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Köcher-Maass Dirichlet Series Corresponding to Jacobi forms and Cohen Eisenstein Series

Dedicated to Professor Y. IHARA for his sixtieth birthday

by

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0. Introduction

0.1. Köcher in [Ko] defined and Maass [Ma] studied in full details certain Dirichlet series corresponding to Siegel's modular forms which are now called the Köchler-Maass Dirichlet series. Maass proved that the Dirichlet series can be continued to meromorphic functions in the whole plane satisfying certain functional equations. In [Ar1, 2] we proved certain formulas for the Köcher-Maass Dirichlet series giving us explicit informations on the poles and their residues.

One of our aim of this paper is to form the Köcher-Maass Dirichlet series attached to Jacobi forms and to obtain explicit formulas giving us the exact location of poles and the exact values of residiues at those poles. Moreover using the correspondence between Jacobi forms and Siegel modular forms of half-integral weights, we intend to define and study the generalized Cohen Eisenstein series in the Kohnen plus space of Siegel modular forms of half integral weights. For instance we describe Siegel's formula for the Cohen Eisenstein series. Finally we define the Köcher-Maass Dirichlet series attached to modular forms of half integral weights and obtain explicit formulas similar to those in the original Siegel modular case. For the proofs we use some convenient properties of the Cohen Eisenstein series.

0.2. We explain our results more precisely. Denote by $J_{k,S}(\Gamma_n)$ the $\mathbb{C}$-vector space of Jacobi forms of degree n, weight k, and of index S with respect to the Siegel modular group $\Gamma_n = Sp_n(\mathbb{Z})$. To each $\phi \in J_{k,S}(\Gamma_n)$, we associate the Köcher-Maass Dirichlet series $D_n(\phi, s)$ based on its Fourier expansion (see (1.15)). The Dirichlet series $D_n(\phi, s)$ can be analytically continued to a meromorphic function of s in the whole s-plane satisfying a certain functional equation (Theorem 8). Moreover under a maximality condition on the index S we show that $D_n(\phi, s)$ multiplied by some gamma factor has a reasonable expression given in Theorem 9 which will give us explicit information on the residues at the poles of the Dirichlet series. For the proof of the main theorem (Theorem 9) we employed the structure theorem for $J_{k,S}(\Gamma_n)$ and certain Eisenstin series which should be called Cohen-Klingen's Eisenstein series on the Siegel upper half plane $\mathfrak{H}_n$ of degree n.

Next we consider the special case of the index S being one. In this case the space

$J_{k,1}(\Gamma_n)$ is naturally isomorphic to the Kohnen plus space $M^+_{k-1/2}(\Gamma_0^{(n)}(4))$ (this fact was observed by Kohnen and Zagier [E-Z] if $n=1$ and by Ikubiyama [Ib] if $n>1$). The natural isomorphism is denoted by $\sigma^{(n)}: J_{k,1}(\Gamma_n)\to M^+_{k-1/2}(\Gamma_0^{(n)}(4))$. Let $C_n^{k-1/2}(\tau)$ denote the image of the Jacobi Eisenstein series of weight k by this isomorphism $\sigma^{(n)}$. This function $C_n^{k-1/2}(\tau)$ may be called the (generalized) Cohen Eisenstein series of degree n and weight $k-1/2$. If $n=1$, $C_1^{k-1/2}(\tau)$ was studied by Cohen [Co] and has a Fourier expansion

$$C_1^{k-1/2}(\tau)=\frac{1}{\zeta(3-2k)}\sum_{d\in\mathbb{Z}_{\geq 0}, d\equiv 0,3 \bmod 4} H(k-1, d)e(d\tau)\,, \tag{0.1}$$

where $H(k-1, d)$ are rational numbers which are explicitly given with the use of special values of certain L-functions (for the explicit values of $H(k-1, d)$ see [Co] and [E-Z]). We obtain certain Siegel's formula for $C_n^{k-1/2}(\tau)$ if k is divisible by 4. Denote by $\mathscr{S}^+_{2k-1}$ the set consisting of positive definite integral symmetric matrices N of size $2k-1$ with determinant 4^{k-1} satisfying the condition:

$$N\equiv -{}^t\lambda\lambda \bmod 4Sym^*_{2k-1}(\mathbb{Z}) \qquad \text{for some} \quad \lambda\in M_{1,2k-1}(\mathbb{Z})\,.$$

Let H_{2k-1} denote the number of the $GL_{2k-1}(\mathbb{Z})$-eqivalence classes in $\mathscr{S}^+_{2k-1}$. We have proved in [Ar5] that $H_7=1$ and $H_{15}=2$ (see also Lemma 12). Let $S_1, S_2, \cdots, S_H$ $(H=H_{2k-1})$ denote a complete set of representatives of the usual $GL_{2k-1}(\mathbb{Z})$-equivalence classes of $\mathscr{S}^+_{2k-1}$. We define a theta series $\theta_S^{(n)}(\tau)$ for each $S\in\mathscr{S}^+_{2k-1}$ to be the sum

$$\theta_S^{(n)}(\tau)=\sum_{G\in M_{2k-1,n}(\mathbb{Z})} e(\mathrm{tr}({}^tGSG\tau)) \qquad (\tau\in\mathfrak{H}_n)\,,$$

which belongs to $M^+_{k-1/2}(\Gamma_0^{(n)}(4))$. The Eisenstein series $C_n^{k-1/2}(\tau)$ can be written as a natural linear combination of the above theta series:

$$C_n^{k-1/2}(\tau)=\frac{1}{M_{2k-1}}\left(\sum_{j=1}^{H}\frac{\theta_{S_j}^{(n)}(\tau)}{\varepsilon(S_j)}\right), \tag{0.2}$$

where $\varepsilon(S)$ for each $S\in\mathscr{S}^+_{2k-1}$ denotes the order of the unit group of S and M_{2k-1} is the mass given by (2.1) (Theorem 14). We see from (0.1), (0.2) that $H(k-1, d)$ for $d>0$ has the following expression

$$\frac{H(k-1, d)}{\zeta(3-2k)}=\frac{1}{M_{2k-1}}\sum_{j=1}^{H}\frac{\#\{G\in M_{2k-1,1}(\mathbb{Z})\,|\,S_j[G]=d\}}{\varepsilon(S_j)}\,,$$

which will give a generalization of the observation given by Eichler-Zagier in [E-Z, p. 85, (6)]. Moreover the Fourier coefficients of $C_n^{k-1/2}(\tau)$ for positive definite $N\in Sym_n(\mathbb{Z})$ can be explicitly written in terms of local densities as is the case in the classical Siegel Eisenstein series (see Theorem 14 and [Si] for the original case).

Finally we attach to each $f\in M^+_{k-1/2}(\Gamma_0^{(n)}(4))$ the Köcher-Maass Dirichlet series $D_n(f, s)$. As an application of Theorem 9, $D_n(f, s)$ can be analytically continued to a meromorphic function of s in the whole plane satisfying a certain functional equation as we shall see in Theorem 15.

Notations

For any commutative ring R we denote by $M_{m,n}(R)$, $M_n(R)$, and $Sym_n(R)$ the set of $m\times n$ matrices, the ring of matrices of size n, and the set of symmetric matrices of size n with entries in R, respectively. Let 1_n denote the identity matrix of size n in $M_n(R)$. For $X\in M_{m,n}(R)$ and $Y\in Sym_m(R)$ we write $Y[X]$ in place of ${}^tXYX$, ${}^tX$ denoting the transposed matrix of X. For square matrices A, B, denote by $A\perp B$ the square matrix $\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}$. Denote by $Sym_n^*(\mathbb{Z})$, $Sym_n^*(\mathbb{Z})^+$, and $Sym_n(\mathbb{R})^+$, the set of all half integral symmetric matrices of size n, the subset of $Sym_n^*(\mathbb{Z})$ consisting of positive definite elements, and the set of all positive definite real symmetric matrices of size n, respectively. We write, for a real symmetric matrix Y, $Y>0$ (resp. $Y\geq 0$), if Y is positive (resp. semi-positive) definite. For any finite set A, let $\sharp A$ denote the cardinality of the set A. We use the symbol $e(z)$ $(z\in\mathbb{C})$ as an abbreviation for $\exp(2\pi i z)$. Moreover let $\zeta(s)$ be the Riemann zeta function.

1. Köcher-Maass Dirichlet series attached to Jacobi forms

1.1. The space of Jacobi forms

The basic notation is almost the same as in [Ar1, 2]. We briefly recall the definition from them.

Let $Sp_n(\mathbb{R})$ denote the real symplectic group of degree n given by

$$Sp_n(\mathbb{R})=\left\{g\in GL_{2n}(\mathbb{R})\,\middle|\,{}^tg\begin{pmatrix} 0 & 1_n \\ -1_n & 0\end{pmatrix}g=\begin{pmatrix} 0 & 1_n \\ -1_n & 0\end{pmatrix}\right\}.$$

We fix a positive integer l. Let $G^J_{n,l}(\mathbb{R})$ be the subgroup of $Sp_{n+l}(\mathbb{R})$ consisting of all elements of the form

$$(1.1)\qquad \begin{pmatrix} 1_l & & \rho & \mu \\ & 1_n & {}^t\mu & 0 \\ & & 1_l & \\ & & & 1_n \end{pmatrix}\begin{pmatrix} 1_l & \lambda & & \\ 0 & 1_n & & \\ & & 1_l & 0 \\ & & -{}^t\lambda & 1_n \end{pmatrix}\begin{pmatrix} 1_l & & & \\ & a & & b \\ & & 1_l & \\ & c & & d \end{pmatrix}$$

$$\left(M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in Sp_n(\mathbb{R}),\ \lambda,\ \mu\in M_{l,n}(\mathbb{R}),\ \rho\in Sym_l(\mathbb{R})\right).$$

The element of the above form (1.1) is simply written as $((\lambda,\mu),\rho)M$. Denote by $H_{n,l}(\mathbb{R})$ the Heisenberg group, a subgroup of $G^J_{n,l}(\mathbb{R})$, consisting of all elements of the form $((\lambda,\mu),\rho)1_{2n}=((\lambda,\mu),\rho)$. Then, for any elements $h=((\lambda,\mu),\rho)$, $h'=((\lambda',\mu'),\rho')\in H_{n,l}(\mathbb{R})$, $M\in Sp_n(\mathbb{R})$, the composition rule is given as follows:

$$hh'=((\lambda+\lambda',\mu+\mu'),\rho+\rho'+\lambda{}^t\mu'+\mu'{}^t\lambda)$$

$$M^{-1}hM=((\lambda^*,\mu^*),\rho+\lambda^{*t}\mu^*-\lambda^t\mu),$$

where $(\lambda^*,\mu^*)=(\lambda,\mu)M$. The group $Sp_n(\mathbb{R})$ acts on the Siegel upper half plane $\mathfrak{H}_n$ of degree n in a usual manner; for $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in Sp_n(\mathbb{R})$ and $\tau\in\mathfrak{H}_n$, set

$$M\langle\tau\rangle=(a\tau+b)(c\tau+d)^{-1} \quad\text{and}\quad J(M,\tau)=c\tau+d.$$

Let $\mathscr{D}_{n,l}$ denote the product space of $\mathfrak{H}_n$ and the set $M_{l,n}(\mathbb{C})$ of all $l\times n$ matrices with entries in $\mathbb{C}$:

$$\mathscr{D}_{n,l}=\mathfrak{H}_n\times M_{l,n}(\mathbb{C}).$$

Then the group $G^J_{n,l}(\mathbb{R})$ naturally acts on the space $\mathscr{D}_{n,l}$:

$$g(\tau,z):=(M\langle\tau\rangle,zJ(M,\tau)^{-1}+\lambda M\langle\tau\rangle+\mu),$$

where $g=((\lambda,\mu),\rho))M\in G^J_{n,l}(\mathbb{R})$, $M\in Sp_n(\mathbb{R})$, and $(\tau,z)\in\mathscr{D}_{n,l}$.

We take a positive definite half integral symmetric matrix S of size l and fix it. Now we define a factor of automorphy for the Jacobi group $G^J_{n,l}(\mathbb{R})$. Let k be a non-negative integer. Set, for $g=((\lambda,\mu),\rho)M\in G^J_{n,l}(\mathbb{R})$, $(\tau,z)\in\mathscr{D}_{n,l}$,

$$\begin{aligned}(1.2)\qquad J_{S,k}(g,(\tau,z)):=&\det(J(M,\tau))^k\\ &\times e(-\mathrm{tr}(S\rho)-\mathrm{tr}(S[\lambda]M\langle\tau\rangle+2^t\lambda SzJ(M,\tau)^{-1}-S[z]J(M,\tau)^{-1}c)),\end{aligned}$$

where $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in Sp_n(\mathbb{R})$. This factor of automorphy has the property:

$$(1.3)\qquad J_{S,k}(g_1g_2,(\tau,z))=J_{S,k}(g_1,g_2(\tau,z))J_{S,k}(g_2,(\tau,z))\qquad(g_1,g_2\in G^J_{n,l}(\mathbb{R})).$$

Let Γ_n denote the Siegel modular group $Sp_n(\mathbb{Z})$. Let Γ^J_n be the usual discrete subgroup of $G^J_{n,l}(\mathbb{R})$ given by

$$\Gamma^J_n=G^J_{n,l}(\mathbb{Z}):=\{((\lambda,\mu),\rho)M\mid M\in\Gamma_n,\ \lambda,\mu\in M_{l,n}(\mathbb{Z}),\ \rho\in Sym_l(\mathbb{Z})\}.$$

Moreover let $H_{n,l}(\mathbb{Z})$ denote the ordinary discrete subgroup of $H_{n,l}(\mathbb{R})$:

$$H_{n,l}(\mathbb{Z})=\{((\lambda,\mu),\rho)\mid\lambda,\mu\in M_{l,n}(\mathbb{Z}),\ \rho\in Sym_l(\mathbb{Z})\}.$$

We introduce as in [Ar1, 2] the subset of $Sym^*_{n+l}(\mathbb{Z})$ which fits very well the framework of Jacobi forms:

$$(1.4)\qquad Sym^*_{n+l}(S;\mathbb{Z}):=\left\{T=\begin{pmatrix} N & {}^tr/2 \\ r/2 & S \end{pmatrix}\,\middle|\, N\in Sym^*_n(\mathbb{Z}),\ r\in M_{l,n}(\mathbb{Z})\right\}.$$

Let $Sym^*_{n+l}(S;\mathbb{Z})^+$ denote the subset of $Sym^*_{n+l}(S;\mathbb{Z})$ consisting of positive definite elements. For each $T=\begin{pmatrix} N & {}^tr/2 \\ r/2 & S \end{pmatrix}\in Sym^*_{n+l}(S;\mathbb{Z})$, we write

$$\tilde{T}=N-\frac{1}{4}{}^trS^{-1}r.$$

Set

$$L^{(n)}=M_{l,n}(\mathbb{Z}) .$$

A holomorphic function $\phi(\tau, z)$ on $\mathscr{D}_{n,l}$ is said to be a Jacobi form of weight k and degree n with respect to Γ_n, if it satisfies the following three conditions:
(i) $\phi(\tau, z+\lambda\tau+\mu)=e(-\mathrm{tr}({}^t\lambda S\lambda+{}^t\lambda Sz))\phi(\tau, z)$ for all $\lambda, \mu\in L^{(n)}$.
(ii) $\phi(M(\tau, z))=e(\mathrm{tr}(S[z]J(M, \tau)^{-1}c))\det J(M, \tau)^k\phi(\tau, z)$ for all $M\in\Gamma_n$.
(iii) (Fourier expansion)

$$\phi(\tau, z)=\sum_{T\in Sym^*_{n+l}(S;\mathbb{Z}),\, T\geq 0} c(T)e(\mathrm{tr}(N\tau+{}^trz)) , \tag{1.5}$$

where $T=\begin{pmatrix} N & {}^tr/2 \\ r/2 & S \end{pmatrix}$ runs over all semi-positive definite symmetric matrices of $Sym^*_{n+l}(S;\mathbb{Z})$. We often write $c(N, r)$ instead of the Fourier coefficient $c(T)$. If $n\geq 2$, this condition is automatically verified.

Denote by $J_{k,S}(\Gamma_n)$ the space consisting of Jacobi forms of weight k and degree n with respect to Γ_n. Moreover the space $J^{\mathrm{cusp}}_{k,S}(\Gamma_n)$ of cusp forms consists of all $\phi\in J_{k,S}(\Gamma_n)$ satisfying the condition that

$$C_{S,k}(\tau, z)^{1/2}\,|\,\phi(\tau, z)\,| \quad \text{is bounded on } \mathfrak{H}_n ,$$

where $C_{S,k}(\tau, z)=(\det\eta)^k\exp(-4\pi\,\mathrm{tr}(S[y]\eta^{-1}))$ for $\eta=\mathrm{Im}(\tau)$ and $y=\mathrm{Im}(z)$. Since this function satisfies the relation

$$C_{S,k}(g(\tau, z))=|\,J_{S,k}(g, (\tau, z))\,|^{-2}C_{S,k}(\tau, z) \qquad \text{for any} \quad g\in G^J_{n,l}(\mathbb{R}) ,$$

$C_{S,k}(\tau,z)^{1/2}\,|\,\phi(\tau, z)\,|$ is invariant under the action of any elements of Γ^J_n.

Let $L^{(n)}/(2S)L^{(n)}$ be the quotient module of $L^{(n)}$ by the submodule $(2S)L^{(n)}$. For each $\mu\in L^{(n)}/(2S)L^{(n)}$, we define the theta series $\theta_\mu(\tau, z)$ to be the sum

$$\sum_{\lambda\in L^{(n)}} e(\mathrm{tr}(S[\lambda+(2S)^{-1}\mu]\tau+2{}^t(\lambda+(2S)^{-1}\mu)Sz)) , \qquad ((\tau, z)\in\mathscr{D}_{n,l}) .$$

We write $\theta^{(n)}_\mu(\tau, z)$ instead of $\theta_\mu(\tau, z)$, if necessary. For a fixed $\tau\in\mathfrak{H}_n$ denote by $\Theta^{(n)}_{S,\tau}$ the space consisting of holomorphic functions $\theta: M_{l,n}(\mathbb{C})\to\mathbb{C}$ satisfying the condition

$$\theta(z+\lambda\tau+\mu)=e(-\mathrm{tr}(S[\lambda]\tau+2{}^t\lambda Sz))\theta(z) \qquad (\text{for all } \lambda, \mu\in L^{(n)}) .$$

Then $\Theta^{(n)}_{S,\tau}$ is a finite dimensional $\mathbb{C}$-vector space. Moreover it is known and easy to see that the theta series $\theta_\mu(\tau, z)$ $(\mu\in L^{(n)}/(2S)L^{(n)})$ form a $\mathbb{C}$-basis of $\Theta^{(n)}_{S,\tau}$. We arrange $\theta_\mu(\tau, z)$ $(\mu\in L^{(n)}/(2S)L^{(n)})$ as a column vector with $L^{(n)}/(2S)L^{(n)}$ being an index set:

$$\Theta(\tau, z):=(\theta_\mu(\tau, z))_{\mu\in L^{(n)}/(2S)L^{(n)}} .$$

The following proposition is known and for instance will follow from Proposition 1.6 of Shintani [Shn].

PROPOSITION 1 (Theta transformation formula). *Set $N=\det(2S)^n$. Then for* *each* $M\in\Gamma_n$,

$$\Theta(M(\tau, z))=J_{S,0}(M, (\tau, z))j_S(M, \tau)\Theta(\tau, z)\,,$$

where $j_S(M, \tau)=(j_S(M, \tau)_{\lambda\mu})_{\lambda,\mu\in L^{(n)}/(2S)L^{(n)}}$ *is an* $N\times N$ *matrix depending only on* M *and* τ *satisfying*

$$\overline{{}^t j_S(M, \tau)}j_S(M, \tau)=|J(M, \tau)|^l\cdot 1_N\,.$$

In another expression,

$$\theta_\lambda(M(\tau, z))=J_{S,0}(M, (\tau, z))\times\sum_{\mu\in L^{(n)}/(2S)L^{(n)}} j_S(M, \tau)_{\lambda\mu}\theta_\mu(\tau, z)\,,$$

where $\{j_S(M, \tau)_{\lambda\mu}\}$ *satisfies*

$$\sum_{\lambda\in L^{(n)}/(2S)L^{(n)}}\overline{j_S(M, \tau)_{\lambda\mu}}j_S(M, \tau)_{\lambda\mu'}=\delta_{\mu,\mu'}|J(M, \tau)|^l\,,$$

$\delta_{\mu,\mu'}$ *denoting the Kronecker symbol.*

We write $j_S^{(n)}(M, \tau)$ instead of $j_S(M, \tau)$, if the degree n should be specified. It immediately follows from this transformation formula that

$$j_S(M_1M_2, \tau)=j_S(M_1, M_2\langle\tau\rangle)j_S(M_2, \tau)\qquad (M_1, M_2\in\Gamma_n)\,. \tag{1.6}$$

Each Jacobi form $\phi\in J_{S,k}(\Gamma_n)$, which as a function of z belongs to $\Theta_{S,\tau}^{(n)}$, has an expression as a linear combination of the theta series:

$$\phi(\tau, z)=\sum_{r\in L^{(n)}/(2S)L^{(n)}} h_r(\tau)\theta_r(\tau, z)\,, \tag{1.7}$$

where each $h_r(\tau)$ has the following Fourier expansion:

$$h_r(\tau)=\sum_{N\in Sym_n^*(\mathbb{Z}),N-(1/4){}^trS^{-1}r\geq 0} c(T)e(\mathrm{tr}(\tilde{T}\tau))\,,$$

$c(T)$ $\left(T=\begin{pmatrix} N & {}^tr/2\\ r/2 & S\end{pmatrix}\right)$ being the Fourier coefficients of ϕ. Moreover we define the subseries $h_r^*(\tau)$ of $h_r(\tau)$ by putting

$$h_r^*(\tau)=\sum_{N\in Sym_n^*(\mathbb{Z}),N-(1/4){}^trS^{-1}r>0} c(T)e(\mathrm{tr}(\tilde{T}\tau))\,. \tag{1.8}$$

We set

$$\boldsymbol{h}(\tau)=(h_r(\tau))_{r\in L^{(n)}/(2S)L^{(n)}}\,,$$

which is viewed as a column vector. Then Proposition 1, (1.6), and the property (ii) of ϕ imply the transformation formula for $\boldsymbol{h}(\tau)$.

PROPOSITION 2. *Let* $M\in\Gamma_n$. *Then*

$${}^t(\boldsymbol{h}(M\langle\tau\rangle))j_S(M, \tau)=\det J(M, \tau)^k\cdot{}^t(\boldsymbol{h}(\tau))\qquad (\textit{for all}\ \ M\in\Gamma_n)\,.$$

Namely,

$$\sum_{q\in L^{(n)}/(2S)L^{(n)}} h_q(M\langle\tau\rangle)j_S(M,\tau)_{qr}=\det J(M,\tau)^k h_r(\tau)\,.$$

We need a lemma on the automorphy factor $j_S(M,\tau)$. Set, for $V\in GL_n(\mathbb{R})$,

$$d_n(V)=\begin{pmatrix} V & 0 \\ 0 & {}^tV^{-1}\end{pmatrix}.$$

Let j be any integer with $0\le j\le n$. For any $M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_j$ we define $l_j^n(M)$, an element of Γ_n, by

$$l_j^n(M)=\begin{pmatrix} a & & b & \\ & 1_{n-j} & & \\ c & & d & \\ & & & 1_{n-j}\end{pmatrix}$$

In particular, $l_0^n(M)=1_{2n}$.

Let $A\in Sym_n(\mathbb{C})$ with $\mathrm{Re}(A)>0$. If l is odd, we employ the ordinary branch of $(\det A)^{l/2}$, a holomorphic function of A which takes positive values if A is a positive definite real symmetric matrix.

LEMMA 3. (i) *For $V\in GL_n(\mathbb{Z})$, $j_S(d_n(V),\tau)=(\delta_{\lambda\mu V^{-1}})_{\lambda,\mu\in L^{(n)}/(2S)L^{(n)}}$. Namely,*

$$j_S(d_n(V),\tau)_{\lambda\mu}=\begin{cases} 0 \cdots \text{ if } \lambda\not\equiv\mu V^{-1} \bmod (2S)L^{(n)}\,, \\ 1 \cdots \text{ if } \lambda\equiv\mu V^{-1} \bmod (2S)L^{(n)}\,. \end{cases}$$

(ii) *For $J_n=\begin{pmatrix} 0 & -1_n \\ 1_n & 0\end{pmatrix}$, we have*

$$j_S(J_n,\tau)=\det(2S)^{-n/2}\det\left(\frac{\tau}{i}\right)^{l/2}\left(e\left(-\frac{1}{2}\mathrm{tr}({}^t\lambda S^{-1}\mu)\right)\right)_{\lambda,\mu\in L^{(n)}/(2S)L^{(n)}}.$$

(iii) *Let $1\le j\le n$. Let $\lambda=(\lambda_1,\lambda_2)\in L^{(n)}/(2S)L^{(n)}$ and $\mu=(\mu_1,\mu_2)\in L^{(n)}/(2S)L^{(n)}$ with $\lambda_1,\mu_1\in L^{(j)}/(2S)L^{(j)}$ and $\lambda_2,\mu_2\in L^{(n-j)}/(2S)L^{(n-j)}$. For $\tau\in\mathfrak{H}_n$, put*

$$\tau_1=\tau\left[\begin{pmatrix} 1_j \\ 0\end{pmatrix}\right]\in\mathfrak{H}_j\,.$$

For any $M\in\Gamma_j$, the (λ,μ)-matrix component of $j_S(l_j^n(M),\tau)$ is given by

$$j_S^{(n)}(l_j^n(M),\tau)_{\lambda\mu}=\begin{cases} 0 & \cdots \text{ if } \lambda_2\not\equiv\mu_2 \bmod(2S)L^{(n-j)}\,, \\ j_S^{(j)}(M,\tau_1)_{\lambda_1\mu_1} & \cdots \text{ if } \lambda_2\equiv\mu_2 \bmod(2S)L^{(n-j)}\,. \end{cases}$$

Proof. The assertion (i) is due to the transformation formula

$$\theta_r(d_n(V)(\tau,z))=\theta_{rV}(\tau,z)\,.$$

In virtue of the Poisson summation formula we have

$$\theta_r(-\tau^{-1}, z\tau^{-1}) = \det(2S)^{-n/2}\det\left(\frac{\tau}{i}\right)^{l/2} e(\mathrm{tr}(\tau^{-1}S[z]))$$
$$\times \sum_{\mu\in L^{(n)}/(2S)L^{(n)}} e\left(-\frac{1}{2}\mathrm{tr}({}^t rS^{-1}\mu)\right)\theta_\mu(\tau, z)\,,$$

from which the assertion (ii) follows. Finally assume that $1\le j<n$. Let $M\in\Gamma_j$. Write

$$(\tau, z) = \left(\begin{pmatrix}\tau_1 & {}^t\tau_{21}\\ \tau_{21} & \tau_2\end{pmatrix}, (z_1, z_2)\right) \qquad \text{with} \quad \tau_1\in\mathfrak{H}_j\,,\ z_1\in M_{l,j}(\mathbb{C})\,.$$

Set $\rho = c\tau_1 + d$. Since

$$l_j^n(M)(\tau, z) = \left(\begin{pmatrix} M\langle\tau_1\rangle & {}^t\rho^{-1}\,{}^t\tau_{21}\\ \tau_{21}\rho^{-1} & \tau_2 - \tau_{21}\rho^{-1}c\,{}^t\tau_{21}\end{pmatrix}, (z_1\rho^{-1}, z_2 - z_1\rho^{-1}c\,{}^t\tau_{21})\right),$$

it is not difficult to see that

$$\theta_r^{(n)}(l_j^n(M)(\tau, z)) = \sum_{\lambda_2\in L^{(n-j)}} \theta_{r_1}^{(j)}(M(\tau_1, z_1 + (\lambda_2 + (2S)^{-1}r_2)\tau_{21}))$$
$$\times e(\mathrm{tr}(S[\lambda_2 + (2S)^{-1}r_2](\tau_2 - \tau_{21}\rho^{-1}c\,{}^t\tau_{21}) + 2\,{}^t(\lambda_2 + (2S)^{-1}r_2)S(z_2 - z_1\rho^{-1}c\,{}^t\tau_{21})))\,,$$

where we put $r = (r_1, r_2)$ with $r_1\in L^{(j)}/(2S)L^{(j)}$, $r_2\in L^{(n-j)}/(2S)L^{(n-j)}$. By the theta transformation formula the right hand side of the above identity is equal to

$$\sum_{\lambda_2\in L^{(n-j)}}\ \sum_{\mu_1\in L^{(j)}/(2S)L^{(j)}} J_{S,0}(M, (\tau_1, z_1)) j_S^{(j)}(M, \tau_1)_{r_1\mu_1}\theta_{\mu_1}^{(j)}(\tau_1, z_1 + (\lambda_2 + (2S)^{-1}r_2)\tau_{21})$$
$$\times e(\mathrm{tr}(S[\lambda_2 + (2S)^{-1}r_2]\tau_2 + 2\,{}^t(\lambda_2 + (2S)^{-1}r_2)Sz_2))\,.$$

Thus after some computation we obtain

$$\theta_r^{(n)}(l_j^n(M)(\tau, z)) = J_{S,0}(M, (\tau_1, z_1)) \sum_{\mu_1\in L^{(j)}/(2S)L^{(j)}} j_S^{(j)}(M, \tau_1)_{r_1\mu_1}\theta_{(\mu_1, r_2)}^{(n)}(\tau, z)\,.$$

If we note that

$$J_{S,0}(M, (\tau_1, z_1)) = J_{S,0}(l_j^n(M), (\tau, z))\,,$$

the above identity completes the proof of the assertion (iii). q.e.d.

Let $\phi\in J_{k,S}(\Gamma_n)$ and let $h_r(\tau)$ be the same as in (1.7). If we put

$$h(\tau) = \sum_{r\in L^{(n)}/(2S)L^{(n)}} h_r(\tau)\,, \tag{1.9}$$

then $h(\tau)$ is a modular form of weight $k - l/2$ with respect to a congruence subgroup of Γ_n. Then $h(\tau)$ and $h_0(\tau)$ have the following relation.

PROPOSITION 4. *We have*

$$h(-\tau^{-1})=c_{n,k,S}\det\left(\frac{\tau}{i}\right)^{k-l/2}h_0(\tau)\,,\qquad \textit{where}\quad c_{n,k,S}=(-1)^{nk/2}\det(2S)^{n/2}\,.$$

Proof. It is easy to see from Proposition 2 that

$$\sum_{r\in L^{(n)}/(2S)L^{(n)}}\sum_{q\in L^{(n)}/(2S)L^{(n)}}h_q(J_n\langle\tau\rangle)j_S(J_n,\tau)_{qr}=\sum_{r\in L^{(n)}/(2S)L^{(n)}}\det J(J_n,\tau)^k h_r(\tau)\,.$$

Since by (ii) of Lemma 3,

$$\sum_{r\in L^{(n)}/(2S)L^{(n)}}j_S(J_n,\tau)_{qr}=\det(2S)^{n/2}\det\left(\frac{\tau}{i}\right)^{l/2}\cdot\delta_{q,0}\,,$$

the equality

$$h_0(-\tau^{-1})=(-1)^{nk/2}\det(2S)^{-n/2}\det\left(\frac{\tau}{i}\right)^{k-l/2}h(\tau)$$

holds true. By changing τ with $-\tau^{-1}$, the assertion follows. q.e.d.

Similarly as in the proof of Maass [Ma, p. 205, Lemma] we can prove without difficulty the estimate for the Fourier coefficients $c(T)$ of $\phi\in J_{k,S}(\Gamma_n)$. Namely, there exists a positive constant C_1 such that

$$|c(T)|\le C_1\det(\tilde{T})^{k-l/2}\qquad\text{for any}\quad T\in Sym^*_{n+l}(S;\mathbb{Z})^+\,. \tag{1.10}$$

1.2. Klingen Eisenstein series

As a typical example of Jacobi forms the Klingen Eisenstein series attached to cusp forms have to be defined for the later use.

First we define the Siegel Φ-operator $\mathscr{S}$ in the case of Jacobi forms (see [Zi], [Ar4]). Set, for any $\phi\in J_{k,S}(\Gamma_n)$ and $(\tau,z)\in\mathscr{D}_{n-1,l}$,

$$\mathscr{S}(\phi)(\tau,z)=\lim_{t\to+\infty}\phi\left(\begin{pmatrix}\tau&0\\0&it\end{pmatrix},(z,0)\right).$$

Then, $\mathscr{S}(\phi)\in J_{k,S}(\Gamma_{n-1})$, and moreover, if ϕ has a Fourier expansion (1.5), then,

$$\mathscr{S}(\phi)(\tau,z)=\sum_{\substack{N\in Sym^*_{n-1}(\mathbb{Z}),r\in M_{l,n-1}(\mathbb{Z})\\N-(1/4)S^{-1}[r]\ge0}}c\left(\begin{pmatrix}N&0\\0&0\end{pmatrix},(r,0)\right)e(\mathrm{tr}(N\tau+{}^trz))\,.$$

For any function $\phi:\mathscr{D}_{n,l}\to\mathbb{C}$ and for $g\in G^J_{n,l}(\mathbb{R})$, we set

$$(\phi\,|_{k,S}g)(\tau,z)=J_{S,k}(g,(\tau,z))^{-1}\phi(g(\tau,z))\,.$$

Let r be an integer with $0\le r\le n$ and let $\phi\in J^{\mathrm{cusp}}_{k,S}(\Gamma_r)$. In case $r=0$, ϕ is assumed to be a constant function. For $(\tau,z)\in\mathscr{D}_{n,l}$, set

$$(\tau,z)^*=(\tau^*,z^*)=\left(\tau\left[\begin{pmatrix}1_r\\0\end{pmatrix}\right],z\begin{pmatrix}1_r\\0\end{pmatrix}\right)\in\mathscr{D}_{r,l}\,.$$

Let $\Gamma_{n,r}$ denote the subgroup of Γ_n consisting of elements whose lower left $r\times(n+r)$

blocks vanish:

$$\Gamma_{n,r}=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_n \,\middle|\, c=\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix},\ d=\begin{pmatrix} d_1 & d_2 \\ 0 & d_4 \end{pmatrix},\ c_1, d_1\in M_r(\mathbb{Z})\right\}.$$

Denote by $\Gamma^J_{n,r}$ the subgroup of Γ^J_n consisting of all elements of the form

$$(((\lambda_1, 0), \mu), \rho)M$$

with $\lambda_1\in M_{l,r}(\mathbb{Z})$, $\mu\in M_{l,n}(\mathbb{Z})$, $\rho\in Sym_l(\mathbb{Z})$, $M\in\Gamma_{n,r}$. We set, for $(\tau, z)\in\mathscr{D}_{n,l}$,

$$E^{k,S}_{n,r}(\phi, (\tau, z))=\sum_{\gamma\in\Gamma^J_{n,r}\backslash\Gamma^J_n}(\tilde{\phi}\,|_{k,S}\gamma)(\tau, z),$$

where $\tilde{\phi}(\tau, z)=\phi(\tau^*, z^*)$ is a function on $\mathscr{D}_{n,l}$. The well-definedness is immediately verified and this infinite series is absolutely convergent for $k>n+r+l+1$, as is shown in [Zi], Theorem 2.5. Moreover $E^{k,S}_{n,r}(\phi, (\tau, z))$ is a Jacobi form of $J_{k,S}(\Gamma_n)$, if $k>n+r+l+1$ ([Zi]). We write $E^{(n)}_{k,S}(\tau, z)$ for $E^{k,S}_{n,r}(\phi, (\tau, z))$, if $r=0$ and $\phi=1$, which is called the Jacobi Eisenstein series of degree n, weight k and index S. Set, for $k>n+r+l+1$,

$$J^{(r)}_{k,S}(\Gamma_n)=\{E^{k,S}_{n,r}(\phi, (\tau, z))\,|\,\phi\in J^{\mathrm{cusp}}_{k,S}(\Gamma_r)\} \qquad (0\le r\le n).$$

Then, $J^{(r)}_{k,S}(\Gamma_n)$ is a $\mathbb{C}$-subspace of $J_{k,S}(\Gamma_n)$. In particular, $J^{(0)}_{k,S}(\Gamma_n)=\mathbb{C}E^{(n)}_{k,S}(\tau, z)$ and $J^{(n)}_{k,S}(\Gamma_n)=J^{\mathrm{cusp}}_{k,S}(\Gamma_n)$. To obtain the structure theorem for the space $J_{k,S}(\Gamma_n)$ we impose the following condition on S.

(1.11) If $S[x]\in\mathbb{Z}$ for any $x\in(2S)^{-1}M_{l,1}(\mathbb{Z})$, then necessarily, $x\in M_{l,1}(\mathbb{Z})$.

It is known by Proposition 4.1 in [Ar4] that, under this condition any $\phi\in J_{k,S}(\Gamma_n)$ is a cusp form, if and only if $\mathscr{S}\phi=0$.

We have proved the following proposition in [Ar4] (see Theorem 4.2 of [Ar4] and its proof).

PROPOSITION 5. *Assume that* $S\in Sym^*_l(\mathbb{Z})^+$ *satisfies the condition* (1.11). *Let* k *be an even positive integer with* $k>2n+l+1$. *Then the following direct sum decomposition for the space* $J_{k,S}(\Gamma_n)$ *holds*:

$$J_{k,S}(\Gamma_n)=\bigoplus_{r=0}^{n} J^{(r)}_{k,S}(\Gamma_n).$$

Let $\varphi\in J^{\mathrm{cusp}}_{k,S}(\Gamma_r)$ $(0\le r\le n-1)$. *We have*

$$\mathscr{S}(E^{k,S}_{n,r}(\varphi, (*, *)))=E^{k,S}_{n-1,r}(\varphi, (*, *)).$$

Dulinski [Du] independently obtained the structure theorem for the space $J_{k,S}(\Gamma_n)$ without any maximality condition on S.

Each Eisenstein series $E^{k,S}_{n,r}(\phi, (\tau, z))$ for $\phi\in J^{\mathrm{cusp}}_{k,S}(\Gamma_r)$ can be written as a linear combination of the theta series:

$$(1.12)\qquad E_{n,r}^{k,S}(\phi,(\tau,z))=\sum_{\mu\in L^{(n)}/(2S)L^{(n)}} C_\mu^{(n)}(\phi,\tau)\theta_\mu^{(n)}(\tau,z)\,.$$

The functions $C_\mu^{(n)}(\phi,\tau)$ may be realized as certain types of Eisenstein series shown in the following Proposition 6. Set

$$(1.13)\qquad C^{(n)}(\phi,\tau)=\sum_{\mu\in L^{(n)}/(2S)L^{(n)}} C_\mu^{(n)}(\phi,\tau)\,.$$

We write, similarly as in (1.7),

$$\phi(\tau,z)=\sum_{\lambda\in L^{(r)}/(2S)L^{(r)}} h_\lambda(\tau)\theta_\lambda^{(r)}(\tau,z)\qquad ((\tau,z)\in\mathscr{D}_{r,l})\,.$$

For each $M\in\Gamma_n$, let $j_S(M,\tau)_{\lambda\mu}$ denote the (λ,μ)-matrix component of $j_S(M,\tau)$ which is the same as in Proposition 1.

PROPOSITION 6. *For each* $\mu\in L^{(n)}/(2S)L^{(n)}$,

$$C_\mu^{(n)}(\phi,\tau)=\sum_{M\in\Gamma_{n,r}\backslash\Gamma_n}\ \sum_{\lambda\in L^{(r)}/(2S)L^{(r)}} \det J(M,\tau)^{-k}h_\lambda(M\langle\tau\rangle^*)j_S(M,\tau)_{\tilde\lambda\mu}\,,$$

where $\tilde\lambda=(\lambda,0)\in L^{(n)}/(2S)L^{(n)}$. *Consequently,*

$$C^{(n)}(\phi,\tau)=\sum_{M\in\Gamma_{n,r}\backslash\Gamma_n}\ \sum_{\lambda\in L^{(r)}/(2S)L^{(r)}}\ \sum_{\mu\in L^{(n)}/(2S)L^{(n)}} \det J(M,\tau)^{-k}h_\lambda(M\langle\tau\rangle^*)j_S(M,\tau)_{\tilde\lambda\mu}\,.$$

Proof. We may choose the set

$$\{(((0,\lambda_2),0),0)M\mid \lambda_2\in L^{(n-r)},\ M\in\Gamma_{n,r}\backslash\Gamma_n\}$$

as a complete set of representatives of $\Gamma_{n,r}^J\backslash\Gamma_n^J$. Then,

$$(1.14)\quad E_{n,r}^{k,S}(\phi,(\tau,z))=\sum_{\lambda_2\in L^{(n-r)}}\ \sum_{M\in\Gamma_{n,r}\backslash\Gamma_n}\ \sum_{\nu\in L^{(r)}/(2S)L^{(r)}}$$
$$J_{S,k}((((0,\lambda_2),0),0),M(\tau,z))^{-1}J_{S,k}(M,(\tau,z))^{-1}h_\nu(M\langle\tau\rangle^*)\theta_\nu^{(r)}(\hat\tau_1,\hat z_1+\lambda_2\hat\tau_{21})\,,$$

where $M(\tau,z)=\left(\begin{pmatrix}\hat\tau_1 & {}^t\hat\tau_{21}\\ \hat\tau_{21} & \hat\tau_2\end{pmatrix},(\hat z_1,\hat z_2)\right)$ with $(\hat\tau_1,\hat z_1)\in\mathscr{D}_{r,l}$. By a straightforward computation and from Proposition 1 we see that

$$\begin{aligned}&\sum_{\lambda_2\in L^{(n-r)}} J_{S,k}((((0,\lambda_2),0),0),M(\tau,z))^{-1}\theta_\nu^{(r)}(\hat\tau_1,\hat z_1+\lambda_2\hat\tau_{21})\\ &=\theta_{(\nu,0)}^{(n)}(M(\tau,z))\\ &=J_{S,0}(M,(\tau,z))\sum_{\mu\in L^{(n)}/(2S)L^{(n)}} j_S^{(n)}(M,\tau)_{\tilde\nu\mu}\theta_\mu^{(n)}(\tau,z)\,,\end{aligned}$$

where we put $\tilde\nu=(\nu,0)\in L^{(n)}/(2S)L^{(n)}$. Therefore substituting the last expression into (1.14), we obtain the assertion of Propositon 6. q.e.d.

1.3. Köcher-Maass Dirichlet series

First we introduce some notations to define Köcher-Maass Dirichlet series associated to Jacobi forms. Let $B_{n,l}(\mathbb{Z})$ denote the subgroup of $GL_{n+l}(\mathbb{Z})$ consisting of all elements of the form

$$\begin{pmatrix} U & 0 \\ y & 1_l \end{pmatrix} \quad \text{with} \quad U \in GL_n(\mathbb{Z}), \quad y \in L^{(n)}.$$

This group $B_{n,l}(\mathbb{Z})$ acts on the set $Sym^*_{n+l}(S;\mathbb{Z})^+$ by

$$T \mapsto T[\gamma] := {}^t\gamma T \gamma \qquad (\gamma \in B_{n,l}(\mathbb{Z}),\ T \in Sym^*_{n+l}(S;\mathbb{Z})^+).$$

We call T, $T' \in Sym^*_{n+l}(S;\mathbb{Z})^+$ in the same class (resp. the same genus) and write $T \sim_{\mathbb{Z}} T'$ (resp. $T \sim_{\text{gen}} T'$), if they are in the same $B_{n,l}(\mathbb{Z})$ orbit (resp. in the same $B_{n,l}(\mathbb{Z}_p)$ orbit for any prime integer p). Then a given genus consists of a finite number of classes. On this notion of classes and genera we refer to [Ar3, 4]. For each $T \in Sym^*_{n+l}(S;\mathbb{Z})^+$, let $O(T;S)(\mathbb{Z})$ denote the unit group of T given by

$$O(T;S)(\mathbb{Z}) = \{\gamma \in B_{n,l}(\mathbb{Z}) \mid T[\gamma] = T\}.$$

Let us define the Köcher-Maass Dirichlet series attached to $\phi \in J_{k,S}(\Gamma_n)$. Let the Fourier expansion of ϕ be the same as in (1.5). We note that, if $T \sim_{\mathbb{Z}} T'$ for T, $T' \in Sym^*_{n+l}(S;\mathbb{Z})^+$, then $c(T) = c(T')$ (see Proposition 1.1 of [Ar4]). We set

$$D_n(\phi, s) = \sum_{T \in Sym^*_{n+l}(S;\mathbb{Z})^+/\sim_{\mathbb{Z}}} \frac{c(T)}{\epsilon(T)\det(\tilde{T})^s}, \tag{1.15}$$

where the summation indicates that T runs over all the $B_{n,l}(\mathbb{Z})$-equivalence classes of elements of $Sym^*_{n+l}(S;\mathbb{Z})^+$, $\epsilon(T) = \sharp(O(T;S)(\mathbb{Z}))$ and $\tilde{T}$ is the same as the previous 1.1. This Dirichlet series is absolutely convergent for $\mathrm{Re}(s) > k - \frac{l}{2} + \frac{n+1}{2}$ in virtue of the estimate (1.10). We define another Dirichlet series $\hat{D}_n(\phi, s)$ by

$$\hat{D}_n(\phi, s) = \sum_{N \in Sym^*_n(\mathbb{Z})^+/\sim} \frac{c(N \perp S)}{\varepsilon(N)(\det N)^s},$$

where N runs over all the $GL_n(\mathbb{Z})$-equivalence classes of positive definite half integral symmetric matrices of size n and $\varepsilon(N)$ denotes the order of the unit group $\{U \in GL_n(\mathbb{Z}) \mid N[U] = N\}$. This series is also absolutely convergent for $\mathrm{Re}(s) > k - \frac{l}{2} + \frac{n+1}{2}$. We now set, for $\phi \in J_{k,S}(\Gamma_n)$,

$$h^*(\tau) = \sum_{r \in L^{(n)}/(2S)L^{(n)}} h^*_r(\tau), \tag{1.16}$$

$h^*_r(\tau)$ being the same as in (1.8). Then, $h^*(\tau)$ is a subseries of the Fourier expansion of $h(\tau)$ (for $h(\tau)$, see (1.9)).

Let $\mathscr{P}_n$ denote the symmetric space of positive definite real symmetric matrices of size n on which the general linear group $GL_n(\mathbb{R})$ acts via $Y \mapsto {}^t g Y g$ ($Y \in \mathscr{P}_n$, $g \in GL_n(\mathbb{R})$). Denote by $dv_n(Y)$ an invariant measure on $\mathscr{P}_n$ normalized by

$$dv_n(Y) = \det(Y)^{-(n+1)/2} \prod_{1 \le i \le j \le n} dy_{ij} \qquad (Y = (y_{ij}) \in \mathscr{P}_n) .$$

We set

$$\xi_n(\phi, s) := \int_{GL_n(\mathbb{Z}) \backslash \mathscr{P}_n} h^*(iY)(\det Y)^s dv_n(Y) .$$

Then this function is related with our Dirichlet series $D_n(\phi, s)$.

PROPOSITION 7. *The above integral is absolutely convergent for* $\mathrm{Re}(s) > k - \frac{l}{2} + \frac{n+1}{2}$ *and moreover*

$$\xi_n(\phi, s) = \gamma_n(s) D_n(\phi, s) ,$$

where the gamma factor $\gamma_n(s)$ *is given by*

$$\gamma_n(s) = 2\pi^{n(n-1)/4}(2\pi)^{-ns} \prod_{j=1}^{n} \Gamma\left(s - \frac{j-1}{2}\right) .$$

Proof. We define the subgroup $B_{n,l}^{\infty}(\mathbb{Z})$ of $B_{n,l}(\mathbb{Z})$ by putting

$$B_{n,l}^{\infty}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1_n & 0 \\ y & 1_l \end{pmatrix} \middle| \, y \in L^{(n)} \right\} .$$

We first note that

$$\begin{aligned} h^*(\tau) &= \sum_{r \in L^{(n)}/(2S)L^{(n)}} \sum_{\substack{N \in Sym_n^*(\mathbb{Z}) \\ N - (1/4){}^t r S^{-1} r > 0}} c(T) e(\mathrm{tr}(\tilde{T}\tau)) \\ &= \sum_{T \in Sym_{n+l}^*(S;\mathbb{Z})^+ / B_{n,l}^{\infty}(\mathbb{Z})} c(T) e(\mathrm{tr}(\tilde{T}\tau)) , \end{aligned}$$

where $Sym_{n+l}^*(S;\mathbb{Z})^+/B_{n,l}^{\infty}(\mathbb{Z})$ is a complete set of representatives of elements of $Sym_{n+l}^*(S;\mathbb{Z})^+$ under the action of $B_{n,l}^{\infty}(\mathbb{Z})$. Then it is easy to see that

$$h^*(\tau) = \sum_{T \in Sym_{n+l}^*(S;\mathbb{Z})^+ / B_{n,l}(\mathbb{Z})} \sum_{\gamma \in O(T;S)(\mathbb{Z}) \backslash B_{n,l}(\mathbb{Z}) / B_{n,l}^{\infty}(\mathbb{Z})} c(T[\gamma]) e(\mathrm{tr}(\widetilde{T[\gamma]}\tau)) ,$$

where γ runs over a complete set of representatives of the double cosets:

$$O(T;S)(\mathbb{Z}) \backslash B_{n,l}(\mathbb{Z}) / B_{n,l}^{\infty}(\mathbb{Z}) .$$

Since $c(T[\gamma]) = c(T)$ and $\epsilon(T) = \sharp O(T;S)(\mathbb{Z})$, we have

$$h^*(\tau)=\sum_{T\in Sym^*_{n+l}(S;\mathbb{Z})^+/B_{n,l}(\mathbb{Z})}\ \sum_{\gamma\in B_{n,l}(\mathbb{Z})/B^\infty_{n,l}(\mathbb{Z})}\frac{c(T)}{\epsilon(T)}e(\mathrm{tr}(\widetilde{T[\gamma]}\tau))$$
$$=\sum_{T\in Sym^*_{n+l}(S;\mathbb{Z})^+/B_{n,l}(\mathbb{Z})}\ \sum_{V\in GL_n(\mathbb{Z})}\frac{c(T)}{\epsilon(T)}e(\mathrm{tr}(\tilde{T}V\tau{}^tV)).$$

We note that $V(GL_n(\mathbb{Z})\backslash\mathscr{P}_n){}^tV$ with V running over all elements of $GL_n(\mathbb{Z})$ doubly cover the space $\mathscr{P}_n$. Therefore we easily have

$$\xi_n(\phi,s)=2\sum_{T\in Sym^*_{n+l}(S;\mathbb{Z})^+/B_{n,l}(\mathbb{Z})}\frac{c(T)}{\epsilon(T)}\int_{\mathscr{P}_n}e^{-2\pi\mathrm{tr}(\tilde{T}\eta)}(\det\eta)^s dv_n(\eta)$$
$$=\gamma_n(s)D_n(\phi,s).$$

Thus we have completed the proof. q.e.d.

We moreover define the integral $\hat{\xi}_n(\phi,s)$ by

$$\hat{\xi}(\phi,s):=\int_{GL_n(\mathbb{Z})\backslash\mathscr{P}_n}h_0^*(iY)(\det Y)^s dv_n(Y),$$

which is also absolutely convergent for $\mathrm{Re}(s)>k-\frac{l}{2}+\frac{n+1}{2}$. Then quite similarly

$$\hat{\xi}_n(\phi,s)=\gamma_n(s)\hat{D}_n(\phi,s).$$

THEOREM 8. *The zeta functions $D_n(\phi,s)$, $\hat{D}_n(\phi,s)$ can be analytically continued to meromorphic functions in the whole s plane which verify the functional equation*

$$\xi_n(\phi,k-l/2-s)=c_{n,k,S}\hat{\xi}_n(\phi,s),$$

where $c_{n,k,S}$ is the constant given in Proposition 4.

The proof is done in a similar manner as in that of [Ma, §15], where invariant differential operators acting on $\mathscr{P}_n$ are used in a skillful way. So we omit it.

The above theorem is inadequate since it cannot give us sufficient information on the residues at poles of $D_n(\phi,s)$ (and of $\xi_n(\phi,s)$). It will be necessary to obtain a formula wich will give us some information on the situation of poles. For that purpose we impose the condition (1.11) on S which is a kind of maximality condition.

Now we formulate our main theorem which gives us an explicit expression for $\xi_n(\phi,s)$. The proof will be postponed to the last paragraph.

Set, for $j\in\mathbb{Z}$,

$$\varepsilon(j)=\begin{cases}1/2 & \cdots j\neq 0\\ 1 & \cdots j=0.\end{cases}$$

THEOREM 9. *Assume that k is an even integer larger than $2n+l+1$ and moreover that S satisfies the condition* (1.11). *Let $\phi\in J_{k,S}(\Gamma_n)$. Then,*

$$(1.17)\qquad \xi_n(\phi,s)=I_n(\phi,s)+\sum_{j=0}^{n-1}\varepsilon(j)v(n-j)\left(\frac{c_{n,k,S}\hat{\xi}_j(\mathscr{S}^{n-j}\phi,\frac{n}{2})}{s-k+\frac{l}{2}+\frac{j}{2}}-\frac{\xi_j(\mathscr{S}^{n-j}\phi,\frac{n}{2})}{s-\frac{j}{2}}\right),$$

$$(1.18)\qquad \hat{\xi}_n(\phi, s)=\hat{I}_n(\phi, s)+\sum_{j=0}^{n-1}\varepsilon(j)v(n-j)\left(\frac{c_{n,k,S}^{-1}\xi_j(\mathscr{S}^{n-j}\phi, \frac{n}{2})}{s-k+\frac{l}{2}+\frac{j}{2}}-\frac{\hat{\xi}_j(\mathscr{S}^{n-j}\phi, \frac{n}{2})}{s-\frac{j}{2}}\right),$$

where we set

$$I_n(\phi, s)=\int_{\substack{GL_n(\mathbb{Z})\backslash\mathscr{P}_n\\ \det\eta\geq 1}}\{(\det\eta)^s h^*(i\eta)+c_{n,k,S}(\det\eta)^{k-l/2-s}h_0^*(i\eta)\}dv_n(\eta)\,,$$

$$\hat{I}_n(\phi, s)=\int_{\substack{GL_n(\mathbb{Z})\backslash\mathscr{P}_n\\ \det\eta\geq 1}}\{(\det\eta)^s h_0^*(i\eta)+c_{n,k,S}^{-1}(\det\eta)^{k-l/2-s}h^*(i\eta)\}dv_n(\eta)\,.$$

Here $I_n(\phi, s)$ and $\hat{I}_n(\phi, s)$ which are absolutely convergent for any s indicate entire functions of s.

2. Modular forms of half integral weights and Cohen Eisenstein series

2.1. Kohnen plus space and Cohen Eisenstein series

In this subparagraph we recall the plus space of Siegel modular forms of half integral weights. The plus space has been invented by Kohnen [Koh] for $n=1$ and by Ibukiyama [Ib] for $n>1$.

Here we restrict ourselves to the case of $l=1$ and $S=1$. We note that the index $S=1$ obviously satisfies the maximality condition (1.11). Let $L=M_{1,n}(\mathbb{Z})=\mathbb{Z}^n$, $\mathbb{Z}^n$ denoting the set of integral row vectors of size n. We may choose the set

$$\{(a_1, \cdots, a_n)\,|\,a_i=0, \text{ or } 1\}$$

as a complete set of representatives of $L/2L$. Then, $\sharp(L/2L)=2^n$.

The special theta series $\theta^{(n)}(\tau)$ is given by

$$\theta^{(n)}(\tau):=\theta_0^{(n)}(\tau, 0)=\sum_{\lambda\in L}e(\lambda\tau{}^t\lambda)\qquad(\tau\in\mathfrak{H}_n)\,.$$

Let $\Gamma_0^{(n)}(4)$ be the congruence subgroup of Γ_n consisting of matrices $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_n$ with $c\equiv 0 \bmod 4$. Denote by $S_n(\mathbb{Z})$ the set of integral symmetric matrices N of size n satisfying the condition

$$N\equiv -\lambda^t\lambda \bmod 4\,Sym_n^*(\mathbb{Z})\qquad\text{for some}\quad \lambda\in L\,.$$

The following plus space is a very convenient subspace of modular forms of half integral weights. Let k be an even positive integer. Let $M_{k-1/2}^+(\Gamma_0^{(n)}(4))$ be the space consisting of holomorphic functions f on $\mathfrak{H}_n$ verifying the conditions (i), (ii):

(i) $f(M\langle\tau\rangle)=\det(c\tau+d)^k\dfrac{\theta^{(n)}(\tau)}{\theta^{(n)}(M\langle\tau\rangle)}f(\tau)$ for any $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0^{(n)}(4)$,

(ii) f has the Fourier expansion

$$f(\tau)=\sum_{N\in Sym_n^*(\mathbb{Z}),N\geq 0}a(N)e(\mathrm{tr}(N\tau))$$

with the condition that $a(N)=0$ unless $N \notin S_n(\mathbb{Z})$.

Any $f \in M^+_{k-1/2}(\Gamma_0^{(n)}(4))$ is said to be a cusp form, if the function

$$(\det \operatorname{Im} \tau)^{(k-1/2)/2} |f(\tau)|$$

is bounded on $\mathfrak{H}_n$. Denote by $S^+_{k-1/2}(\Gamma_0^{(n)}(4))$ the subspace of $M^+_{k-1/2}\Gamma_0^{(n)}(4))$ consisting of cusp forms. It is remarkable that the space $M^+_{k-1/2}(\Gamma_0^{(n)}(4))$ is isomorphic to the space $J_{k,1}(\Gamma_n)$ of Jacobi forms of degree n, weight k, and index 1. This isomorphism was discovered by Kohnen, Eichler-Zagier (see [E-Z]) in case $n=1$ and by Ibukiyama [Ib] in case $n>1$.

THEOREM 10 (Kohnen, Eichler-Zagier, Ibukiyama). *Let k be an even positive integer. Then the space $J_{k,1}(\Gamma_n)$ is isomorphic to the space $M^+_{k-1/2}(\Gamma_0^{(n)}(4))$ via the linear map $\sigma^{(n)}: \phi(\tau, z) \mapsto f(\tau) := h(4\tau) = \sum_{r \in L/2L} h_r(4\tau)$, where we write ϕ as a linear combination of the theta series: $\phi(\tau, z) = \sum_{r \in L/2L} h_r(\tau)\theta_r(\tau, z)$. Moreover the space $J^{\text{cusp}}_{k,1}(\Gamma_n)$ of cusp forms corresponds bijectively to the space $S^+_{k-1/2}(\Gamma_0^{(n)}(4))$ via this map.*

Let r be an integer with $0 \le r \le n$ and $f \in S^+_{k-1/2}(\Gamma_0^{(r)}(4))$. Then f corresponds to the unique $\phi \in J^{\text{cusp}}_{k,1}(\Gamma_r)$ via $\sigma^{(r)}$. Namely, $\sigma^{(r)}\phi = f$. If $r=0$, then we understand that $\sigma^{(0)}$ is the identity map of $\mathbb{C}$ and that f is a constant. We define a function $C^{k-1/2}_{n,r}(f, \tau)$ by the image of the Klingen Eisenstein series $E^{k,1}_{n,r}(\phi, (\tau, z))$ via the above isomorphism $\sigma^{(n)}$:

$$C^{k-1/2}_{n,r}(f, \tau) = \sigma^{(n)}(E^{k,1}_{n,r}(\phi, (\tau, z))) .$$

If $r=0$ and $f=1$, we simply write $C^{k-1/2}_n(\tau)$ instead of $C^{k-1/2}_{n,0}(1, \tau)$, namely

$$C^{k-1/2}_n(\tau) = \sigma^{(n)}(E^{(n)}_{k,1}(\tau, z)) .$$

We may call $C^{k-1/2}_n(\tau)$ the Cohen Eisenstein series of degree n and $C^{k-1/2}_{n,r}(f, \tau)$ the Cohen-Klingen Eisenstein series, since in the case of $n=1$, $C^{k-1/2}_1(\tau)$ was studied by Cohen [Co].

Let $M^{+(r)}_{k-1/2}$ denote the subspace of $M^+_{k-1/2}(\Gamma_0^{(n)}(4))$ spanned by Cohen-Klingen's Eisenstein Series $C^{k-1/2}_{n,r}(f, \tau)$ with f varying in $S^+_{k-1/2}(\Gamma_0^{(r)}(4))$:

$$M^{+(r)}_{k-1/2} = \{ C^{k-1/2}_{n,r}(f, \tau) \mid f \in S^+_{k-1/2}(\Gamma_0^{(r)}(4)) \} .$$

We know by Proposition 5 that the space $J_{k,1}(\Gamma_n)$ is spanned by Klingen's Eisenstein series and hence by the above theorem the space $M^+_{k-1/2}(\Gamma_0^{(n)}(4))$ is decomposed in the following manner.

PROPOSITION 11. *Assume that k is an even integer with $k>2n+2$. Then the direct sum decomposition holds*:

$$M^+_{k-1/2}(\Gamma_0^{(n)}(4)) = \bigoplus_{r=0}^{n} M^{+(r)}_{k-1/2} .$$

2.2. Siegel's formula for the Cohen Eisenstein series

For a positive integer m let $Sym^*_{m+1}(1;\mathbb{Z})$ denote the subset of $Sym^*_{m+1}(\mathbb{Z})$

consisting of $Q \in Sym^*_{m+1}(\mathbb{Z})$ of the form $Q = \begin{pmatrix} M & {}^tq/2 \\ q/2 & 1 \end{pmatrix}$ with $M \in Sym^*_m(\mathbb{Z})$, $q \in M_{1,m}(\mathbb{Z})$ (see (1.4)). Let $Sym^*_{m+1}(1;\mathbb{Z})^+$ denote the subset of $Sym^*_{m+1}(1;\mathbb{Z})$ consisting of positive definite elements of $Sym^*_{m+1}(1;\mathbb{Z})$. Note that

$$S_m(\mathbb{Z}) = \{4\tilde{Q} \mid Q \in Sym^*_{m+1}(1;\mathbb{Z})\}, \qquad \tilde{Q} \text{ being } M - \frac{1}{4}{}^tqq .$$

Assume k is divisible by 4. Denote by $\mathscr{S}^+_{2k-1}$ the set consisting of positive definite integral symmetric matrices N of size $2k-1$ satisfying the conditions

(i) $N \equiv -{}^t\lambda\lambda \bmod 4\, Sym^*_{2k-1}(\mathbb{Z})$ for some $\lambda \in M_{1,2k-1}(\mathbb{Z})$,

(ii) $\det N = 4^{k-1}$.

As is given in [Ar4], 6.1, set

$$US^+_{2k} = \{Q \in Sym^*_{2k}(1;\mathbb{Z})^+ \mid \det(2Q) = 1\} .$$

Obviously, $\mathscr{S}^+_{2k-1} = \{4\tilde{Q} \mid Q \in US^+_{2k}\}$. In US^+_{2k} we consider the $B_{2k-1,1}(\mathbb{Z})$-equivalence classes as in 1.3, while we consider in $\mathscr{S}^+_{2k-1}$ the ordinary classes and genera.

Lemma 12. *The $B_{2k-1,1}(\mathbb{Z})$-equivalence class in US^+_{2k} corresponds one to one onto the $GL_{2k-1}(\mathbb{Z})$-equivalence class in $\mathscr{S}^+_{2k-1}$ via the map: $Q \mapsto 4\tilde{Q}$. Moreover $\mathscr{S}^+_{2k-1}$ consists of a single genus.*

Proof. If $Q \in US^+_{2k}$, then, $4\tilde{Q} \in \mathscr{S}^+_{2k-1}$. It is easy to see that, if $Q \in US^+_{2k}$ is $B_{2k-1,1}(\mathbb{Z})$-equivalent with $Q' \in US^+_{2k}$, then $4\tilde{Q}$ is $GL_{2k-1}(\mathbb{Z})$-equivalent with $4\tilde{Q}'$. Moreover the correspondence is easily seen to be surjective. Let N and N' be $GL_{2k-1}(\mathbb{Z})$-equivalent elements in $\mathscr{S}^+_{2k-1}$, namely $N' = {}^tUNU$ with some $U \in GL_{2k-1}(\mathbb{Z})$. Choose

$$Q = \begin{pmatrix} M & {}^tq/2 \\ q/2 & 1 \end{pmatrix} \in US^+_{2k} \qquad \left(\text{resp. } Q' = \begin{pmatrix} M' & {}^tq'/2 \\ q'/2 & 1 \end{pmatrix} \in US^+_{2k}\right)$$

with the condition $4\tilde{Q} = N$ (resp. $4\tilde{Q}' = N'$). Then, $4M' - {}^tq'q' = {}^tU(4M - {}^tqq)U$, from which $q' - qU \in 2M_{1,2k-1}(\mathbb{Z})$. Hence if we put $r = (q' - qU)/2 \in M_{1,2k-1}(\mathbb{Z})$ and $\gamma = \begin{pmatrix} U & 0 \\ r & 1 \end{pmatrix} \in B_{2k-1,1}(\mathbb{Z})$, then $T' = {}^t\gamma T\gamma$.

It has been proved in [Ar5] that US^+_{2k} consists of a single genus in the sense given in the subparagraph 1.3. Therefore via the above bijective correspondence the set $\mathscr{S}^+_{2k-1}$ consists of a single genus in the ordinary sense. q.e.d.

Let $S_1, S_2, \cdots, S_H$ be a complete set of representatives of the ordinary $GL_{2k-1}(\mathbb{Z})$-equivalence classes of $\mathscr{S}^+_{2k-1}$. Let M_{2k-1} be the mass of $\mathscr{S}^+_{2k-1}$ given by

$$M_{2k-1} = \sum_{j=1}^{H} \frac{1}{\varepsilon(S_j)} ,$$

where $\varepsilon(S_j) = \#(O(S_j)(\mathbb{Z}))$, $O(S_j)(\mathbb{Z})$ denoting the unit group $\{U \in GL_{2k-1}(\mathbb{Z}) \mid {}^tUS_jU = S_j\}$ of S_j. The exact value of M_{2k-1} is given in Theorem 0.1 (see also Theorem 2.3) of [Ar5] via the above lemma:

$$M_{2k-1}=2^{3-2k}\prod_{j=1}^{k-1}\frac{B_j}{j}, \tag{2.1}$$

where B_j's are the Bernoulli numbers given by

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{r=1}^{\infty}(-1)^{r+1}B_r\frac{x^{2r}}{(2r)!}.$$

The theta series $\theta_S^{(n)}(\tau)$ attached to each S of $\mathscr{S}_{2k-1}^+$ is defined by

$$\theta_S^{(n)}(\tau)=\sum_{G\in M_{2k-1,n}(\mathbb{Z})}e(\mathrm{tr}({}^tGSG\tau)),$$

which is an element of $M_{k-1/2}^+(\Gamma_0^{(n)}(4))$ as we see later. Moreover we have defined in [Ar3] the theta series $\theta_Q^{(n)}(\tau, z)$ attached to $Q\in US_{2k}^+$ which is an element of $J_{k,1}(\Gamma_n)$. Namely for $Q=\begin{pmatrix} M & {}^tq/2 \\ q/2 & 1\end{pmatrix}\in US_{2k}^+$,

$$\theta_Q^{(n)}(\tau, z)=\sum_{G\in M_{2k,n}(\mathbb{Z})}e(\mathrm{tr}(Q[G]\tau+{}^tz(q, 2)G)) \qquad ((\tau, z)\in\mathscr{D}_{n,1}). \tag{2.2}$$

As we have mentioned in [Ar4, 6.2], the theta series $\theta_S^{(n)}(\tau)$ corresponds to $\theta_Q^{(n)}(\tau, z)$ via the isomorphism $\sigma^{(n)}$. We exhibit this fact as a lemma.

LEMMA 13. *Let $Q\in US_{2k}^+$. Then,*

$$\sigma^{(n)}(\theta_Q^{(n)}(*, *))(\tau)=\theta_{4\tilde{Q}}^{(n)}(\tau).$$

Moreover, $\theta_S^{(n)}(\tau)$ for any $S\in\mathscr{S}_{2k-1}^+$ is an element of $M_{k-1/2}^+(\Gamma_0^{(n)}(4))$.

Proof. We may rewrite (2.2) into the form:

$$\begin{aligned}\theta_Q^{(n)}(\tau, z)&=\sum_{G_1\in M_{2k-1,n}(\mathbb{Z})}\ \sum_{G_2\in M_{1,n}(\mathbb{Z})}e\left(\mathrm{tr}\left(\left(M-\frac{1}{4}{}^tqq\right)[G_1]\tau\right)\right)\\ &\quad\times e\left(\mathrm{tr}\left({}^t\left(G_2+\frac{1}{2}qG_1\right)\left(G_2+\frac{1}{2}qG_1\right)\tau+2{}^tz\left(G_2+\frac{1}{2}qG_1\right)\right)\right)\\ &=\sum_{r\in L/2L}h_r(\tau)\theta_r^{(n)}(\tau, z)\end{aligned}$$

with

$$h_r(\tau)=\sum_{G_1\in M_{2k-1,n}(\mathbb{Z}),\, qG_1\equiv r \bmod 2L}e\left(\mathrm{tr}\left(\left(M-\frac{1}{4}{}^tqq\right)[G_1]\tau\right)\right).$$

Thus,

$$\sigma^{(n)}(\theta_Q^{(n)}(*, *))(\tau)=\sum_{r\in L/2L}h_r(4\tau)=\theta_{4\tilde{Q}}^{(n)}(\tau).$$

The latter assertion is a direct consequence of the former one and Theorem 10.

q.e.d.

To describe some significant properties of $C_n^{k-1/2}(\tau)$ we have to recall the notion of local densities. Assume that m, n are positive integers with $m \geq n$. Let $M \in Sym_m(\mathbb{Z})^+$ and $N \in Sym_n(\mathbb{Z})^+$. For each prime integer p, let $\alpha_p(M, N)$ denote the ordinary local density given by

$$\alpha_p(M, N) = \lim_{\nu \to \infty} p^{-\nu(mn - n(n+1)/2)} \# \{ G \in M_{m,n}(\mathbb{Z}/p^\nu \mathbb{Z}) \mid M[G] \equiv N \bmod p^\nu \, Sym_n^*(\mathbb{Z}) \} .$$

Moreover for $M \in Sym_m(\mathbb{R})^+$ and $N \in Sym_n(\mathbb{R})^+$ the local density $\alpha_\infty(M, N)$ at the infinity is given by

$$\alpha_\infty(M, N) = (\det M)^{-n/2} (\det N)^{(m-n-1)/2} \gamma_{mn}$$

with

$$\gamma_{mn} = \alpha_\infty(1_m, 1_n) = \frac{\pi^{mn/2}}{(4\pi)^{n(n-1)/4} \prod_{j=1}^n \Gamma(\frac{m-j+1}{2})} .$$

Let $Q \in Sym_{m+1}^*(1; \mathbb{Z})^+$, $T \in Sym_{n+1}^*(1; \mathbb{Z})^+$. Let $\alpha_p(Q; T)$ be the local density defined in [Ar3]:

$$\alpha_p(Q; T) = \lim_{\nu \to \infty} p^{-\nu(mn - n(n+1)/2)} A_{p^\nu}(Q; T) ,$$

where

$$A_{p^\nu}(Q; T) = \# \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in M_{m+1,n}(\mathbb{Z}/p^\nu \mathbb{Z}) \,\middle|\, Q\left[\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \right] \equiv T \bmod p^\nu \, Sym_{n+1}^*(\mathbb{Z}) \right\} .$$

It is easily verified that, if $p > 2$, then,

$$\alpha_p(Q; T) = \alpha_p(4\tilde{Q}, 4\tilde{T}) .$$

Moreover the local density $\alpha_\infty(Q; T)$ is defined by

$$\alpha_\infty(Q; T) = 2^{-n} \alpha_\infty(\tilde{Q}, \tilde{T}) = 2^{-n} (\det \tilde{Q})^{-n/2} (\det \tilde{T})^{(m-n-1)/2} \gamma_{mn} .$$

The Cohen Eisenstein series enjoys the following beautiful expressions.

THEOREM 14. *Assume that $k > 2n+2$ and that k is divisivle by 4.*

(i) *The Siegel formula for the Cohen Eisenstein series holds*:

$$C_n^{k-1/2}(\tau) = \frac{1}{M_{2k-1}} \left(\sum_{j=1}^{H} \frac{\theta_{S_j}^{(n)}(\tau)}{\varepsilon(S_j)} \right) .$$

(ii) *Choose any $S \in \mathscr{S}_{2k-1}^+$. The Eisenstein series $C_n^{k-1/2}(\tau)$ has the Fourier expansion*

$$C_n^{k-1/2}(\tau)=\sum_{N\in S_n(\mathbb{Z}),N\geq 0} C_{k-1/2,n}(N)e(\mathrm{tr}(N\tau))\,,$$

where the coefficient $C_{k-1/2,n}(N)$ for positive definite $N\in S_n(\mathbb{Z})$ is given by

$$C_{k-1/2,n}(N)=\tilde{\alpha}(S,N):=\prod_{v\leq\infty}\tilde{\alpha}_v(S,N)\,.$$

Here if $v=p>2$, then $\tilde{\alpha}_p(S,N)$ is equal to the ordinary local density $\alpha_p(S,N)$. If $v=2$, ∞, then $\tilde{\alpha}_v(S,N)$ equals $\alpha_v(Q;T)$, where $Q\in US_{2k}^+$ and $T\in Sym_{n+1}^*(1;\mathbb{Z})^+$ are chosen so that $4\tilde{Q}=S$ and $4\tilde{T}=N$, respectively.

Proof. Let $Q_1, Q_2, \cdots, Q_H$ be a complete set of representatives of the $B_{2k-1,1}(\mathbb{Z})$-equivalence classes in US_{2k}^+. Via Lemma 12 we may set

$$S_i=4\tilde{Q}_i \qquad (1\leq i\leq H)\,.$$

Siegel's formula (Theorem 0.2 in [Ar4]) for the Jacobi Eisenstein series $E_{k,1}^{(n)}(\tau,z)$ implies that

$$E_{k,1}^{(n)}(\tau,z)=\frac{1}{M_{2k-1}}\left(\sum_{j=1}^{H}\frac{\theta_{Q_j}^{(n)}(\tau,z)}{\epsilon(Q_j)}\right).$$

By operating the map $\sigma^{(n)}$ on the both sides of the above identity, we have the identity in the first assertion.

Choose any $Q\in US_{2k}^+$. We have proved in [Ar3] that the Jacobi Eisenstein series $E_{k,1}^{(n)}(\tau,z)$ has the Fourier expansion:

$$E_{k,1}^{(n)}(\tau,z)=\sum_{T\in Sym_{n+1}^*(1;\mathbb{Z}),T\geq 0} e_{k,1}^{(n)}(T)e(\mathrm{tr}(N\tau+{}^t rz))$$

with the expression

$$e_{k,1}^{(n)}(T)=\prod_{p\leq\infty}\alpha_p(Q;T) \qquad \text{for} \quad T>0$$

(see Proposition 4.2, (5.5), Theorem 5.6 and its proof in [Ar3]). From this expression the assertion (ii) immediately follows. q.e.d.

2.3. Köcher-Maass Dirichlet series corresponding to modular forms of half integral weights

Finally we define the Köcher-Maass Dirichlet series attached to modular forms of $M_{k-1/2}^+(\Gamma_0^{(n)}(4))$. Let $f\in M_{k-1/2}^+(\Gamma_0^n(4))$ such that $\sigma^{(n)}(\phi)=f$ with $\phi\in J_{k,1}(\Gamma_n)$, where $\sigma^{(n)}$ is the isomorphism given in Theorem 10. Let

$$f(\tau)=\sum_{N\in S_n(\mathbb{Z}),N\geq 0} a(N)e(\mathrm{tr}(N\tau)) \qquad (\tau\in\mathfrak{H}_n)$$

be a Fourier expansion of f. We set, similarly as in the original case,

$$D_n(f, s) := \sum_{N \in S_n(\mathbb{Z})^+/\sim} \frac{a(N)}{\varepsilon(N)(\det N)^s},$$

where the summation indicates that T runs over all the $GL_n(\mathbb{Z})$-equivalence classes of integral symmetric matrices of $S_n(\mathbb{Z})^+$, $S_n(\mathbb{Z})^+$ denoting the set consisting of positive definite elements of $S_n(\mathbb{Z})$. Then $D_n(f, s)$ is absolutely convergent for $\mathrm{Re}(s) > k + n/2$. Let the Fourier expansion of ϕ be the same as in (1.5). Since $f(\tau) = h(4\tau) = \sum_{r \in L/2L} h_r(4\tau)$, immediately, $a(N) = c(T)$, where $T \in Sym^*_{n+1}(1; \mathbb{Z})$ is chosen so that $4\tilde{T} = N$. At this step note that $\epsilon(T) = \varepsilon(N)$. Similarly as in the proof of Lemma 12, the $B_{n,1}(\mathbb{Z})$-equivalence class in $Sym^*_{n+1}(1; \mathbb{Z})$ corresponds one to one onto the $GL_n(\mathbb{Z})$-equivalence class in $S_n(\mathbb{Z})$ via the map: $T \to 4\tilde{T}$. Thus,

$$D_n(f, s) = 4^{-ns} D_n(\phi, s).$$

The operation of Siegel's Φ-operator on f is defined in a usual manner:

$$(\mathscr{S}f)(\tau) = \lim_{t \to \infty} f\begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix} \qquad (\tau \in \mathfrak{H}_{n-1}).$$

Obviously, $\sigma^{(n-1)}(\mathscr{S}\phi) = \mathscr{S}f = \mathscr{S}(\sigma^{(n)}\phi) \in M^+_{k-1/2}(\Gamma_0^{(n-1)}(4))$.

Set

$$\hat{f}(\tau) = (-1)^{nk/2} 2^{-n/2} \det\left(\frac{\tau}{i}\right)^{-k+1/2} f((-4\tau)^{-1}).$$

Then we have, by Proposition 4,

$$\hat{f}(\tau) = (-1)^{nk/2} 2^{-n/2} \det\left(\frac{\tau}{i}\right)^{-k+1/2} h(-\tau^{-1}) = h_0(\tau),$$

from which we see that $\hat{f}(\tau)$ has the Fourier expansion of the form

$$\hat{f}(\tau) = \sum_{N \in Sym^*_n(\mathbb{Z}), N \geq 0} b(N) e(\mathrm{tr}(N\tau)).$$

Exactly, $b(N) = c\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$. We define the zeta function $D_n(\hat{f}, s)$ by putting

$$D_n(\hat{f}, s) = \sum_{N \in Sym_n(\mathbb{Z})^+/\sim} \frac{b(N)}{\varepsilon(N)(\det N)^s}.$$

Immediately, $D_n(\hat{f}, s) = \hat{D}_n(\phi, s)$. Multiplying some gamma factors, we define

$$\xi_n(f, s) = 4^{ns} \gamma_n(s) D_n(f, s) \quad \text{and} \quad \hat{\xi}_n(f, s) = \gamma_n(s) D_n(\hat{f}, s).$$

Moreover we set

$$f^*(\tau) = h^*(4\tau) \quad \text{and} \quad \hat{f}^*(\tau) = h_0^*(\tau).$$

The following theorem is an immediate consequence of Theorems 8, 9.

THEOREM 15. *The zeta functions $D_n(f, s)$, $D_n(\hat{f}, s)$ can be continued to meromorphic* *functions of s in the whole s plane satisfying the functional equation*

$$\xi_n(f, s) = c_{n,k}\hat{\xi}_n(f, k-1/2-s),$$

where we put $c_{n,k} = (-1)^{nk/2}2^{n/2}$. *Moreover*

$$\xi_n(f, s) = I_n(f, s) + \sum_{j=0}^{n-1} \varepsilon(j)v(n-j)\left(\frac{c_{n,k}\hat{\xi}_j(\mathscr{S}^{n-j}f, \frac{n}{2})}{s-k+\frac{1}{2}+\frac{j}{2}} - \frac{\xi_j(\mathscr{S}^{n-j}f, \frac{n}{2})}{s-\frac{j}{2}}\right),$$

where

$$I_n(f, s) = \int_{\substack{GL_n(\mathbb{Z})\backslash\mathscr{P}_n \\ \det\eta\geq 1}} \{(\det\eta)^s f^*(i\eta/4) + c_{n,k}(\det\eta)^{k-1/2-s}\hat{f}^*(i\eta)\}dv_n(\eta).$$

3. Proof of Theorem 9

In this paragraph we shall give a proof of our Theorem 9. The proof will be done in a manner similar to that of Theorem in [Ar1].

From now on we assume that the index S satisfies the maximality condition (1.11). We set, for simplicity,

$$R_S^{(r)} = L^{(r)}/(2S)L^{(r)} = M_{l,r}(\mathbb{Z})/(2S)M_{l,r}(\mathbb{Z})$$

for each positive integer r. Let $\phi \in J_{k,S}(\Gamma_n)$ and let its Fourier expansion be the same as in (1.5). Moreover let $h(\tau)$ and $h^*(\tau)$ be the same as in (1.9) and (1.16), respectively. If necessary, we write $h(\tau, \phi)$ and $h^*(\tau, \phi)$ in place of $h(\tau)$ and $h^*(\tau)$, respectively. For each integer j with $0 \leq j \leq n$, let $h(\tau)^{(j)}$ be the rank j-part of the Fourier expansion of $h(\tau)$. Namely,

$$h(\tau)^{(j)} = \sum_{r\in R_S^{(n)}} \sum_{\substack{N\in Sym_n^*(\mathbb{Z}) \\ \tilde{T}\geq 0, \mathrm{rank}(\tilde{T})=j}} c(T)e(\mathrm{tr}(\tilde{T}\tau)) = \sum_{\substack{T\in Sym_{n+l}^*(S;\mathbb{Z})/B_{n,l}^{\infty}(\mathbb{Z}) \\ \tilde{T}\geq 0, \mathrm{rank}(T)=j}} c(T)e(\mathrm{tr}(\tilde{T}\tau)).$$

Note that $h^*(\tau) = h(\tau)^{(n)}$. We write, for simplicity,

$$\Delta_n = GL_n(\mathbb{Z})$$

and for each j with $0 \leq j \leq n$ define the subgroup $\Delta_{j,n-j}$ by

$$\Delta_{j,n-j} = \left\{\begin{pmatrix} V & X \\ 0 & W \end{pmatrix} \middle| V\in\Delta_j,\ W\in\Delta_{n-j},\ X\in M_{j,n-j}(\mathbb{Z})\right\}.$$

LEMMA 16. *We have*

$$h(\tau)^{(j)} = \sum_{U\in\Delta_n/\Delta_{j,n-j}} h^*\left(\tau\left[U\begin{pmatrix} E_j \\ 0 \end{pmatrix}\right], \mathscr{S}^{n-j}\phi\right).$$

Proof. Take any $T \in Sym_{n+l}^*(\mathbb{Z})$ with $\tilde{T} \geq 0$ and $\mathrm{rank}(\tilde{T}) = j$. Then T can be uniquely written as

$$T=\begin{pmatrix} U & {}^t\xi \\ 0 & 1_l \end{pmatrix}\begin{pmatrix} N & 0 & {}^tr/2 \\ 0 & 0 & 0 \\ r/2 & 0 & S \end{pmatrix}\begin{pmatrix} {}^tU & 0 \\ \xi & 1_l \end{pmatrix}$$

with $\xi\in L^{(n)}$, $U\in\Delta_n/\Delta_{j,n-j}$, $r\in R_S^{(j)}$, $N\in Sym_j(\mathbb{Z})$, and $N-\frac{1}{4}{}^trS^{-1}r>0$. If $j=0$, we put $T=0$. Since

$$c(T)=c\begin{pmatrix} N & 0 & {}^tr/2 \\ 0 & 0 & 0 \\ r/2 & 0 & S \end{pmatrix}=c\left(\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}, (r, 0)\right)$$

and

$$\tilde{T}=U\begin{pmatrix} N-\frac{1}{4}{}^trS^{-1}r & 0 \\ 0 & 0 \end{pmatrix}{}^tU,$$

we have

$$\begin{aligned} h(\tau)^{(j)}= & \sum_{U\in\Delta_n/\Delta_{j,n-j}}\ \sum_{r\in R_S^{(j)}}\ \sum_{\substack{N\in Sym_j^*(\mathbb{Z}) \\ N-(1/4){}^trS^{-1}r>0}} \\ & c\left(\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}, (r, 0)\right)e\left(\operatorname{tr}\left(\left(N-\frac{1}{4}{}^trS^{-1}r\right)\left(({}^tU\tau U)\begin{bmatrix} 1_j \\ 0 \end{bmatrix}\right)\right)\right) \\ = & \sum_{U\in\Delta_n/\Delta_{j,n-j}} h^*\left(\tau\left[U\begin{pmatrix} E_j \\ 0 \end{pmatrix}\right], \mathscr{S}^{n-j}\phi\right). \end{aligned}$$

q.e.d.

Set, for $0\le r\le n$,

$$G_n^r=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_n \,\middle|\, \operatorname{rank}((0\quad 1_{n-r})c)=n-r\right\},$$

$$G_n^{*r}=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_n \,\middle|\, \operatorname{rank}((0\quad 1_{n-r})c)=\operatorname{rank}((0\quad 1_{n-r})d)=n-r\right\}.$$

Let $\phi\in J_{k,S}^{\mathrm{cusp}}(\Gamma_r)$. Assume k is an even integer with $k>n+r+l+1$. Moreover let $C^{(n)}(\phi, \tau)$ and $C_0^{(n)}(\phi, \tau)$ be the functions given in (1.12), (1.13). Denote by $C^{(n)}(\phi, \tau)^*$ and $C_0^{(n)}(\phi, \tau)^*$ the non-singular part of $C^{(n)}(\phi, \tau)$ and $C_0^{(n)}(\phi, \tau)$, respectively (see (1.8) and (1.16)). The following expressions of $C^{(n)}(\phi, \tau)^*$ and $C_0^{(n)}(\phi, \tau)^*$ correspond to those of $C^{(n)}(\phi, \tau)$ and $C_0^{(n)}(\phi, \tau)$ in Proposition 6.

PROPOSITION 17. *We have*

$$C^{(n)}(\phi, \tau)^*=\sum_{M\in\Gamma_{n,r}\backslash G_n^r}\ \sum_{\lambda\in R_S^{(r)}}\ \sum_{\mu\in R_S^{(n)}} \det J(M, \tau)^{-k}h_\lambda(M\langle\tau\rangle^*)j_S^{(n)}(M, \tau)_{\tilde{\lambda}\mu}$$

and

$$C_0^{(n)}(\phi, \tau)^* = \sum_{M \in \Gamma_{n,r} \backslash G_n^r} \sum_{\lambda \in R_S^{(r)}} \det J(M, \tau)^{-k} h_\lambda(M\langle\tau\rangle^*) j_S^{(n)}(M, \tau)_{\tilde{\lambda}0} .$$

Proof. If $r=n$, then

$$C^{(n)}(\phi, \tau) = C^{(n)}(\phi, \tau)^* = \sum_{\lambda \in R_S^{(n)}} h_\lambda(\tau) .$$

In this case the first identity holds true. We prove by induction on n. If $n=1$ and $r=0$, then, ϕ is a constant function. Let $\phi=1$. Then,

$$C^{(1)}(1, \tau) = \sum_{M \in \Gamma_{1,0} \backslash \Gamma_1} \sum_{\mu \in R_S^{(1)}} \det J(M, \tau)^{-k} j_S^{(1)}(M, \tau)_{0\mu} ,$$

from which we easily have the first identity for $n=1$, $r=0$. Let $\phi \in J_{k,S}^{\mathrm{cusp}}(\Gamma_r)$ be fixed. We assume that the first identity for $C^{(j)}(\phi, \tau)^*$ is true for any j with $r \le j < n$. Let $C^{(n)}(\phi, \tau)^{(j)}$ denote the rank j-part of $C^{(n)}(\phi, \tau)$. Lemma 16 and the induction assumption imply that

$$C^{(n)}(\phi, \tau)^{(j)} = \sum_{U \in \Delta_n \backslash \Delta_{j,n-j}} C^{(j)}\left(\phi, \tau\left[U\begin{pmatrix} 1_j \\ 0 \end{pmatrix}\right]\right)^*$$

$$= \sum_{U \in \Delta_n / \Delta_{j,n-j}} \sum_{M \in \Gamma_{j,r} \backslash G_j^r} \sum_{\lambda \in R_S^{(r)}} \sum_{\mu \in R_S^{(j)}} \det J(M, \hat{\tau})^{-k} h_\lambda(M\langle\hat{\tau}\rangle^*) j_S^{(j)}(M, \hat{\tau})_{\tilde{\lambda}\mu} ,$$

where we put $\hat{\tau} = \tau\left[U\begin{pmatrix} 1_j \\ 0 \end{pmatrix}\right]$ and $\tilde{\lambda} = (\lambda, 0) \in R_S^{(j)}$. Thus we have, with the help of Lemma 3, (iii),

$$\sum_{\lambda \in R_S^{(r)}} \sum_{\mu \in R_S^{(j)}} \det J(M, \hat{\tau})^{-k} h_\lambda(M\langle\hat{\tau}\rangle^*) j_S^{(j)}(M, \hat{\tau})_{\tilde{\lambda}\mu}$$

$$= \sum_{\lambda \in R_S^{(r)}} \sum_{\mu \in R_S^{(n)}} \det J(l_j^n(M), {}^tU\tau U)^{-k} h_\lambda((l_j^n(M)\langle {}^tU\tau U\rangle)^*) j_S^{(n)}(l_j^n(M), {}^tU\tau U)_{(\lambda,0),\mu}$$

Therefore by a little more computation with (1.6) and Lemma 3, (i),

$$C^{(n)}(\phi, \tau)^{(j)} = \sum_{M \in \Gamma_{j,r} \backslash G_j^r} \sum_{U \in \Delta_n / \Delta_{j,n-j}} \det J(l_j^n(M) d_n({}^tU), \tau)^{-k}$$

$$\times \sum_{\lambda \in R_S^{(r)}} \sum_{\mu \in R_S^{(n)}} h_\lambda((l_j^n(M) d_n({}^tU)\langle\tau\rangle)^*) j_S^{(n)}(l_j^n(M) d_n({}^tU), \tau)_{(\lambda,0),\mu}$$

Set

$$G_n^{r,j} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n \,\middle|\, \mathrm{rank}((0 \quad 1_{n-r})c) = j - r \right\} .$$

Then by Lemma 2 of [Ar1],

$$l_j^n(M)\begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix} \qquad \text{with} \quad M \in \Gamma_{j,r} \backslash G_j^r \quad \text{and} \quad U \in \Delta_n/\Delta_{j,n-j}$$

give rise to a complete set of representatives of the cosets $\Gamma_{n,r} \backslash G_n^{r,j}$. Since

$$G_n^r \cup \left(\bigcup_{j=r}^{n-1} \Gamma_{n,r} \backslash G_n^{r,j} \right) \qquad \text{(disjoint union)}$$

gives a complete set of representatives of the cosets $\Gamma_{n,r} \backslash \Gamma_n$ and

$$C^{(n)}(\phi, \tau)^* = C^{(n)}(\phi, \tau) - \sum_{j=r}^{n-1} C^{(n)}(\phi, \tau)^{(j)},$$

we obtain the first identity. The second one is similarly verified. q.e.d.

Let $\phi \in J_{k,S}^{\mathrm{cusp}}(\Gamma_r)$. Taking the expressions in Proposition 17 into account, we define the functions $P_{n,r}^k(\tau, \phi)$, $Q_{n,r}^k(\tau, \phi)$ as follows:

$$P_{n,r}^k(\tau, \phi) = \sum_{M \in \Gamma_{n,r} \backslash G_n^{*r}} \sum_{\lambda \in R_S^{(r)}} \sum_{\mu \in R_S^{(n)}} \det J(M, \tau)^{-k} h_\lambda(M\langle\tau\rangle^*) j_S(M, \tau)_{\tilde{\lambda}\mu},$$

$$Q_{n,r}^k(\tau, \phi) = \sum_{M \in \Gamma_{n,r} \backslash G_n^{*r}} \sum_{\lambda \in R_S^{(r)}} \det J(M, \tau)^{-k} h_\lambda(M\langle\tau\rangle^*) j_S(M, \tau)_{\tilde{\lambda}0}.$$

If $r = n$, we have

$$P_{n,n}^k(\tau, \phi) = h(\tau) = h^*(\tau), \quad Q_{n,n}^k(\tau, \phi) = h_0(\tau) = h_0^*(\tau).$$

Then we have the transformation formula for exchanging $\tau \to -\tau^{-1}$.

Proposition 18. *Let $0 \le r \le n$ and $\phi \in J_{k,S}^{\mathrm{cusp}}(\Gamma_r)$. Then,*

$$P_{n,r}^k(-\tau^{-1}, \phi) = c_{n,k,S} \det\left(\frac{\tau}{i}\right)^{k-l/2} Q_{n,r}^k(\tau, \phi).$$

Proof. It follows from the definition that

$$P_{n,r}^k(-\tau^{-1}, \phi) = \det J(J_n, \tau)^k$$
$$\times \sum_{M \in \Gamma_{n,r} \backslash G_n^{*r}} \sum_{\lambda \in R_S^{(r)}} \sum_{\mu \in R_S^{(n)}} \det J(MJ_n, \tau)^{-k} h_\lambda((MJ_n\langle\tau\rangle)^*) j_S(M, J_n\langle\tau\rangle)_{\tilde{\lambda}\mu}.$$

Note that

$$j_S(M, J_n\langle\tau\rangle) = j_S(MJ_n, \tau) j_S(J_n, \tau)^{-1}.$$

Since Lemma 3 implies that

$$j_S(J_n, \tau)^{-1} = \det\left(\frac{\tau}{i}\right)^{-l/2} \det(2S)^{-n/2} \left(e\left(\frac{1}{2}\mathrm{tr}({}^t\lambda S^{-1}\mu)\right) \right)_{\lambda, \mu \in R_S^{(n)}},$$

we have, by a little computation,

$$\sum_{\mu\in R_S^{(n)}} j_S(M, J_n\langle\tau\rangle)_{\tilde{\lambda},\mu} = \det(2S)^{n/2}\det\left(\frac{\tau}{i}\right)^{-l/2} j_S(MJ_n, \tau)_{\tilde{\lambda},0}\,.$$

Thus using the fact $G_n^{*r}J_n = G_n^{*r}$, we have the transformation formula. q.e.d.

Then the functions $C^{(n)}(\phi, \tau)^*$, $C_0^{(n)}(\phi, \tau)^*$ can be expressed in terms of the functions $P_{n,r}^k$, $Q_{n,r}^k$.

PROPOSITION 19. *Let* $0 \le r \le n-1$ *and* $\phi \in J_{k,S}^{\mathrm{cusp}}(\Gamma_r)$. *Then,*

$$C^{(n)}(\phi, \tau)^* = P_{n,r}^k(\tau, \phi)$$
$$+ c_{n,k,S}\det\left(\frac{\tau}{i}\right)^{-k+l/2}\sum_{j=r}^{n-1}\sum_{U\in\Delta_n/\Delta_{n-j,j}} Q_{j,r}^k\left(-(\tau[U])^{-1}\begin{bmatrix}0\\1_j\end{bmatrix}, \phi\right),$$

$$C_0^{(n)}(\phi, \tau)^* = Q_{n,r}^k(\tau, \phi)$$
$$+ c_{n,k,S}^{-1}\det\left(\frac{\tau}{i}\right)^{-k+l/2}\sum_{j=r}^{n-1}\sum_{U\in\Delta_n/\Delta_{j,n-j}} P_{j,r}^k\left(((-\tau)^{-1}[U])\begin{bmatrix}1_j\\0\end{bmatrix}, \phi\right).$$

Proof. For any subset $\mathfrak{G}$ of Γ_n, set $\mathfrak{G}J_n = \{MJ_n \mid M\in\mathfrak{G}\}$. On the right hand side of the first identity in Proposition 17 we replace M with MJ_n. Then we have

$$C^{(n)}(\phi, \tau)^* = \sum_{M\in\Gamma_{n,r}\backslash G_n^rJ_n}\sum_{\lambda\in R_S^{(r)}}\sum_{\mu\in R_S^{(n)}} \det J(MJ_n, \tau)^{-k} h_\lambda((MJ_n\langle\tau\rangle)^*) j_S(MJ_n, \tau)_{\tilde{\lambda}\mu}\,.$$

We know from Lemma 3 that

$$\sum_{\lambda\in R_S^{(n)}} j_S(MJ_n, \tau)_{\tilde{\lambda}\mu} = \det(2S)^{n/2}\det\left(\frac{\tau}{i}\right)^{l/2} j_S(M, J_n\langle\tau\rangle)_{\tilde{\lambda}0}\,.$$

Set, for each j with $r \le j \le n-1$,

$$\mathfrak{R}_n^{r,j} = \left\{ l_j^n(M)\begin{pmatrix}{}^tU & 0\\ 0 & U^{-1}\end{pmatrix} \middle| M\in\Gamma_{j,r}\backslash G_j^{*r},\ U\in\Delta_n/\Delta_{j,n-j}\right\}.$$

Lemma 4 of [Ar1] asserts that the disjoint union

$$(\Gamma_{n,r}\backslash G_n^{*r}) \cup \left(\bigcup_{j=r}^{n-1}\mathfrak{R}_n^{r,j}\right)$$

gives rise to a complete set of representatives of the cosets $\Gamma_{n,r}\backslash G_n^rJ_n$. Therefore,

$$C^{(n)}(\phi, \tau)^*$$
$$= \sum_{M\in\Gamma_{n,r}\backslash G_n^{*r}}\sum_{\lambda\in R_S^{(r)}}\sum_{\mu\in R_S^{(n)}} \det J(MJ_n, \tau)^{-k} h_\lambda((MJ_n\langle\tau\rangle)^*) j_S^{(n)}(MJ_n, \tau)_{\tilde{\lambda}\mu}\,.$$
$$+ c_{n,k,S}\det\left(\frac{\tau}{i}\right)^{-k+l/2}$$
$$\times\sum_{j=r}^{n-1}\sum_{M\in\mathfrak{R}_n^{r,j}}\sum_{\lambda\in R_S^{(r)}} \det J(M, -\tau^{-1})^{-k} h_\lambda((M\langle-\tau^{-1}\rangle)^*) j_S(M, -\tau^{-1})_{\tilde{\lambda}0}\,,$$

from which the first identity of the proposition follows. The second one is quite similarly verified. q.e.d.

Let $\phi \in J_{k,S}(\Gamma_n)$. By the structure theorem (Propositon 5) of the space $J_{k,S}(\Gamma_n)$, ϕ is written in a linear combination of the Klingen Eisenstein series:

$$\phi(\tau, z) = \sum_{r=0}^{n} E_{n,r}^{k,S}(\phi_r, (\tau, z)) \tag{3.1}$$

with some $\phi_r \in J_{k,S}^{\mathrm{cusp}}(\Gamma_r)$. This expression is uniquely determined by ϕ. Set

$$P_n(\tau, \phi) = \sum_{r=0}^{n} P_{n,r}^{k}(\tau, \phi_r) \quad \text{and} \quad Q_n(\tau, \phi) = \sum_{r=0}^{n} Q_{n,r}^{k}(\tau, \phi_r) .$$

In virtue of Proposition 18, 19 these functions satisfy the following properties.

PROPOSITION 20. *Assume $k > 2n + l + 1$ and k is even. Let $\phi \in J_{k,S}(\Gamma_n)$. Then,*

$$P_n(-\tau^{-1}, \phi) = c_{n,k,S} \det\left(\frac{\tau}{i}\right)^{k-l/2} Q_n(\tau, \phi) .$$

Moreover,

$$h^*(\tau) = P_n(\tau, \phi) + c \cdot \det\left(\frac{\tau}{i}\right)^{-k+l/2} \sum_{j=0}^{n-1} \sum_{U \in \Delta_n/\Delta_{j,n-j}} Q_j\left(((-\tau)^{-1}[U])\begin{bmatrix} 1_j \\ 0 \end{bmatrix}, \mathscr{S}^{n-j}\phi\right),$$

$$h_0^*(\tau) = Q_n(\tau, \phi) + c^{-1} \cdot \det\left(\frac{\tau}{i}\right)^{-k+l/2} \sum_{j=0}^{n-1} \sum_{U \in \Delta_n/\Delta_{n-j,j}} P_j\left(-(\tau[U])^{-1}\begin{bmatrix} 0 \\ 1_j \end{bmatrix}, \mathscr{S}^{n-j}\phi\right),$$

where $h_0^*(\tau)$, $h^*(\tau)$ are the functions determined by (1.8), (1.16) *respectively from ϕ and we put, for simplicity, $c = c_{n,k,S}$.*

REMARK. We note here that, for any function F on $\mathfrak{H}_j$.

$$\sum_{U \in \Delta_n/\Delta_{j,n-j}} F\left(((-\tau)^{-1}[U])\begin{bmatrix} 1_j \\ 0 \end{bmatrix}\right) = \sum_{U \in \Delta_n/\Delta_{n-j,j}} F\left(-(\tau[U])^{-1}\begin{bmatrix} 0 \\ 1_j \end{bmatrix}\right),$$

if the concerned infinite series is absolutely convergent.

Proof. The first transformation formula is immediately derived from Proposition 18. We write ϕ in the form of (3.1). Then

$$h(\tau) = \sum_{r=0}^{n} C^{(n)}(\phi_r, \tau) \quad \text{and} \quad h^*(\tau) = \sum_{r=0}^{n} C^{(n)}(\phi_r, \tau)^* .$$

Note by the property of the operator $\mathscr{S}$ (Proposition 5) that

$$(\mathscr{S}^{n-j}\phi)(\tau, z) = \sum_{r=0}^{j} E_{j,r}^{k,S}(\phi_r, (\tau, z)) \qquad ((\tau, z) \in \mathscr{D}_{j,l}) .$$

Then by Proposition 19 and the definition of $Q_j(*, \mathscr{S}^{n-j}\phi)$,

$$h^*(\tau)=\sum_{r=0}^{n} P_{n,r}^k(\tau,\phi_r)+c\cdot\det\left(\frac{\tau}{i}\right)^{-k+l/2}\sum_{r=0}^{n-1}\sum_{j=r}^{n-1}\sum_{U} Q_{j,r}^k\left(-(\tau[U])^{-1}\begin{bmatrix}0\\1_j\end{bmatrix},\phi_r\right)$$
$$=P_n(\tau,\phi)+c\cdot\det\left(\frac{\tau}{i}\right)^{-k+l/2}\sum_{j=0}^{n-1}\sum_{U}\sum_{r=0}^{j} Q_{j,r}^k\left(-(\tau[U])^{-1}\begin{bmatrix}0\\1_j\end{bmatrix},\phi_r\right)$$
$$=P_n(\tau,\phi)+c\cdot\det\left(\frac{\tau}{i}\right)^{-k+l/2}\sum_{j=0}^{n-1}\sum_{U} Q_j\left(-(\tau[U])^{-1}\begin{bmatrix}0\\1_j\end{bmatrix},\mathscr{S}^{n-j}\phi\right),$$

where the summation with respect to U indicates that U runs through all representatives of $\Delta_n/\Delta_{n-j,j}$. Thus we obtain the formula for $h^*(\tau)$. The last one is similarly verified. q.e.d.

Finally we reformulate Theorem 9 in a convenient form and give it a proof.

THEOREM 21. *Assume that the index S satisfies the maximality condition* (1.11) *and that k is an even integer with $k>2n+l+1$. Let $\phi\in J_{k,S}(\Gamma_n)$. Then the functions $\xi_n(\phi,s)$, $\hat{\xi}_n(\phi,s)$ can be continued analytically to meromorphic functions in the whole s-plane and have the integral expressions in the vertical strip $\frac{n-1}{2}<\mathrm{Re}(s)<k-\frac{l}{2}-\frac{n-1}{2}$:*

$$\xi_n(\phi,s)=\int_{\Delta_n\backslash\mathscr{P}_n}(\det\eta)^s P_n(i\eta,\phi)dv_n(\eta),$$
$$\hat{\xi}_n(\phi,s)=\int_{\Delta_n\backslash\mathscr{P}_n}(\det\eta)^s Q_n(i\eta,\phi)dv_n(\eta),$$

where the integrals are absolutely convergent in the same strip. Moreover, the expressions (1.17) *and* (1.18) *for $\xi_n(\phi,s)$ and $\hat{\xi}_n(\phi,s)$ are valid for any $s\in\mathbb{C}$.*

Proof. The proof is done by induction on n. If $n=1$, then the assertion can be shown in the same manner as in the original elliptic modular case of Hecke. Assume that the assertions are valid for any Jacobi forms of degree less than n. In a usual manner we have

$$\xi_n(\phi,s)=\int_{\substack{\Delta_n\backslash\mathscr{P}_n\\ \det\eta\geq 1}} h^*(i\eta)(\det\eta)^s dv_n(\eta)+\int_{\substack{\Delta_n\backslash\mathscr{P}_n\\ \det\eta\geq 1}} h^*(i\eta^{-1})(\det\eta)^{-s}dv_n(\eta)$$
$$=I_n(\phi,s)+\int_{\substack{\Delta_n\backslash\mathscr{P}_n\\ \det\eta\geq 1}}\{h^*(i\eta^{-1})-c_{n,k,S}(\det\eta)^{k-l/2}h_0^*(i\eta)\}(\det\eta)^{-s}dv_n(\eta).$$

By using Proposition 20, the difference $h^*(i\eta^{-1})-c_{n,k,S}(\det\eta)^{k-l/2}h_0^*(i\eta)$ in the last integral is equal to

$$c_{n,k,S}(\det\eta)^{k-l/2}\sum_{j=0}^{n-1}\sum_{U\in\Delta_n/\Delta_{j,n-j}} Q_j\left(i(\eta[U])\begin{bmatrix}1_j\\0\end{bmatrix},\mathscr{S}^{n-j}\phi\right)$$
$$-\sum_{j=0}^{n-1}\sum_{U\in\Delta_n/\Delta_{n-j,j}} P_j\left(i(\eta[U])^{-1}\begin{bmatrix}0\\1_j\end{bmatrix},\mathscr{S}^{n-j}\phi\right).$$

Quite in the same manner as in the proof of Lemma 5 in [Ar1], we can evaluate the explicit values of the concerned integrals under the induction assumption.

LEMMA 22. *Let j be any integer with* $0 \leq j \leq n-1$.
(i) *If* $\mathrm{Re}(s) > j/2$, *then*

$$\int_{\substack{\Delta_n \backslash \mathscr{P}_n \\ \det\eta \geq 1}} (\det\eta)^{-s} \sum_{U \in \Delta_n/\Delta_{n-j,j}} P_j\left(i(\eta[U])^{-1}\begin{bmatrix} 0 \\ 1_j \end{bmatrix}, \mathscr{S}^{n-j}\phi \right) dv_n(\eta)$$
$$= \varepsilon(j)v(n-j)\frac{1}{s-j/2}\xi_j\left(\mathscr{S}^{n-j}\phi, \frac{n}{2}\right)$$

and

$$\int_{\substack{\Delta_n \backslash \mathscr{P}_n \\ \det\eta \geq 1}} (\det\eta)^{-s} \sum_{U \in \Delta_n/\Delta_{n-j,j}} Q_j\left(i(\eta[U])^{-1}\begin{bmatrix} 0 \\ 1_j \end{bmatrix}, \mathscr{S}^{n-j}\phi \right) dv_n(\eta)$$
$$= \varepsilon(j)v(n-j)\frac{1}{s-j/2}\hat{\xi}_j\left(\mathscr{S}^{n-j}\phi, \frac{n}{2}\right).$$

(ii) *If* $\mathrm{Re}(s) > -j/2$, *then*

$$\int_{\substack{\Delta_n \backslash \mathscr{P}_n \\ \det\eta \geq 1}} (\det\eta)^{-s} \sum_{U \in \Delta_n/\Delta_{j,n-j}} Q_j\left(i(\eta[U])\begin{bmatrix} 1_j \\ 0 \end{bmatrix}, \mathscr{S}^{n-j}\phi \right) dv_n(\eta)$$
$$= \varepsilon(j)v(n-j)\frac{1}{s+j/2}\hat{\xi}_j\left(\mathscr{S}^{n-j}\phi, \frac{n}{2}\right)$$

and

$$\int_{\substack{\Delta_n \backslash \mathscr{P}_n \\ \det\eta \geq 1}} (\det\eta)^{-s} \sum_{U \in \Delta_n/\Delta_{j,n-j}} P_j\left(i(\eta[U])\begin{bmatrix} 1_j \\ 0 \end{bmatrix}, \mathscr{S}^{n-j}\phi \right) dv_n(\eta)$$
$$= \varepsilon(j)v(n-j)\frac{1}{s+j/2}\xi_j\left(\mathscr{S}^{n-j}\phi, \frac{n}{2}\right).$$

Thus using Lemma 22, under the induction assumption we have, for $\mathrm{Re}(s) > k - l/2$, the identity (1.17) and quite similarly (1.18). On the other hand we see from Proposition 20 and Lemma 22 that, if $\frac{n-1}{2} < \mathrm{Re}(s) < k - \frac{l}{2} - \frac{n-1}{2}$,

$$\int_{\Delta_n \backslash \mathscr{P}_n} (\det\eta)^s P_n(i\eta, \phi) dv_n(\eta)$$
$$= \int_{\substack{\Delta_n \backslash \mathscr{P}_n \\ \det\eta \geq 1}} \{(\det\eta)^s P_n(i\eta, \phi) + c_{n,k,S}(\det\eta)^{k-l/2-s} Q_n(i\eta, \phi)\} dv_n(\eta)$$
$$= I_n(\phi, s) + \sum_{j=0}^{n-1} \varepsilon(j)v(n-j)\left(\frac{c_{n,k,S}\hat{\xi}_j(\mathscr{S}^{n-j}\phi, \frac{n}{2})}{s-k+\frac{l}{2}+\frac{j}{2}} - \frac{\xi_j(\mathscr{S}^{n-j}\phi, \frac{n}{2})}{s-\frac{j}{2}} \right)$$
$$= \xi_n(\phi, s).$$

Quite similarly we have the integral expression for $\hat{\xi}_n(\phi, s)$ in the strip $\frac{n-1}{2} < \mathrm{Re}(s) < k - \frac{l}{2} - \frac{n-1}{2}$. Thus we have completed the proof. q.e.d.

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