alpha_k ,$$

where δ and δ' range over the elements of Φ_m such that $\gamma(\delta) - \gamma(\delta') \in \sum_{1 \le k \le u} \boldsymbol{R}\alpha_k$.

LEMMA 6.3. *For a positive integer m, one has an isomorphism of K-modules*

$$\{S^m(\mathfrak{p}_+) \otimes v_\lambda\}_K \simeq \bigoplus_{\delta \in \Phi_m} (\tau_{\lambda+\gamma(\delta)}, V_{\lambda+\gamma(\delta)}) \tag{6.6}$$

if the lowest highest weight λ fulfills the condition:

$$\frac{2(\lambda, \alpha_k)}{(\alpha_k, \alpha_k)} > \kappa_m \tag{6.7}$$

for every simple root α_k $(1 \le k \le u)$ of Δ_c^+.

Proof. We introduce on the real vector space $\sqrt{-1}\mathfrak{t}_0^* = \sum_{\beta \in \Delta} \boldsymbol{R}\beta$ a lexicographic order $>$ for which $\Delta^+ = \{\beta \in \Delta \mid \beta > 0\}$, and arrange the elements of Φ_m as $\delta_0, \delta_1, \cdots, \delta_r$ so that $\gamma(\delta_0) \ge \gamma(\delta_1) \ge \cdots \ge \gamma(\delta_r)$. Let $\mathfrak{b} := \mathfrak{t} + \sum_{\alpha \in \Delta_c^+} \mathfrak{g}_\alpha$ be a Borel subalgebra of $\mathfrak{k}$. Note that the subspace $S^m(\mathfrak{p}_+) \otimes v_\lambda \subset S^m(\mathfrak{p}) \otimes V_\lambda$ is $\mathfrak{b}$-stable. Setting

$$A_j := \sum_{0 \le i \le j} \boldsymbol{C}X(\delta_i) \otimes v_\lambda \subset S^m(\mathfrak{p}_+) \otimes v_\lambda \tag{6.8}$$

for $0 \le j \le r$, we get a $\mathfrak{b}$-stable flag $\{A_j\}_j$ of $A_r = S^m(\mathfrak{p}_+) \otimes v_\lambda$.

Let $\{A_j\}_K$ be the K-submodule of $S^m(\mathfrak{p}) \otimes V_\lambda$ generated by subspace A_j. One sees that $\{A_j\}_K = U(\mathfrak{n}_-)A_j$ with $\mathfrak{n}_- := \sum_{\alpha \in \Delta_c^+} \mathfrak{g}_{-\alpha}$, by bearing in mind the decomposition $\mathfrak{k} = \mathfrak{b} + \mathfrak{n}_-$. Moreover, if $X(\delta_j) \otimes v_\lambda$ is not contained in $\{A_{j-1}\}_K$, this vector naturally gives rise to a non-zero highest vector of quotient K-module $\{A_j\}_K / \{A_{j-1}\}_K$, and one has

$$\{A_j\}_K / \{A_{j-1}\}_K \simeq (\tau_{\lambda+\gamma(\delta_j)}, V_{\lambda+\gamma(\delta_j)}) \tag{6.9}$$

as K-modules. So, to complete the proof, it is enough to show that

$$X(\delta_j) \otimes v_\lambda \notin \{A_{j-1}\}_K = U(\mathfrak{n}_-)A_{j-1}$$

for every j under the assumption (6.7) on λ.

Suppose by contraries that $X(\delta_j) \otimes v_\lambda \in U(\mathfrak{n}_-)A_{j-1}$ for some j. Then there exist elements $D_i \in U(\mathfrak{n}_-)$ $(i < j)$ such that

$$X(\delta_j) \otimes v_\lambda = \sum_{i<j} D_i (X(\delta_i) \otimes v_\lambda) , \tag{6.10}$$

and that

$$[Z, D_i] = (\gamma(\delta_j) - \gamma(\delta_i))(Z) D_i \qquad (Z \in \mathfrak{t}). \tag{6.11}$$

If i_0 is the smallest number i with $D_i \neq 0$, the right hand side of (6.10) turns out to be

$$X(\delta_{i_0}) \otimes D_{i_0} v_\lambda \mod S^m(\mathfrak{p}) \otimes \left\{ \sum_{\mu > \lambda + \gamma(\delta_j) - \gamma(\delta_{i_0})} V_\lambda(\mu) \right\}, \tag{6.12}$$

where $V_\lambda(\mu)$ is the weight space of V_λ of weight μ. We thus find from (6.10) that $D_{i_0} v_\lambda = 0$.

It then follows from the celebrated Bernstein-Gelfand-Gelfand resolution of V_λ in terms of Verma $U(\mathfrak{k})$-modules (cf. [12, Lemma 2.2.10]) that the weight $\gamma(\delta_j) - \gamma(\delta_{i_0})$ of D_{i_0} must be of the form

$$-\frac{2(\lambda, \alpha_l)}{(\alpha_l, \alpha_l)} \alpha_l - \sum_{1 \leq k \leq u} n_k \alpha_k \qquad (n_k \geq 0, \in \boldsymbol{Z}) \tag{6.13}$$

for some simple root $\alpha_l \in \Delta_c^+$. This contradicts the assumption (6.7). Q.E.D.

Now let K^C be the complexification of compact group K, and B be the Borel subgroup of K^C with Lie algebra $\mathfrak{b}$. For a holomorphic B-module F, we write $H^i(F)$ ($i = 0, 1, \cdots$) for the i-th cohomology group of K^C/B with coefficients in the sheaf of germs of holomorphic sections of vector bundle $K^C \times_B F$ (see e.g., [15, 3.1.2] for the definition of $H^i(F)$). Then $H^i(F)$ admits a natural K^C-module structure.

For a T-integral linear form μ on $\mathfrak{t}_0 = \mathrm{Lie}(T)$, let $C(\mu)$ denote the one-dimensional B-module with differential μ extended to $\mathfrak{b}$ trivially on the nilradical $\mathfrak{n} = \sum_{\alpha \in \Delta_c^+} \mathfrak{g}_\alpha$.

The following lemma, due to Hotta-Parthasarathy, describes the K-module in the left hand side of (6.2) by means of cohomology group of K^C/B.

Lemma 6.4. ([8, Lemma 5.2]) *The K-module* $\mathrm{Ker}^m(\mathrm{Gr}[\mathscr{D}_\lambda])$ *is isomorphic to the cohomology group* $H^t(S^m(\mathfrak{p}_+) \otimes C(\lambda + 2\rho_c))$ ($t := \dim K^C/B$) *for every integer* $m \geq 0$, *provided that the parameter λ is far from the walls in the sense of* (5.9).

The above two lemmas enable us to prove Theorem 6.2, as follows

Proof of Theorem 6.2. Fix a non-negative integer m, and set

$$\tilde{A}_j := \sum_{i \leq j} \boldsymbol{C} X(\delta_i) \otimes C(\lambda + 2\rho_c) \subset S^m(\mathfrak{p}_+) \otimes C(\lambda + 2\rho_c) \qquad (0 \leq j \leq r),$$

where $\{X(\delta_j)\}_{0 \leq j \leq r}$ is the basis of $S^m(\mathfrak{p}_+)$ defined in (6.4). Then one has a flag of B-modules

$$(0) \subset \tilde{A}_0 \subset \tilde{A}_1 \subset \cdots \subset \tilde{A}_{j-1} \subset \tilde{A}_j \subset \cdots \subset \tilde{A}_r = S^m(\mathfrak{p}_+) \otimes C(\lambda + 2\rho_c) \tag{6.14}$$

such that

$$0 \to \tilde{A}_{j-1} \to \tilde{A}_j \to C(\lambda + \gamma(\delta_j) + 2\rho_c) \to 0 \qquad \text{as } B\text{-modules}. \tag{6.15}$$

Now assume that λ satisfies the regularity condition (6.7) in Lemma 6.3. Then one finds that the linear form $\lambda+\gamma(\delta_j)$ is Δ_c^+-dominant for every j. It follows from the Borel-Weil-Bott theorem (see e.g., [15, Th. 3.1.2.2]) that

$$H^p(C(\lambda+\gamma(\delta_j)+2\rho_c))\simeq\begin{cases} V_{\lambda+\gamma(\delta_j)} & (p=t)\,, \\ (0) & (p\neq t)\,, \end{cases} \tag{6.16}$$

as K^C-modules. Hence (6.15) induces an exact sequence of K^C-modules

$$0\to H^t(\tilde{A}_{j-1})\to H^t(\tilde{A}_j)\to V_{\lambda+\gamma(\delta_j)}\to 0\,, \tag{6.17}$$

and consequently we deduce

$$H^t(S^m(\mathfrak{p}_+)\otimes C(\lambda+2\rho_c))\simeq\bigoplus_{\delta\in\Phi_m}(\tau_{\lambda+\gamma(\delta)},V_{\lambda+\gamma(\delta)})\,. \tag{6.18}$$

This together with Proposition 6.1, Lemmas 6.3 and 6.4 completes the proof of Theorem 6.2. Q.E.D.

6.2. Annihilator ideal $\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$

For a subset E of $\mathfrak{g}$, let $\mathscr{I}(E)$ denote the ideal of $S(\mathfrak{g})$ determined by E:

$$\mathscr{I}(E):=\{f\in S(\mathfrak{g})\,|\,f(X)=0\ \ (\forall X\in E)\}\,. \tag{6.19}$$

Two results in 6.1 allow us to establish the following

THEOREM 6.5. *Let $\lambda=\Lambda-\rho_c+\rho_n$ be the lowest highest weight of discrete series H_Λ. Then one has*

$$\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])\subset\mathscr{I}(K_C\mathfrak{p}_+)\,. \tag{6.20}$$

Moreover there exists a positive constant c such that the equality holds in (6.20) *provided that $(\lambda,\alpha)\geq c$ for all $\alpha\in\Delta_c^+$.*

This theorem together with Theorem 5.2 immediately yields

COROLLARY 6.6. *If the lowest highest weight λ is sufficiently Δ_c^+-regular, the annihilator ideal of graded* $S(\mathfrak{g})$*-module* $\mathrm{Gr}\,H_\Lambda$ *(see* 5.1*) coincides with its radical.*

Proof of Theorem 6.5. The inclusion (6.20) follows immediately from Proposition 6.1. To prove the second assertion, note at first that $\mathscr{I}(K_C\mathfrak{p}_+)$ is a graded ideal of $S(\mathfrak{g})$ containing $\mathfrak{k}S(\mathfrak{g})$. Since $S(\mathfrak{g})$ ia a Noetherian ring, there exist a finite number of homogeneous elements $D_j\in S(\mathfrak{p})$ $(1\leq j\leq r)$ such that

$$\mathscr{I}(K_C\mathfrak{p}_+)=\mathfrak{k}S(\mathfrak{g})+S(\mathfrak{g})D_1+\cdots+S(\mathfrak{g})D_r\,.$$

Let c_j be the positive constants in Theorem 6.2 associated to $d_j:=\deg D_j\,(1\leq j\leq r)$, and put $c:=\max_j(c_j)$. Then (6.2) tells us that, if $(\lambda,\alpha)\geq c$ $(\forall\alpha\in\Delta_c^+)$, then each D_j is identically zero on $\mathrm{Ker}^{d_j}(\mathrm{Gr}[\mathscr{D}_\lambda])$. One easily sees from this fact that D_j annihilates all the vectors in $\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$. We thus conclude $\mathscr{I}(K_C\mathfrak{p}_+)=\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ as desired. Q.E.D.

7. Completion of the proof of Theorem 3.1

By virtue of Theorems 5.3 and 6.5, we find that

$$\mathscr{V}(H_\Lambda^*)=K_C\mathfrak{p}_+\,, \tag{7.1}$$

if the corresponding lowest highest weight λ is sufficiently Δ_c^+-regular. Here it should be noticed that $K_C\mathfrak{p}_+$ is (Zariski) closed in $\mathfrak{p}$, because $\mathfrak{p}_+$ is stable under the Borel subgroup B while the flag variety K_C/B is complete. A standard argument of Zuckerman's translation principle, given below, allows us to show that (7.1) is always true for any $\Lambda\in\Xi$. This together with (5.1) will complete the proof of Theorem 3.1.

To be more precise, we use the following proposition.

PROPOSITION 7.1. (Cf. [2, Prop. 1.3]) *Let H be a finitely generated $U(\mathfrak{g})$-module, and L be a $U(\mathfrak{g})$-submodule of H. Then L is finitely generated over $U(\mathfrak{g})$, and its associated variety $\mathscr{V}(L)$ is contained in $\mathscr{V}(H)$.*

Proof. Take a finite-dimensional subspace H_0 of H such that $H=U(\mathfrak{g})H_0$, and set $H_k:=U_k(\mathfrak{g})H_0$, $L_k:=H_k\cap L$ for each integer $k\geq 0$. Then $(L_k)_{k=0,1,\cdots}$ gives an increasing filtration of $U(\mathfrak{g})$-module L such that $U_m(\mathfrak{g})L_k\subset L_{m+k}$, and correspondingly one has graded $S(\mathfrak{g})$-modules

$$\mathrm{gr}(H;H_0)=\bigoplus_k M_k \ \ (\text{see } (1.1)) \quad \text{and} \quad \mathrm{gr}(L)=\bigoplus_k N_k$$

with $M_k:=H_k/H_{k-1}$ and $N_k:=L_k/L_{k-1}$ respectively. Note that the latter $\mathrm{gr}(L)$ can be looked upon as an $S(\mathfrak{g})$-submodule of the former $\mathrm{gr}(H;H_0)$ in the canonical way. Since the ring $S(\mathfrak{g})$ is Noetherian and since $\mathrm{gr}(H;H_0)$ is of finite type over $S(\mathfrak{g})$, we find that $\mathrm{gr}(L)$ also is finitely generated as an $S(\mathfrak{g})$-module. Hence there exists an integer $j\geq 0$ such that $\mathrm{gr}(L)=S(\mathfrak{g})(N_0+\cdots+N_j)$ and so we obtain $L=U(\mathfrak{g})L_j$. This shows that L is finitely generated over $U(\mathfrak{g})$.

The second claim $\mathscr{V}(L)\subset\mathscr{V}(H)$ can be shown just as in the proof of [20, Th. 2.2]. Actually, for the above j we can prove

$$N_k=S^{k-j}(\mathfrak{g})N_j\,,\qquad L_k=U_{k-j}(\mathfrak{g})L_j\quad(k=j,j+1,\cdots)\,, \tag{7.2}$$

which imply that

$$\sqrt{\mathrm{Ann}_{S(\mathfrak{g})}\,\mathrm{gr}(L)}=\sqrt{\mathrm{Ann}_{S(\mathfrak{g})}\,\mathrm{gr}(L;L_j)}\,. \tag{7.3}$$

Here $\mathrm{gr}(L;L_j)$ is the graded $S(\mathfrak{g})$-module defined through the filtration $(U_k(\mathfrak{g})L_j)_k$ of L, and $\sqrt{J}$ denotes the radical of an ideal J of $S(\mathfrak{g})$. The inclusion $\mathscr{V}(L)\subset\mathscr{V}(H)$ now follows by taking the varieties associated to the ideals in (7.3), since $\sqrt{\mathrm{Ann}_{S(\mathfrak{g})}\,\mathrm{gr}(L)}$ includes the ideal $\sqrt{\mathrm{Ann}_{S(\mathfrak{g})}\,\mathrm{gr}(H;H_0)}$ which defines the variety $\mathscr{V}(H)$. Q.E.D.

We are now in a position to complete the proof of Theorem 3.1.

In view of (5.1) it suffices to show (7.1) for $\Lambda=\lambda+\rho_c-\rho_n\in\Xi$ when the lowest highest weight λ is *not* sufficiently Δ_c^+-regular. For such a Λ, we can take an irreducible finite-dimensional $(\mathfrak{g}, K)$-module Y_μ with Δ^+-highest weight μ for which the linear form $\lambda+\mu$ is sufficiently Δ_c^+-regular and so (7.1) holds for $\Lambda+\mu$.

By making the use of Zuckerman's translation functors (see e.g., [19, I, 3.4]), we deduce that $H^*_{\Lambda+\mu}$ (resp. H^*_Λ) is isomorphic to a $(\mathfrak{g}, K)$-submodule of the tensor product $H^*_\Lambda \otimes Y^*_\mu$ (resp. $H^*_{\Lambda+\mu} \otimes Y_\mu$). It then follows from Proposition 7.1 together with [3, Lemma 4.1] that

$$\begin{aligned}\mathscr{V}(H^*_{\Lambda+\mu}) &\subset \mathscr{V}(H^*_\Lambda \otimes Y^*_\mu) = \mathscr{V}(H^*_\Lambda)\\ &\subset \mathscr{V}(H^*_{\Lambda+\mu} \otimes Y_\mu) = \mathscr{V}(H^*_{\Lambda+\mu})\,.\end{aligned}$$

We thus conclude from (7.1) applied to $\Lambda+\mu$, that $\mathscr{V}(H^*_\Lambda) = K_{\mathbf{C}}\mathfrak{p}_+$ as desired. Now Theorem 3.1 is completely proved.

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