COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 47, No. 1 1998

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

Another Look at the Differential Operators on Quantum Matrix Spaces and its Applications

Dedicated to Professor Kiyosato Okamoto on the occasion of his sixtieth birthday

by

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(Received November 29, 1997)

Abstract. The present paper gives a new viewpoint for differential operators with respect to the coordinates of quantum matrix spaces. Special emphasis is put on the inductive construction of these differential operators from the q-difference operators defined on each column (or row). This idea for understanding our operators provides us with two important applications, (1) a construction of the q-oscillator representation of the quantized enveloping algebra $U_q(\mathfrak{sp}_{2N})$ of the symplectic Lie algebra $\mathfrak{sp}_{2N}$, and a construction of the quantum dual pair $(\mathfrak{sp}_{2N}, \mathfrak{o}_n)$ through the tensor power of the q-oscillator representation, and (2) a new definition of a quantum analogue of hypergeometric equations of many variables. In addition to these, we give an explanation of the spectral parameter in the quantum Capelli Identity for $GL_q(n)$ discussed in [NUW1].

0. Introduction

The aim of the present paper is to make further investigation of the differential operators with respect to the coordinates of the quantum matrix space introduced in [HiW] and [NUW1]. Our study meets here two other works on the analysis on quantum homogeneous spaces, [NUW2, 3] and [N1].

This paper provides a new point of view for these operators. The definition given in [NUW1] is based on the following idea: focussing on the relations among three kinds of operators, i.e., polarization-, multiplication- and differential-operators, in the classical situation, we regard them as linear equations with differential operators as unknowns, and solved the corresponding equations to get the quantum group counterparts. We will show that, in contrast to this original definition, our differential operators can be described inductively by operators defined on quantum matrix spaces of smaller sizes. In this way we see our operators are *woven* (see §3 for precise

[1] Partially supported by Grant-in-Aid for Scientific Research No. 06221237, the Ministry of Education, Science and Culture of Japan.

[2] Partially supported by Grant-in-Aid for Scientific Research (B) No. 08454010, the Ministry of Education, Science and Culture of Japan.

statement) from the much better understood q-difference operators (cf. [NUW1], §5).

Applications include a construction of q-oscillator representations of the quantized enveloping algebra $U_q(\mathfrak{sp}_{2N})$ of the symplectic Lie algebra on the quantum matrix space and a formulation of quantum version of the Gelfand hypergeometric equations associated with Grassmannians. The former is a necessary basic step for further study of a quantum dual pair $(\mathfrak{sp}_{2N}, \mathfrak{so}_n)$, the most important dual pair. This is a direct continuation of our recent study in [NUW2, 3] on the quantum dual pair $(\mathfrak{sl}_2, \mathfrak{so}_n)$, spherical harmonics, and the Capelli identity. To go further to define and to investigate systematically general quantum dual pairs, it is indispensable to understand the interrelations among our differential operators on quantum matrix spaces of different sizes.

The latter application mentioned above is expected to provide another basic foundation for harmonic analysis on quantum homogeneous spaces (cf. [N1]). Our operators enable us to give a natural definiton for the q-hypergeometric equations in the sense of the quantum group version of the generalized hypergeometric equations treated in [G] and [GZK].

The paper is organized as follows: In §1, we review basic facts concerning q-difference operators and their connection with the quantum enveloping algebra $U_q(\mathfrak{gl}_n)$. In §2, after recalling the definition of differential operators with constant coefficients on the quantum square matrix spaces introduced in [NUW1], we generalize this definition to the rectangular matrix case. Further, in §3, we show that these operators can be reconstructed from the q-difference operators defined on each column (resp., row). §4 is devoted to a natural construction of the q-oscillator representation and its tensor power. We also discuss a commutant of the n-fold tensor power of the q-oscillator representation of $U_{q^2}(\mathfrak{sp}_{2N})$ and see tha algebra $U_q(\mathfrak{o}_n)$ (in the sense of [GaK], NUW3, N2]) appearing naturally in it. We formulate a quantum version of Gelfand's hypergeometric equations in §5. In the Supplement, besides the above applications, we explain why the spectral parameter arises in the quantum Capelli Identity for $GL_q(n)$ discussed in [NUW1]. This formula gives a quantum matrix counterpart of the classical Boole formula in [B],

$$x^m\left(\frac{d}{dx}\right)^m = \vartheta(\vartheta-1)\cdots(\vartheta-m+1),$$

where $\vartheta = x(d/dx)$. The formula in Theorem S.2 suggests the meaning of the spectral parameter appearing in our differential operator. Its classical limit is intimately related to the lower order Capelli Identities discussed in [Ca], [HU].

The authors would like to express their thanks to Professor M. Noumi for his valuable comments.

1. From q-difference operators to the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$

Throughout the paper the ground field $\mathbb{K}$ is fixed as the rational function field $\mathbb{Q}(q)$ of one variable q. In this section we make an elementary observation which explains how the Hopf algebra structure of $U_q(\mathfrak{gl}_n)$ arises naturally in the framework of the q-difference operators.

1.1. Partial q-difference operators

Let $\mathscr{A}=\mathbb{K}[t_1, \cdots, t_n]$ denote the q-commutative ring with the relations $t_i t_j = q t_j t_i$ for $i<j$. We define the mutually commutative automorphisms γ_i $(i=1, \cdots, n)$ as follows:

$$(\gamma_i\varphi)(t_1, \cdots, t_i, \cdots, t_n)=\varphi(t_1, \cdots, qt_i, \cdots, t_n) \qquad \text{for} \quad \varphi\in\mathscr{A} .$$

Using γ_i we define two types of (partial) q-difference operators as

$$\partial_i^L\varphi=t_i^{-1}\frac{\gamma_i-\gamma_i^{-1}}{q-q^{-1}}\varphi , \qquad \partial_i^R\varphi=\left(\frac{\gamma_i-\gamma_i^{-1}}{q-q^{-1}}\varphi\right)t_i^{-1} .$$

In other words, these operators are taken in the forms

$$t_i\partial_i^L=\{\gamma_i\} , \qquad t_i^\circ\partial_i^R=\{\gamma_i\} . \tag{1.1.1}$$

Here t_i and t_i° respectively represent the left and right multiplication operators by the element t_i, and also $\{\gamma_i\}$ represents the difference operator (of symmetric form) given by

$$\{\gamma_i\}=\frac{\gamma_i-\gamma_i^{-1}}{q-q^{-1}} . \tag{1.1.2}$$

The following commutation relations are easily derived from the definition:

$$\begin{aligned}
&t_it_j-qt_jt_i=0 , \quad \partial_i^L\partial_j^L-q\partial_j^L\partial_i^L=0 \qquad (i<j) ,\\
&t_i^\circ t_j^\circ-q^{-1}t_j^\circ t_i^\circ=0 , \quad \partial_i^R\partial_j^R-q^{-1}\partial_j^R\partial_i^R=0 \qquad (i<j) ,\\
&t_it_j^\circ-t_j^\circ t_i=0 , \qquad \partial_i^L\partial_j^R-\partial_j^R\partial_i^L=0 \qquad (\forall i,j) ,\\
&\partial_i^Lt_i-q^{\mp1}t_i\partial_i^L=\gamma_i^{\pm1} , \quad \partial_i^Rt_i^\circ-q^{\mp1}t_i^\circ\partial_i^R=\gamma_i^{\pm1} \qquad (\forall i) ,\\
&\partial_i^Lt_j-q^{\mp1}t_j\partial_i^L=0 , \quad \partial_i^Rt_j^\circ-q^{\pm1}t_j^\circ\partial_i^R=0 \qquad (i\lessgtr j) .
\end{aligned}$$

Also we see for any i $(1\le i\le n)$

$$\begin{aligned}
&\partial_i^Lt_i^\circ-q^{\mp1}t_i^\circ\partial_i^L=\gamma_i^{\pm1}\,\mathrm{Int}(t_i) ,\\
&\partial_i^Rt_i-q^{\mp1}t_i\partial_i^R=\gamma_i^{\pm1}\,\mathrm{Int}(t_i)^{-1} ,
\end{aligned}$$

where

$$\mathrm{Int}(t_i)=t_i^{-1}t_i^\circ=t_i^\circ t_i^{-1}=\gamma_1\cdots\gamma_{i-1}\gamma_{i+1}^{-1}\cdots\gamma_n^{-1} . \tag{1.1.3}$$

Note that if $i\neq j$, the corresponding operators commute:

$$\partial_i^Lt_j^\circ-t_j^\circ\partial_i^L=0 , \qquad \partial_i^Rt_j-t_j\partial_i^R=0 .$$

It is convenient to write the operators involved in our discussion in the following forms:

$$\partial_i^L t_i=\{q\gamma_i\}\,, \qquad \partial_i^R t_i^\circ=\{q\gamma_i\}\,,$$
$$t_i\partial_i^R=\mathrm{Int}(t_i)^{-1}\{\gamma_i\}\,, \qquad \partial_i^R t_i=\mathrm{Int}(t_i)^{-1}\{q\gamma_i\}\,,$$
$$t_i^\circ\partial_i^L=\mathrm{Int}(t_i)\{\gamma_i\}\,, \qquad \partial_i^L t_i^\circ=\mathrm{Int}(t_i)\{q\gamma_i\}\,.$$

1.2. Realization of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$

Motivated by the classical situation in which the general linear group $GL(n)$ acts on the polynomial ring with n variables, we introduce the following operators acting on $\mathscr{A}$:

$$\left\{\begin{array}{ll} e_i=\gamma_{i+1}t_i^\circ\partial_{i+1}^R & (1\le i<n)\,, \\ f_i=\gamma_{i+1}^{-1}t_{i+1}^\circ\partial_i^R & (1\le i<n)\,, \\ \gamma_i^{\pm 1}=\gamma_i^{\pm 1} & (1\le i\le n)\,. \end{array}\right. \tag{1.2}$$

A simple calculation then shows that these operators give a representation of $U_q(\mathfrak{gl}_n)$ under the assignment

$$\hat{e}_i\leftarrow e_i\,, \quad \hat{f}_i\leftarrow f_i\,, \quad q^{\pm\varepsilon_i}\leftarrow\gamma_i^{\pm 1}\,.$$

Here the q-deformed algebra $U_q(\mathfrak{gl}_n)$ is, as usual, defined as an associative algebra generated by the symbols $q^{\pm\varepsilon_i}$ $(1\le i\le n)$, $\hat{e}_j$, $\hat{f}_j$ $(1\le j<n)$ with the following relations:

$$q^0=1\,, \quad q^\lambda q^\mu=q^{\lambda+\mu}\,,$$
$$q^\lambda\hat{e}_jq^{-\lambda}=q^{\langle\lambda,\varepsilon_j-\varepsilon_{j+1}\rangle}\hat{e}_j\,, \quad q^\lambda\hat{f}_jq^{-\lambda}=q^{-\langle\lambda,\varepsilon_j-\varepsilon_{j+1}\rangle}\hat{f}_j\,,$$
$$\hat{e}_i\hat{f}_j-\hat{f}_j\hat{e}_i=\delta_{ij}\frac{q^{\varepsilon_j-\varepsilon_{j+1}}-q^{-\varepsilon_j+\varepsilon_{j+1}}}{q-q^{-1}}\,,$$
$$\hat{e}_i\hat{e}_j=\hat{e}_j\hat{e}_i\,, \quad \hat{f}_i\hat{f}_j=\hat{f}_j\hat{f}_i \quad (|i-j|>1)\,,$$
$$\hat{e}_i^2\hat{e}_j-(q+q^{-1})\hat{e}_i\hat{e}_j\hat{e}_i+\hat{e}_j\hat{e}_i^2=0 \quad (|i-j|=1)\,,$$
$$\hat{f}_i^2\hat{f}_j-(q+q^{-1})\hat{f}_i\hat{f}_j\hat{f}_i+\hat{f}_j\hat{f}_i^2=0 \quad (|i-j|=1)\,,$$

where $\langle\ ,\ \rangle$ represents the canonical symmetric bilinear form such that $\langle\varepsilon_i,\varepsilon_j\rangle=\delta_{ij}$, and $q^\lambda=q^{a_1\varepsilon_1}\cdots q^{a_n\varepsilon_n}$ for an integral linear combination $\lambda=a_1\varepsilon_1+\cdots+a_n\varepsilon_n$. The last two cubic equations in the above set of relations are sometimes referred to as the Serre relations.

The algebra $U_q(\mathfrak{gl}_n)$ also has a Hopf algebra structure. We take here the following convention for the comultiplication Δ:

$$\Delta(q^\lambda)=q^\lambda\otimes q^\lambda\,,$$
$$\Delta(\hat{e}_j)=\hat{e}_j\otimes q^{-\varepsilon_j+\varepsilon_{j+1}}+1\otimes\hat{e}_j \quad (1\le j<n)\,,$$
$$\Delta(\hat{f}_j)=\hat{f}_j\otimes 1+q^{\varepsilon_j-\varepsilon_{j+1}}\otimes\hat{f}_j \quad (1\le j<n)\,.$$

The counit ε and antipode S are respectively given for the generators by

$$\varepsilon(q^{\lambda})=1\,, \quad \varepsilon(\hat{e}_j)=\varepsilon(\hat{f}_j)=0\,,$$

$$S(q^{\lambda})=q^{-\lambda}\,, \quad S(\hat{e}_j)=-\hat{e}_j q^{\varepsilon_j-\varepsilon_{j+1}}\,, \quad S(\hat{f}_j)=-q^{-\varepsilon_j+\varepsilon_{j+1}}\hat{f}_j\,, \qquad \text{for} \quad 1\leq j<n\,.$$

Although the above action on $\mathscr{A}$ is *not* a faithful representation of $U_q(\mathrm{gl}_n)$, it gives us some insight into the natural coalgebra structure of $U_q(\mathrm{gl}_n)$. We will explain this idea below.

The first point to note is that, with the comultiplication rule above, the multiplication $\mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ is a $U_q(\mathrm{gl}_n)$ homomorphism. Classically, this is called the Leibniz rule. Conversely if we require this situation to hold under representation of $U_q(\mathrm{gl}_n)$ by the difference operators above, the natural "Leibniz rule" should be compatible with the comultiplication rule. Actually, the Leibniz rule suggests the latter. One is then naturally forced to define the action $\{\gamma_i\}$ on the product of two functions in $\mathscr{A}$ in the following way:

$$\begin{aligned}\Delta(\{\gamma_i\})&=\gamma_i\otimes\{\gamma_i\}+\{\gamma_i\}\otimes\gamma_i^{-1}\\&=\{\gamma_i\}\otimes\gamma_i+\gamma_i^{-1}\otimes\{\gamma_i\}\,.\end{aligned}$$

We note here two possibilities in choosing the action $\{\gamma_i\}$ above. Let us take, for example, the first of these. For $\varphi,\psi\in\mathscr{A}$, we observe

$$\begin{aligned}t_i^{\circ}(\varphi\psi)&=\varphi\psi t_i=\varphi(t_i^{\circ}\psi)\\&=\varphi t_i(t_i^{-1}t_i^{\circ}\psi)=(t_i^{\circ}\varphi)(\mathrm{Int}(t_i)\psi)\,.\end{aligned}$$

This formula suggests

$$\Delta(t_i^{\circ-1})=1\otimes t_i^{\circ-1}=t_i^{\circ-1}\otimes\gamma_1^{-1}\cdots\gamma_{i-1}^{-1}\gamma_{i+1}\cdots\gamma_n\,.$$

These formulas then give the Leibniz rule $\Delta(\partial_i^R)$ for the difference operator of ∂_i^R as

$$\begin{aligned}\Delta(\partial_i^R)&=\Delta(t_i^{\circ-1})\Delta(\{\gamma_i\})\\&=\gamma_i\otimes\partial_i^R+\partial_i^R\otimes\gamma_1^{-1}\cdots\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i+1}\cdots\gamma_n\,.\end{aligned}$$

With these we are finally led to

$$\begin{aligned}&\Delta(t_{i+1}^{\circ}\partial_i^R)\\&\quad=(1\otimes t_{i+1}^{\circ})\gamma_i\otimes\partial_i^R+(t_{i+1}^{\circ}\otimes\gamma_1\cdots\gamma_i\gamma_{i+2}^{-1}\cdots\gamma_n^{-1})\partial_i^R\otimes\gamma_1^{-1}\cdots\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i+1}\cdots\gamma_n\\&\quad=\gamma_i\otimes t_{i+1}^{\circ}\partial_i^R+t_{i+1}^{\circ}\partial_i^R\otimes\gamma_{i+1}\,,\end{aligned}$$

or

$$\Delta(f_i)=\Delta(\gamma_{i+1}^{-1})\Delta(t_{i+1}^{\circ}\partial_i^R)=\gamma_i\gamma_{i+1}^{-1}\otimes f_i+f_i\otimes 1\,.$$

This is really the comultiplication of $\hat{f}_j$ described above. We obtain the comultiplication rules for other operators in a similar manner.

REMARK. We note that the quantized enveloping algebra $U_q(\mathrm{gl}_n)$ is, by definition, the Hopf algebra generated by the elements $k_j^{\pm1}$, $\hat{e}_j$ and $\hat{f}_j$ $(1\leq j\leq n-1)$, where $k_i=q^{\varepsilon_i-\varepsilon_{i+1}}$, rather than q^{ε_i} in the fundamental relations of $U_q(\mathrm{gl}_n)$.

1.3. L-operators

Throughout this paper, we make extensive use of the so-called L-operators $L^+_{ij}, L^-_{ij} \in U_q(\mathfrak{gl}_n)$ as in [RTF], [NUW1]; these play a role analogous to that of root vectors of the Lie algebra $\mathfrak{gl}_n$. As is shown in [J], there exists a unique family of elements $\hat{E}_{ij}$ $(1 \leq i \neq j \leq n)$ in $U_q(\mathfrak{gl}_n)$ such that

$$\hat{E}_{j,j+1}=\hat{e}_j\,, \quad \hat{E}_{j+1,j}=\hat{f}_j\,,$$
$$\hat{E}_{ij}=\hat{E}_{ik}\hat{E}_{kj}-q^{\pm 1}\hat{E}_{kj}\hat{E}_{ik} \qquad (i \lessgtr k \lessgtr j)\,.$$

Using these elements, we define the elements $L^\pm_{ij} \in U_q(\mathfrak{gl}_n)$ as follows:

$$L^+_{ij}=\begin{cases} q^{\varepsilon_j} & (i=j) \\ (q-q^{-1})q^{\varepsilon_j}\hat{E}_{ji} & (i<j) \\ 0 & (i>j) \end{cases} \qquad L^-_{ij}=\begin{cases} q^{-\varepsilon_j} & (i=j) \\ -(q-q^{-1})q^{-\varepsilon_i}\hat{E}_{ji} & (i>j) \\ 0 & (i<j)\,. \end{cases}$$

Concerning the Hopf algebra structure of $U_q(\mathfrak{gl}_n)$, we should remark that, in matrix form, the L-operators $L^\pm=(L^\pm_{ij})_{1\leq i,j\leq n}$ satisfy

$$\Delta(L^\pm)=L^\pm \otimes L^\pm\,, \tag{1.3.1}$$

$$S(L^\pm)L^\pm=L^\pm S(L^\pm)=1 \quad \text{and} \quad \varepsilon(L^\pm)=1\,. \tag{1.3.2}$$

The square S^2 of the antipode S, which is an automorphism of the Hopf algebra $U_q(\mathfrak{gl}_n)$, is explicitly given by $S^2(a)=q^{-2\rho}aq^{2\rho}$ for any $a \in U_q(\mathfrak{gl}_n)$. Here $q^{2\rho}$ is the group-like element of $U_q(\mathfrak{gl}_n)$ corresponding to the sum of positive roots $2\rho=\sum_{j=1}^n 2(n-j)\varepsilon_j$.

For later use, we remark that the description of the action of $U_q(\mathfrak{gl}_n)$ on $\mathscr{A}$ above is given by L-operators in the following matrix form:

$$\frac{1}{q-q^{-1}}L(1)\cdot\varphi=({}^t[\partial_1^R \;\; \partial_2^R \;\cdots\; \partial_n^R]\varphi)[t_1 \;\; t_2 \;\cdots\; t_n]\,. \tag{1.3.3}$$

Here the matrix form of L-operators with the spectral parameter λ is defined by

$$L(\lambda)=(L_{ij}(\lambda))_{1\leq i,j\leq n}\,, \quad L_{ij}(\lambda)=\lambda L^+_{ij}-\lambda^{-1}L^-_{ij} \in U_q(\mathfrak{gl}_n)\otimes \mathbb{K}[\lambda,\lambda^{-1}]\,. \tag{1.3.4}$$

2. Differential operators on quantum rectangular matrix spaces

In this section, we extend the quantum analogue of differential operators ∂_{ij} on the square matrices constructed in [NUW1], to the case of rectangular matrices cases.

We first recall briefly some facts on quantum R-matrices related to the L-operators defined in the preceding section. The description of the quantum matrix space will be given not simply for square matrices, but for the general rectangular case.

2.1. Quantum matrix spaces

Let (ρ, V) be the vector representation of $U_q(\mathfrak{gl}_n)$. More specifically, V is an n-dimensional vector space over $\mathbb{K}$ with canonical basis $(v_1, \cdots, v_n)$ and its left $U_q(\mathfrak{gl}_n)$

module structure is given by

$$\begin{cases} \rho(L_{ii}^{+})v_k = q^{\delta_{ki}}v_k\,, & \rho(L_{ij}^{+})v_k = (q-q^{-1})\delta_{ki}v_j \qquad (i<j)\,, \\ \rho(L_{ii}^{-})v_k = q^{-\delta_{ki}}v_k\,, & \rho(L_{ij}^{-})v_k = -(q-q^{-1})\delta_{ki}v_j \qquad (i>j)\,. \end{cases}$$

We put $R^{\pm} = \sum_{i,j=1}^{n} e_{ij} \otimes \rho(L_{ij}^{\pm})$, where $e_{ij} \in \mathrm{End}_{\mathbb{K}}(V)$ $(1 \le i, j \le n)$ are the matrix units with respect to the basis $(v_j)_{1 \le j \le n}$ and $\otimes$ is the Kronecker product of matrices. More precisely these matrices can be written as

$$R^{\pm} = \sum_{i,j=1}^{n} q^{\pm\delta_{ij}} e_{ii} \otimes e_{jj} \pm (q-q^{-1}) \sum_{i \lessgtr j} e_{ij} \otimes e_{ji}\,. \tag{2.1.1}$$

It is well known that $R^{\pm}$ satisfy the Yang-Baxter equation

$$R_{12}^{\varepsilon} R_{13}^{\varepsilon} R_{23}^{\varepsilon} = R_{23}^{\varepsilon} R_{13}^{\varepsilon} R_{12}^{\varepsilon} \quad (\varepsilon = \pm) \tag{2.1.2}$$

in $\mathrm{End}_{\mathbb{K}}(V_1 \otimes_{\mathbb{K}} V_2 \otimes_{\mathbb{K}} V_3)$ with $V_a = V$. Here the subscripts a and b of $R_{a,b}^{\varepsilon}$ indicate the pair of components this operator acts on non-trivially. We also recall that $R^{+} - R^{-} = (q-q^{-1})P$, with the matrix $P = \sum_{i,j} e_{ij} \otimes e_{ji}$ representing the flip operator: $v \otimes w \mapsto w \otimes v$. Further note that $(R_{12}^{+})^{-1} = R_{21}^{-}$ and ${}^{t}R_{12}^{\pm} = R_{21}^{\pm}$, where ^{t} denotes the transposition of matrix. Using these R-matrices we can rewrite the fundamental relations of $U_q(\mathfrak{gl}_n)$ in the following way:

$$R^{+} L_1^{\varepsilon} L_2^{\varepsilon} = L_2^{\varepsilon} L_1^{\varepsilon} R^{+} \quad (\varepsilon = \pm) \quad \text{and} \quad R^{+} L_1^{+} L_2^{-} = L_2^{-} L_1^{+} R^{+}\,. \tag{2.1.3}$$

To specify the size of the matrices $R^{\pm}$ and $L^{\pm}$, we will write them, if necessary, as $R^{[n]\pm}$ and $L^{[n]\pm}$.

Recall the definition of the coordinate ring $\mathscr{A}(\mathrm{Mat}_q(m \times n))$ of the quantum matrix space $\mathrm{Mat}_q(m \times n)$. It is the associative algebra generated by the "canonical coordinate" t_{rs} $(1 \le r \le m,\ 1 \le s \le n)$ with the following commutation relations:

$$\begin{aligned} t_{rj} t_{sj} &= q t_{sj} t_{rj} && (1 \le r < s \le k,\ 1 \le j \le n)\,, \\ t_{ri} t_{rj} &= q t_{rj} t_{ri} && (1 \le r \le k,\ 1 \le i < j \le n)\,, \\ t_{rj} t_{si} &= t_{si} t_{rj} && (1 \le r < s \le k,\ 1 \le i < j \le n)\,, \\ t_{ri} t_{sj} - t_{sj} t_{ri} &= (q-q^{-1}) t_{rj} t_{si} && (1 \le r < s \le k,\ 1 \le i < j \le n)\,. \end{aligned}$$

If we put $T = (t_{rs})_{1 \le r \le m, 1 \le s \le n}$, then the commutation relations above are equivalently written as the Yang-Baxter equation

$$R^{[m]+} T_2 T_1 = T_1 T_2 R^{[n]+}\,. \tag{2.1.4}$$

In the case $m = n$, we will denote $\mathscr{A}(\mathrm{Mat}_q(n \times n))$ by $\mathscr{A}(\mathrm{Mat}_q(n))$. This algebra $\mathscr{A}(\mathrm{Mat}_q(n))$ has a distinguished central element called the *quantum determinant*:

$$\det_q = \det_q(T) \overset{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} (-q)^{l(\sigma)} t_{\sigma(1)1} \cdots t_{\sigma(n)n}\,.$$

Here $\mathfrak{S}_n$ is the permutation group of the index set $\{1, 2, \cdots, n\}$, and for each $\sigma \in \mathfrak{S}_n$, $l(\sigma)$ represents the number of inversions in σ. Further, the coordinate ring of $\mathscr{A}(GL_q(n))$

of the quantum general linear group $GL_q(n)$ is then defined to be the localization $\mathscr{A}(\mathrm{Mat}_q(n))[\det_q(T)^{-1}]$ of $\mathscr{A}(\mathrm{Mat}_q(n))$ with respect to $\det_q(T)$. This algebra naturally possesses a Hopf algebra structure such that $\Delta(t_{ij})=\sum_{k=1}^{n} t_{ik}\otimes t_{kj}$ and $\varepsilon(t_{ij})=\delta_{ij}$, for $1\le i,j\le n$. Note also that $\det_q(T)$ is a group-like element. The antipode of $\mathscr{A}(GL_q(n))$, also denoted by S, is a $\mathbb{K}$ algebra anti-automorphism satisfying $S(T)T=TS(T)=I$ and $S(\det_q(T))=\det_q(T)^{-1}$, where $S(T)=(S(t_{ij}))_{1\le i,j\le n}$.

Since $\mathscr{A}(GL_q(n))$ has a natural two-sided comodule structure over itself, the algebra $\mathscr{A}(GL_q(n))$ becomes a $U_q(\mathfrak{gl}_n)$-bimodule through a pairing (,) defined by

$$(L_1^{\pm}, T_2)=R^{\pm}\,, \qquad (L^{\pm}, \det_q(T))=q^{\pm}I\,.$$

To be more precise, the left and right actions are given by

$$a\,.\,\varphi=(id\otimes a)\circ\Delta(\varphi)\in\mathscr{A}(GL_q(n))\,,$$
$$\varphi\,.\,a=(a\otimes id)\circ\Delta(\varphi)\in\mathscr{A}(GL_q(n))$$

for $a\in U_q(\mathfrak{gl}_n)$ and $\varphi\in\mathscr{A}(GL_q(n))$, where the symbol id denotes the identity operator on the algebra $\mathscr{A}(GL_q(n))$. It should also be noted that the algebra $\mathscr{A}(\mathrm{Mat}_q(m\times n))$ has a natural structure of a two-sided comodule over the pair of Hopf algebras $(\mathscr{A}(GL_q(n)), \mathscr{A}(GL_q(m)))$. Accordingly, it becames a bimodule over the pair of Hopf algebras $(U_q(\mathfrak{gl}_n), U_q(\mathfrak{gl}_m))$. One can write down these actions of the L-operators on $T\in\mathscr{A}(\mathrm{Mat}_q(m\times n))$ in a concrete form:

$$L_1^{[n]\pm}\,.\,T_2=T_2R^{[n]\pm}\,, \qquad T_2\,.\,L_1^{[m]\pm}=R^{[m]\pm}T_2\,. \tag{2.1.5}$$

We note also $L^{\pm}.\det_1(T)=\det_q(T).L^{\pm}=q^{\pm}\det_q(T)$ for $n=m$.

2.2. Differential operators on quantum rectangular matrix spaces

We now recall briefly the definition of the (quantized) differential operators on the coordinate ring $\mathscr{A}(GL_q(n))$, similar objects to the partial differentiation (differential operators with constant coefficients) ∂/∂_{ij} ([NUW1], Theorem 2.1).

THEOREM & DEFINITION. *There exists a unique family of linear operators* $\partial_{ij}(\lambda)=\lambda\partial_{ij}^{+}-\lambda^{-1}\partial_{ij}^{-}\in\mathrm{End}_{\mathbb{K}}(\mathscr{A}(GL_q(n)))\otimes\mathbb{K}[\lambda,\lambda^{-1}]$ $(1\le i,j\le n)$, *with the spectral parameter* λ, *satisfying both of the following equations*:

$$L_{ji}(\lambda)=(q-q^{-1})\sum_{k=1}^{n} t_{ki}^{\circ}\partial_{kj}(\lambda)\,, \tag{2.2.1}$$

$$L_{ji}(\lambda)^{\circ}=(q-q^{-1})\sum_{k=1}^{n} t_{jk}\partial_{ik}(\lambda)\,, \tag{2.2.2}$$

where t_{ki}° *and* $L_{ji}(\lambda)^{\circ}$ *represent the actions from the right. We define the operator* $\partial_{ij}=\partial_{ij}(1)$ *as the differential operator with respect to the coordinates on* $\mathrm{Mat}_q(n)$. □

It should be remarked that the operators ∂_{ij} defined above are, in fact, acting not only on $\mathscr{A}(GL_q(n))$ but also on $\mathscr{A}(\mathrm{Mat}_q(n))$ (see Proposition 5.1 in [NUW1]). This property is essential to the discussion below.

From this point, we consider the rectangular matrix case. Namely, we have the following proposition which is concerned with the existence of "differential operators" with constant coefficients on the space $\mathrm{Mat}_q(m\times n)$.

PROPOSITION 2.2. *For the natural action of $U_q(\mathrm{gl}_n)$ (resp., $U_q(\mathrm{gl}_m)$) from the left (resp., right) on $\mathscr{A}(\mathrm{Mat}_q(m\times n))$ there exists a unique family of linear operators $\bar{\partial}_{ij}$ $(1\le i\le m,\ 1\le j\le n)\in \mathrm{End}_{\mathbb{K}}(\mathscr{A}(\mathrm{Mat}_q(m\times n)))$ such that (in matrix form)*

$$L^{[n]}.\varphi=(q-q^{-1})({}^t\bar{\partial}\varphi)T\,,$$

$$\varphi.L^{[m]}=(q-q^{-1})T({}^t\bar{\partial}\varphi)\,,$$

for the matrix $\bar{\partial}=(\bar{\partial}_{ij})_{1\le i\le m,1\le j\le n}$. Here

$$L^{[n]}=L^{[n]}(1)=L^{[n]+}-L^{[n]-}\,.$$

If $m=n$ this implies the above theorem for $\lambda=1$.

Furthermore, these definitions are compatible with the embedding of $\mathscr{A}(\mathrm{Mat}_q(m\times n))$ into $\mathscr{A}(\mathrm{Mat}_q(N))$ for any $N\ge\max(m,n)$. More precisely, the restriction of ∂_{ij} $(\in \mathrm{End}_{\mathbb{K}}(\mathscr{A}(GL_q(N))))$ to $\mathscr{A}(\mathrm{Mat}_q(m\times n))$ coincides with $\bar{\partial}_{ij}$:

$$\partial_{ij}\big|_{\mathscr{A}(\mathrm{Mat}_q(m\times n))}=\bar{\partial}_{ij}\,. \quad \square$$

REMARK. If we specialize as $m=1$ in Proposition 2.2, then the first formula above reduces to (1.3.3) and the second one reduces to

$$\{\gamma\}=\sum_{i=1}^{n} t_i\partial_i^R\,,$$

which is derived from the very definition of Euler's degree operator.

For the proof of this proposition, we need some more knowledge on the basic properties of ∂_{ij}, and we give the proof in the next subsection.

2.3. Fundamental properties of differential operators on $\mathrm{Mat}_q(m\times n)$

We first write down several commutation relations among the multiplication operators and the differential operators which are disccussed in [NUW1] for the case $m=n$.

PROPOSITION 2.3. *As linear operators on $\mathscr{A}(GL_q(n))$ we have*

(1) *The operators $\partial_{ij}(\lambda)$ satisfy the commutation relations*

$$R_{21}^{\pm}\partial_2(\lambda)\partial_1(\lambda)=\partial_1(\lambda)\partial_2(\lambda)R_{12}^{\pm}\,,\quad R_{12}^{+}\partial_2^{+}\partial_1^{-}=\partial_1^{-}\partial_2^{+}R_{21}^{-}\,.$$

(2) *The commutation relations for $\partial_{ij}^{\pm}$ with the left and right multiplication operators are*

$$\partial_1^{\pm}T_2=T_2\partial_1^{\pm\,t_1}R_{12}^{\pm}\,,\quad \partial_1^{\pm}T_2^{\circ}={}^{t_1}R_{12}^{\pm}T_2^{\circ}\partial_1^{\pm}\,,$$

where ${}^{t_1}R_{12}^{\pm}$ represents the matrices obtained from $R_{12}^{\pm}$ by the transposition in the first factor. $\square$

Note that the above commutation relations follow directly from the Leibniz rule for $\partial_{ij}(\lambda)$:

Leibniz rule ([NUW1] Proposition 4.1): For $\varphi_1, \varphi_2 \in \mathscr{A}(GL_q(n))$,

$$
{}^t\partial(\lambda\mu)(\varphi_1\varphi_2) = \mu^{\pm 1}({}^t\partial(\lambda)\varphi_1)(L^{\pm\circ}.\varphi_2) + \lambda^{\mp 1}(L^{\mp}.\varphi_1)({}^t\partial(\mu)\varphi_2)\,.
$$

Moreover, if we write down the commutation relations for the ∂_{ij} themselves (which act on $\mathscr{A}(\mathrm{Mat}_q(n))$) and for these opperators together with the multiplication operators t_{ij}, we have the componentwise relations

$$
\begin{aligned}
\partial_{rj}\partial_{sj} &= q\partial_{sj}\partial_{rj} \qquad (1 \le r < s \le n,\ 1 \le j \le n)\,,\\
\partial_{ri}\partial_{rj} &= q^{-1}\partial_{rj}\partial_{ri} \qquad (1 \le r \le n,\ 1 \le i < j \le n)\,,\\
\partial_{ri}\partial_{sj} &= \partial_{sj}\partial_{ri} \qquad (1 \le r < s \le n,\ 1 \le i < j \le n)\,,\\
\partial_{rj}\partial_{si} - \partial_{si}\partial_{rj} &= (q-q^{-1})\partial_{ri}\partial_{sj} \qquad (1 \le r < s \le n,\ 1 \le i < j \le n)\,,
\end{aligned}
$$

and

(2.3.1) $$\partial_{ij}t_{kl} - q^{-\delta_{lj}}t_{kl}\partial_{ij} + (q-q^{-1})\delta_{lj}\sum_{\alpha<j} t_{k\alpha}\partial_{i\alpha} = \delta_{lj}L_{ki}^{+\circ}\,,$$

(2.3.2) $$\partial_{ij}t_{kl} - q^{\delta_{lj}}t_{kl}\partial_{ij} - (q-q^{-1})\delta_{lj}\sum_{\alpha>j} t_{k\alpha}\partial_{i\alpha} = \delta_{lj}L_{ki}^{-\circ}\,,$$

(2.3.3) $$\partial_{ij}t_{kl}^{\circ} - q^{-\delta_{ik}}t_{kl}^{\circ}\partial_{ij} + (q-q^{-1})\delta_{ik}\sum_{i<\alpha} t_{\alpha l}^{\circ}\partial_{\alpha j} = \delta_{ik}L_{jl}^{+}\,,$$

(2.3.4) $$\partial_{ij}t_{kl}^{\circ} - q^{\delta_{ik}}t_{kl}^{\circ}\partial_{ij} - (q-q^{-1})\delta_{ik}\sum_{i>\alpha} t_{\alpha l}^{\circ}\partial_{\alpha j} = \delta_{ik}L_{jl}^{-}\,.$$

REMARK. The above commutation relations for the ∂_{ij} among themselves are the same as those for the t_{ij} if we reverse the order of the numbering of columns of the matrix $\partial = (\partial_{ij})$.

Proof of Proposition 2.2. It is clear that the embedding of $\mathscr{A}(\mathrm{Mat}_q(m\times n))$ into a larger algebra $\mathscr{A}(\mathrm{Mat}_q(N))$ can be described by a pair of an m-tuple I and an n-tuple J given by

$$
I = \{i_1, \cdots, i_m\}\ (1 \le i_1 < \cdots < i_m \le N)\,, \qquad J = \{j_1, \cdots, j_n\}\ (1 \le j_1 < \cdots < j_n \le N)
$$

for any fixed N satisfying $N \ge \max(m, n)$. We denote such an embedding by ρ_{IJ}. Then it is also clear that the restriction of L-operators for this embedding is compatible with the actions on $\mathscr{A}(\mathrm{Mat}_q(m\times n))$,

$$
L_{ji}^{[N]\pm}\big|_{\rho_{IJ}(\mathscr{A}(\mathrm{Mat}_q(mn)))} = \begin{cases} L_{ji}^{[m]\pm} & (i \in I, j \in J)\,,\\ 0 & (\text{otherwise})\,, \end{cases}
$$

for the left action. A similar formula holds for the right action. On the other hand, using the commutation relations for ∂_{ij} with the multiplication operators, we can show by a simple inductive argument that ∂_{ji} defines the zero operator on the space $\rho_{IJ}(\mathscr{A}(\mathrm{Mat}_q(m\times n)))$, except when $i \in I$ and $j \in J$. Thus, recalling the defining relations of ∂_{ij} on $\mathscr{A}(GL_q(N))$, we find for $i \in I, j \in J$, that

(2.3.5) $$L_{ji}^{[m]}=(q-q^{-1})\sum_{k\in I} t_{ki}^{\circ}\partial_{ji}\big|_{\rho_{IJ}(\mathscr{A}(\mathrm{Mat}_q(m\times n)))},$$

(2.3.6) $$L_{ji}^{[n]\circ}=(q-q^{-1})\sum_{k\in J} t_{jk}\partial_{ik}\big|_{\rho_{IJ}(\mathscr{A}(\mathrm{Mat}_q(m\times n)))}.$$

These relations guarantee the operators $\bar{\partial}_{ij}$ are well-defined.

The uniqueness of the operators can also be shown by induction. In fact, by Proposition 5.2 in [NUW1], we know that the restriction of our operator ∂_{ij} (and hence $\bar{\partial}_{ij}$) defines the q-difference operator on each column (resp., row). The uniqueness of the operator on each column (resp., row) follows immediately from this fact. The Leibniz rule

(2.3.7) $$\partial_{ji}(\varphi_1\varphi_2)=\sum_k(\partial_{ki}\varphi_1)(L_{kj}^{\pm\circ}.\varphi_2)+\sum_k(L_{ik}^{\mp}.\varphi_1)(\partial_{jk}\varphi_2)$$

guarantees the uniqueness of the operator $\bar{\partial}_{ij}$ in the general case. This proves the assertion of Proposition 2.2. □

REMARK. In the proof above, we note that the uniqueness of the operators $\bar{\partial}_{ij}$ follows simply from two facts: (1) The restriction of $\bar{\partial}_{ij}$ either on each column or row gives a q-difference operator, and (2) the relations (2.3.5) and (2.3.6) hold (in fact only one of these is needed).

3. Weaving q-difference operators into differential operators on quantum matrix spaces

In [NUW1] we showed that the restriction of our differential operators to each column (resp., row) gave q-difference operators. In this section we will conversely show that these differential operators are obtained from q-difference operators as a kind of normal product of operators.

We consider the following situation. We regard $\mathscr{A}(\mathrm{Mat}_q(m\times n))$ as the m-fold tensor product of $\mathscr{A}=\mathbb{K}[t_1,\cdots,t_n]$ via the multiplication of $\mathscr{A}(\mathrm{Mat}_q(m\times n))$:

$$\mathscr{A}^{\otimes m}=\mathscr{A}\otimes\cdots\otimes\mathscr{A}\xrightarrow{\sim}\mathscr{A}(\mathrm{Mat}_q(m\times n)).$$

In this paragraph, for simplicity, we denote by ∂_{rs}^L and ∂_{rs}^R the q-difference operators with respect to the variable t_{rs}:

$$\partial_{rs}^L\varphi=t_{rs}^{-1}\frac{\gamma_{rs}-\gamma_{rs}^{-1}}{q-q^{-1}}\varphi,\qquad \partial_{rs}^R\varphi=\left(\frac{\gamma_{rs}-\gamma_{rs}^{-1}}{q-q^{-1}}\varphi\right)t_{rs}^{-1}.$$

Here γ_{rs} represents the automorphism defined by the q-shift operator $t_{ij}\to q^{\delta_{ir}\delta_{js}}t_{ij}$. We now give the *weaving* formula.

THEOREM 3.1. *Via the natural embedding* $\mathscr{A}^{\otimes m}={}^t(\mathscr{A}\otimes\cdots\otimes\mathscr{A})\hookrightarrow\mathscr{A}(\mathrm{Mat}_q(m\times n))$, *the differential operators* $\bar{\partial}_{ij}$ *and the right multiplication operators* $\bar{t}_{ij}^{\circ}$ *on* $\mathscr{A}^{\otimes m}$ *are given by the following formulas*:

$$
\begin{aligned}
\bar{\partial}_{ij}= & \sum_{l\geq 0}(q-q^{-1})^l \sum_{\substack{1\leq i_1<i_2<\cdots<i_l<i\\ j<j_1<j_2<\cdots<j_l\leq i}} \gamma_{ij}\otimes\cdots\otimes\gamma_{i_1-1j}\otimes t^{\circ}_{i_1j_1}\partial^R_{i_1j} \\
& \otimes\gamma_{i_1+1j_1}\otimes\cdots\otimes\gamma_{i_2-1j_1}\otimes t^{\circ}_{i_2j_2}\partial^R_{i_2j_1}\otimes\gamma_{i_2+1j_2}\otimes\cdots\otimes\gamma_{i_3-1j_2} \\
& \otimes t^{\circ}_{i_3j_3}\partial^R_{i_3j_2}\otimes\gamma_{i_3+1j_3}\otimes\cdots\otimes\gamma_{i_l-1j_{l-1}}\otimes t^{\circ}_{i_lj_l}\partial^R_{i_lj_{l-1}}\otimes\gamma_{i_l+1j_l}\otimes\cdots \\
& \otimes\gamma_{i-1j_l}\otimes\partial^R_{ij_l}\otimes\underbrace{1\otimes\cdots\otimes 1}_{(m-i)\text{-times}},
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{t}^{\circ}_{ij}= & \overbrace{1\otimes\cdots\otimes 1}^{(i-1)\text{-times}}\otimes\sum_{l\geq 0}\{-(q-q^{-1})\}^l \sum_{\substack{i<i_1<i_2<\cdots<i_l\leq m\\ j<j_l<j_{l-1}<\cdots<j_1\leq n}} t^{\circ}_{ij_1}\otimes\gamma^{-1}_{i+1j_1} \\
& \otimes\cdots\otimes\gamma^{-1}_{i_1-1j_1}\otimes t^{\circ}_{i_1j_2}\partial^R_{i_1j_1}\otimes\gamma^{-1}_{i_1+1j_2}\otimes\cdots\otimes\gamma^{-1}_{i_2-1j_2}\otimes t^{\circ}_{i_2j_3}\partial^R_{i_2j_2} \\
& \otimes\gamma^{-1}_{i_2+1j_3}\otimes\cdots\otimes\gamma^{-1}_{i_3-1j_3}\otimes t^{\circ}_{i_3j_4}\partial^R_{i_3j_3}\otimes\gamma^{-1}_{i_3+1j_4}\otimes\cdots\otimes\gamma^{-1}_{i_l-1j_l} \\
& \otimes t^{\circ}_{i_lj}\partial^R_{i_lj_l}\otimes\gamma^{-1}_{i_l+1j}\otimes\cdots\otimes\gamma^{-1}_{mj}. \quad \square
\end{aligned}
$$

This theorem is proved by induction using the following key proposition in a step-by-step discussion. We introduce an ad-hoc notation on the operators for the proof as follows. According to the division of an $m\times n$ matrix into two n-column parts each consisting of consecutive rows, say (*Top*) and (*Bottom*), we have in $\mathscr{A}=\mathscr{A}(\mathrm{Mat}_q(m\times n))$ two subalgebras $\bar{\mathscr{A}}$ and $\underline{\mathscr{A}}$ which are respectively generated by the coordinates for the (*Top*)- and (*Bottom*)-indices. If an operator $a\in\mathrm{End}_{\mathbb{K}}(\mathscr{A}\ (\mathrm{Mat}_q(m\times n)))$ also stabilizes $\bar{\mathscr{A}}$ or $\underline{\mathscr{A}}$, then we write the operator a whose action is restricted to $\bar{\mathscr{A}}$ or $\underline{\mathscr{A}}$ respectively as $\bar{a}$ or $\underline{a}$ to distinguish where it acts. When a acts on the whole of $\mathscr{A}$, we denote it by $\underline{\bar{a}}$ to emphasize this point. We are concerned with a description of operators on $\mathscr{A}$ which can be reduced to the form $\underline{\bar{a}}=\sum_i\bar{b}_i\otimes\underline{c}_i\in\mathrm{End}_{\mathbb{K}}(\bar{\mathscr{A}})\otimes\mathrm{End}_{\mathbb{K}}(\underline{\mathscr{A}})$.

PROPOSITION 3.2. *Using the convention for the notation given above, we have an expression for the right multiplication operator* t°_{lj} *as*

$$
\begin{cases}
\underline{\bar{t}}^{\circ}_{lj}=\bar{t}^{\circ}_{lj}\otimes\underline{\gamma}_j^{-1}-(q-q^{-1})\sum_{j<\alpha,\lambda\in(Bottom)}\bar{t}^{\circ}_{l\alpha}\otimes\underline{t}^{\circ}_{\lambda j}\underline{\partial}_{\lambda\alpha} & \text{for } l\in(Top) \\
\underline{\bar{t}}^{\circ}_{lj}=1\otimes\underline{t}^{\circ}_{lj} & \text{for } l\in(Bottom),
\end{cases}
$$

and for the differential operator ∂_{li} as

$$
\begin{cases}
\underline{\bar{\partial}}_{li}=\bar{\partial}_{li}\otimes 1 & \text{for } l\in(Top) \\
\underline{\bar{\partial}}_{li}=\bar{\gamma}_i\otimes\underline{\partial}_{li}+(q-q^{-1})\sum_{i<\beta,\kappa\in(Top)}\bar{t}^{\circ}_{\kappa\beta}\bar{\partial}_{\kappa i}\otimes\underline{\partial}_{l\beta} & \text{for } l\in(Bottom).
\end{cases}
$$

Proof. For the right multiplication operators, it is clear that the above formula for $l\in(Bottom)$ is equivalent to the following:

$$\bar{\underline{t}}^{\circ}_{lj}=\bar{t}^{\circ}_{lj}\otimes\underline{\gamma}_j^{-1}-(q-q^{-1})\sum_{j<\alpha}\bar{t}^{\circ}_{l\alpha}\otimes\underline{L}_{\alpha j}$$

$$=\sum_{j\leq\alpha}\bar{t}^{\circ}_{l\alpha}\otimes\underline{L}^{-}_{\alpha j}\,.$$

We will first show this form. Using the natural graded structure of $\mathscr{A}(\mathrm{Mat}_q(m\times n))$, we prove this assertion by induction on the degree of the polynomials.

Let $\psi\in\mathscr{A}(\mathrm{Mat}_q(m\times n))$ and $\varphi\in\mathscr{A}(Bottom)$. Then it suffices to prove that

$$t^{\circ}_{lj}(\psi\varphi)=(\psi\varphi)t_{lj}=\sum_{j\leq\alpha}(\psi t_{l\alpha})L^{-}_{\alpha j}.\varphi\,,\qquad(\text{for } l\in(Top))\,. \tag{3.1}$$

Suppose that $\varphi=t_{pk}$ for $k\in(Bottom)$. Then, if $p\leq j$, a simple commutation relation between t_{lj} and t_{pk} implies

$$t^{\circ}_{lj}(\psi t_{pk})=q^{-\delta_{jk}}(\psi t_{lj})t_{pk}=(\psi t_{lj})\gamma_j^{-1}.t_{pk}\,. \tag{3.2}$$

For $p>j$, the relation

$$t_{pk}t_{lj}=t_{lj}t_{pk}-(q-q^{-1})t_{lk}t_{pj}$$

implies

$$t^{\circ}_{lj}(\psi t_{pk})=(\psi t_{lj})t_{pk}-(q-q^{-1})\psi t_{lk}t_{pj}\,.$$

Since $L^{-}_{jj}=\gamma_j^{-1}$ and $L^{-}_{ji}.t_{pq}=-(q-q^{-1})t_{pi}\delta_{jq}$ $(j>i)$, those formulas respectively amount to the desired relation for polynomials of degree 1. For general $\varphi\in\mathscr{A}(Bottom)$, it then suffices to show that if the formula (3.2) holds for two elements $\varphi_1,\varphi_2\in\mathscr{A}(Bottom)$, it also holds the product $\varphi_1\varphi_2$:

$$\begin{aligned}
t^{\circ}_{lj}(\psi\varphi_1\varphi_2)&=\sum_{j\leq\alpha}((\psi\varphi_1)t_{l\alpha})(L^{-}_{\alpha j}.\varphi_2)\\
&=\sum_{j\leq\alpha}\left(\sum_{\alpha\leq\beta}(\psi t_{l\beta})(L^{-}_{\beta\alpha}.\varphi_1)\right)(L^{-}_{\alpha j}.\varphi_2)\\
&=\sum_{j\leq\beta}(\psi t_{l\beta})\sum_{j\leq\alpha\leq\beta}(L^{-}_{\beta\alpha}.\varphi_1)(L^{-}_{\alpha j}.\varphi_2)\\
&=\sum_{j\leq\beta}(\psi t_{l\beta})L^{-}_{\beta j}.(\varphi_1\varphi_2)\,.
\end{aligned}$$

The last equality here follows immediately from the comultiplication formula of L-operators $\{L^{-}_{\beta j}\}$: $\Delta(L^{-}_{\beta j})=\sum_{j\leq\alpha\leq\beta}L^{-}_{\beta\alpha}\otimes L^{-}_{\alpha j}$. This completes the proof of the assertion for right multiplication operators.

Next we prove the assertion for differential operators. It of course suffices to show that for $l\in(Bottom)$

$$\bar{\underline{\partial}}_{li}=\sum_{i\leq\beta}\bar{L}^{+}_{i\beta}\otimes\underline{\partial}_{l\beta}\,. \tag{3.3}$$

For the moment, we denote by p_{li} the right-hand side of (3.3) with $l \in (Bottom)$ and put $p_{li} = \underline{\bar{\partial}}_{li}$ for $l \in (Top)$. Then we see

$$
\begin{aligned}
&(q-q^{-1})\sum_{l} \underline{\bar{t}}^{\circ}_{lj} p_{li} \\
&= (q-q^{-1}) \sum_{l \in (Top)} \underline{\bar{t}}^{\circ}_{lj} p_{li} + (q-q^{-1}) \sum_{l \in (Bottom)} \underline{\bar{t}}^{\circ}_{lj} p_{li} \\
&= (q-q^{-1}) \left\{ \sum_{l \in (Top)} \left(\sum_{j \le \alpha} \bar{t}^{\circ}_{l\alpha} \otimes \underline{L}^{-}_{\alpha j} \right) (\bar{\partial}_{li} \otimes 1) + \sum_{l \in (Bottom)} (1 \otimes \underline{t}^{\circ}_{li}) \left(\sum_{i \le \beta} \bar{L}^{+}_{i\beta} \otimes \underline{\partial}_{l\beta} \right) \right\} \\
&= (q-q^{-1}) \left\{ \sum_{l \in (Top)} \sum_{j \le \alpha} \bar{t}^{\circ}_{l\alpha} \bar{\partial}_{li} \otimes \underline{L}^{-}_{\alpha j} + \sum_{l \in (Bottom)} \sum_{i \le \beta} \bar{L}^{+}_{i\beta} \otimes \underline{t}^{\circ}_{lj} \underline{\partial}_{l\beta} \right\} \\
&= \sum_{j \le \alpha} \bar{L}_{i\alpha} \otimes \underline{L}^{-}_{\alpha j} + \sum_{i \le \beta} \bar{L}^{+}_{i\beta} \otimes \underline{L}_{\beta j}\,.
\end{aligned}
$$

This implies that the right-hand side coincides with the comultiplication of $\underline{\bar{L}}_{ij}$, that is, the action of L_{ij} on $\mathrm{End}_{\mathbb{K}}(\mathscr{A}(Top)) \otimes \mathrm{End}_{\mathbb{K}}(\mathscr{A}(Bottom))$. Hence by simple induction we obtain

$$(q-q^{-1})\sum_{l} \underline{\bar{t}}^{\circ}_{lj} p_{li} = \underline{\bar{L}}_{ij}\,.$$

By the uniqueness of operators, which is guaranteed by Proposition 2.2 (see also the Remark at the end of §2.2), we have $p_{ij} = \underline{\bar{\partial}}_{ij}$. This completes the proof of the proposition. □

REMARK. As we have various combinations of multiplications (left and right), of difference operators (left and right) and the direction of weaving, i.e., the manner of dividing tensor space (e.g., horizontally or vertically), we can discuss a number of weaving formulas similar to those presented above. We do not need to consider each of them individually.

4. $U_q(\mathfrak{sp}_{2N})$ and its oscillator representation

From this point, we use simply a symbol ∂_{ij} in place of $\bar{\partial}_{ij}$ for differential operators on $\mathrm{Mat}_q(m \times n)$.

In this section we construct the quantized oscillator representation of $U_{q^2}(\mathfrak{sp}_{2N})$ and its tensor product representation explicitly by using the differential operators ∂_{ij}. Moreover, as the rank one case discussed in [NUW3], we will see that the quantized algebra $U_q(\mathfrak{o}_m)$ in the sense of Gavrilik and Klimyk [GaK1] appears naturally in the commutant of the tensor power.

4.1. Hopf algebra $U_q(\mathfrak{sp}_{2N})$

First, let us recall the definition of a quantized enveloping algebra $U_q(\mathfrak{sp}_{2N})$ of the symplectic Lie algebra $\mathfrak{sp}_{2N}$. The algebra $U_q(\mathfrak{sp}_{2N})$ is a Hopf algebra generated by the elements $\hat{e}_i, \hat{f}_i, k_i^{\pm 1}$ $(1 \le i \le N)$ with $k_i = q^{\varepsilon_i - \varepsilon_{i+1}}$ $(1 \le i \le N-1)$ and $k_N = q^{2\varepsilon_N}$, subject to the following three types of relations:

(1) $$\hat{e}_i, \hat{f}_i, k_i^{\pm 1} \ (1 \le i \le N-1) \quad \text{generate} \quad U_q(\mathfrak{sl}_N),$$

(2) $$\begin{cases} k_N \hat{e}_N k_N^{-1} = q^2 \hat{e}_N, \quad k_N \hat{e}_{N-1} k_N^{-1} = q^{-1} \hat{e}_{N-1}, \quad k_{N-1} \hat{e}_N k_{N-1}^{-1} = q^{-1} \hat{e}_N, \\ k_N \hat{f}_N k_N^{-1} = q^{-2} \hat{f}_N, \quad k_N \hat{f}_{N-1} k_N^{-1} = q \hat{f}_{N-1}, \quad k_{N-1} \hat{f}_N k_{N-1}^{-1} = q \hat{f}_N, \\ \hat{e}_N \hat{f}_N - \hat{f}_N \hat{e}_N = \dfrac{k_N - k_N^{-1}}{q - q^{-1}}, \\ \hat{e}_N^2 \hat{e}_{N-1} - (q + q^{-1}) \hat{e}_N \hat{e}_{N-1} \hat{e}_N + \hat{e}_{N-1} \hat{e}_N^2 = 0, \\ \hat{e}_{N-1}^3 \hat{e}_N - (q^2 + 1 + q^{-2})(\hat{e}_{N-1}^2 \hat{e}_N \hat{e}_{N-1} - \hat{e}_{N-1} \hat{e}_N \hat{e}_{N-1}^2) - \hat{e}_N \hat{e}_{N-1}^3 = 0, \\ \hat{f}_N^2 \hat{f}_{N-1} - (q + q^{-1}) \hat{f}_N \hat{f}_{N-1} \hat{f}_N + \hat{f}_{N-1} \hat{f}_N^2 = 0, \\ \hat{f}_{N-1}^3 \hat{f}_N - (q^2 + 1 + q^{-2})(\hat{f}_{N-1}^2 \hat{f}_N \hat{f}_{N-1} - \hat{f}_{N-1} \hat{f}_N \hat{f}_{N-1}^2) - \hat{f}_N \hat{f}_{N-1}^3 = 0, \end{cases}$$

and for $1 \le j \le N-2$

(3) $$\begin{cases} k_j \hat{e}_N = \hat{e}_N k_j, \quad k_j \hat{f}_N = \hat{f}_N k_j, \\ \hat{e}_N \hat{e}_j = \hat{e}_j \hat{e}_N, \quad \hat{e}_j \hat{e}_N = \hat{e}_N \hat{e}_j, \\ \hat{f}_N \hat{f}_j = \hat{f}_j \hat{f}_N, \quad \hat{f}_j \hat{f}_N = \hat{f}_N \hat{f}_j. \end{cases}$$

The comultiplication, which we also denote by Δ, is given by

$$\Delta(\hat{e}_j) = \hat{e}_j \otimes k_j^{-1} + 1 \otimes \hat{e}_j, \quad \Delta(\hat{f}_j) = \hat{f}_j \otimes 1 + k_j \otimes \hat{f}_j,$$
$$\Delta(k_j^{\pm 1}) = k_j^{\pm 1} \otimes k_j^{\pm 1}, \qquad \text{for} \quad 1 \le j \le N.$$

The counit ε is given by

$$\varepsilon(\hat{e}_j) = \varepsilon(\hat{f}_j) = 0, \quad \varepsilon(k_j^{\pm 1}) = 1, \qquad \text{for} \quad 1 \le j \le N.$$

4.2. The q-oscillator representation of $U_{q^2}(\mathfrak{sp}_{2N})$

Let us consider the q-commutative polynomial ring $\mathscr{A} = \mathbb{K}[t_1, \cdots, t_N]$, where the relations $t_i t_j = q t_j t_i$ $(i<j)$ hold among the variables t_j. Then the (q-)oscillator representation of $U_{q^2}(\mathfrak{sp}_{2N})$ is realized on $\mathscr{A}$.

THEOREM 4.1. *The action ω on the generators of $U_{q^2}(\mathfrak{sp}_{2N})$ defined below gives a left module structure on $\mathscr{A} = \mathbb{K}[t_1, \cdots, t_N]$:*

$$\begin{cases} \omega(\hat{e}_i) = \gamma_{i+1} t_i^{\circ} \partial_{i+1}^R, \quad \omega(\hat{f}_i) = \gamma_{i+1}^{-1} t_{i+1}^{\circ} \partial_i^R, \\ \omega(k_i) = \gamma_i \gamma_{i+1}^{-1} \quad (1 \le i \le N-1) \\ \omega(\hat{e}_N) = \dfrac{1}{[2]} t_N^{\circ 2}, \quad \omega(\hat{f}_N) = -\dfrac{1}{[2]} \partial_N^{R^2}, \quad \omega(k_N) = q \gamma_N^2, \end{cases}$$

where $[l]$ stands for the q-integer: $[l] = \dfrac{q^l - q^{-l}}{q - q^{-1}}$. □

Proof. The first three actions $\omega(\hat{e}_i)$, $\omega(\hat{f}_i)$, $\omega(k_i)$ $(1 \leq i \leq N-1)$ just correspond to the actions of the subalgebra $U_{q^2}(\mathfrak{sl}_N)$ of $U_{q^2}(\mathfrak{sp}_{2N})$. Moreover, as in the case of $U_{q^2}(\mathfrak{sl}_2)$ (cf. [NUW3]), we easily find that

$$[\partial_N^{R^2}, t_N^{\circ 2}] = [2] \frac{q\gamma_N^2 - q^{-1}\gamma_N^{-2}}{q - q^{-1}}.$$

Also it is easy to see that the relations among the operators of the commuting family are maintained. It therefore suffices to show the Serre relations (with respect to the base q^2),

$$\hat{e}_N^2 \hat{e}_{N-1} - (q^2 + q^{-2}) \hat{e}_N \hat{e}_{N-1} \hat{e}_N + \hat{e}_{N-1} \hat{e}_N^2 = 0,$$
$$\hat{e}_{N-1}^3 \hat{e}_N - (q^4 + 1 + q^{-4})(\hat{e}_{N-1}^2 \hat{e}_N \hat{e}_{N-1} - \hat{e}_{N-1} \hat{e}_N \hat{e}_{N-1}^2) - \hat{e}_N \hat{e}_{N-1}^3 = 0$$

between the operators $\hat{e}_{N-1}$ and $\hat{e}_N$, and those of $\hat{f}_{N-1}$ and $\hat{f}_N$, respectively. The proof is done by direct calculation. In fact, by the basic commutation relations among partial q-difference operators and multiplication operators in 1.1. of §1, we see

$$[\hat{e}_N, \hat{e}_{N-1}]_{q^2} = \frac{1}{[2]} \gamma_N t_{N-1}^{\circ} (t_N^{\circ 2} \partial_N^R - q^2 \partial_N^R t_N^{\circ 2}) = -q t_{N-1}^{\circ} \gamma_N^2 t_N^{\circ},$$

where $[A, B]_t = AB - tBA$. Hence we have

$$[\hat{e}_N, [\hat{e}_N, \hat{e}_{N-1}]_{q^2}]_{q^{-2}} = -q \frac{1}{[2]} (t_N^{\circ 2} t_{N-1}^{\circ} \gamma_N^2 t_N^{\circ} - q^{-2} t_{N-1}^{\circ} \gamma_N^2 t_N^{\circ 3}) = 0.$$

This implies the first relation in question. As the remaining relations can be checked in the same way, we omit their explicit proofs. □

We call the representation ω on $\mathscr{A} = \mathbb{K}[t_1, \cdots, t_N]$ the *(q-)oscillator representation* of $U_{q^2}(\mathfrak{sp}_{2N})$. This is not irreducible but breaks into two irreducible components which consist of the polynomials of even and odd degrees respectively. Since the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to $U_q(\mathfrak{sp}_2)$ as a Hopf algebra, the oscillator representation of $U_{q^2}(\mathfrak{sl}_2)$ constructed in [NUW2] and [NUW3] is a special (rank 1) case of the above.

REMARK. In the paper [Ha], Hayashi constructed explicitly the oscillator representations of $U_q(\mathfrak{sp}_{2N})$ on a "commutative" polynomial ring. Our description of the operators utilizes the natural ring structure of representation spaces. This has an advantage when considering their tensor power representations (see Theorem 4.2 below).

Recall the identification via the natural multiplication of $\mathscr{A}(\mathrm{Mat}_q(m \times N))$:

$$\mathscr{A}^{\otimes m} = \mathscr{A} \otimes \cdots \otimes \mathscr{A} \xrightarrow{\sim} \mathscr{A}(\mathrm{Mat}_q(m \times N)).$$

Through this identification we naturally obtain the m-fold tensor product representation $\omega^{\otimes m}$ of the oscillator representation ω of $U_{q^2}(\mathfrak{sp}_{2N})$ by the comultiplication Δ:

THEOREM 4.2. *The actions $\omega^{\otimes m}$ of $U_{q^2}(\mathfrak{sp}_{2N})$ on $\mathscr{A}(\mathrm{Mat}_q(m\times N))$ are given by the following formulas*:

$$\omega^{\otimes m}(L_{ji})=(q-q^{-1})\sum_{k=1}^{m} t_{ki}^{\circ}\partial_{kj} \quad (1\le i, j\le N),$$

$$\omega^{\otimes m}(\hat{e}_N)=\frac{1}{[2]}\sum_{k=1}^{m} q^{k-m}t_{kN}^{\circ 2},$$

$$\omega^{\otimes m}(\hat{f}_N)=-\frac{1}{[2]}\sum_{k=1}^{m} q^{k-1}\partial_{kN}^{2},$$

$$\omega^{\otimes m}(k_N)=q^{m+2\varepsilon_N}.$$

The first formula here indicates the $\mathfrak{sl}_N$-part of $\mathfrak{sp}_{2N}$.

Proof. The $\mathfrak{sl}_N$-part is easily seen from the comultiplication rule (1.3.1). Also the last formula for $\omega^{\otimes m}(k_N)$ is clear.

We prove the remaining of the assertions by induction on m. For $\hat{f}_N$, we utilize the identification

$$\mathscr{A}^{\otimes m}\simeq\mathscr{A}^{\otimes m-1}\otimes\mathscr{A}\xrightarrow{\sim}\mathscr{A}(\mathrm{Mat}_q((m-1)\times N)\otimes\mathscr{A}.$$

We first recall its comultiplication:

$$\Delta(\hat{f}_N)=\hat{f}_N\otimes 1+k_N\otimes\hat{f}_N.$$

Then we have from this and our induction assumption that

$$\begin{aligned}\omega^{\otimes m}(\hat{f}_N)&=\omega^{\otimes m-1}(\hat{f}_N)\otimes 1+\omega^{\otimes m-1}(k_N)\otimes\omega(\hat{f}_N)\\&=-\frac{1}{[2]}\sum_{k=1}^{m-1} q^{k-1}\overline{\partial_{kN}^{2}}\otimes 1-q^{m-1}\overline{\gamma_N^{2}}\otimes\left(\frac{1}{[2]}\underline{\partial}_{mN}^{2}\right).\end{aligned}$$

On the other hand, from Proposition 3.2 we have

$$\bar{\underline{\partial}}_{li}=\bar{\partial}_{li}\otimes 1 \qquad \text{for} \quad l<m, \quad \text{and} \quad \bar{\underline{\partial}}_{mi}=\bar{\gamma}_i\otimes\underline{\partial}_{mi},$$

so that

$$\sum_{k=1}^{m} q^{k-1}\partial_{kN}^{2}=\sum_{k=1}^{m-1} q^{k-1}\bar{\partial}_{kN}^{2}\otimes 1+q^{m-1}\bar{\gamma}_N^{2}\otimes\underline{\partial}_{mN}^{2}.$$

Combining this with the above expression for $\omega^{\otimes m}(\hat{f}_N)$, we reach the desired conclusion.

For $\hat{e}_N$, utilize the identification

$$\mathscr{A}^{\otimes m}\simeq\mathscr{A}\otimes\mathscr{A}^{\otimes m-1}\xrightarrow{\sim}\mathscr{A}\otimes\mathscr{A}(\mathrm{Mat}_q((m-1)\times N)),$$

in place of that given above for $\mathscr{A}^{\otimes m}$. Then the proof parallels that for the case of $\hat{f}_N$. □

By the realization of the tensor power of the q-oscillator representation above, we naturally come to an important observation on the operators which commute with the actions of $U_{q^2}(\mathfrak{sp}_{2N})$. To describe such operators, we recall the q-deformed algebra $U_q(\mathfrak{o}_m)$, which was introduced in [GaK1] and treated in detail in [NUW3]. Let $U_q(\mathfrak{o}_m)$ be an associative algebra with $m-1$ generators Π_j ($j=1, 2, \cdots, m-1$) subject to the relations

$$\begin{cases} [\Pi_i, \Pi_j]=0 \quad \text{if} \quad |i-j|>1\,, \\ \Pi_i^2\Pi_j-(q+q^{-1})\Pi_i\Pi_j\Pi_i+\Pi_j\Pi_i^2=-\Pi_j \quad \text{if} \quad |i-j|=1\,. \end{cases}$$

Put

$$\Psi_j=(q-q^{-1})^{-1}\{S^{-1}(L^+_{jj+1})-qL^-_{j+1\,j}\}q^{\varepsilon_j}\,,$$

or in other words,

$$\Psi_j=-(q-q^{-1})^{-1}\{q^{-1}L^+_{jj+1}q^{-\varepsilon_{j+1}}+qL^-_{j+1\,j}q^{\varepsilon_j}\}\,.$$

In the notation used in [NUW3], this Ψ_j has an expression $\Psi_j=S^{-1}(\Theta_j)$. Since S^{-1} is an anti-automorphism of $U_q(\mathfrak{gl}_m)$, it is clear from Theorem 7.4 in [NUW3] that the map

$$\Psi:\ U_q(\mathfrak{o}_m)\ni\Pi_j\longmapsto\Psi_j^\circ\in U_q(\mathfrak{gl}_m)$$

can be extended to an algebra homomorphism of $U_q(\mathfrak{o}_m)$ to $U_q(\mathfrak{gl}_m)$. Thus we have a $U_q(\mathfrak{o}_m)$ action on $\mathscr{A}(\mathrm{Mat}_q(m\times N))$ which is expected to commute with $U_{q^2}(\mathfrak{sp}_{2N})$.

The following is a generalization of a result in [NUW3] that the quantized pair $(\mathfrak{sp}_{2N}, \mathfrak{o}_m)$ for arbitrary N forms a dual pair.

THEOREM 4.3. *The action of $U_q(\mathfrak{o}_m)$ through the map Ψ commutes with actions under the representation $\omega^{\otimes m}$ of $U_{q^2}(\mathfrak{sp}_{2N})$.*

Proof. Since the Ψ_j° are given by the right action of $U_q(\mathfrak{gl}_m)$, it is clear that they commute with the $\mathfrak{sl}_N$-part of $\mathfrak{sp}_{2N}$ in Theorem 4.2 and with $\omega^{\otimes m}(k_N)$. Hence we have only to show that

$$[\omega^{\otimes m}(\hat{e}_N), \Psi_j^\circ]=0\,, \qquad [\omega^{\otimes m}(\hat{f}_N), \Psi_j^\circ]=0\,.$$

To prove the first expression we introduce

$$Q_N=\sum_{k=1}^{m}q^{k-m}t_{kN}^2\,.$$

Note that $\omega^{\otimes m}(\hat{e}_N)=\dfrac{1}{[2]}Q_N^\circ$. Then since

$$[\Psi_j^\circ, Q_N^\circ]\,.\,\varphi=\Psi_j^\circ\,.(\varphi Q_N)-(\Psi_j^\circ\,.\,\varphi)Q_N \qquad \text{for} \quad \varphi\in\mathscr{A}(\mathrm{Mat}_q(m\times N))\,,$$

the comultiplication rule of Ψ_j° (that is, $\Delta(\Psi_j^\circ)=\Psi_j^\circ\otimes 1+q^{\varepsilon_j-\varepsilon_{j+1}\circ}\otimes\Psi_j^\circ$) implies $[\Psi_j^\circ, Q_N^\circ]\,.\,\varphi=(q^{\varepsilon_j-\varepsilon_{j+1}\circ}\,.\,\varphi)(\Psi_j^\circ\,.\,Q_N)$. Hence it is enough to show that $\Psi_j^\circ\,.\,Q_N=0$. This is easily checked by a simple calculation as follows. Since $\partial_{is}(t_{kN}^2)=[2]\delta_{ik}\delta_{sN}t_{kN}$ we have

$$\begin{aligned}\Psi_j^\circ . Q_N &= -\left\{q^{-1}q^{-\varepsilon_{j+1}\circ}\sum_{l=1}^{N} t_{jl}\partial_{j+1l} - qq^{\varepsilon_j\circ}\sum_{l=1}^{N} t_{j+1l}\partial_{jl}\right\}\cdot\left(\sum_{k=1}^{m} q^{k-m}t_{kN}^2\right)\\ &= -q^{-1}q^{-\varepsilon_{j+1}\circ}.\{t_{jN}\partial_{j+1N}(q^{j+1-m}t_{j+1N}^2)\} + qq^{\varepsilon_j\circ}.\{t_{j+1N}\partial_{jN}(q^{j-m}t_{jN}^2)\}\\ &= [2]\{-q^{j-m}q^{-\varepsilon_{j+1}\circ}.(t_{jN}t_{j+1N}) + q^{j-m+1}q^{\varepsilon_j\circ}.(t_{j+1N}.t_{jN})\} = 0\,.\end{aligned}$$

Next we prove the second commutativity of operators by induction on the degree of elements in $\mathscr{A}(\mathrm{Mat}_q(m\times N))$. For this points we write $\hat{f}_N$ and k_N for $\omega^{\otimes m}(\hat{f}_N)$, $\omega^{\otimes m}(k_N)$. Suppose that $\varphi_1, \varphi_2 \in \mathscr{A}(\mathrm{Mat}_q(m\times N))$ are of degree 1 and n respectively. It is obvious that $\hat{f}_N.\varphi = 0$. Assume $[\Psi_j^\circ, \hat{f}_N].\varphi_2 = 0$. Noting that the operators Ψ_j°, $q^{\varepsilon_j\circ}$ keep the degree of $\varphi \in \mathscr{A}(\mathrm{Mat}_q(m\times N))$ invariant, we see by successive use of the comultiplication rules for Ψ_j° and $\hat{f}_N$ that

$$\begin{aligned}&[\Psi_j^\circ, \hat{f}_N].(\varphi_1\varphi_2)\\ &= \Psi_j^\circ.\{(\hat{f}_N.\varphi_1)\varphi_2 + (k_N.\varphi_1)(\hat{f}_N.\varphi_2)\} - \hat{f}_N.\{(\Psi_j^\circ.\varphi_1)\varphi_2 + (q^{\varepsilon_j-\varepsilon_{j+1}\circ}.\varphi_1)(\Psi_j^\circ.\varphi_2)\}\\ &= (\Psi_j^\circ k_N.\varphi_1)(\hat{f}_N.\varphi_2) + (q^{\varepsilon_j-\varepsilon_{j+1}\circ}k_N.\varphi_1)(\Psi_j^\circ\hat{f}_N.\varphi_2)\\ &\quad - \{(k_N\Psi_j^\circ.\varphi_1)(\hat{f}_N.\varphi_2) + (k_N q^{\varepsilon_j-\varepsilon_{j+1}\circ}.\varphi_1)(\hat{f}_N\Psi_j^\circ.\varphi_2)\}\\ &= (q^{\varepsilon_j-\varepsilon_{j+1}\circ}k_N.\varphi_1)[\Psi_j^\circ, \hat{f}_N].\varphi_2 = 0\,.\end{aligned}$$

This completes the proof. □

5. Quantum grassmannians and q-hypergeometric series

As another application of our construction of the q-differential operators, we briefly give a formulation of q-hypergeometric equations. Using our differential operators, we can give a more direct analogue of Gelfand's hypergeometric equation than that given in [N1]. The difference between the formulation in [N1] and ours is implicit as long as the $2\times n$ case is treated.

We review the main features of Gelfand's generalized hypergeometric equations introduced in [G] (see also [GZ] and [GZK]) associated with the Grassmannian $G_{m,n}$. Let us consider the matrix space $\mathrm{Mat}(m\times n)$, on which GL_m and GL_n act from the left and the right, respectively. The system of equations (the Gelfand Hypergeometric Equation) consists of three parts, (1) the relative invariance under GL_m from the left, (2) the homogeneity under the diagonal subgroup of GL_n from the right, and (3) the second order differential equations defined by a family of (pseudo-) Laplacians which bears a finite-dimensional representation of GL_n.

With the same spirit, a quantum version of hypergeometric equations was introduced in [N1]. The most difficult point there was part (3) above. In that paper, instead of considering the operators $\Delta_{ij}^{rs} = \partial^2/\partial t_{ri}\partial t_{sj} - \partial^2/\partial t_{si}\partial t_{rj}$, Noumi used the oprators $\sum_{1\le r\le s\le m}(t_{ri}t_{sj} - t_{si}t_{rj})\Delta_{ij}^{rs}$ as the classical counterparts. For $m=2$, the difference between the two is just the multiplication of $t_{1i}t_{2j} - t_{2i}t_{1j}$. The reason why Noumi adopted these operators is that the latter operators are written in terms of the infinitesimal generators of the right action of GL_n, so that these operators can have quantum counterparts in the action of $U_q(\mathfrak{gl}_n)$. Along this line, Nakatani studied the case $m=3$, $n=6$ in [Na]. More general cases are discussed in [NaN] with a slightly different formulation by using affinized coordinates. But these transitions are seemingly different from Gelfand's definition, and the relation between these two is not clear even in the classical case.

In the remainder of this paper, we propose a direct quantum analogue of the hypergeometric equations in terms of our differential operators.

5.1. The operators $\square_{ij}^{sr}$:

As explained above, our concern is to consider a direct analogue of the operators $\partial^2/\partial t_{ri}\partial t_{sj} - \partial^2/\partial t_{si}\partial t_{rj}$. Two points are to be checked here, (1) the compatibility of this family of operators and the other actions of GL_m and the diagonal subgroup of GL_n, and (2) the way in which the infinitesimal action of GL_n gives the shift of the parameters. The second of these points gives the so-called contiguity relations. To see these points for our quantum case, we first calculate the commutation relations between our operators and left and right actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$, respectively. From the representation theoretic point of view, this should be given by the behavior of the differential operators ∂_{ij} under suitable adjoint actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$. Let us define two adjoint actions ad and $\widetilde{ad}$ as follows: for $a \in U_q(\mathfrak{gl}_n)$ (resp., $a \in U_q(\mathfrak{gl}_m)$) and $\varphi \in \mathrm{End}_{\mathbb{K}}(\mathscr{A}(\mathrm{Mat}_q(m \times n)))$,

$$ad(a)\varphi = \sum_i a_i^{(1)}\varphi S(a_i^{(2)}) , \qquad \widetilde{ad}(a)\varphi = \sum_i a_i^{(2)\circ}\varphi S(a_i^{(1)})^\circ ,$$

where the comultiplication is given in the form $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$. Here we distinguish the left and the right actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ by the superscript $^\circ$ for elements $a \in U_q(\mathfrak{gl}_m)$. It is immediately seen that the former adjoint action ad gives a left action, but the latter, $\widetilde{ad}$, defines a right action. Note that under these two adjoint actions, the algebra $\mathrm{End}_{\mathbb{K}}(\mathscr{A}(\mathrm{Mat}_q(m \times n)))$ becomes an algebra with both $U_q(\mathfrak{gl}_n)$- and $U_q(\mathfrak{gl}_m)$-symmetries. This means that the unit $\mathbb{K} \to \mathrm{End}_{\mathbb{K}}(\mathscr{A}(\mathrm{Mat}_q(m \times n)))$ and the composition of endomorphisms $\mathrm{End}_{\mathbb{K}}(\mathscr{A}(\mathrm{Mat}_q(m \times n))) \otimes \mathrm{End}_{\mathbb{K}}(\mathscr{A}(\mathrm{Mat}_q(m \times n))) \to \mathrm{End}_{\mathbb{K}}(\mathscr{A}(\mathrm{Mat}_q(m \times n)))$ are $U_q(\mathfrak{gl}_n)$- and $U_q(\mathfrak{gl}_m)$-homomorphisms. Explicitly, these adjoint actions of L-operators are respectively given by

$$\text{(5.1.1)} \qquad ad(K_{ij}^{\pm})\varphi = \sum_k L_{ik}^{\pm}\varphi S(L_{kj}^{\pm}) , \qquad \widetilde{ad}(L_{ij}^{\pm})\varphi = \sum_k L_{kj}^{\pm\circ}\varphi S(L_{ik}^{\pm})^\circ .$$

It is often convenient to write these actions in matrix form as follows:

$$ad(L^{\pm})\varphi = L^{\pm}\varphi S(L^{\pm}) , \qquad \widetilde{ad}({}^tL^{\pm})\varphi = {}^tL^{\pm}\varphi {}^tS(L^{\pm})^\circ .$$

PROPOSITION 5.1. *The differential operators ∂_{ij} bear the contragredient of vector representation under the adjoint actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ both from the right and the left. The explicit form of the actions are*:

$$\begin{cases} ad(L_1^{\pm})^t\partial_2 = (R_{12}^{\pm})^{-1t}\partial_2\,, \\ \widetilde{ad}({}^tL_1^{\pm})^t\partial_2 = {}^t\partial_2(R_{12}^{\pm})^{-1}\,. \end{cases}$$

Proof. Since the definition of the differential operators is compatible with the restriction to smaller matrix spaces, we may assume $m=n$ by embedding them into a square matrix of larger size if necessary. Thus the proof is obtained from the Yang-Baxter equation for L-operators: for $\varphi \in \mathscr{A}(\mathrm{Mat}_q(m\times n))$ and $\varepsilon = \pm$,

$$\begin{aligned} ad(L_1^{\pm})^t\partial_2^{\varepsilon}\varphi &= L_1^{\pm t}\partial_2^{\varepsilon}S(L_1^{\pm})\varphi \\ &= (q-q^{-1})^{-1}L_1^{\pm}.(S(T_2)(L_2^{\varepsilon\circ}.(S(L_1^{\pm})\varphi))) \\ &= (q-q^{-1})^{-1}L_1^{\pm}.(S(T_2))(L_1^{\pm}L_2^{\varepsilon\circ}.(S(L_1^{\pm})\varphi)) \\ &= (q-q^{-1})^{-1}(R_{12}^{\pm})^{-1}S(T_2)L_2^{\varepsilon\circ}\varphi \\ &= (R_{12}^{\pm})^{-1t}\partial_2^{\varepsilon}\varphi\,. \end{aligned}$$

Since $\partial = \partial^{+} - \partial^{-}$, this calculation clearly gives our first assertion. The proof of the second assertion is similar. $\square$

For $1\le r<s\le m$, $1\le i<j\le n$, we put

$$\square_{ij}^{rs} = \partial_{ri}\partial_{sj} - q^{-1}\partial_{rj}\partial_{si}\,. \tag{5.1.2}$$

Then under the adjoint actions of $U_q(\mathrm{gl}_n)$ and $U_q(\mathrm{gl}_m)$, any operator of the form $\square_{ij}^{rs}$ generates a finite-dimensional representation. This is clearly seen from Proposition 5.1 and the fact that $\square_{ij}^{rs}$ is the square tensor of the ∂_{ij}. We will give a more explicit formula for this representation below.

PROPOSITION 5.2. *Let W be the linear space spanned by the $\square_{ij}^{rs}$ $(1\le r<s\le m$, $1\le i<j\le n)$, Then W is closed under the joint adjoint actions of $U_q(\mathrm{gl}_n)$ and $U_q(\mathrm{gl}_m)$. More precisely, W is equivalent to $\lambda_n^{*(2)}\otimes {}^t\lambda_m^{*(2)}$ as a $U_q(\mathrm{gl}_n)\otimes U_q(\mathrm{gl}_m)$-module. Here $\lambda_n^{*(2)}$ and ${}^t\lambda_m^{*(2)}$ respectively stand for the exterior square of the contragredient of vector representation of $U_q(\mathrm{gl}_n)$ and $U_q(\mathrm{gl}_m)$.*

Proof. Considering the q-exterior algebra and Proposition 5.1 above, we can prove the assertion using the same technique as in §3 of [NUW1]. In fact, since ad (resp., $\widetilde{ad}$) gives a structure of $U_q(\mathrm{gl}_n)$-symmetry (resp., $U_q(\mathrm{gl}_m)$-symmetry), from Proposition 5.1 we see

$$ad(L_0^{\pm})^t\partial_1{}^t\partial_2 = (R_{01}^{\pm})^{-1}(R_{02}^{\pm})^{-1t}\partial_1{}^t\partial_2\,,\quad \widetilde{ad}({}^tL_0^{\pm})^t\partial_1{}^t\partial_2 = {}^t\partial_1{}^t\partial_2(R_{01}^{\pm})^{-1}(R_{02}^{\pm})^{-1}\,.$$

Thus, rather than showing the detailed calculations, we simply present the explicit results:

$$\begin{aligned} ad(L_{\alpha\beta}^{+})\square_{ij}^{rs} &= q^{-\delta_{\alpha i}-\delta_{\alpha j}}\delta_{\alpha\beta}\square_{ij}^{rs} \\ &\quad -(q-q^{-1})\{\delta_{\alpha<i}\delta_{\beta i}\square_{\alpha j}^{rs} - q\delta_{\alpha<i}\delta_{\beta j}\square_{\alpha i}^{rs} + \delta_{i<\alpha<j}\delta_{\beta j}\square_{i\alpha}^{rs}\}\,, \end{aligned} \tag{5.1.3}$$

$$\begin{aligned} ad(L_{\alpha\beta}^{-})\square_{ij}^{rs} &= q^{\delta_{\alpha i}+\delta_{\alpha j}}\delta_{\alpha\beta}\square_{ij}^{rs} \\ &\quad +(q-q^{-1})\{\delta_{\alpha>j}\delta_{\beta j}\square_{i\alpha}^{rs} - q^{-1}\delta_{\alpha>j}\delta_{\beta i}\square_{j\alpha}^{rs} + \delta_{j>\alpha>i}\delta_{\beta i}\square_{\alpha j}^{rs}\}\,, \end{aligned} \tag{5.1.4}$$

$$(5.1.5)\qquad \widetilde{ad}(L^+_{\alpha\beta})\Box^{rs}_{ij}=q^{-\delta_{\beta r}-\delta_{\beta s}}\delta_{\alpha\beta}\Box^{rs}_{ij}$$
$$-(q-q^{-1})\{\delta_{\beta>s}\delta_{\alpha s}\Box^{r\beta}_{ij}-q\delta_{\beta>s}\delta_{\alpha r}\Box^{s\beta}_{ij}+\delta_{s>\beta>r}\delta_{\alpha r}\Box^{\beta s}_{ij}\}\,,$$

$$(5.1.6)\qquad \widetilde{ad}(L^-_{\alpha\beta})\Box^{rs}_{ij}=q^{\delta_{\beta r}+\delta_{\beta s}}\delta_{\alpha\beta}\Box^{rs}_{ij}$$
$$+(q-q^{-1})\{\delta_{\beta<r}\delta_{\alpha r}\Box^{\beta s}_{ij}-q^{-1}\delta_{\beta<r}\delta_{\alpha s}\Box^{\beta r}_{ij}+\delta_{s>\beta>r}\delta_{\alpha s}\Box^{r\beta}_{ij}\}\,,$$

where we put

$$\delta_{\alpha>\beta}=\begin{cases}1 & \text{for}\quad \alpha>\beta\,,\\ 0 & \text{otherwise}\end{cases}$$

etc. The proposition follows immediately. $\Box$

REMARK. Since $\Box^{rs}_{ij}=-q^{-1}(\partial_{rj}\partial_{si}-q\partial_{sj}\partial_{ri})$, the correspondence between left and right actions are given by the replacement

$$\alpha\longleftrightarrow\beta\,;\quad q\longleftrightarrow q^{-1}\,,\quad +\longleftrightarrow-\,;\quad i\longleftrightarrow r\,,\quad j\longleftrightarrow s\,,$$

and of roles of super- and sub-scripts of $\Box^{rs}_{ij}$ coming from indices of columns and rows, respectively.

To define a quantum version of the Gelfand hypergeometric system of equations in Subsection 5.2 we will need the following commutation relations.

PROPOSITION 5.3. *The following commutation relations hold between L-operators* $\{L^{\pm}_{ij}\}$ *and pseudo Laplacians* $\{\Box^{rs}_{ij}\}$:

$$\text{(L1)}\qquad \Box^{rs}_{ij}L^+_{\alpha\beta}-q^{\delta_{\alpha i}+\delta_{\alpha j}}L^+_{\alpha\beta}\Box^{rs}_{ij}$$
$$=(q-q^{-1})\{\delta_{i>\alpha}L^+_{i\beta}\Box^{rs}_{\alpha j}+\delta_{j>\alpha>i}L^+_{j\beta}\Box^{rs}_{i\alpha}-q^{-1}\delta_{i>\alpha}L^+_{j\beta}\Box^{rs}_{\alpha i}\}$$

$$\text{(L2)}\qquad \Box^{rs}_{ij}L_{\alpha\beta}-q^{-\delta_{\alpha i}-\delta_{\alpha j}}L^-_{\alpha\beta}\Box^{rs}_{ij}$$
$$=(q-q^{-1})\{-\delta_{\alpha>j}L^-_{j\beta}\Box^{rs}_{i\alpha}-\delta_{j>\alpha>i}L^-_{i\beta}\Box^{rs}_{\alpha j}+q\delta_{\alpha>j}L^-_{i\beta}\Box^{rs}_{j\alpha}\}\,,$$

for the left actions of $L^{\pm}\in U_q(\mathrm{gl}_n)$,

$$\text{(R1)}\qquad \Box^{rs}_{ij}L^{+\circ}_{\alpha\beta}-q^{\delta_{r\beta}+\delta_{s\beta}}L^{+\circ}_{\alpha\beta}\Box^{rs}_{ij}$$
$$=(q-q^{-1})\{\delta_{s<\beta}L^{+\circ}_{\alpha s}\Box^{r\beta}_{ij}+\delta_{r<\beta<s}L^{+\circ}_{\alpha r}\Box^{\beta s}_{ij}-q^{-1}\delta_{s<\beta}L^{+\circ}_{\alpha r}\Box^{s\beta}_{ij}\}$$

$$\text{(R2)}\qquad \Box^{rs}_{ij}L^{-\circ}_{\alpha\beta}-q^{-\delta_{r\beta}-\delta_{s\beta}}L^{-\circ}_{\alpha\beta}\Box^{rs}_{ij}$$
$$=(q-q^{-1})\{-\delta_{r>\beta}L^{-\circ}_{\alpha r}\Box^{rs}_{ij}-\delta_{s>\beta>r}L^{-\circ}_{\alpha s}\Box^{r\beta}_{ij}+q\delta_{r>\beta}L^{-\circ}_{\alpha s}\Box^{\beta r}_{ij}\}\,,$$

for the right actions of $L^{\pm}\in U_q(\mathrm{gl}_m)$. $\Box$

The proofs of the above commutation relations are performed with short calculations by using the following commutation relations:

LEMMA 5.4. *The commutation relations between the L-operators and the differential operators are given as follows*:

$$R^{\pm}L_1^{\pm t}\partial_2 = {}^t\partial_2 L_1^{\pm}\,, \qquad L_1^{\pm\circ t}\partial_2 R^{\pm} = {}^t\partial_2 L_1^{\pm\circ}\,.$$

Proof. These are proved by direct calculation from the definition. Since their proofs are similar to that of Proposition 4.3 in [NUW1], we leave them to the reader. □

REMARKS. (1) When we apply this lemma to the proof of the formulas (L1), (L2), (R1) and (R2), the following paraphrased formulas are used:

$$\partial_{\beta i}L_{j\alpha}^{\pm} = q^{\pm\delta_{ij}}L_{j\alpha}^{\pm}\partial_{\beta i} \pm (q-q^{-1})\delta_{j\lessgtr i}L_{i\alpha}^{\pm}\partial_{\beta j}\,,$$

$$\partial_{\beta i}L_{j\alpha}^{\pm\circ} = q^{\pm\delta_{\alpha\beta}}L_{j\alpha}^{\pm\circ}\partial_{\beta i} \pm (q-q^{-1})\delta_{\beta\lessgtr\alpha}L_{j\beta}^{\pm\circ}\partial_{\alpha i}\,.$$

(2) For the proof of Proposition 5.3, the first formula is for the size n (left action), and the latter for m (right action), respectively.

(3) Let $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ be the comultiplication formula for $a \in U_q(\mathrm{gl}_n)$. Then for an operator $w \in \mathrm{End}_{\mathbb{K}}(\mathscr{A}(\mathrm{Mat}_q(m \times n)))$, we have

$$w \circ a = \sum_i a_i^{(2)} \circ ad(S^{-1}(a_i^{(1)}))(w)\,.$$

From this formula, we can prove Proposition 5.3 by using formulas of two adjoint actions of L-operators given in the proof of Proposition 5.2. Actually, the commutation relations (L1), (L2), (R1) and (R2) can be seen from the explicit descriptions of the adjoint actions of another L-operator $\tilde{L}^{\pm}$ defined by $\tilde{L}_{ij}^{\pm} = S(L_{ji}^{\mp})$ (cf. [NUW1], Appendix B). That is, (L1) and (L2) (resp., (R1) and (R2)) are directly reduced from the explicit formulas of the $\square_{ij}^{rs}$ under the two actions $ad(\tilde{L}^{\pm})$ (resp., $\widetilde{ad}({}^t\tilde{L}^{\pm})$). However, comparing the explicit forms, it will soon be found that the latter formulas (L1), (L2), (R1) and (R2) are more convenient than the formulas of the adjoint actions of the $L_{ij}^{\pm}$, at least in proving the theorem.

5.2. Quantum hypergeometric equation

As Proposition 5.3 implies both the compatibility and the contiguity relations, we propose the following equations as quantum version of the (generalized) hypergeometric equations.

DEFINITION. *(QHGE)*

(1) $$L_{rs}^{\pm\circ}.\Phi = \delta_{rs}q^{\mp\delta_{rs}}\Phi \quad (1 \le r, s \le m)\,,$$

(2) $$L_{ii}^{\pm}.\Phi = q^{\pm\lambda_i}\Phi \quad (1 \le i \le n)\,,$$

(3) $$\square_{ij}^{rs}\Phi = 0 \quad (1 \le r, s \le m,\ 1 \le i, j \le n)\,,$$

where the parameters λ_i satisfy $\sum_{i=1}^n \lambda_i = m$. □

REMARK. The relations (R1) and (R2) in Proposition 5.3 guarantee the compatibility of the equations (1) and (3). The relations (L1) and (L2) in Proposition 5.3 show that the algebra $U_q(\mathrm{gl}_n)$ stabilizes the set of the solutions to the equations (1) and (3). The commutation relations of $L_{ii}^{\pm}$ and $L_{ab}^{\pm}$ in $U_q(\mathrm{gl}_n)$ imply the contiguity relations, i.e., the manner in which the parameters λ_i become shifted under the action of $U_q(\mathfrak{gl}_n)$. The last remark is the same as that clarified in [N1]. In a similar way, we may also formulate the contiguity relations of a quantum version of the generalized hypergeometric system of "the confluent type" using the other maximal abelian subalgebra of $U_q(\mathfrak{gl}_n)$ in place of the equation (2), as discussed for the classical case in [KHT].

REMARK. The relations among the $\square_{ij}^{rs}$ and L-operators reduce the numbers of equations in (3). In fact, we can verify the following assertions:

LEMMA 5.5. *Suppose $m \leq n$. For fixed i, j, the following implications hold*:

$$\square_{ij}^{r\,r+1}\Phi=0 \quad \text{for } r=1,2,\cdots,m-1 \Longrightarrow \square_{ij}^{rs}\Phi=0 \quad \text{for any } 1\leq r,s\leq m\,.$$

Also for fixed r, we have

$$\square_{j\,j+1}^{rm}\Phi=0 \quad \text{for } j=m+1,\cdots,n-1 \Longrightarrow \square_{ij}^{rm}\Phi=0 \quad \text{for any } 1\leq i,j\leq n\,. \quad \square$$

Moreover, if we consider the actions from the right and left, we obtain a version of second order Capelli identities:

PROPOSITION 5.6. *Suppose that $1\leq r<s\leq m$, $1\leq i<j\leq n$. Let $L(\lambda)=\lambda L^{+}-\lambda^{-1}L^{-}$ denote the L-operator with spectral parameter λ. Then we have*

$$L_{ir}(q)L_{js}(1)-q^{-1}L_{is}(q)L_{jr}(1)=q(q-q^{-1})^2\sum_{1\leq\alpha<\beta\leq m}\det\nolimits_q\begin{bmatrix} t_{\alpha r} & t_{\alpha s}\\ t_{\beta r} & t_{\beta s}\end{bmatrix}^{\circ}\square_{ij}^{\alpha\beta}\,,$$

$$L_{ir}(q)^{\circ}L_{js}(1)^{\circ}-qL_{is}(q)^{\circ}L_{jr}(1)^{\circ}=q(q-q^{-1})^2\sum_{1\leq\alpha<\beta\leq n}\det\nolimits_q\begin{bmatrix} t_{i\alpha} & t_{i\beta}\\ t_{j\alpha} & t_{j\beta}\end{bmatrix}\square_{\alpha\beta}^{rs}\,,$$

where ${}^{\circ}$ indicates the action from the right.

Proof. The second identity is easily obtained by the same method using the q-exterior algebra, as in the proof of the Capelli identity (see [NUW1], §3). In contrast with this, the first identity can still be proven in a similar way, but we use the q^{-1}-exterior algebra instead. The detailed calculations are left to the reader. $\square$

REMARKS. (1) Put

$$C^{rs}(\lambda)=L_{rr}(q\lambda)L_{ss}(\lambda)-q^{-1}L_{rs}(q\lambda)L_{sr}(\lambda)\,,$$

$$C_{ij}(\lambda)=L_{ii}(q\lambda)^{\circ}L_{jj}(\lambda)^{\circ}-qL_{ji}(q\lambda)^{\circ}L_{ij}(\lambda)^{\circ}\,,$$

where λ is a spectral parameter. Then, in particular, the above proposition asserts:

$$C^{rs}(1)=q(q-q^{-1})^2\sum_{1\leq\alpha<\beta\leq m}\det\nolimits_q\begin{bmatrix} t_{\alpha r} & t_{\alpha s}\\ t_{\beta r} & t_{\beta s}\end{bmatrix}^{\circ}\square_{rs}^{\alpha\beta}\,,$$

$$C_{ij}(1)=q(q-q^{-1})^2\sum_{1\leq\alpha<\beta\leq n}\det\nolimits_q\begin{bmatrix} t_{i\alpha} & t_{i\beta}\\ t_{j\alpha} & t_{j\beta}\end{bmatrix}^{\circ}\square_{\alpha\beta}^{ij}\,,$$

respectively. From these formulas, we see that solution to our equations also satisfy the equations defined in [N1] and [Na], because there, the equations $C_{ij}(1)\Phi=0$ $(1\le i<j\le n)$ are employed instead of (3) in QHGE.

(2) Higher order variants of Capelli-like identities, similar to those in Proposition 5.6, for the shifted quantum determinant of L-operators can be obtained in the same way as shown above.

Supplement

In this supplement we derive another expression of the right hand side of the quantum Capelli Identity for $GL_q(n)$ discussed in [NUW1]. In particular, Lemma S.1 below provides in part a relevant meaning of the spectral parameter appearing in the differential operators $\partial_{ij}(\lambda)$.

Recall the defining relation of $\partial_{ij}^{\pm}$,

$$L_{ji}^{\pm\circ}=(q-q^{-1})\sum_{k=1}^{n} t_{jk}\partial_{ik}^{\pm}\ .$$

Now apply both sides of this formula to $\det_q(T)^m$. Then, since we have $L_{ji}^{\pm\circ}.\det_q(T)=q^{\pm1}\det_q(T)\delta_{ij}$, by the comultiplication rules of $L_{ji}^{\pm\circ}$ it is easy to see that $L_{ji}^{\pm\circ}.\det_q(T)^m=q^{\pm m}\det_q(T)^m\delta_{ij}$. Hence we obtain

$$(q^m-q^{-m})\delta_{ij}\det_q(T)^m=(q-q^{-1})\sum_{k=1}^{n} t_{jk}\partial_{ik}(\det_q(T)^m)\,,$$

where we put $\partial_{ij}=\partial_{ij}(1)=\partial_{ij}^{+}-\partial_{ij}^{-}$. By this it is immediately seen that

$$\partial_{ij}(\det_q(T)^m)=[m]S(t_{ji})\det_q(T)^m\,, \tag{S.1}$$

where $[m]=\dfrac{q^m-q^{-m}}{q-q^{-1}}$.

Using this simple relation, one can obtain the following lemma, which asserts that the spectral parameter λ appearing in the differential operators $\partial_{ij}(\lambda)$ simply plays the role of a shift operation.

LEMMA S.1. *If we put $\lambda=q^u$, then $\partial_{ij}(\lambda)$ is given by*

$$\partial_{ij}(q^u)=\det_q(T)^{-u}\partial_{ij}(1)\det_q(T)^u\ . \tag{S.2}$$

Proof. For simplicity we put $D=\det_q(T)$. Then for any $\varphi\in\mathscr{A}(\mathrm{Mat}_q(n))$, by the Leibniz rule of ∂_{ij} and the definition of ∂_{ij}, we observe

$$\begin{aligned}\partial_{ij}(D^m\varphi)&=\sum_{k=1}^{n}\{\partial_{kj}(D^m)L_{ki}^{\pm\circ}.\varphi+L_{jk}^{\mp}.D^m\partial_{ik}\varphi\}\\&=[m]D^m\sum_{k=1}^{n}S(t_{jk})L_{ki}^{\pm\circ}.\varphi+q^{\mp m}D^m\partial_{ij}\varphi\\&=(q-q^{-1})[m]D^m\partial_{ij}^{\pm}\varphi+q^{\mp m}D^m\partial_{ij}\varphi\ .\end{aligned}$$

Thus we have

$$D^{-m}\partial_{ij}D^m=(q^m-q^{-m})\partial^+_{ij}+q^{-m}\partial_{ij}$$
$$=q^m\partial^+_{ij}-q^{-m}\partial^-_{ij}=\partial_{ij}(q^m)\,.$$

This completes the proof of the lemma. □

REMARK. Letting q tend to 1 in (S.2), we have

$$\frac{\partial}{\partial t_{ij}}+uS(t_{ij})=(\det T)^{-u}\frac{\partial}{\partial t_{ij}}(\det T)^u\,, \tag{S.3}$$

in the classical situation (see (2.6) in [NUW1]). We may rewrite this in the following equivalent form:

$$\left(\sum_{i=1}^n t_{ki}\frac{\partial}{\partial t_{ij}}\right)+u\delta_{kj}=(\det T)^{-u}\left(\sum_{i=1}^n t_{ki}\frac{\partial}{\partial t_{ij}}\right)(\det T)^u\,. \tag{S.4}$$

This is a matrix analogue of the simple relation $t^{-u}\vartheta t^u=\vartheta+u$, where $\vartheta=t\dfrac{d}{dt}$.

Let us now recall the quantum Capelli Identity for $GL_q(n)$, which is a complete expression of the quantum analogue of invariant differential operators by the central elements of $U_q(\mathfrak{gl}_n)$ explicitly (see the Theorem in [NUW1], p. 580):

$$z(\lambda q^{n-1})=q^{\binom{n}{2}}\det_q(T)\det_{q^{-1}}({}^t\partial(\lambda))\,,$$

where

$$z(\lambda)=(q-q^{-1})^{-n}\sum_{\sigma\in\mathfrak{S}_n}(-q)^{l(\sigma)}L(\lambda q^{-n+1})_{\sigma(n)n}\cdots L(\lambda)_{\sigma(1)1}\,,$$

and

$$\det_{q^{-1}}({}^t\partial(\lambda))=\sum_{\sigma\in\mathfrak{S}_n}(-q^{-1})^{l(\sigma)}\partial(\lambda)_{\sigma(1)1}\cdots\partial(\lambda)_{\sigma(n)n}\,.$$

If we put $\lambda=q^m$, then by Lemma S.1 we have

$$\det_{q^{-1}}({}^t\partial(q^m))=D^{-m}\sum_{\sigma\in\mathfrak{S}_n}(-q^{-1})^{l(\sigma)}(D^m\partial(q^m)_{\sigma(1)1}D^{-m})\cdots(D^m\partial(q^m)_{\sigma(n)n}D^{-m})D^m$$
$$=D^{-m}\sum_{\sigma\in\mathfrak{S}_n}(-q^{-1})^{l(\sigma)}\partial_{\sigma(1)1}\cdots\partial_{\sigma(n)n}D^m=D^{-m}\det_{q^{-1}}({}^t\partial)D^m\,.$$

This implies that the quantum Capelli Identity for $GL_q(n)$ may be rewritten in the following form (cf. [He, p. 339]).

THEOREM S.2. *Put $\lambda=q^u$. Then*

$$z(q^{u+n-1})=q^{\binom{n}{2}}\det_q(T)^{1-u}\det_{q^{-1}}({}^t\partial)\det_q(T)^u\,. \quad \square \tag{S.5}$$

REMARKS. (1) From the identity (S.5) above it is easy to see that

$$z(q^{n-m})z(q^{n-m+1})\cdots z(q^{n-1})=q^{m\binom{n}{2}}\det_q(T)^m\det_{q^{-1}}({}^t\partial)^m . \tag{S.6}$$

This formula actually may be seen as a quantum matrix analogue of the classical Boole formula in the Introduction (see [B]).

(2) Letting q tend to 1 in the above theorem we see that

$$\det(E_{n-j+1\,n-i+1}+(j-1+u)\delta_{ij})=\det(t_{ij})^{1-u}\det\left(\frac{\partial}{\partial t_{ji}}\right)\det(t_{ij})^u . \tag{S.7}$$

(see also, Remark (1) in [NUW1], p. 581.) From this expression one might derive the lower order Capelli Identities discussed in [Ca], [HU].

(3) Using the comultiplication rule for $L^{\pm}$ and the action of $L^{\pm}$ on $(\det_q(T))^m$, i.e., $L^{\pm}.(\det_q(T))^m=q^{\pm m}(\det_q(T))^m$, we have the relation of operators

$$(\det_q(T))^{-m}L(q^u)(\det_q(T))^m=L(q^{u+m}) .$$

From this we obtain alternative direct proofs of Lemma S.1 and Theorem S.2, because $\det_q(T)$ is central in $\mathscr{A}(\mathrm{Mat}_q(n))$.

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